E C O N O M I C S B U L L E T I N

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M. Hossein Partovi
California State University, Sacramento

Michael Caputo
University of Central Florida, Orlando

Abstract

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Principal Portfolios: Recasting the Efficient Frontier

M. Hossein Partovi

Department of Physics and Astronomy,
California State University, Sacramento, California 95819-6041

Michael R. Caputo

Department of Economics, University of Central Florida, Orlando, Florida 32816-1400

Abstract

A new method of analyzing the efficient portfolio problem under the assumption that short sales are allowed is presented. It is based on the remarkable finding that the original asset set can be re-organized as a set of *uncorrelated* portfolios, here named *principal portfolios.* The original problem of portfolio selection from the existing, *correlated* assets is thereby traded for the reduced problem of choosing from a set of *uncorrelated* portfolios. These portfolios constitute a new investment environment of uncorrelated assets, thereby providing significant conceptual and practical simplification in any portfolio optimization process such as the determination of the efficient frontier. The principal portfolio analysis of the efficient frontier reveals new features of the volatility structure of the optimal portfolios.

1. Introduction

About half a century ago, Markowitz (1952, 1959) ushered in the modern era of portfolio theory when he introduced the mean-variance formulation of the efficient portfolio problem. His approach was based on the assumption that short-time changes in risky asset prices are best described as random fluctuations, an idea that goes back at least as far as Bachelier (1900). Markowitz treated risky asset prices as normally distributed, correlated random variables, and characterized each asset price by a mean value and a variance, and the mutual interaction of each pair of assets by its covariance, hence the designation "mean-variance" formulation. The addition of a riskless asset to the picture by Tobin (1958) a few years later completed the formulation of the basic model and produced a simple and intuitively appealing solution to the portfolio selection problem. However, a practical implementation of this solution for a set of N risky assets required an input set consisting of N estimated rates of return together with $N(N+1)/2$ estimated values for the corresponding variances and correlation coefficients, as well as a numerical inversion of the resulting $N \times N$ covariance matrix for the asset set. In response to these difficulties, a simplified model of stock price movements, namely the single-index model, was subsequently proposed by Sharpe (1963). This model significantly reduces the estimation and computation work involved in finding the efficient frontier, and in practice yields results that are actually better than one might have expected. Consequently, it has come to be used widely as a tool of portfolio analysis, with its jargon routinely employed in discussions of investment strategies. The simplification achieved by the introduction of this model, on the other hand, raised the question of the nature and economic significance of the single index which occurs in the model and on which all asset prices are anchored. The answer came in the celebrated work of Sharpe (1964), Lintner (1965), and Mossin (1966), the standard CAPM model, in which the requirement of equilibrium in the capital markets was invoked to identify the said index as derived from the market portfolio and the prevailing prices. An analytic derivation of the efficient frontier, developed by Merton (1972), provided a detailed verification of the properties of efficient portfolios. These foundational developments in portfolio theory, as well as many others such as option pricing theory and its derivatives which have followed since (Ingersoll, 1987), were either direct outgrowths of Markowitz' original work or inspired by it.

The purpose of this communication is to present a reformulation of the efficient portfolio problem which greatly simplifies its structure and leads to a conceptually more transparent solution procedure as well as a calculationally simpler problem. It is based on the fact that portfolio analysis would be greatly simplified if asset prices were uncorrelated. While actual market assets such as stocks or mutual fund shares are in fact significantly correlated, one can inquire whether certain mixtures of them might be completely uncorrelated. If this possibility could be realized, one would simply trade the original problem of stock selection from the existing, *correlated assets* for the much simpler problem of choosing from a set of *uncorrelated portfolios*. Indeed for all intents and purposes, these uncorrelated portfolios would function as a new, totally uncorrelated asset set. We shall refer to such special mixtures of the original set as *principal portfolios*. Remarkably, it turns out that when short sales are allowed, it is always possible to reorganize the entire asset set as an equivalent set of principal portfolios. Mathematically, this reorganization amounts to a change of basis from the initial asset set to the set of principal portfolios, with the collection of all possible asset mixtures constituting the underlying linear space. A standard result in linear algebra then guarantees that the new basis vectors can always be so chosen as to diagonalize the covariance matrix, i.e., eliminate all correlations. Thus the new basis greatly simplifies the general problem of portfolio optimization. An example is Eq. (9) which demonstrates the volatility reduction property of the efficient frontier as the number of assets is increased.

A second point worth emphasizing here is that a typical principal portfolio is in general leveraged[1] as well as hedged when short sales are allowed. A measure of the volatility reduction feature of principal portfolios can be gleaned from the following qualitative argument. We will show below that *the mean weighted variance of the principal portfolios is equal to the mean variance of the original asset set.* This implies that the variance of a typical principal portfolio is about the same as that of a single asset in the original set. Since all *covariance* terms ($N^2 - N$ in number) among the principal portfolios vanish, it follows that a major reduction of price volatility is already achieved by these portfolios without any optimization. It is important to realize that hedging and leveraging are not the defining or intended characteristics of principal portfolios but simply properties that are concomitant with the elimination of correlations.

2. Casting Assets as Principal Portfolios

To motivate the transformation from the market assets to principal portfolios, let us start with the standard portfolio selection problem essentially as set forth by Markowitz in 1952. Consider the problem of selecting an efficient portfolio from a set of N risky financial assets s_i under idealized market conditions, including the possibility of a riskless asset and short sales. Treating returns as random variables, the model seeks to find the portfolio which achieves a prescribed level of expected return with a minimum variance. It can thus be formulated as the optimization problem (Markowitz 1952, 1959; see also Fama and Miller 1972, Elton and Gruber 1991)

$$\min_{\mathbf{x}} \mathbf{x}^\dagger \sigma \mathbf{x} \;\; s.t. \;\; 1 = \sum_{i=1}^{N} x_i \text{ and } \mathcal{R} = \mathbf{r}^\dagger \mathbf{x}, \tag{1}$$

where $-\infty < x_i < +\infty$ represents the fraction of the total investment allocated to asset s_i, σ stands for the covariance matrix of the asset set, r_i represents the expected rate of return for asset s_i, $\mathcal{R}$ is the prescribed expected rate of return for the portfolio, and "$\dagger$" signifies matrix transposition. The possibility of short sales implies that the vector $\mathbf{x}$ is unrestricted in the sign and magnitude of its components, being only subject to the pair of constraints in Eq. (1). Note that we have adopted here the unrestricted interpretation of short sales according to which the proceeds from the sale are immediately available for investment. The problem posed in Eq. (1) and variants of it have been extensively analyzed by various methods during the half century since its introduction. Our purpose here is to solve this problem in two steps: first, we shall reorganize the asset set into principal portfolios, then we shall select the optimal portfolio from the set of principal portfolios.

That the first step in the above program is actually implementable simply follows from the fact that any real, symmetric, $N \times N$ matrix is orthogonally diagonalizable and possesses a complete set of orthogonal eigenvectors. Thus we are assured that σ admits a set of N orthogonal eigenvectors $\mathbf{e}^\mu$, $\mu = 1, 2, \ldots, N$, so that $\sigma \mathbf{e}^\mu = v_\mu^2 \mathbf{e}^\mu$, where $v_\mu^2 \geq 0$ are the

[1] We use the term "leveraged" here to imply that the portfolio contains borrowed assets, e.g., short-sold positions.

eigenvalues of the covariance matrix. With no loss in generality, we will take these vectors to be of unit length, so that $\mathbf{e}^{\mu\dagger}\mathbf{e}^{\nu} = \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Kronecker symbol. The covariance matrix itself can then be represented in terms of its eigenvalues and eigenvectors in the form $\sigma_{ij} = \sum_{\mu=1}^{N} v_\mu^2 e_i^\mu e_j^\mu$, according to a well-known result in linear algebra. The principal portfolios are now defined by $S_\mu \stackrel{\text{def}}{=} \sum_{i=1}^{N} e_i^\mu s_i / W_\mu$, i.e., the principal portfolio S_μ is defined to be a mix which contains an amount e_i^μ / W_μ of asset s_i, where $W_\mu \stackrel{\text{def}}{=} \sum_{i=1}^{N} e_i^\mu$. Note that W_μ represents the relative investment weight of the μth eigenvector, a quantity which is in general different from unity and may even be negative if asset (short) sales dominate asset purchases in the mix that constitutes the portfolio.

In order to avoid dealing with negative weights and at the same time remove the remaining arbitrariness[2] in the definition of $\mathbf{e}^\mu$, we shall stipulate that the latter be chosen so that $W_\mu \geq 0$. Observe that $W_\mu = 0$ implies that purchases and (short) sales are equal in the portfolio, i.e., the portfolio is totally leveraged so that in any given portfolio asset purchases are financed by the proceeds from the short sales. If N is sufficiently large, one would expect a typical W_μ to be of the order of $N^{\frac{1}{2}}$ for portfolios that solely consist of purchased assets. To see the basis of this estimate, recall that the condition $\mathbf{e}^{\mu\dagger}\mathbf{e}^{\nu} = \delta_{\mu\nu}$ stipulated above implies that $1 = \sum_{i=1}^{N} (e_i^\mu)^2$. Using the latter and the self-evident inequality $\sum_{i=1}^{N} (e_i^\mu - W_\mu/N)^2 \geq 0$, we find that $W_\mu \leq N^{\frac{1}{2}}$, where the upper limit of $N^{\frac{1}{2}}$ is reached for a uniformly allocated portfolio with $e_i^\mu = N^{-\frac{1}{2}}$. This inequality, together with the condition $W_\mu \geq 0$ stipulated above, in turn implies that $0 \leq W_\mu \leq N^{\frac{1}{2}}$, hence the above-stated estimate for the case where all contributions to W_μ are positive.

Returning to principal portfolios, we note that each is characterized by an expected rate of return $R_\mu = \sum_{i=1}^{N} e_i^\mu r_i / W_\mu$ and a variance ${V_\mu}^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} e_i^\mu \sigma_{ij} e_j^\mu / {W_\mu}^2 = v_\mu^2 / {W_\mu}^2$, to be referred to as *expected principal rate of return* and *principal variance*, respectively. Note that the expected principal rate of return is unrestricted in magnitude and sign, so that a negative R_μ is entirely possible. A general portfolio, initially specified by x_i, is now described by X_μ, which, by definition, is the fraction of the total investment allocated to principal portfolio S_μ. Having chosen the relative weights to be non-negative, we are assured that a positive X_μ corresponds to a net purchase of assets, whereas a negative one designates a net sale.

Clearly, the allocation vectors $\mathbf{x}$ and $\mathbf{X}$ are fully equivalent and related by the following change of basis in the linear space of allocation vectors: $x_i = \sum_{\mu=1}^{N} e_i^\mu \tilde{X}_\mu$ and $\tilde{X}_\mu = \sum_{i=1}^{N} e_i^\mu x_i$, where $\tilde{X}_\mu \stackrel{\text{def}}{=} X_\mu / W_\mu$. The pair of constraints $1 - \sum_{i=1}^{N} x_i = 0$ and $\mathcal{R} - \mathbf{r}^\dagger \mathbf{x} = 0$, on the other hand, are transformed into $1 - \sum_{\mu=1}^{N} X_\mu = 0$ and $\mathcal{R} - \mathbf{R}^\dagger \mathbf{X} = 0$, respectively. Moreover, observe that matrices σ and V have equal traces by virtue of being similar, and furthermore that the sum of the variances of the original asset set, which is given by $\mathrm{tr}(\sigma) \stackrel{\text{def}}{=} \sum_{i=1}^{N} \sigma_{ii}$, is equal to the sum of the eigenvalues of the original covariance matrix, namely $\sum_{\mu=1}^{N} v_\mu^2$. Using the latter equality, we arrive at the result that $N^{-1} \sum_{\mu=1}^{N} {W_\mu}^2 {V_\mu}^2 = N^{-1} \sum_{i=1}^{N} \sigma_{ii}$, i.e., the weighted average of the principal variances is equal to the mean of the original variances, as noted earlier.

We may summarize the above transformation by saying that the original *investment environment* $\{s_i, r_i, \sigma_{ij}\}_{i,j=1}^{N}$ has been transformed into the equivalent, *principal environment*

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[2] If the spectrum of the covariance matrix is nondegenerate, then the definition given here uniquely determines the principal portfolios. Otherwise, one can arbitrarily fix these portfolios, subject to the stipulated conditions.

$\{S_\mu, R_\mu, \mathsf{V}_{\mu\nu}\}_{\mu,\nu=1}^N$ where the transformed covariance matrix V has a diagonal form.

A riskless asset, if present, would constitute a principal portfolio by itself, with a vanishing variance and a certain return rate of return. As is well known, the efficient frontier in the presence of a riskless asset is a simple linear combination of the latter and a particular efficient portfolio formed of risky assets. Accordingly, we will continue dealing with risky assets, excluding riskless assets from consideration until explicitly introduced in §3B. We will thus assume the principal variances to be positive definite, i.e. ${V_\mu}^2 > 0$, since a principal portfolio with a vanishing principal variance is essentially equivalent to a riskless asset and can be treated separately as stipulated.

At this juncture it is useful to summarize the main properties of principal portfolios:

Proposition 1. *Every investment environment $\{s_i, r_i, \sigma_{ij}\}_{i,j=1}^N$ which allows short sales can be recast as a principal portfolio environment $\{S_\mu, R_\mu, \mathsf{V}_{\mu\nu}\}_{\mu,\nu=1}^N$ where the principal covariance matrix V is diagonal. The weighted mean of the principal variances equals the mean variance of the original environment. In general, a typical principal portfolio is hedged and leveraged.*

3. The Efficient Frontier

Having described the construction and properties of principal portfolios, we now turn to the second step of determining the composition of the efficient frontier within the principal portfolio environment. We start our analysis by considering risky assets only.

A. Riskless Asset Excluded

In this section we will assume that the principal portfolios of the original (risky) asset set have already been determined and the transformation from the investment environment $\{s_i, r_i, \sigma_{ij}\}_{i,j=1}^N$ to the principal portfolio environment $\{S_\mu, R_\mu, \mathsf{V}_{\mu\nu}\}_{\mu,\nu=1}^N$ has been effected.

The efficient frontier in the principal portfolio environment is defined by the constrained optimization problem

$$\min_{\mathbf{X}} \sum_{\mu=1}^N {V_\mu}^2 {X_\mu}^2 \;\; s.t. \;\; 1 - \sum_{\mu=1}^N X_\mu = 0 \text{ and } \mathcal{R} - \mathbf{R}^\dagger \mathbf{X} = 0, \tag{2}$$

where ${V_\mu}^2 > 0$ are the principal variances as already defined, and $-\infty < X_\mu < +\infty$, $\mu = 1, 2, \ldots, N$. This is the principal-portfolio version of the problem posed in Eq. (1), to which it is identical in form except for the absence of correlations here.

The solution to the problem in Eq. (2) is most conveniently expressed in terms of rescaled vectors. These are the *volatility-adjusted return* vector $\check{R}_\mu \stackrel{\text{def}}{=} R_\mu / V_\mu$, the inverse volatility vector $Z_\mu \stackrel{\text{def}}{=} 1/V_\mu$, and the vector $\check{X}_\mu \stackrel{\text{def}}{=} X_\mu V_\mu$. In terms of these rescaled vectors, the Lagrange function appears as

$$L(\check{\mathbf{X}}, \check{\mathbf{R}}, \mathbf{Z}, \mathcal{R}) \stackrel{\text{def}}{=} \check{\mathbf{X}}^\dagger \check{\mathbf{X}} + \lambda_1 (1 - \tilde{\mathbf{X}}^\dagger \mathbf{Z}) + \lambda_2 (\mathcal{R} - \check{\mathbf{R}}^\dagger \tilde{\mathbf{X}}), \tag{3}$$

where λ_1 and λ_2 are the Lagrange multipliers.

The solution to this problem is readily found as

$$\tilde{\mathbf{X}}(\check{\mathbf{R}}, \mathbf{Z}, \mathcal{R}) = (\lambda_1 \mathbf{Z} + \lambda_2 \check{\mathbf{R}})/2. \tag{4}$$

Note that since $X_\mu = (\lambda_1 + \lambda_2 R_\mu)/2{V_\mu}^2$, the efficient portfolio buys or sells (short) each principal portfolio in a risk-adjusted manner, that is, in inverse proportion to the principal variance ${V_\mu}^2$ of that portfolio and as an increasing linear function of the portfolio's expected return, R_μ. This will of course ensure that high-volatility and low-return portfolios are relatively suppressed, exactly as expected of a portfolio selection algorithm whose objective is volatility reduction subject to an expected level of return.

It should also be noted that Eq. (4) displays the two-mutual-fund theorem of Merton (1972) in the principal portfolio environment. The two mutual funds are defined in terms the vectors $\mathbf{Z}$ and $\check{\mathbf{R}}$ which are entirely determined by the parameters of the principal portfolio environment. The optimal portfolio chooses from these "mutual funds" according to Eq. (4), with the mix determined by the Lagrange multipliers λ_1 and λ_2. Inasmuch as the composition of the optimum portfolio depends on $\mathcal{R}$ only through the Lagrange multipliers, one can see that a specific optimum mix of the two mutual funds is determined by the expected portfolio return $\mathcal{R}$, exactly as expected.

The pair of constraint equations can now be used to find the multipliers. These are given by

$$\lambda_1{=}2(\check{\mathbf{R}}^\dagger\check{\mathbf{R}} - \mathcal{R}\check{\mathbf{R}}^\dagger\mathbf{Z})/\mathcal{D},\ \ \lambda_2{=}2(\mathcal{R}\mathbf{Z}^\dagger\mathbf{Z} - \check{\mathbf{R}}^\dagger\mathbf{Z})/\mathcal{D},$$

where

$$\mathcal{D} \stackrel{\text{def}}{=} (\check{\mathbf{R}}^\dagger\check{\mathbf{R}})(\mathbf{Z}^\dagger\mathbf{Z}) - (\check{\mathbf{R}}^\dagger\mathbf{Z})^2 \geq 0.$$

Using these relations, we can rewrite Eq. (4) in terms of the allocation vector $\check{\mathbf{X}}$ as a function of $\check{\mathbf{R}}$, $\mathbf{Z}$, and $\mathcal{R}$. The result is

$$\check{\mathbf{X}}(\check{\mathbf{R}}, \mathbf{Z}, \mathcal{R}) = \frac{\check{\mathbf{R}}^\dagger\check{\mathbf{R}} - \mathcal{R}\check{\mathbf{R}}^\dagger\mathbf{Z}}{(\check{\mathbf{R}}^\dagger\check{\mathbf{R}})(\mathbf{Z}^\dagger\mathbf{Z}) - (\check{\mathbf{R}}^\dagger\mathbf{Z})^2}\mathbf{Z} + \frac{\mathcal{R}\mathbf{Z}^\dagger\mathbf{Z} - \check{\mathbf{R}}^\dagger\mathbf{Z}}{(\check{\mathbf{R}}^\dagger\check{\mathbf{R}})(\mathbf{Z}^\dagger\mathbf{Z}) - (\check{\mathbf{R}}^\dagger\mathbf{Z})^2}\check{\mathbf{R}}. \tag{5}$$

This equation specifies the composition of the efficient portfolio in terms of the principal investment environment.

The minimized value of the objective function, which we shall refer to as the *efficient variance* and denote by ${V_{eff}}^2$, can now be determined from the above equations as follows:

$${V_{eff}}^2(\check{\mathbf{R}}, \mathbf{Z}, \mathcal{R}) = \frac{(\check{\mathbf{R}} - \mathcal{R}\mathbf{Z})^\dagger(\check{\mathbf{R}} - \mathcal{R}\mathbf{Z})}{(\check{\mathbf{R}}^\dagger\check{\mathbf{R}})(\mathbf{Z}^\dagger\mathbf{Z}) - (\check{\mathbf{R}}^\dagger\mathbf{Z})^2}. \tag{6}$$

This equation defines the efficient frontier in terms of the inverse volatility and volatility-adjusted return vectors of the principal investment environment. Observe that the quadratic dependence of the efficient variance ${V_{eff}}^2$ on the expected return $\mathcal{R}$, a well-known result of portfolio analysis, has emerged in a simple and general manner from our analysis. Solving Eq. (6) for $\mathcal{R}$ will thus yield the familiar (double-valued) expected return-volatility relation for the efficient portfolio.

The minimum value of ${V_{eff}}^2$ with respect to $\mathcal{R}$, on the other hand, occurs at $\mathcal{R}^\star \stackrel{\text{def}}{=} \check{\mathbf{R}}^\dagger\mathbf{Z}/\mathbf{Z}^\dagger\mathbf{Z}$, as can be readily verified. Its value, on the other hand is given by

$$\min{}_{\mathcal{R}}\ {V_{eff}}^2(\check{\mathbf{R}}, \mathbf{Z}, \mathcal{R}) = {V^\star_{eff}}^2(\mathbf{Z}) = (\mathbf{Z}^\dagger\mathbf{Z})^{-1} = 1/\textstyle\sum_{\mu=1}^N {V_\mu}^{-2}. \tag{7}$$

This result can be restated in the form

$$\frac{1}{{V^\star_{eff}}^2} = \frac{1}{{V_1}^2} + \frac{1}{{V_2}^2} + \ldots + \frac{1}{{V_N}^2}, \tag{8}$$

a remarkably simple result expressing the fact that at the minimum volatility point of the efficient frontier $(\mathcal{R}^\star, V^\star_{eff})$, the inverse variance of the efficient portfolio is the sum of the inverse principal variances. The relative suppression of the contribution of high-volatility portfolios pointed out earlier is particularly evident in this expression. It is also evident from Eq. (8) that

$$\frac{{V_{min}}^2}{N} \leq {V^\star_{eff}}^2 \leq \frac{{V_{max}}^2}{N}, \tag{9}$$

where ${V_{min(max)}}^2$ denotes the minimum (maximum) of the principal variances. This equation expresses the volatility eduction property of the efficient frontier in a particularly transparent manner.

B. Riskless Asset Included

Thus far we have excluded riskless assets from consideration. We will now add a riskless asset, s_0, with expected (and, by assumption, *certain*) return r_0, to the original environment. In keeping with our basic assumptions, we will allow both lending and borrowing of the riskless asset.

As observed earlier, a riskless asset is already a principal portfolio, which we shall designate as S_0, with a unit weight $W_0 = 1$, a vanishing (principal) variance $V_0 = 0$, and a principal return $R_0 = r_0$. Therefore, the optimization problem now appears as

$$\min_{X_0, \mathbf{X}} \sum_{\mu=1}^{N} {V_\mu}^2 {X_\mu}^2 \;\; s.t. \;\; 1 - \sum_{\mu=0}^{N} X_\mu = 0 \text{ and } \mathcal{R} - \sum_{\mu=0}^{N} R_\mu X_\mu = 0 \tag{10}$$

where $-\infty < X_\mu < +\infty$, $0 \leq \mu \leq N$, as stipulated. Because the variance V_0 corresponding to the riskless asset X_0 vanishes, a simpler solution than the one above obtains here, much in the same manner as in the textbook treatments of this problem. It therefore suffices to state the results here. The allocation rule for an efficient portfolio with unlimited short selling and use of a riskless asset is found to be

$$X_\mu(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}) = A(R_\mu - R_0){Z_\mu}^2, \; 1 \leq \mu \leq N, \tag{11}$$

and

$$X_0(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}) = 1 - \sum_{\mu=1}^{N} X_\mu(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}), \tag{12}$$

where A is given by

$$A \stackrel{\text{def}}{=} (\mathcal{R} - R_0)/[\sum_{\mu=1}^{N} (R_\mu - R_0)^2 {Z_\mu}^2].$$

Note that the expected portfolio return $\mathcal{R}$ would ordinarily be set at a level higher than the riskless return R_0 in which case A would be a positive quantity proportional to the expected *return surplus* $\mathcal{R} - R_0$ of the efficient portfolio relative to the riskless asset. Thus with $\mathcal{R} - R_0 > 0$, the allocation rule given in Eq. (11) would simply require that each principal portfolio (other than the riskless asset S_0) be bought, or sold short, according to whether it produces an expected return surplus, or deficit, relative to the riskless asset, and in a quantity which is in direct proportion to the said surplus or deficit and in inverse proportion to the variance of the principal portfolio. For the low-return branch which corresponds to the condition $\mathcal{R} - R_0 < 0$, the allocation rule simply reverses the role of the surplus and deficit producing portfolios. Thus every principal portfolio is put to work, as it were, the high-return ones by purchase and the low-return ones by short sale (assuming the upper branch).

It should be noted here that Eqs. (11) and (12) contain the two-mutual-fund theorem of Merton (1972) in the present context. Note that here the riskless asset itself constitutes one of the two mutual funds.

The efficient frontier in the presence of a riskless asset is a two-branch linear function, as is well known. Here these branches appear as

$$\mathcal{R} = R_0 \pm [\sum_{\mu=1}^{N} (R_\mu - R_0)^2 {Z_\mu}^2]^{\frac{1}{2}} V_{eff}, \tag{13}$$

with the upper branch being the high return choice. Both branches can be written in the equivalent form

$$\frac{(\mathcal{R} - R_0)^2}{{V_{eff}}^2} = \frac{(R_1 - R_0)^2}{{V_1}^2} + \frac{(R_2 - R_0)^2}{{V_2}^2} + \ldots + \frac{(R_N - R_0)^2}{{V_N}^2}, \tag{14}$$

another simple result with an intuitively appealing content. At this point we summarize the properties of the efficient portfolio in the presence of a riskless asset:

Proposition 2. *In the presence of a riskless asset, the upper branch of the efficient portfolio is formed by buying or selling each principal portfolio in proportion to its expected return surplus or deficit and in inverse proportion to its variance, and conversely for the lower branch. When so formed, the square of the volatility-adjusted return surplus or deficit for the efficient portfolio is the sum of the squares of the volatility-adjusted return or deficit for each risky principal portfolio for either branch.*

4. Conclusion

Being free of correlations, principal portfolios are the natural instruments for portfolio analysis when short sales are allowed. The portfolio selection rules found above are simple and intuitive, and directly reflect the underlying objective of achieving a given return with a minimum of volatility. In particular, Eqs. (8) and (14) display the volatility structure of the optimum portfolio in a strikingly simple form. Similarly, Eq. (9) establishes the volatility reduction feature of the optimum portfolio with the increasing size of the asset set in a transparent and general manner. On the other hand, the principal variances in terms of which these relations are expressed are subject to the quadratic sum rule expressed in Proposition 1. These findings clearly show that the return-volatility structure of the efficient frontier is more simply related to the the principal portfolio environment than the original asset set. This confirms the expectation that a correlation-free representation of the original asset set is the natural environment for analyzing portfolio selection problems.

The next step in this investigation is the application of the principal portfolio analysis to the single-index model of stock prices. The properties found above for the principal portfolios and the volatility structure of the efficient frontier can then be studied in more detail and in terms of the familiar parameters of that model.

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