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PRINCIPAL SERIES WHITTAKER FUNCTIONS ON $Sp(2; \boldsymbol{R})$, II

TAKUYA MIYAZAKI AND TAKAYUKI ODA

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Abstract. We consider Whittaker model for generalized principal series representations of the real sympletic group of degree 2. We obtain an integral formula for the radial part of the vector of with an extreme K-type in the Whittaker model.

Introduction. In our previous papers [O], [M-O], we investigated Whittaker functions of the large discrete series representations, and of the principal series representations of the real symplectic group $Sp(2; \boldsymbol{R})$ of rank 2, respectively.

In this paper we shall obtain explicit integral formulae for the radial part of the Whittaker functions on $G = Sp(2; \boldsymbol{R})$, belonging to the principal series representations associated with the Jacobi parabolic subgroup P_1 of G.

Let (π, H_π) be an irreducible admissible representation. Denote by N a maximal unipotent subgroup of G. For a continuous character $\eta: N \to \boldsymbol{C}^*$ of N, let $C_\eta^\infty(N \backslash G)$ be the space of complex-valued C^∞-functions f on G satisfying

$$f(ng) = \eta(n) \cdot f(g) \qquad \text{for any} \quad n \in N\,,\ g \in G\,.$$

Consider $C_\eta^\infty(N \backslash G)$ as a $(\mathfrak{g}, K)$-module via the right regular action of G. Then the intertwining space

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi, C_\eta^\infty(N \backslash G))$$

is the space of algebraic Whittaker vectors. When π is a principal series representation with a generic parameter μ of $\mathfrak{a}_{\boldsymbol{C}}^*$, the dimension of the above space is known and equals the order of the (little) Weyl group, i.e. 8 in our case (cf. Kostant [Kos, §5]). Here $\mathfrak{a}_{\boldsymbol{C}}^*$ is the dual of the complexification of the Lie algebra $\mathfrak{a}$ of A.

Choose a K-type (τ, V_τ), $\tau \in \hat{K}$, which occurs with multiplicity one in H_π, and let $i: V_\tau \hookrightarrow H_\pi$ be an injective K-homomorphism which is unique up to nonzero scalar multiple. Then we call the elements of the image of the restriction map

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi, C_\eta^\infty(N \backslash G)) \to \mathrm{Hom}_K(V_\tau, C_\eta^\infty(N \backslash G)) \simeq C_\eta^\infty(N \backslash G) \otimes_K V_\tau^*\,,$$

Whittaker functions with K-type τ^* belonging to the representation π.

Now consider the standard maximal parabolic subgroup P_1 of G associated to the long simple root. In this paper we call this parabolic subgroup the Jacobi

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parabolic subgroup. Let π be a generalized principal series representation induced from P_1, i.e.

$$\pi=\mathrm{Ind}_{P_1}^{G}(\sigma_1\otimes e^{\nu_1}\otimes 1_{N_1})$$

with σ_1 a discrete series representation of M_1, ν_1 a complex-valued linear form on the Lie algebra $\mathfrak{a}_1$ of A_1, where $P_1=M_1A_1N_1$ is the Langlands decomposition. Let τ be the "corner" K-type in π. Then the radial part of the Whittaker functions with K-type τ^* belonging to π is a solution of a holonomic system of rank 4 on A.

This is a situation completely similar to the case of the large discrete series representations and their minimal K-types, discussed in [O]. Therefore, applying the method of [O], we can obtain an integral expression for these Whittaker functions. It is the main result of this paper.

Now let us explain the contents of this paper. In §1, we recall some notation which were used in [O]. In §2, we recall the definition of the principal series representations of G which are induced from a maximal parabolic subgroup and their decomposition into K-types. We recall some structure of the unipotent radical N of a minimal parabolic subgroup and the definition of generic characters η of it in §3. Definitions of the Whittaker functions and Schmid operators are given in §4. We obtain a fundamental formula which is used for the computation of the radial part of the Schmid operators in §5. In §6, we give an expression of the Casimir element acting on $C_\eta^\infty(N\backslash G)\otimes_K V_\tau^*$ and also determine the eigenvalue of the element on the space of the principal series representation under consideration. In §7, the action of the shift operator on the minimal K-type vector is explicitly computed and we obtain explicit formulae of the differential equations for Whittaker functions in Propositions 7.1 and 7.3. Finally in §8, we obtain integral representations of the radial parts of the Whittaker functions with minimal K-type, Theorems 8.1 and 8.2.

The observation to start this paper, namely, that the "shape" of the K-types of $\mathrm{Ind}_{P_1}^{G}(\sigma\otimes e^{\nu_1}\otimes 1_{N_1})$ is the same as that of the large discrete series, is due to the first named author.

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1. Notation. We use the same notation as in the previous paper [O]. The symplectic group of degree 2 is given by

$$G=Sp(2;\boldsymbol{R})=\left\{g\in SL_4(\boldsymbol{R})\,\middle|\,{}^t gJg=J=\begin{pmatrix}0 & 1_2\\ -1_2 & 0\end{pmatrix}\right\}.$$

Here ${}^t g$ denotes the transpose of g and 1_2 the unit matrix of size 2. A maximal compact subgroup K of G is given by

$$K=\left\{\begin{pmatrix}A & B\\ -B & A\end{pmatrix}\in Sp(2;\boldsymbol{R})\,\middle|\,A,B\in M_2(\boldsymbol{R})\right\},$$

which is isomorphic to the unitary group $U(2)$ of size 2.

The Lie algebra of G is given by

$$\mathfrak{g}=\mathfrak{sp}(2;\boldsymbol{R})=\{X\in M_4(\boldsymbol{R})\,|\,JX+{}^tXJ=0\}\,,$$

and that of K

$$\mathfrak{u}(2)\simeq\mathfrak{k}=\left\{X=\begin{pmatrix} A & B\\ -B & A\end{pmatrix}\,\middle|\, A,\,B\in M_2(\boldsymbol{R});\,{}^tA=-A,\,{}^tB=B\right\}.$$

We have the associated Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$.

An $\boldsymbol{R}$-basis of $\mathfrak{u}(2)$ is given by

$$\sqrt{-1}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sqrt{-1}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad Y=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad Y'=\sqrt{-1}\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Let $\mathfrak{u}(2)_{\boldsymbol{C}}=\mathfrak{u}(2)\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Then a basis of $\mathfrak{u}(2)_{\boldsymbol{C}}$ is given by

$$Z=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad H'=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

$$X=\frac{1}{2}(Y-\sqrt{-1}Y')=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \bar{X}=\frac{1}{2}(-Y-\sqrt{-1}Y')=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Via the isomorphism $\mathfrak{k}_{\boldsymbol{C}}\overset{\sim}{\to}\mathfrak{u}(2)_{\boldsymbol{C}}$, the preimage of the above basis of $\mathfrak{u}(2)_{\boldsymbol{C}}$ is given by

$$Z=(-\sqrt{-1})\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1\\ \hline -1 & & & \\ & -1 & & \end{array}\right);\qquad H'=(-\sqrt{-1})\left(\begin{array}{cc|cc} & & 1 & \\ & & & -1\\ \hline -1 & & & \\ & 1 & & \end{array}\right);$$

$$Y=\left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & 1\\ & & -1 & 0\end{array}\right);\qquad Y'=\left(\begin{array}{cc|cc} & & & 1\\ & & 1 & \\ \hline & -1 & & \\ -1 & & & \end{array}\right).$$

A compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by $\mathfrak{h}=\boldsymbol{R}(\sqrt{-1}Z)+\boldsymbol{R}(\sqrt{-1}H')$.

Put $H_1'=1/2(Z+H')$, $H_2'=1/2(Z-H')$. We consider a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For a linear form $\beta:\mathfrak{h}\to\boldsymbol{C}$, we write $\beta_i=\beta(\sqrt{-1}H_i')\in\boldsymbol{C}$. For each $\beta\in\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h},\boldsymbol{C})$, set $\mathfrak{g}_\beta=\{X\in\mathfrak{g}_{\boldsymbol{C}}=\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}\,|\,[H,X]=\beta(H)X,\ {}^\forall H\in\mathfrak{h}\}$. Then the roots of $(\mathfrak{g},\mathfrak{h})$ are given by

$$\Sigma=\{\beta=(\beta_1,\beta_2)\neq(0,0)\,|\,\mathfrak{g}_\beta\neq 0\}=\sqrt{-1}\{\pm(2,0),\,\pm(0,2),\,\pm(1,1),\,\pm(1,-1)\}\,.$$

We write root vectors X_β in $\mathfrak{g}_\beta$ which are given in Table 1 of [O, p. 265]. Then $\mathfrak{k}_{\boldsymbol{C}}=\mathfrak{h}_{\boldsymbol{C}}+\boldsymbol{C}X_{(1,-1)}+\boldsymbol{C}X_{(-1,1)}$. Set

$$\mathfrak{p}_+ = CX_{(2,0)} + CX_{(1,1)} + CX_{(0,2)}, \quad \mathfrak{p}_- = CX_{-(2,0)} + CX_{-(1,1)} + CX_{-(0,2)},$$

then $\mathfrak{g}_C = \mathfrak{k}_C \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$. For each root $\beta = (\beta_1, \beta_2)$, we put $\|\beta\| = \sqrt{|\beta_1|^2 + |\beta_2|^2}$. The set $\{c \cdot \|\beta\|(X_\beta + X_{-\beta}), c \cdot \sqrt{-1}\|\beta\|(X_\beta - X_{-\beta}), \beta \in \Sigma_n^+\}$ forms an orthonormal basis of $\mathfrak{p} = \mathfrak{p}_{\boldsymbol{R}}$ with respect to the Killing form for some constant c. Here $\Sigma_n^+ = \{(2, 0), (1, 1), (0, 2)\}$ is the set of non-compact positive roots. Also we denote $\Sigma_c^+ = \{(1, -1)\}$ the set of compact positive roots.

⟨Restricted roots and the Iwasawa decomposition.⟩ We choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Set

$$H_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & 0 \end{array}\right) \quad \text{and} \quad H_2 = \left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & -1 \end{array}\right),$$

then these form a basis of $\mathfrak{a}$.

Let $\{e_1 = (1, 0), e_2 = (0, 1)\}$ be a standard basis of the 2-dimensional Euclidean plane $\boldsymbol{R}^2$. Then the root system Ψ of $(\mathfrak{g}, \mathfrak{a})$ is $\Psi = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$ with a positive root system $\Psi_+ = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}$. Then $\mathfrak{n} = \sum_{\alpha \in \Psi_+} \mathfrak{g}_\alpha$ is a nilradical of a minimal parabolic subalgebra. We choose generators E_α of $\mathfrak{g}_\alpha$ $(\alpha \in \Psi_+)$ as in [O, p. 266]. In $\mathfrak{g}_C$, the Iwasawa decomposition of the root vectors $\{X_\beta; \beta \in \Sigma\}$ are given as follows, which is obtained by direct computation.

Lemma 1.1 ([O, Lemma 1.1]).

$$X_{(2,0)} = H_1' + H_1 + 2\sqrt{-1}E_{2e_1}; \quad X_{(-2,0)} = -H_1' + H_1 - 2\sqrt{-1}E_{2e_1};$$

$$X_{(1,1)} = 2 \cdot \bar{X} + 2 \cdot E_{e_1 - e_2} + 2\sqrt{-1}E_{e_1 + e_2};$$

$$X_{(-1,-1)} = -2 \cdot X + 2 \cdot E_{e_1 - e_2} - 2\sqrt{-1}E_{e_1 + e_2};$$

$$X_{(0,2)} = H_2' + H_2 + 2\sqrt{-1}E_{2e_2}; \quad X_{(0,-2)} = -H_2' + H_2 - 2\sqrt{-1}E_{2e_2}.$$

⟨The Jacobi parabolic subgroup.⟩

Definition 1.2. We call the standard maximal parabolic subgroup P_1 corresponding to the long simple root of Ψ, the *Jacobi* parabolic subgroup of G.

The Langlands decomposition $P_1 = M_1 A_1 N_1$ of P_1 is given by

$$M_1 = \left\{ \left(\begin{array}{cc|cc} \varepsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ \hline 0 & 0 & \varepsilon & 0 \\ 0 & c & 0 & d \end{array}\right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \boldsymbol{R}), \varepsilon \in \{\pm 1\} \right\},$$

$$A_1 = \{\mathrm{diag}(t, 1, t^{-1}, 1) \mid t \in \boldsymbol{R}_{>0}\},$$

and the unipotent radical

$$N_1=\left\{\left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{array}\right)\right\}.$$

Here $\mathrm{diag}(a_1, a_2, a_3, a_4)$ denotes the diagonal matrix whose (i, i)-components are given by a_i. The unipotent radical N_1 is the 3-dimensional Heisenberg group.

2. Generalized principal series associated with the Jacobi parabolic subgroup. In this section we review some results on the K-types of the principal series representations associated with the maximal parabolic subgroup P_1. Let us start with the definition of these representations.

A discrete series representation σ of the semisimple part $M_1 \simeq \{\pm 1\} \times SL(2; \boldsymbol{R})$ of P_1 is given as a pair (ε, ξ), where $\varepsilon: \{\pm 1\} \to \boldsymbol{C}^*$ is a character, and ξ is a discrete series representation of $SL(2; \boldsymbol{R})$. For an element $\nu_1 \in \mathfrak{a}_{1,\boldsymbol{C}}^*$, let $\exp(\nu_1): A_1 \to \boldsymbol{C}^*$ be a character of A_1, and we can define a representation $\sigma \otimes \nu_1$ of P_1 by

$$\sigma \otimes \nu_1(p_1) = \sigma(m_1)a_1^{\nu_1}, \qquad \text{for} \quad p_1 = m_1 a_1 n_1 \in P_1 = M_1 A_1 N_1 .$$

Then the representation $\pi(P_1; \sigma; \nu_1)$ of the principal series associated with the Jacobi group P_1 is defined as the induced representation $\mathrm{Ind}_{P_1}^G(\sigma \otimes (\nu_1 + \rho_{P_1}))$. Here $\rho_{P_1} = (1/2)\{(e_1 - e_2) + 2e_1 + (e_1 + e_2)\} = 2e_1$.

In order to formulate some results on the K-types of the above representations, we have to recall the parameterization of the discrete series of $SL(2; \boldsymbol{R})$. The weight lattice of $SL(2; \boldsymbol{R})$ is identified with $\boldsymbol{Z}$. Then the Harish-Chandra parameters of the discrete series representations of $SL(2; \boldsymbol{R})$ consist of $\boldsymbol{Z} \setminus \{0\}$.

For a given Harish-Chandra parameter $m \in \boldsymbol{Z} \setminus \{0\}$, its Blattner parameter k is given by $k = m + 1$, if $m > 0$, $k = m - 1$, if $m < 0$. We denote by D_k^+ the discrete series representation with Blattner parameter k, if $k \geq 0$ ($k \geq 2$, in fact). Similarly we set D_k^- to be contragradient discrete series representation for D_k^+, $k \geq 2$. Hence the Blattner parameter of D_k^- is equal to $-k$.

⟨K-types of the principal series representation.⟩ We describe the K-types of the principal series representation $\pi(P_1; \sigma; \nu_1)$ associated with P_1, $\sigma = (\varepsilon, D_k^\pm)$. The irreducible finite-dimensional representations of the Lie algebra $\mathfrak{k}_{\boldsymbol{C}} \simeq \mathfrak{gl}(2, \boldsymbol{C})$ are parameterized by the set $\{\lambda = (\lambda_1, \lambda_2) \in \boldsymbol{Z}^{\oplus 2} \mid \lambda_1 \geq \lambda_2$, i.e. λ is dominant$\}$. We denote $(\tau_\lambda, V_\lambda)$ the representation associated with λ in the above set. Let $d = \lambda_1 - \lambda_2$, then the dimension of V_λ is $d + 1$. Let $\gamma_{2e_1} = \mathrm{diag}(-1, 1, -1, 1)$ in M_1.

PROPOSITION 2.1. *Let $\pi(P_1; \sigma; \nu_1)$ be the principal series representation of G associated with P_1, $\sigma = (\varepsilon, D_k^\pm)$ and $\nu_1 \in \mathfrak{a}_{1,\boldsymbol{C}}^*$. Then for a dominant integral weight $\lambda = (\lambda_1, \lambda_2)$ the irreducible representation τ_λ of K occurs in $\pi(P_1; \sigma; \nu_1)$ with multiplicity*

$$\#\{m\in \boldsymbol{Z}\mid m\equiv k\ (\mathrm{mod}\ 2),\ m\cdot \mathrm{sgn}(D_k^{\pm})-k\geq 0\ ,\\ (-1)^{\lambda_1+\lambda_2-m}=\varepsilon(\gamma_{2e_1}),\ \max(\lambda_2, k)\leq m\leq \lambda_1\}\ .$$

Here we set $\mathrm{sgn}(D_k^+)=+1$ *and* $\mathrm{sgn}(D_k^-)=-1$. *In particular,*

(i) *if* $\varepsilon(\gamma_{2e_1})=(-1)^k$ *and* $\sigma=(\varepsilon, D_k^-)$, *then each of* $\tau_{(l,l)}$ $(l\in \boldsymbol{Z},\ l\equiv k\ (\mathrm{mod}\ 2),\ l\leq -k)$ *or* $\tau_{(l,-k)}$ $(l\in \boldsymbol{Z},\ l\equiv k\ (\mathrm{mod}\ 2),\ l\geq -k)$ *occurs in* $\pi(P_1; \sigma; \nu_1)$ *with multiplicity one;*

(ii) *if* $\varepsilon(\gamma_{2e_1})=-(-1)^k$ *and* $\sigma=(\varepsilon, D_k^-)$, *then each of* $\tau_{(l,l-1)}$ $(l\in \boldsymbol{Z},\ l\leq -k)$ *or* $\tau_{(l,-k-1)}$ $(l\in \boldsymbol{Z},\ l\equiv k\ (\mathrm{mod}\ 2),\ l\geq -k)$ *occurs in* $\pi(P_1; \sigma; \nu_1)$ *with multiplicity one.*

PROOF. Consider the restriction of $\sigma=(\varepsilon, D_k^{\pm})$ to $K\cap M_1$:

$$\sigma|_{K\cap M_1}=\sum_{\omega\in(K\cap M_1)^\wedge}[\sigma:\omega]\omega\ .$$

Here $[\sigma:\omega]$ is the multiplicity of ω in $\sigma|_{K\cap M_1}$. Since $K\cap M_1\simeq\{\pm 1\}\times SO(2)$, any $\omega\in(K\cap M_1)^\wedge$ is specified by its value $\omega(\gamma_{2e_1})$ at $\gamma_{2e_1}=\mathrm{diag}(-1, 1, -1, 1)$ and the restriction $\omega|_{SO(2)}$. We define a character χ_m $(m\in \boldsymbol{Z})$ of $SO(2)$ by

$$\chi_m(\gamma_\theta)=\exp(\sqrt{-1}m\theta)\ ,$$

where $r_\theta\in SO(2)$ is the rotation with angle θ. Then the K-type theorem for $D_k^{\pm}$ implies that the multiplicity of $\omega=(\omega(\gamma_{2e_1}), \chi_m)$ is given by

$$[\sigma:\omega]=\begin{cases}1\ , & \text{if}\quad m\equiv k\ (\mathrm{mod}\ 2),\ m\cdot\mathrm{sgn}(D_k^{\pm})-k\geq 0,\ \omega(\gamma_{2e_1})=\varepsilon(\gamma_{2e_1})\ ;\\ 0\ , & \text{otherwise}\ .\end{cases}$$

The Frobenius reciprocity implies that the multiplicity of $\tau_\lambda\in\hat{K}$ in $\pi(P_1; \sigma; \nu_1)$ is given by

$$[\pi(P_1; \sigma; \nu_1):\tau_{(\lambda_1,\lambda_2)}]=\sum_{\omega\in(K\cap M_1)^\wedge}[\sigma|_{K\cap M_1}:\omega]\cdot[\tau|_{K\cap M_1}:\omega]$$

(cf. Knapp [Kn, Chap. 8, Prop. 8.4, p. 207] or Vogan [V, Chap. 4, formula (4.1.15), p. 145]).

Since the irreducible decomposition of $\tau_{(\lambda_1,\lambda_2)}|_{K\cap M_1}$ is given by

$$\tau_{(\lambda_1,\lambda_2)}|_{K\cap M_1}=\bigoplus_{\lambda_2\leq m\leq\lambda_1}((-1)^{\lambda_1+\lambda_2-m}, \chi_m)\ ,$$

together with the above formula of $[\sigma:\omega]$, we have the former part of the proposition. To show the statement on the multiplicity one of the latter part, note that $(-1)^m=(-1)^k$ and that $(-1)^{\lambda_1+\lambda_2-m}=\varepsilon(\gamma_{2e_1})$ are equivalent to $(-1)^{\lambda_1}=(-1)^{\lambda_2}\cdot(-1)^k\cdot\varepsilon(\gamma_{2e_1})$. Then the rest of the proof is elementary. q.e.d.

3. Characters of the unipotent radical. Put $N=\exp(\mathfrak{n})$. Then N is written as

$$N=\left\{\left(\begin{array}{cc|cc}1 & n_0 & & \\ 0 & 1 & & \\ \hline & & 1 & 0\\ & & -n_0 & 1\end{array}\right)\cdot\left(\begin{array}{c|cc}1_2 & n_1 & n_2\\ & n_2 & n_3\\ \hline & \multicolumn{2}{c}{1_2}\end{array}\right)\ \middle|\ n_0, n_1, n_2, n_3\in \boldsymbol{R}\right\}.$$

The commutator group $[N, N]$ of N is given by

$$[N,N]=\left\{\left(\begin{array}{c|c} 1_2 & \begin{matrix} n_1 & n_2 \\ n_2 & 0 \end{matrix} \\ \hline & 1_2 \end{array}\right) \middle| \; n_1, n_2 \in \boldsymbol{R} \right\}.$$

Hence a unitary character η of N is written as

$$\left(\begin{array}{c|c} \begin{matrix} 1 & n_0 \\ 0 & 1 \end{matrix} & \\ \hline & \begin{matrix} 1 & 0 \\ -n_0 & 1 \end{matrix} \end{array}\right) \cdot \left(\begin{array}{c|c} 1_2 & \begin{matrix} n_1 & n_2 \\ n_2 & n_3 \end{matrix} \\ \hline & 1_2 \end{array}\right) \longmapsto \exp\{2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3)\}$$

for some real numbers $c_0, c_3 \in \boldsymbol{R}$.

We denote by the same letter η, the derivative of η

$$\eta : \mathfrak{n} \longrightarrow \boldsymbol{C}.$$

Since $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] = \boldsymbol{R}E_{e_1-e_2} \oplus \boldsymbol{R}E_{2e_2}$, η is determined by the purely imaginary numbers

$$\eta_{e_1-e_2} = \eta(E_{e_1-e_2}) \quad \text{and} \quad \eta_{2e_2} = \eta(E_{2e_2}).$$

Here $\eta_{e_1-e_2} = 2\pi\sqrt{-1}c_0$ and $\eta_{2e_2} = 2\pi\sqrt{-1}c_3$.

ASSUMPTION 3.1. Throughout this paper, we assume that the character η of N is non-degenerate, i.e. both $\eta_{e_1-e_2}$ and η_{2e_2} are non-zero.

4. Whittaker functions and Schmid operators. Let $\eta : N = \exp(\mathfrak{n}) \to \boldsymbol{C}^*$ be a unitary character. Then we denote by $C_\eta^\infty(N\backslash G)$ the space

$$C_\eta^\infty(N\backslash G) = \{\phi : G \to \boldsymbol{C},\ C^\infty\text{-function} \mid \phi(ng) = \eta(n)\phi(g),\ (n, g) \in N \times G\}.$$

By the right regular action of G, $C_\eta^\infty(N\backslash G)$ is a smooth G-module, and a $(\mathfrak{g}_{\boldsymbol{C}}, K)$-module.

For any finite-dimensional K-module (τ, V), we put

$$C_{\eta,\tau}^\infty(N\backslash G/K) = \{F : G \to V,\ C^\infty\text{-function} \mid F(ngk^{-1}) = \eta(n)\tau(k)F(g),\ (n, g, k) \in N \times G \times K\}.$$

Let (π, H_π) be an irreducible admissible representation of G, and denote its associated $(\mathfrak{g}, K)$-module by the same symbol. Consider a homomorphism ψ_π of $(\mathfrak{g}, K)$-modules

$$\psi_\pi : H_\pi \longrightarrow C_\eta^\infty(N\backslash G).$$

Let (τ^*, V_{τ^*}) be a K-type of H_π and $i : V_{\tau^*} \hookrightarrow H_\pi$ an injection of K-modules. Then the restriction of ψ_π to V_{τ^*} via i defines an element $\psi_{\pi,\tau,i}$ in $C_{\eta,\tau}^\infty(N\backslash G/K) = C_\eta^\infty(N\backslash G) \otimes_K V_\tau \cong \mathrm{Hom}_K(V_{\tau^*}, C_\eta^\infty(N\backslash G))$. Here (τ, V_τ) is the contragradient representation of τ^*.

DEFINITION 4.1. We call $\psi_{\pi,\tau,i}$ a *Whittaker function* of K-type τ belonging to the representation π. The function $\psi_{\pi,\tau,i}$ is determined by its restriction to A. We denote this by the same symbol $\psi_{\pi,\tau,i}$.

Now let us recall the definition of Schmid operators. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and $\mathrm{Ad}=\mathrm{Ad}_{\mathfrak{p}_{\boldsymbol{C}}}$ the adjoint representation of K on $\mathfrak{p}_{\boldsymbol{C}}$. Then we have a canonical covariant differential operator $\nabla_{\eta,\lambda}$ from $C^{\infty}_{\eta,\tau}(N\backslash G/K)$ to $C^{\infty}_{\eta,\tau\otimes\mathrm{Ad}}(N\backslash G/K)$:

$$\nabla_{\eta,\tau}F=\sum_i R_{X_i}F(\cdot)\otimes X_i\,,\quad F\in C^{\infty}_{\eta,\tau}(N\backslash G/K)\,,$$

where $\{X_i\}_i$ is any fixed orthonormal basis of $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$, and $R_{X_i}F(g)=(d/dt)F(g\cdot\exp(tX_i))|_{t=0}$, $g\in G$.

Let $P_{\tau'}\colon V_\tau\otimes\mathfrak{p}_{\boldsymbol{C}}\to V_{\tau'}$ be the projection to an irreducible component of the representation $V_\tau\otimes\mathfrak{p}_{\boldsymbol{C}}$ of K. Then for a Whittaker function $\psi_{\pi,\tau,i}\in C^{\infty}_{\eta,\tau}(N\backslash G/K)$, the $V_{\tau'}$-valued function $\phi'=P_{\tau'}\circ\nabla_{\eta,\tau}(\psi_{\pi,\tau,i})\in C^{\infty}_{\eta,\tau'}(N\backslash G/K)$ is also a Whittaker function belonging to π, because the coefficients of ϕ' are linear combination of the derivations of the coefficients of $\psi_{\pi,\tau,i}$ with respect to elements in $\mathfrak{p}_{\boldsymbol{C}}$.

5. Radial part of Schmid operators. Put $A=\exp(\mathfrak{a})$, i.e.

$$A=\left\{\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1}\end{pmatrix}\,\middle|\, a_1,a_2\in\boldsymbol{R},\,a_1>0,\,a_2>0\right\}.$$

Then we have the Iwasawa decomposition $G=NAK$ of $Sp(2;\boldsymbol{R})$. The value of $F\in C^{\infty}_{\eta,\tau}(N\backslash G/K)$ is determined by its restriction $\phi=F|_A$ to A.

We compute the radial part of the Schmid operators $\nabla_{\eta,\lambda}=\nabla_{\eta,\tau_\lambda}$. As an orthogonal basis of $\mathfrak{p}$, we take $C\|\beta\|(X_\beta+X_{-\beta})$, $C\|\beta\|\sqrt{-1}(X_\beta-X_{-\beta})$, $\beta\in\Sigma_n^+$ with some $C>0$ depending on the Killing form. Then

$$\nabla_{\eta,\lambda}F=2C^2\sum_{\beta\in\Sigma_n^+}\|\beta\|^2R_{X_{-\beta}}F\otimes X_\beta+2C^2\sum_{\beta\in\Sigma_n^+}\|\beta\|^2R_{X_\beta}F\otimes X_{-\beta}\,.$$

We define

$$\nabla^+_{\eta,\lambda}F=\frac{1}{4}\Sigma\|\beta\|^2\cdot R_{X_{-\beta}}F\otimes X_\beta\,;\quad \nabla^-_{\eta,\lambda}F=\frac{1}{4}\Sigma\|\beta\|^2\cdot R_{X_\beta}F\otimes X_{-\beta}\,.$$

In order to write $R(\nabla^{\pm}_{\eta,\lambda})$, we introduce some symbols. Set ∂_i to be R_{H_i} restricted to functions on A $(i=1,2)$, and define linear differential operators $\mathscr{L}_i^{\pm}$ and $\mathscr{S}^{\pm}$ on $C^\infty(A,V_\lambda)$ by

$$\begin{cases}\mathscr{L}_i^{\pm}\phi=(\partial_i\pm2\sqrt{-1}a_i^2\eta(E_{2e_i}))\phi \quad (i=1,2)\,;\\ \mathscr{S}^{\pm}\phi=\{a_1a_2^{-1}\eta(E_{e_1-e_2})\pm\sqrt{-1}a_1a_2\eta(E_{e_1+e_2})\}\phi\,.\end{cases}$$

Then we have

PROPOSITION 5.1 ([O, Proposition 6.1]). *The radial part of the shift operators, $R(\nabla^{\pm}_{\eta,\lambda})\colon C^\infty(A, V_\lambda)\to C^\infty(A, V_\lambda\otimes\mathfrak{p}_{\pm})$, are expressed as*

(i)
$$\begin{aligned}R(\nabla^{+}_{\eta,\lambda})\phi=&(\mathscr{L}_1^- +\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_+}(H_1')-4)(\phi\otimes X_{(2,0)})\\&+(\mathscr{S}^- +\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_+}(X))(\phi\otimes X_{(1,1)})\\&+(\mathscr{L}_2^- +\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_+}(H_2')-2)(\phi\otimes X_{(0,2)}),\end{aligned}$$

(ii)
$$\begin{aligned}R(\nabla^{-}_{\eta,\lambda})\phi=&(\mathscr{L}_1^+ -\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_-}(H_1')-4)(\phi\otimes X_{(-2,0)})\\&+(\mathscr{S}^+ -\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_-}(\bar{X}))(\phi\otimes X_{(-1,-1)})\\&+(\mathscr{L}_2^+ -\tau_\lambda\otimes \mathrm{Ad}_{\mathfrak{p}_-}(H_2')-2)(\phi\otimes X_{(0,-2)}).\end{aligned}$$

6. Casimir operator. We shall investigate the action of the Casimir operator on the Whittaker functions belonging to the principal series representations $\pi(P_1;\sigma;\nu_1)$ associated with the Jacobi parabolic subgroup.

To obtain the value of the infinitesimal character of π at the Casimir element

$$\begin{aligned}L=&H_1^2+H_2^2-4H_1-2H_2+2E_{e_1-e_2}E_{-e_1+e_2}+4E_{2e_1}E_{-2e_1}\\&+2E_{e_1+e_2}E_{-e_1-e_2}+4E_{2e_2}E_{-2e_2},\end{aligned}$$

we recall how the discrete series of $SL(2;\boldsymbol{R})$ are obtained as sub-quotient of the principal series representation.

⟨The representations of $SL(2;\boldsymbol{R})$.⟩ The principal series representations of $S=SL(2;\boldsymbol{R})$ are given as follows. Set

$$Q=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\in S\right\},$$

$$M_Q=\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\},\quad A_Q=\{\mathrm{diag}(a,a^{-1})\mid a\in\boldsymbol{R}_{>0}\}.$$

Also we write $K'=SO(2)$. Let $\varepsilon_Q\colon M_Q\to\boldsymbol{C}^*$ be a character of M_Q, $\lambda\in\mathfrak{a}^*_{Q,\boldsymbol{C}}$, and $\rho_Q\in\mathfrak{a}^*_Q$ is a half of the positive root. Then $\xi=\mathrm{Ind}_Q^S(\varepsilon_Q\otimes(\lambda+\rho_Q))$ is a principal series representations of S. We denote by V_ξ the subspace consisting of K'-finite vectors in the representation space of ξ. We identify $\mathfrak{a}^*_{Q,\boldsymbol{C}}$ with $\boldsymbol{C}$ via

$$\lambda\colon \log\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)=\lambda'\cdot\log a.$$

Here λ' is the complex number corresponding to λ, and ρ_Q is identified with 1. We denote by $\mathfrak{s}$ the Lie algebra of $SL(2;\boldsymbol{R})$.

PROPOSITION 6.1 (cf. Vogan [V, Proposition (1.3.3), p. 59]). *Let us work in the category of $(\mathfrak{s}, K')$-modules.*

(i) *If λ is a positive integer with parity $-\varepsilon_Q$, then ξ contains the discrete series $(\mathfrak{s}, K')$-modules $D^{\pm}_{\lambda+1}$ as submodules. Then quotient $V_\xi/(D^{+}_{\lambda+1} \oplus D^{-}_{\lambda+1})$ is a finitedimensional representation of weight $\lambda-1$.*

(ii) *If λ is a negative integer with parity $-\varepsilon_Q$. Then ξ contains as its unique irreducible submodule, the finite-dimensional representation of weight $-\lambda-1$. The quotient of V_ξ by this is isomorphic to $D^{+}_{-\lambda+1} \oplus D^{-}_{-\lambda+1}$.*

By the above proposition the discrete series representation $D^{\pm}_k$ of S is embedded in a principal series representation $\mathrm{Ind}_Q^S(\varepsilon_{Q,k} \otimes (k-1+\rho_Q))$. Here $\varepsilon_{Q,k} \in \hat{M}_Q$ is given by $\varepsilon_{Q,k}(\pm)=(\pm)^k$. By the transitivity of parabolic induction (see Vogan [V, Chap. 4] for example), $\pi=\mathrm{Ind}_{P_1}^G((\varepsilon, D^{\pm}_k) \otimes (\nu_1+\rho_{P_1}))$ is a submodule of

$$\mathrm{Ind}_{P_1}^G((\varepsilon, \mathrm{Ind}_Q^S(\varepsilon_{Q,k} \otimes (k-1+\rho_Q))) \otimes (\nu_1+\rho_{P_1})) \cong \mathrm{Ind}_P^G(\sigma_M \otimes (\mu'+\rho_P)),$$

where $\sigma_M \in \hat{M}$ is specified by $\sigma(\gamma_{2e_1})=\varepsilon(\gamma_{2e_1})$ and $\sigma(\gamma_{2e_2})=\varepsilon_{Q,k}(-1_2)$ and $\mu' \in \mathfrak{a}^*_{\mathbf{C}}$ is given by $\mu'=(\nu_1, k-1)$. Since the principal series representation $\pi'=\mathrm{Ind}_P^G(\sigma_M \otimes (\mu'+\rho_P))$ is quasi-simple, the eigenvalue of C_2 on π is equal to that on π'. Similarly, π is obtained as a quotient $(\mathfrak{g}, K)$-module of $\pi''=\mathrm{Ind}_P^G(\sigma_M \otimes (\mu''+\rho_P))$, where σ_M is the same as above and $\mu''=(\nu_1, -k+1)$.

Now we can write the action of the Casimir operator on the Whittaker functions.

⟨The even case.⟩ Assume that $\varepsilon(\gamma_{2e_1})=(-1)^k$ in this case. The K-type $\tau_{(-k,-k)}$ occurs with multiplicity one in π. Let i: $W_{\tau_{(-k,-k)}} \to V_\pi$ be the injective K-homomorphism unique up to constant multiple. Then there is a K-homomorphism j: $W_{\tau_{(-k,-k)}} \to V_{\pi''}$ which is a lifting of i with respect to the natural surjection $V_{\pi''} \to V_\pi$, which is also unique up to constant multiple because of multiplicity one (cf. [M-O, Proposition 3.1]). Then the space of Whittaker functions of type $\tau_{k,k}$ belonging to π is contained in the space of Whittaker functions of type $\tau_{k,k}$ belonging to π''.

PROPOSITION 6.2. (i) *Let $R(L)$ be the radial part of the Casimir operator L, and set $r(L)=a^{-\rho}R(L)a^{\rho}$. Then for $I \in C^\infty(A)$*

$$r(L)I=\{\partial_1^2+\partial_2^2+2\eta^2_{e_1-e_2}(a_1/a_2)^2+4\eta^2_{2e_2}a_2^4+4k\sqrt{-1}\,\eta_{e_1-e_2}a_2^2-5\}I.$$

(ii) *Let $\phi(a) \in C^\infty(A)$ be the radial part of the Whittaker function of type $\tau_{k,k}$ belonging to π, and let $I(a)=a^{-\rho}\phi(a)$. Then*

$$\{r(L)+5\}I=(\nu_1^2+(k-1)^2)I.$$

⟨The odd case.⟩ Assume that $\varepsilon(\gamma_{2e_1})=-(-1)^k$. By Proposition 2.1, the K-type $(\tau_{(-k,-k-1)}, V)$ occurs in π with multiplicity one. We use the realization of $(\tau_\lambda, V_\lambda)$ with a basis $\{v_j\}_{0 \le j \le d}$ in [O, §3]. We call the basis as the standard basis for τ_λ. We also use freely the formulae in [O, Lemmas 3.1, 3.2, 3.3]. Let $\{v_0, v_1\}$ be the standard basis of V, and V^* the dual space of V. Fix an injective K-homomorphism i: $V^* \to H_{\pi,K}$ unique up to constant multiple. Then we can find again a lifting j: $V^* \to H_{\pi'',K}$ of i so that $i=p \cdot j$ where p: $H_{\pi'',K} \to H_{\pi,K}$ is the natural surjection. As in the even case, Lemmas

7.4 and 7.5 of [M-O] implies the following.

PROPOSITION 6.3. (i) *Let $R(L)$ be the radial part of the Casimir operator L on $C^\infty(A)\otimes V$. Then the operator $r(L)=a^{-\rho}R(L)a^{\rho}$ reads*

$$r(L)=\{(P+4(k+1)\sqrt{-1}\eta_{2e_2}a_2^2)b_0(a)-2\eta_{e_1-e_2}(a_1/a_2)b_1(a)\}v_0$$
$$+\{2\eta_{e_1-e_2}(a_1/a_2)b_0(a)+(P+4k\sqrt{-1}\eta_{2e_2}a_2^2)b_1(a)\}v_1-5I$$

for $I(a)=b_0(a)v_0+b_1(a)v_1\in C^\infty(A)\otimes_{\mathbf{C}}V$, $b_i(a)\in C^\infty(A)$, $i=0,\ 1$. Here

$$P=\partial_1^2+\partial_2^2+2\eta_{e_1-e_2}^2(a_1/a_2)^2+4\eta_{2e_2}^2a_2^4\ .$$

(ii) *Let $\phi\in C^\infty(A)\otimes V\cong C^\infty_{\eta,\tau_{(k+1,k)}}(N\backslash G/K)$ be a Whittaker function with K-type $\tau_{(k+1,k)}$ belonging to π. Set $I=a^{-\rho}\phi$. Then $I(a)$ satisfies*

$$\{r(L)+5\}I=\{v_1^2+(k-1)^2\}I\ .$$

7. Holonomic systems for Whittaker functions. In this section we compute explicit formula of differential equations for Whittaker functions of the fundamental series $\pi(P_1;(\varepsilon,D_k^-);\nu_1+\rho_{P_1})$ with K-type $\tau_{(k,k)}$ or $\tau_{(k+1,k)}$.

7.1. The even case. We consider the case $\varepsilon(\gamma_{2e_1})=(-1)^k$ in the first place. In this case π has K-type $\tau_{(-k,-k)}$ with multiplicity one (cf. Proposition 2.1). Here is the main result in this subsection for the even case.

PROPOSITION 7.1. *Let $\phi\in C^\infty(A)\cong C^\infty_{\eta,\tau_{(k,k)}}(N\backslash G/K)$ be a Whittaker function with K-type $\tau_{(k,k)}$ belonging to π. Define $h(a)\in C^\infty(A)$ by*

$$\phi(a_1,a_2)=a_1^{k+1}a_2^k\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)h(a_1,a_2)\ .$$

Then $h(a)$ satisfies

(i) $(\partial_1\partial_2-\mathscr{S}^2)h=0$;

(ii) $\{(\partial_1+\partial_2)^2+2(k-1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+(k-1)^2-\nu_1^2\}h=0$.

The system (i), (ii) *make a holonomic system of rank* 4.

PROOF. First we show (i). Let

$$\mathscr{D}_k^{\mathrm{down}}:\ C^\infty_{\eta,\tau_{(k,k)}}(N\backslash G/K)\to C^\infty_{\eta,\tau_{(k-2,k-2)}}(N\backslash G/K)$$

be the down shift operator defined in [M-O, §8, (8.3)]. Then, since $\tau_{(-k+2,-k+2)}$ does not occur in π, the Whittaker function ϕ in $C^\infty_{\eta,\tau_{(k,k)}}(N\backslash G/K)$ satisfies $\mathscr{D}_k^{\mathrm{down}}\phi=0$, i.e. for $\phi(a)$

$$\mathscr{S}_k^{\mathrm{down}}(\phi)=[\{\partial_1-(k+1)\}\{\partial_2+2\sqrt{-1}a_2^2\eta_{2e_2}-k\}-\mathscr{S}^2]\phi=0\ .$$

Define $h(a)$ as in the statement. Then the above equation leads to the first equation in the proposition.

Now we rewrite the action of the Casimir element (cf. Proposition 6.2). Then we have

$$(\partial_1^2+\partial_2^2-2\mathcal{S}^2-4\eta_{2e_2}^2a_2^4+4k\sqrt{-1}\eta_{2e_2}a_2^2)I=\{\nu_1^2+(k-1)^2\}I$$

for $I=a_1^{k-1}a_2^{k-1}\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)h$. Rewriting this equation for h, we have

$$\{(\partial_1+k-1)^2+(\partial_2+k-1-2\sqrt{-1}\eta_{2e_2}a_2^2)^2-2\mathcal{S}^2-4\eta_{2e_2}^2a_2^4$$
$$+4k\sqrt{-1}\eta_{2e_2}a_2^2\}h=\{\nu_1^2+(k-1)^2\}h\,.$$

Note that $\mathcal{S}^2h=\partial_1\partial_2h$ by the equation (i). Then the above equation leads to

$$\{(\partial_1+\partial_2)^2+2(k-1)(\partial_1+\partial_2)+2(k-1)^2-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2\}h=\{\nu_1^2+(k-1)^2\}h\,.$$

This is the second equation (ii) in the proposition. q.e.d.

7.2. The odd case. We consider the corresponding result for the odd case, i.e. $\varepsilon(\gamma_{2e_1})=-(-1)^k$. By Proposition 2.1, π has K-type $(\pi_{(-k,-k-1)}, V^*)$ with multiplicity one.

Let $\phi(a)=c_0(a)u_0+c_1(a)v_1\in C^\infty(A)\otimes_{\mathbf{C}}V$ be a Whittaker function with K-type $\tau_{(k+1,k)}$ belonging to π. Recall the down shift operator

$$\mathcal{E}_k^{\mathrm{down}}\colon C^\infty_{\eta,\tau_{(k+1,k)}}(N\backslash G/K)\to C^\infty_{\eta,\tau_{(k,k-1)}}(N\backslash G/K)$$

defined in §9 of [M-O], which is the composite of the operator $\nabla^+_{\eta,\tau_{(k+1,k)}}$ and the projector onto $\tau_{(k,k-1)}$. Because the K-type $\tau_{(-k+1,-k)}$ does not occur in π, we have $\mathcal{E}_k^{\mathrm{down}}(\phi)=0$. If we write $I(a)=a^{-\rho}\phi(a)=b_0(a)v_0+b_1(a)v_1$, then by Remark 9.2, (3) of [M-O], this condition is equivalent to the two equations:

$$\text{(7.2.1)}\qquad \begin{cases}\eta_{e_1-e_2}(a_1/a_2)b_0(a)+(\partial_1-k)b_1(a)=0\,;\\ (\partial_2+2\sqrt{-1}\eta_{2e_2}a_2^2-k)b_0(a)+\eta_{e_1-e_2}(a_1/a_2)b_1(a)=0\,.\end{cases}$$

Definition 7.2. We define functions $h_0(a)$ and $h_1(a)$ by

$$\begin{cases}b_0(a)=a_1^{k+1}a_2^k\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)h_0(a)\,;\\ b_1(a)=a_1^ka_2^{k-1}\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)h_1(a)\,.\end{cases}$$

It is easy to check that $h_i(a)$, $i=0,1$, satisfy

$$\eta_{e_1-e_2}a_1^2h_0(a)+\partial_1h_1(a)=0\,,\quad a_2^2\partial_2h_0(a)+\eta_{e_1-e_2}h_1(a)=0\,,$$

which are derived from (7.2.1). In particular, they satisfy

$$(\partial_1\partial_2-\mathcal{S}^2)h_i(a)=0\qquad(i=0,1)\,.$$

Now recall the differential equations arising from the Casimir operator, i.e. Proposition 7.2 of [M-O]. By using (7.2.1), this is equivalent to the system of equations:

$$\{\partial_1^2+\partial_2^2+2\eta_{e_1-e_2}^2(a_1/a_2)^2+4\eta_{2e_2}^2a_2^4+4(k+1)\sqrt{-1}\eta_{2e_2}a_2^2$$
$$+2(\partial_2+2\sqrt{-1}\eta_{2e_2}a_2^2-k)\}b_0(a)=\{\nu_1^2+(k-1)^2\}b_0(a)$$

and

$$\{\partial_1^2+\partial_2^2+2\eta_{e_1-e_2}^2(a_1/a_2)^2+4\eta_{2e_2}^2a_2^4+4k\sqrt{-1}\eta_{2e_2}a_2^2-2(\partial_1-k)\}b_1(a)$$
$$=\{v_1^2+(k-1)^2\}b_1(a)\,.$$

We want to write this in terms of $h_i(a)$, $i=0, 1$. These two equations are transformed to

$$\{(\partial_1+k+1)^2+(\partial_2+k-2\sqrt{-1}\eta_{2e_2}a_2^2)^2+2\mathscr{S}^2$$
$$+4\eta_{2e_2}^2a_2^4+4(k+1)\sqrt{-1}\eta_{2e_2}a_2^2+2\partial_2\}h_0(a)=dh_0(a)\,,$$
$$\{(\partial_1+k)^2+(\partial_2+k-1-2\sqrt{-1}\eta_{2e_2}a_2^2)^2+2\mathscr{S}^2$$
$$+4\eta_{2e_2}^2a_2^4+4k\sqrt{-1}\eta_{2e_2}a_2^2-2\partial_1\}h_1(a)=dh_1(a)\,.$$

Here $d=v_1^2+(k-1)^2$. Direct computations show that these are reduced to

$$\{\partial_1^2+\partial_2^2+2\mathscr{S}^2+2(k+1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+(k+1)^2+k^2\}h_0(a)=dh_0(a),$$
$$\{\partial_1^2+\partial_2^2+2\mathscr{S}^2+2(k-1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+(k-1)^2+k^2\}h_1(a)=dh_1(a).$$

Recall that $\partial_1\partial_2h_i=\mathscr{S}^2h_i$, $i=0, 1$. Together with these, the above equations give the following:

PROPOSITION 7.3. *Let ϕ be a Whittaker function with K-type $\tau_{(k+1,k)}$ belonging to π. Write $\phi=a^\rho I$ and*

$$I(a)=b_0(a)v_0+b_1(a)v_1\in C^\infty(A)\otimes_{\mathbf{C}}V$$

with respect to the standard basis $\{v_0, v_1\}$ of V. Define $h_i(a)$ as in Definition 7.2. Then $h_i(a)$ satisfy the differential equations:

(i) $$\eta_{e_1-e_2}a_1^2h_0(a)+\partial_1h_1(a)=0\,,$$

(ii) $$a_2^2\partial_2h_0(a)+\eta_{e_1-e_2}h_1(a)=0\,,$$

accordingly

(iii) $$(\partial_1\partial_2-\mathscr{S}^2)h_i(a)=0\,,\qquad i=0, 1\,.$$

Moreover they satisfy

(iv) $$\{(\partial_1+\partial_2)^2+2\mathscr{S}^2+2(k+1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2\}h_0(a)=(v_1^2-k^2-4k)h_0(a)$$

(v) $$\{(\partial_1+\partial_2)^2+2\mathscr{S}^2+2(k-1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2\}h_1(a)=(v_1^2-k^2)h_1(a).$$

8. Integral formula for the Whittaker function.

8.1. The even case. Now we want to solve the equations in Proposition 7.1:

(i) $$(\partial_1\partial_2-\mathscr{S}^2)h=0\,,$$

(ii) $$\{(\partial_1+\partial_2)^2+2(k-1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+(k-1)^2-v_1^2\}h=0\,,$$

where $\mathscr{S}=\eta_{e_1-e_2}(a_1/a_2)$.

We first find the form of the solution by formal computation. Assume that h is represented as a Laplace transform

$$h(a_1, a_2)=\int_{\mathbf{R}^2} \Phi(u_1, u_2)\exp(u_1 a_1^2+u_2 a_2^{-2})du_1 du_2 .$$

Then we have

$$\partial_1\partial_2 h(a_1, a_2)=\int_{\mathbf{R}^2}(-4u_1u_2)\left(\frac{a_1}{a_2}\right)^2 \Phi(u_1, u_2)\exp(u_1 a_1^2+u_2 a_2^{-2})du_1 du_2 .$$

Therefore the equation (i) above implies that

$$(4u_1u_2+\eta_{e_1-e_2}^2)\Phi(u_1, u_2)=0 .$$

Hence the support of a generalized function Φ is on the hyperbola $4u_1u_2=-\eta_{e_1-e_2}^2$ (>0, if η is unitary.) Thus taking a function ϕ on $\mathbf{R}\setminus\{0\}$, we can write h as

$$h(a_1, a_2)=\int_{\mathbf{R}} \phi(u)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u},$$

where c is a constant ± 1. This is a general solution of the equation (i). Now we note that

$$\partial_1 h(a_1, a_2)=\int_{\mathbf{R}}\left(-\frac{c\eta_{e_1-e_2}^2}{2u}a_1^2\right)\phi(u)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u},$$

and

$$\partial_2 h(a_1, a_2)=\int_{\mathbf{R}}\left(-2\frac{cu}{a_2^2}\right)\phi(u)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u}.$$

Assume that $\exp\{c(u/a_2^2-\eta_{e_1-e_2}^2 a_1^2/4u)\}\to 0$ as $u\to 0$ or $u\to\infty$. Then integration by parts shows that

$$\begin{aligned}(\partial_1+\partial_2)h(a_1, a_2)&=\int_{\mathbf{R}}(-2)\left(\frac{c\eta_{e_1-e_2}^2}{4u}a_1^2+\frac{cu}{a_2^2}\right)\phi(u)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u}\\ &=\int_{\mathbf{R}}(-2)\phi(u)u\frac{\partial}{\partial u}\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u}\\ &=\int_{\mathbf{R}}\left(2u\frac{\partial}{\partial u}\phi(u)\right)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u}.\end{aligned}$$

Hence from the equation (ii), we get a differential equation for ϕ,

$$\{(2\partial_u)^2+2(k-1)(2\partial_u)+8\sqrt{-1}c\eta_{2e_2}u+(k-1)^2-\nu_1^2\}\phi(u)=0, \tag{8.1.1}$$

where $\partial_u=u(\partial/\partial u)$. Assume that ϕ has support in $\{u\in\mathbf{R}\mid u>0\}$. Then we should choose $c=-1$ to justify the integration by parts. We set $\phi(u)=v^{-1/2-k+1}\psi(v)$ with $v=\sqrt{u}$. Then $2u(\partial/\partial u)=v(\partial/\partial v)$, and the above equation (8.1.1) is written for $\psi(v)$ as

(8.1.2)
$$v^2\frac{\partial^2\psi}{\partial v^2}+\left(\frac{1}{4}-\nu_1^2-8\sqrt{-1}\eta_{2e_2}v^2\right)\psi=0\,.$$

On the other hand, when $\mathrm{Re}(\kappa-1/2-m)\leq 0$, the function

$$W_{k,m}(z)=\frac{e^{-1/2z}\cdot z^{\kappa}}{\Gamma(1/2-\kappa+m)}\int_0^{\infty}t^{-\kappa-1/2+m}\left(1+\frac{t}{z}\right)^{\kappa-1/2+m}\cdot e^{-t}dt$$

defined for $z\notin(-\infty,0)$ satisfies the Whittaker differential equation

$$\frac{d^2}{dz^2}W+\left\{-\frac{1}{4}+\frac{\kappa}{z}+\frac{1/4-m^2}{z^2}\right\}W=0\,,$$

and $W_{\kappa,m}(z)$ is a unique solution which rapidly decreases if $z\to+\infty$. Set $\kappa=0$, and $m=\nu_1$ in above $W_{\kappa,m}$. We finally obtain that

$$\psi(v)=W_{0,\nu_1}(\sqrt{32|\eta_{2e_2}|}\,v)$$

satisfies the differential equation (8.1.2). This gives an integral representation of the function $h(a_1, a_2)$.

THEOREM 8.1. *Let $\pi=\pi(P_1;\sigma;\nu_1)$, $\sigma=(\varepsilon, D_k^-)$, be a principal series representation of $Sp(2;\boldsymbol{R})$ as before. Assume that the character $\eta: N_m\to\boldsymbol{C}$ is unitary and generic. Then*:

(i) *π has a Whittaker model for η if and only if* $\mathrm{Im}(\eta_{2e_2})<0$.

(ii) *In this case, the function $h(a_1, a_2)$ has integral representation*

$$h(a_1,a_2)=$$
$$\mathrm{const.}\times\int_0^{\infty}t^{-k+(1/2)}W_{0,\nu_1}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta_{e_1-e_2}^2\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}\,.$$

Then we have an integral representation of the radial part of the Whittaker vector $\phi(a_1, a_2)$ as

$$\phi(a_1,a_2)=\mathrm{const.}\times a_1^{k+1}a_2^{k}\cdot\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)$$
$$\times\int_0^{\infty}t^{-k+(1/2)}W_{0,\nu_1}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta_{e_1-e_2}^2\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}\,.$$

It is determined uniquely up to constant multiple under the condition that it rapidly decreases when $a_1/a_2\to\infty$ and $a_2\to\infty$.

PROOF. We replace t by a_1t in the above integral representation of $h(a_1, a_2)$. Then

$$h(a_1, a_2) = \text{const.} \times \int_0^\infty \left(\frac{a_1}{a_2} \cdot a_2 \cdot t\right)^{-k+(1/2)} W_{0,\nu_1}\left(\frac{a_1}{a_2} \cdot a_2 \cdot t\right)$$
$$\times \exp\left\{-\frac{1}{32\sqrt{-1}\eta_{2e_2}}\left(\frac{a_1}{a_2}\right)^2 \cdot t^2 + 8\sqrt{-1}\eta_{e_1-e_2}^2\eta_{2e_2} \cdot t^{-2}\right\}\frac{dt}{t}.$$

If $\mathrm{Im}(\eta_{2e_2})<0$, then $-1/32\sqrt{-1}\eta_{2e_2}<0$ and $8\sqrt{-1}\eta_{e_1-e_2}^2\eta_{2e_2}<0$. Therefore the integrand above rapidly decreases when $t\to 0$ and when $t\to+\infty$. Hence the above integral converges, and as a function in a_1, a_2, it rapidly decreases when $a_1/a_2\to+\infty$ and when $a_2\to+\infty$. q.e.d.

8.2. The odd case. As in the even case, we can obtain a solution of the system of differential equations in Proposition 7.3.

(i) $(\partial_1\partial_2-\mathcal{S}^2)h_i(a)=0,\quad i=0, 1$,

(ii) $\{(\partial_1+\partial_2)^2+2(k+1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+k^2+4k-\nu_1^2\}h_0(a)=0$,

(iii) $\{(\partial_1+\partial_2)^2+2(k-1)(\partial_1+\partial_2)-4\sqrt{-1}\eta_{2e_2}a_2^2\partial_2+k^2-\nu_1^2\}h_1(a)=0$.

We can obtain an integral formula for $h_i(a_1, a_2)$, $i=0, 1$, in the following manner. The argument is similar to the even case. From the equations (i) for $i=0, 1$, we can write $h_i(a_1, a_2)$, $i=0, 1$ as

$$h_i(a_1, a_2)=\int_{\mathbf{R}}\phi_i(u)\exp\left\{c\left(\frac{u}{a_2^2}-\frac{\eta_{e_1-e_2}^2}{4u}a_1^2\right)\right\}\frac{du}{u},$$

where c is a constant ± 1. By the same way as in the even case in §8.1, we can obtain the differential equations for $\phi_i(u)$, $i=0, 1$ from the equations (ii), (iii) by integration by parts.

$$\{(2\partial_u)^2+2(k+1)(2\partial_u)+8\sqrt{-1}c\eta_{2e_2}u+k^2+4k-\nu_1^2\}\phi_0(u)=0, \tag{8.2.1}$$

$$\{(2\partial_u)^2+2(k-1)(2\partial_u)+8\sqrt{-1}c\eta_{2e_2}u+k^2-\nu_1^2\}\phi_1(u)=0. \tag{8.2.2}$$

We set

$$\phi_0(u)=v^{-1/2-k-1}\psi_0(v),\quad \phi_1(u)=v^{-1/2-k+1}\psi_0(v),$$

with $v=\sqrt{u}$. Then the above equations (8.2.1) and (8.2.2) for each $\phi_i(u)$ are rewritten for $\psi_i(v)$ as

$$v^2\frac{\partial^2\psi_0}{\partial v^2}+\left(\frac{1}{4}-(\nu_1^2+1-2k)-8\sqrt{-1}\eta_{2e_2}v^2\right)\psi_0=0,$$

$$v^2\frac{\partial^2\psi_1}{\partial v^2}+\left(\frac{1}{4}-(\nu_1^2+1-2k)-8\sqrt{-1}\eta_{2e_2}v^2\right)\psi_1=0.$$

Here we choose $c=-1$ as before.

THEOREM 8.2. *Let $\pi=\pi(P_1;\sigma;\nu_1)$, $\sigma=(\varepsilon, D_k^-)$ be a principal series representation of $Sp(2;\boldsymbol{R})$ as before. Assume that the character $\eta: N_m\to\boldsymbol{C}$ is unitary and generic. Then*:

(i) *π has a Whittaker model for η if and only if* $\mathrm{Im}(\eta_{2e_2})<0$.

(ii) *In this case, the functions $h_i(a_1,a_2)$ have integral representations with some constant C as follows*:

$$h_0(a_1,a_2)=C\times\{-16\sqrt{-1}\eta^2_{e_1-e_2}\eta_{2e_2}\}$$
$$\times\int_0^\infty t^{-(1/2)-k-1}W_{0,\sqrt{\nu_1^2+1-2k}}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta^2_{e_1-e_2}\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}.$$

$$h_1(a_1,a_2)=C$$
$$\times\int_0^\infty t^{-(1/2)-k+1}W_{0,\sqrt{\nu_1^2+1-2k}}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta^2_{e_1-e_2}\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}.$$

These are determined uniquely up to constant multiple under the same condition as in the even case.

Finally we obtain the radial parts of Whittaker vector $\phi=c_0(a)v_0+c_1(a)v_1$ as follows:

$$c_0(a_1,a_2)=C\times\{-16\sqrt{-1}\eta_{e_1-e_2}\eta_{2e_2}\}\times a_1^{k+3}a_2^{k+1}\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)$$
$$\times\int_0^\infty t^{-(1/2)-k-1}W_{0,\sqrt{\nu_1^2+1-2k}}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta^2_{e_1-e_2}\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}.$$

$$c_1(a_1,a_2)=C\times a_1^{k+2}a_2^k\exp(-\sqrt{-1}\eta_{2e_2}a_2^2)$$
$$\times\int_0^\infty t^{-(1/2)-k+1}W_{0,\sqrt{\nu_1^2+1-2k}}(t)\exp\left(-\frac{t^2}{32\sqrt{-1}\eta_{2e_2}a_2^2}+\frac{8\sqrt{-1}\eta^2_{e_1-e_2}\eta_{2e_2}a_1^2}{t^2}\right)\frac{dt}{t}.$$

By direct calculation one can show that these solutions satisfy the system of differential equations (7.2.1).

REMARK. Here we do not discuss the irreducibility of the generalized principal series representations π associated with P_1. Regardless of whether it is irreducible or not, the image of the "corner" K-type vector in π by the unique element Ψ (if exists) in $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi,\mathcal{A}_\eta(N\backslash G))$ satisfies the integral expression given above.

Recently Hayata [Ha] obtained a similar integral expression for Whittaker functions with corner K-type belonging to the generalized principal series representations of $SU(2,2)$.

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Department of Mathematics
Tokyo Metropolitan University
Tokyo 192–0397
Japan

Graduate School of Mathematical Sciences
The University of Tokyo
8-1 Komaba 3-chome, Meguro-ku, Tokyo 153–8914
Japan