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ABSTRACT

Let $f \in \mathbb{Q}[X_1, \dots, X_n]$ be a polynomial of degree D. We consider the problem of computing the real dimension of the real algebraic set defined by $f = 0$. Such a problem can be reduced to quantifier elimination. Hence it can be tackled with Cylindrical Algebraic Decomposition within a complexity that is doubly exponential in the number of variables. More recently, denoting by d the dimension of the real algebraic set under study, deterministic algorithms running in time $D^{O(d(n-d))}$ have been proposed. However, no implementation reflecting this complexity gain has been obtained and the constant in the exponent remains unspecified.

We design a probabilistic algorithm which runs in time which is essentially cubic in $D^{d(n-d)}$. Our algorithm takes advantage of genericity properties of *polar varieties* to avoid computationally difficult steps of quantifier elimination. We also report on a first implementation. It tackles examples that are out of reach of the state-of-the-art and its practical behavior reflects the complexity gain.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algorithms*; F.2.2 [**Theory of Computation**]: Analysis of algorithm and problem complexity—*Non numerical algorithms and problems: Geometrical problems and computation*

Keywords

Real dimension; Real solutions; Polynomial systems; Real Geometry.

General Terms

Algorithms; Theory.

1. INTRODUCTION

This paper is devoted to the design and the implementation of an algorithm for computing the real dimension d of an algebraic set $V \cap \mathbb{R}^n$ defined by one polynomial equation $f = 0$ with rational coefficients and degree D. Recall that when $V \cap \mathbb{R}^n$ is empty, its real dimension d is -1 by convention, when it is non-empty but finite it is 0 else it is the largest integer d such that there is a projection of $V \cap \mathbb{R}^n$ over a d-dimensional affine subspace of coordinates with a non-empty interior.

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Motivations. Computing the real dimension is a question of first importance since it is a basic topological invariant. It encodes the number of independent motions that are allowed on a geometric body or the number of independent parameters that may vary independently. Hence, computing the real dimension of semi-algebraic sets has many applications in engineering sciences (see e.g. [26] and references therein). It also has some algorithmic interest since the knowledge of the real dimension can be exploited to accelerate other algorithms studying real algebraic or semi-algebraic sets (see e.g. [8, Section 13.3] or [6, 7]).

State-of-the-art. Quantifier elimination (QE) over the reals plays a central role for computing the real dimension since it allows to obtain semi-algebraic descriptions of projections of semi-algebraic sets. Hence, it allows to decide if the interior of such a projection is empty. Consequently, Cylindrical Algebraic Decomposition (CAD) due to Collins [12] can be used for computing the real dimension. However, the arithmetic complexity of this algorithm is doubly exponential in the total number of variables. Several software implementing variants and improvements of CAD have been designed (Mathematica, Maple, QEPCAD, RedLog, etc.) but because of this doubly exponential complexity they are rather limited to 3 or 4 variables on a wide range of examples.

The current algorithms within the best known complexity class are due to Basu, Pollack and Roy [9] (see also [8, Chapter 14]) following previous work of Koiran [28] and Vorobjov [40]. Let $S \subset \mathbb{R}^n$ be a real algebraic set defined by a polynomial equation of degree D with rational coefficients.

These algorithms use QE techniques that essentially allow to compute the projection of S on a i-dimensional linear subspace in time $D^{O(i(n-i))}$ arithmetic operations [8, Thm 14.16]. Then, the arithmetic complexity of these algorithms is bounded by $D^{O(d(n-d))}$. These algorithms are deterministic and the complexity is output sensitive since it depends on d. They also allow to handle general semi-algebraic sets in time $(sD)^{O(d(n-d))}$ (where s is the number of inequalities). However several questions remain open.

1. What is the complexity constant hidden in the exponent of the above complexity estimates?
2. Can we obtain an efficient implementation that reflects the complexity gain compared to doubly exponential algorithms?

Main results. We provide answers to both questions: we obtain a probabilistic algorithm whose arithmetic complexity is essentially cubic in $D^{d(n-d)}$; a first implementation shows that it can tackle examples that are out of reach of the state-of-the-art. We give more details on our methodology below.

In the whole paper, let $f \in \mathbb{Q}[X_1, \dots, X_n] \setminus \{0\}$ of degree D and let $V \subset \mathbb{C}^n$ be the algebraic set defined by $f = 0$. Our technique is still based on the investigation of projections of $V \cap \mathbb{R}^n$. Let π_i

be the canonical projection $(\mathbf{x}_1, \dots, \mathbf{x}_n) \to (\mathbf{x}_1, \dots, \mathbf{x}_i)$. Remark that in order to decide if $\pi_i(V \cap \mathbb{R}^n)$ has an empty interior, there is no need to compute a semi-algebraic description of this projection using general QE techniques. Indeed, it is sufficient to compute *(i)* a polynomial that defines a hypersurface containing the boundary of $\pi_i(V \cap \mathbb{R}^n)$, *(ii)* compute sample points in each connected component of the complementary of the real trace of that hypersurface and *(iii)* for each such sample point y, decide if the fiber $\pi_i^{-1}(y) \cap V \cap \mathbb{R}^n$ is empty. If there is no non-empty real fiber then the interior of $\pi_i(V)$ is empty.

This process has already been identified and formalized in [23, 24] where a dedicated projection step has been designed for quantifier elimination over the reals under some conditions on the input. These are regularity conditions (the algebraic set defined by $f = 0$ must be smooth) and properness conditions (for any $y \in \mathbb{R}^i$, there is a closed ball B containing y such that $\pi_i^{-1}(B) \cap V \cap \mathbb{R}^n$ is closed and bounded).

Note that when V is smooth, if $V \cap \mathbb{R}^n \neq \emptyset$, then its real dimension is $n-1$ by the implicit function theorem. Thus, in our context, this regularity condition is a strong obstruction since on all examples where the real dimension does not coincide with the complex one, this condition is not satisfied.

Moreover, the properness of the restriction of π_i to $V \cap \mathbb{R}^n$ cannot be always ensured, especially when $V \cap \mathbb{R}^n$ is not bounded and i is less than the dimension of $V \cap \mathbb{R}^n$.

Hence, results in [23, 24] are not sufficient and need to be generalized for our purpose. To do that we investigate polar varieties of a deformation V_ε of V defined by $f - \varepsilon = 0$ (where ε is an infinitesimal encoding a small perturbation of the constant coefficient in f). This allows us to retrieve a regular situation (V_ε is actually smooth). Next, we show that properness assumptions in [23, 24] can be substituted with properness assumptions on polar varieties of V_ε as in [35].

Our algorithm is probabilistic because its correctness depends on some changes of coordinates that are performed randomly. Indeed, we prove that for such a generic choice these properness assumptions on polar varieties are satisfied. Finally, letting the deformation ε tend to 0, this allows us to obtain a hypersurface defining the boundary of $\pi_i(V)$. We finally get a routine for deciding the emptiness of the interior of $\pi_i(V \cap \mathbb{R}^n)$ in generic coordinates. Also, an extra outcome of the paper is a generalization of several results in [23, 24].

These geometric steps can be eliminated using many algebraic elimination routines. To estimate the complexity we mainly rely on [17]. We use the arithmetic complexity model over $\mathbb{Q}$: we count arithmetic operations over $\mathbb{Q}$ as a unit. Below, $\tilde{O}(x)$ means $O(x \log(x)^a)$ for some $a > 0$. We can now state our main result.

THEOREM 1. *Let f be a polynomial in $\mathbb{Q}[X_1, \dots, X_n] \setminus \{0\}$ of degree D and let d be the real dimension of the real algebraic set defined by $f = 0$. There exists a probabilistic algorithm which computes d in time $\tilde{O}(n^{16}(1 + D)^{3d(n-d)+5n+5})$.*

Also, note that when a real algebraic set is defined by a polynomial system $f_1 = \dots = f_p = 0$ with coefficients in $\mathbb{Q}$, our algorithm can be used with input $f_1^2 + \dots + f_p^2$.

We also report on the practical performances of an implementation of our algorithm. We have used as benchmarks sums of squares of random dense polynomials, discriminants of characteristic polynomials of linear symmetric matrices and series of polynomials that are known to be non-negative over the reals. For all these polynomials, the dimensions of the real algebraic sets they define may vary. We find that our implementation allows to tackle polynomials that are out of reach of the best CAD implementations. As importantly, we emphasize that, in practice, the behaviour of our implementation is output sensitive. Indeed, for families of fixed dimension, timings seem to show a behaviour of type $D^{O(n)}$ but computations performed better when d or $n - d$ are small w.r.t $n/2$.

Related works. As already mentioned, algorithms in [8, 28, 40] are the first ones with a singly exponential complexity for computing the real dimension of semi-algebraic sets.

The use of *polar varieties* in symbolic computation appears first in [1] to compute sample points in smooth equidimensional real algebraic sets (see also [2, 3, 4, 5] and reference therein). There are also used for global optimization and for computing roadmaps (see [18, 19, 36] and references therein).

Properness properties of the restriction of a projection to a polar variety are introduced in [35] and used in [23, 24].

Our complexity estimates rely on complexity results on the geometric resolution algorithm; we refer to [17, 29, 37] and references therein for a description of these algorithms and their parametric variants.

Structure of the paper.

Section 2 is devoted to preliminaries and notation used throughout the paper. Section 3 is devoted to the description of the algorithm, the proof of its correctness and its complexity analysis. Section 4 is devoted to the proof of geometric results on which the algorithm relies. The last section reports on practical experiments.

2. PRELIMINARIES

We start with basic notions and some notation on algebraic sets.

Algebraic sets. Let $\mathbb{K}$ be a field of characteristic 0, $\overline{\mathbb{K}}$ be its algebraic closure. Let I be an ideal of $\mathbb{K}[X_1, \dots, X_n]$ generated by $(g_1, \dots, g_s)$. The $\mathbb{K}$-algebraic set associated to I is the $\mathbb{K}$-algebraic set defined by the polynomial equations $g_1 = \dots = g_s = 0$; we denote it by $V(I)$.

Let $W \subset \overline{\mathbb{K}}^n$ be a $\mathbb{K}$-algebraic set defined by polynomial equations in $\mathbb{K}[X_1, \dots, X_n]$. The dimension of W is defined as the *Krull dimension* of its associated ideal $I(W) = \{g \in \mathbb{K}[X_1, \dots, X_n] \mid \forall x \in W, g(x) = 0\}$ (see e.g. [14]). This notion of dimension coincides with other notions inspired by differential or algebraic geometry (see e.g. [14, Part II]). Roughly speaking, it is the number of generic hypersurfaces such that their intersection with W is a finite set of points. If W' is another algebraic set and $W \subset W'$, then the dimension of W is at most the dimension of W'.

The $\mathbb{K}$-algebraic set W is said to be $\mathbb{K}$-*irreducible* if it cannot be decomposed as the union of two different $\mathbb{K}$-algebraic sets. Any $\mathbb{K}$-irreducible algebraic set W is uniquely decomposed as a finite union of $\mathbb{K}$-irreducible algebraic sets; these are called the *irreducible components* of W.

When all the irreducible components of W have the same dimension, we say that W is *equidimensional*.

Let W be an $\mathbb{K}$-algebraic set, equidimensional of co-dimension c and let $(f_1, \dots, f_p)$ be a set of generators of its associated ideal. A point $x \in W$ is called *regular* if the Jacobian matrix $\left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1 \le i \le p \\ 1 \le j \le n}}$ associated to $(f_1, \dots, f_p)$ has rank c at x. The kernel of this Jacobian matrix at x is the tangent space to W at x; we denote it by $T_x W$. The points in W that are not regular are *singular* by definition. An algebraic set with no singular points is *smooth*.

Algebraic sets are closed sets of the Zariski topology. Let $W \subset \overline{\mathbb{K}}^n$. The Zariski closure of W is the smallest algebraic set that contains it; we denote it by $\overline{W}$.

Most of the time, the field $\mathbb{K}$ will be clear from the context and will be omitted in the above terminology.

Fields of Puiseux series. We follow the notation of [8, Chap. 2] to define the *field of Puiseux series* $\mathbb{K}\langle\varepsilon\rangle = \{\Sigma_{i \ge i_0} a_i \varepsilon^{i/q} \mid a_i \in \mathbb{K}, q \in \mathbb{N}^*,\ i_0 \in \mathbb{Z}\}$ where ε is an infinitesimal over $\mathbb{K}$.

Let $S \subset \mathbb{R}^n$ be a semi-algebraic set; it is the real solution set of polynomial equations and inequalities with coefficients in $\mathbb{R}$. We denote by $\mathrm{Ext}(S, \mathbb{K}\langle\varepsilon\rangle)$ the set of solutions of this system in $\mathbb{R}\langle\varepsilon\rangle^n$.

We say that $y = \sum_{i \ge i_0} a_i \varepsilon^{i/q}$ in $\mathbb{K}\langle\varepsilon\rangle$ is *bounded over* $\mathbb{K}$ if $i_0 \ge 0$. We say that $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{K}\langle\varepsilon\rangle^n$ is *bounded over* $\mathbb{K}$ if each coordinate y_i is bounded over $\mathbb{K}$. Given a bounded element $y \in \mathbb{K}\langle\varepsilon\rangle$ and $\mathbf{y} \in \mathbb{K}\langle\varepsilon\rangle^n$, then $\lim_{\varepsilon \to 0}(y)$ denotes a_0 in $\mathbb{K}$ and $\lim_{\varepsilon \to 0}(\mathbf{y})$ denotes the point $(\lim_{\varepsilon \to 0}(y_1), \dots, \lim_{\varepsilon \to 0}(y_n)) \in \mathbb{K}^n$. Given a subset $A \subset \mathbb{K}\langle\varepsilon\rangle^n$, we denote by $\lim_{\varepsilon \to 0}(A)$ the set

$$\{\lim_{\varepsilon \to 0}(y) \mid y \in A \text{ and } y \text{ is bounded.}\}$$

We say that A is bounded over $\mathbb{K}$ if evry point in A is bounded over $\mathbb{K}$. By [8, Prop 2.99] the application $\lim_{\varepsilon\to 0}$ is a ring homomorphism. We recall the following result in [31, Lemma 3.5] that we will use repeatedly in the sequel.

LEMMA 2. *[31, Lemma 3.5] Let $f \in \mathbb{Q}[X_1,\dots,X_n]$ be a non-zero polynomial, let $V_\varepsilon \subset \mathbb{C}\langle\varepsilon\rangle^n$ be algebraic sets defined by the equation $f-\varepsilon=0$. Then, V_ε is either empty or smooth and equidimensional of codimension 1.*

Projections and Polar varieties. Let $W \subset \overline{\mathbb{K}}^n$ be an equidimensional algebraic set and let $(f_1,\dots,f_p)$ be a set of generators of the ideal associated to W. We denote by π_i the canonical projection $(x_1,\dots,x_n) \to (x_1,\dots,x_i)$ for $1 \le i \le n$.
A regular point $x \in W$ is a *critical point* of the restriction of the projection π_i to W if $\pi_i(T_xW) \neq \overline{\mathbb{K}}^i$. These are the regular points of W at which the truncated Jacobian matrix $\left(\frac{\partial f_j}{\partial X_k}\right)_{\substack{1\le j\le p\\ i+1\le k\le n}}$ is rank defective.
The *polar variety* associated to π_i and W is the Zariski closure of the critical locus defined above (we refer to [36, Section 2.1]).
Consider now the polynomial $f \in \mathbb{Q}[X_1,\dots,X_n]$ that is to be given as input to our algorithm and the algebraic set $V \subset \mathbb{C}^n$ defined by $f=0$. We denote by $V_\varepsilon \subset \mathbb{C}\langle\varepsilon\rangle^n$ the algebraic set defined by $f=\varepsilon$. By Lemma 2, it is smooth. We will consider the polar varieties associated to π_i and V_ε. They are defined as the zero set in $\mathbb{C}\langle\varepsilon\rangle^n$ of

$$f-\varepsilon = \frac{\partial f}{\partial X_{i+1}} = \cdots = \frac{\partial f}{\partial X_n} = 0.$$

It will be denoted by $W_{\varepsilon,i}$.

Changes of variables and topological notions. We repeatedly use linear changes of variables and projections in the sequel.
The set of invertible matrices with entries in $\mathbb{K}$ is denoted by $\mathrm{GL}_n(\mathbb{K})$. Let $\mathbf{A} \in \mathrm{GL}_n(\mathbb{K})$, $g \in \mathbb{K}[X_1,\dots,X_n]$. We denote by $g^{\mathbf{A}}$ the polynomial $g(\mathbf{AX})$ (with $\mathbf{X}=[X_1,\dots,X_n]$). For any set of polynomials $G \subset \mathbb{K}[\mathbf{X}]$, we denote by $G^{\mathbf{A}}$ the set $\{g^{\mathbf{A}} \mid g \in G\}$.
Let $V \subset \overline{\mathbb{K}}^n$ be an algebraic set. We denote by $V^{\mathbf{A}} \subset \overline{\mathbb{K}}^n$ the image of V by the map $x \mapsto \mathbf{A}^{-1}x$. This notation is naturally extended to semi-algebraic sets when $\mathbb{K}$ is a real closed field.
Assume that $\mathbb{K}$ is equipped with a Euclidean topology. Let U be a subset of $\mathbb{K}^n$, we denote by $\mathsf{Int}(U)$ the interior of U for the Euclidean topology. We denote by $\mathsf{Bd}(U)$ the Euclidean boundary of U defined as the closure of U without its interior.
The properness of a projection is defined as in [35, Section 1] : A map $\pi : A \subset \mathbb{K}^n \to \mathbb{K}^i$ is *proper* at $y \in \mathbb{K}^i$ if there exists a neighborhood $\mathcal{O}$ of y such that $\pi^{-1}(\overline{\mathcal{O}})$ is closed in $\mathbb{K}^n$ and bounded over $\mathbb{K}^n$, where $\overline{\mathcal{O}}$ denotes the closure of $\mathcal{O}$ for the Euclidean topology. If π is proper at every point of $\pi(A)$, we simply say that π is proper.

3. ALGORITHM

3.1 Descriptions

We start with the description of the main subroutines. The two following ones are rather standard.

HasRealSolutions: it takes as input a polynomial equation with rational coefficients. It returns true if there exists at least one real solution and false otherwise (see [32, Theorem 4]).

PointsPerComponents: it takes as input a polynomial inequation $g \neq 0$ with rational coefficients. It returns a set of points meeting each connected component of the semi-algebraic set defined by $g \neq 0$ (see [16, Section 4] and [34]).

We describe now the third subroutine. As sketched in the introduction, we need a subroutine that allows to decide if the projection of some real algebraic set has an empty interior. This third subroutine performs this task under some assumptions.

Let $f \in \mathbb{Q}[X_1,\dots,X_n]$ and $0<i<n$ an integer. Below, for $g \in \mathbb{Q}[X_1,\dots,X_n]$, we denote by $I_i(g)$ the ideal $\left\langle \frac{\partial g}{\partial X_{i+1}},\dots,\frac{\partial g}{\partial X_n}\right\rangle : \left\langle \frac{\partial g}{\partial X_1},\dots,\frac{\partial g}{\partial X_i}\right\rangle^\infty + \langle g\rangle$ (see [13, Section 4.4] for the definition of saturated ideals).

HasEmptyInterior:
Input: a polynomial $f \in \mathbb{Q}[X_1,\dots,X_n]\setminus\{0\}$, an integer i such that $0<i<n$ and a matrix $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$ s.t. $I_i(f^{\mathbf{A}}) \cap \mathbb{Q}[X_1,\dots,X_i]$ is not empty and $\mathsf{Bd}\left(\pi_i\left(V^{\mathbf{A}} \cap \mathbb{R}^n\right)\right) \subset V(I_i(f^{\mathbf{A}}) \cap \mathbb{Q}[X_1,\dots,X_i])$.
Output: true if $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) = \emptyset$, false otherwise.

1. compute $g \neq 0$ in the ideal $I_i(f^{\mathbf{A}}) \cap \mathbb{Q}[X_1,\dots,X_i]$
2. let $L = \mathsf{PointsPerComponents}(g \neq 0)$.
3. for $(\alpha_1,\dots,\alpha_i) \in L$ do
   (a) let $f^{\mathbf{A}}_\alpha = f^{\mathbf{A}}(\alpha_1,\dots,\alpha_i,X_{i+1},\dots,X_n)$
   (b) if $\mathsf{HasRealSolutions}(f^{\mathbf{A}}_\alpha = 0) =$ true then return false
4. return true

We now describe our main algorithm that is called RealDimension which takes as input $f \in \mathbb{Q}[X_1,\dots,X_n]$. In the following, $\lceil \frac{n}{2} \rceil$ denotes the first integer greater than $\frac{n}{2}$. The algorithm starts by checking that the real algebraic set defined by $f=0$ has solutions. When this is the case, it chooses randomly a linear change of variables and performs successive calls to HasEmptyInterior.

Algorithm RealDimension:
Input: A polynomial $f \in \mathbb{Q}[X_1,\dots,X_n]$.
Output: The real dimension of $V \cap \mathbb{R}^n$.

1. if $\mathsf{HasRealSolutions}(f=0) =$ false then return -1
2. choose a random matrix $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$
3. for $i = 1$ to $\lceil \frac{n}{2} \rceil$ do
   (a) if $\mathsf{HasEmptyInterior}(f,i,\mathbf{A}) =$ true then return $i-1$
   (b) if $\mathsf{HasEmptyInterior}(f,n-i,\mathbf{A}) =$ false then return $n-i$

3.2 Correctness

Correctness of Algorithm RealDimension. The correctness proof of Algorithm RealDimension relies on the following results.

PROPOSITION 3. *[27, Section 2] Let $W \subset \mathbb{C}^n$ be an algebraic set. Assume that $W \cap \mathbb{R}^n$ has real dimension $d>0$. Then there exists a non-empty Zariski open set $\Gamma_1 \subset \mathrm{GL}_n(\mathbb{C})$ such that for any $\mathbf{A} \in \Gamma_1 \cap \mathrm{GL}_n(\mathbb{Q})$ and $0<i\le d$, $\mathsf{Int}(\pi_d(W^{\mathbf{A}} \cap \mathbb{R}^n)) \neq \emptyset$.*

THEOREM 4. *Let $0<i<n$ be an integer. There exists a non-empty Zariski open set $\Gamma_2 \subset \mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A} \in \Gamma_2 \cap \mathrm{GL}_n(\mathbb{Q})$, $\mathsf{HasEmptyInterior}(f,i,\mathbf{A})$ returns true if $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ is empty and false otherwise.*

We can now prove the correctness of Algorithm RealDimension with input $f \in \mathbb{Q}[X_1,\dots,X_n]$. We denote by d the real dimension of the real algebraic set defined by $f=0$. We make the assumption that this set is non-empty (the empty case is correctly handled at Step 1). Hence, we assume that $d \ge 0$.
We make the assumption that the matrix $\mathbf{A}$ chosen at step 2 lies in $\Gamma_1 \cap \Gamma_2$ where Γ_1 and Γ_2 are the non-empty Zariski open subsets of $\mathrm{GL}_n(\mathbb{C})$ defined in Proposition 3 and Theorem 4.
When the dimension d is 0, we enter in the loop at Step 3 and the call to $\mathsf{HasEmptyInterior}(f,i,\mathbf{A})$ returns true (since $\pi_1(V^{\mathbf{A}})$ is a finite set of points) and 0 is the returned value.
Assume now that $d>0$. Since $\mathbf{A} \in \Gamma_1$ by assumption, Proposition 3 implies that for any $i \le d$, $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \neq \emptyset$. Also, by

definition of the real dimension of a real algebraic set, for any $i > d$, $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) = \emptyset$ holds.
As long as $i \leq d$ and $n - i > d$, the calls to HasEmptyInterior in Step 3a and Step 3b return respectively false and true and the loop goes on by increasing i.
Finally if $d < \frac{n}{2}$ and $i = d + 1$, then Step 3a returns $i - 1 = d$. If $d \geq \frac{n}{2}$ and $i = n - d \leq \frac{n}{2}$, the call to HasEmptyInterior at Step 3a returns false and Step 3b returns $n - i = d$. □

Correctness of HasEmptyInterior**.** Assume for the moment the following proposition (we prove it in Section 4).

PROPOSITION 5. *There exists a Zariski open set $\Gamma' \subset \mathrm{GL}_n(\mathbb{C})$ such that for any $\mathbf{A} \in \Gamma' \cap \mathrm{GL}_n(\mathbb{Q})$ and for any $1 \leq i < n$, the following holds. Let $I_i(f)$ be the ideal defined in Subsection 3.1.*

1. *Let $x \in V\left(\left\langle \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n} \right\rangle\right) - V\left(\left\langle \frac{\partial f^{\mathbf{A}}}{\partial X_1}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_i} \right\rangle\right)$. Then the Jacobian matrix associated to $\left(\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$ at x has maximal rank and $V(I_i(f^{\mathbf{A}}))$ is either empty or equidimensional of dimension $i - 1$.*
2. $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ *is contained in* $\pi_i(V(I_i(f^{\mathbf{A}})) \cap \mathbb{R}^n)$.

We can now prove Theorem 4. As above, $f \in \mathbb{Q}[X_1, \ldots, X_n]$ is the polynomial given in the input and $V \subset \mathbb{C}^n$ is the algebraic set defined by $f = 0$.
Let $\Gamma' \subset \mathrm{GL}_n(\mathbb{C})$ be the non-empty Zariski open set defined in Proposition 5. Let $\mathbf{A} \in \Gamma' \cap \mathrm{GL}_n(\mathbb{Q})$. Then by assertion (2) of Proposition 5, $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \subset \pi_i(V(I_i(f^{\mathbf{A}})) \cap \mathbb{R}^n)$.
By the elimination theorem [13], we deduce that $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ is contained in $V(I_i(f^{\mathbf{A}}) \cap \mathbb{Q}[X_1, \ldots, X_i]) \cap \mathbb{R}^i$. Recall also that assertion (1) of Proposition 5 implies that $V(I_i(f^{\mathbf{A}}) \cap \mathbb{Q}[X_1, \ldots, X_i])$ has codimension ≥ 1.
So the set L computed at Step 2 contains at least one point in each connected component of the semi-algebraic set defined by $g \neq 0$. Since $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \subset \pi_i(V(I_i(f^{\mathbf{A}})) \cap \mathbb{R}^n)$, L contains at least one point in each connected component of $\mathbb{R}^i - \mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$.
The final step of HasEmptyInterior decides the emptiness of $V^{\mathbf{A}} \cap \mathbb{R}^n \cap \pi_i^{-1}(\alpha)$ for every point $\alpha \in L$.
If $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) = \emptyset$, then $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$ is a subset of its boundary. Then for all $\alpha \in \mathbb{R}^i - \mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$, the set $V^{\mathbf{A}} \cap \mathbb{R}^n \cap \pi_i^{-1}(\alpha)$ is empty and HasEmptyInterior returns true as requested.
If $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \neq \emptyset$, then $\mathsf{Int}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ contains at least one connected component of $\mathbb{R}^i - \mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$. So there exists $\alpha \in L$, such that α lies in this component. We deduce that α lies in the interior of $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$. In other words, we have $\pi_i^{-1}(\alpha) \cap V^{\mathbf{A}} \cap \mathbb{R}^n \neq \emptyset$ and HasEmptyInterior returns false as requested. □

3.3 Complexity analysis

Our complexity analysis relies mainly on the use of algebraic elimination routines from [17]. The complexity of these routines depends polynomially on geometric degrees of algebraic sets. We investigate below the degrees of the geometric objects manipulated by our algorithm. In the whole paragraph, $f \in \mathbb{Q}[X_1, \ldots, X_n]$ denotes the input of RealDimension; it has degree D and the equation $f = 0$ defines the real algebraic set $V \cap \mathbb{R}^n$.

Degrees of algebraic sets. Let $W \subset \mathbb{C}^n$ be a non-empty irreducible algebraic set. The degree $\deg(W)$ of W is defined in [21, Section 2] as the maximal cardinality of a finite set which is obtained by intersecting W with a linear affine subspace. The degree of a reducible closed set is the sum of the degree of its irreducible components.
The complexity of RealDimension and HasEmptyInterior depends on the degree of the objects under study. Let i be an integer such that $0 < i < n$. Let $I_i(f)$ be the ideal defined in Subsection 3.1. In this paragraph we analyze the degree of the algebraic sets defined respectively by the ideal $I_i(f)$, $I_i(f) \cap \mathbb{Q}[X_1, \ldots, X_n]$ and $\pi_i^{-1}(y) \cap V \cap \mathbb{R}^n$ where y is a point returned in Step 2 of HasEmptyInterior.

LEMMA 6. *Let D be the degree of f. Then for $1 \leq i \leq n-1$, the degree of $V(I_i(f))$ and $\pi_i(V(I_i(f)))$ is bounded by D^{n-i+1} and for all $\mathbf{x}_i \in \mathbb{R}^i$, the degree of $V \cap \mathbb{R}^n \cap \pi_i^{-1}(\mathbf{x}_i)$ is bounded by D.*

PROOF. The degree of f is D so $\deg(V) \leq D$. We denote by R_i and J_i the ideals $\left\langle \frac{\partial f}{\partial X_{i+1}}, \ldots, \frac{\partial f}{\partial X_n} \right\rangle$ and $R_i : \left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i} \right\rangle^{\infty}$.
By [13, Section 4.4, Thm. 7 and Prop. 10],

$$V(J_i) = \bigcap_{l=1}^{i} \overline{V(R_i) - V\left(\left\langle \frac{\partial f}{\partial X_l} \right\rangle\right)}.$$

Then the degree of $V(J_i)$ is bounded by the product of the degrees of $\overline{V(R_i) - V\left(\left\langle \frac{\partial f}{\partial X_l} \right\rangle\right)}$, each one bounded by $\deg(V(R_i))$.
With Bezout's inequality [21], the degree of $V(R_i)$ is bounded by $\deg\left(\bigcap_{r=i+1}^{n} V\left(\left\langle \frac{\partial f}{\partial X_r} \right\rangle\right)\right) \leq D^{n-i}$.
Since $I_i(f) = J_i + \langle f \rangle$, then the degree of $V(I_i(f))$ is bounded by $D^{n-i} \cdot D = D^{n-i+1}$ and the bound holds for $\deg(\pi_i(V(I_i(f))))$.
Finally, the degree of a fiber above a point $P = (x_1, \ldots, x_i)$ is the degree of $V(\langle f, X_1 - x_1, \ldots, X_i - x_i \rangle)$ which is D. □

Complexity estimates. Our goal is to establish the following result.

THEOREM 7. *We assume that Algorithm* RealDimension *chooses $\mathbf{A}$ in $\Gamma_1 \cap \Gamma_2 \cap \mathrm{GL}_n(\mathbb{Q})$ where Γ_1 and Γ_2 are the non-empty Zariski open subsets of $\mathrm{GL}_n(\mathbb{C})$ defined in Proposition 3 and Theorem 4. Let d be the real dimension of $V \cap \mathbb{R}^n$, then the number of arithmetic operations needed to compute d is bounded by*

$$\tilde{O}(n^{16}(1 + D)^{3d(n-d)+5n+5}).$$

PROOF. In the sequel, we omit superscripts $\mathbf{A}$ indicating the changes of variables to keep notation simple. Also, the extra cost induced by these changes of variables are negligible compared to the cost of all other steps of the algorithm.
We start by estimating the complexity of HasEmptyInterior(f, i, Id) for an integer $0 < i < n$.

Step 1. By Proposition 5, $I_i(f)$ has dimension $i - 1$ at most. We deduce that $I_i(f) \cap \mathbb{Q}[X_1, \ldots, X_i]$ is the intersection of ideals $\tilde{I}_k$ such that $\tilde{I}_k \cap \mathbb{Q}[X_1, \ldots, X_i]$ has co-dimension k for $1 \leq k \leq i$. Below, we show how to compute $g_k \in \tilde{I}_k \cap \mathbb{Q}[X_1, \ldots, X_i]$ with $\deg(g_k) \leq \deg(V(\tilde{I}_k))$. Since $\deg(V(I_i(f))) = \sum_{k=1}^{i} \deg(V(\tilde{I}_k))$, we deduce that $g = g_1 \cdots g_i$ has degree less than or equal to D^{n-i+1} (Lemma 6). By Proposition 5, the Jacobian matrix associated to $\left(\frac{\partial f}{\partial X_{i+1}}, \ldots, \frac{\partial f}{\partial X_n}\right)$ at x has maximal rank at any point of $W = V\left(\left\langle \frac{\partial f}{\partial X_{i+1}}, \ldots, \frac{\partial f}{\partial X_n} \right\rangle\right) - V\left(\left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i} \right\rangle\right)$. This implies that we can apply lifting algorithms in [17] at points of the above constructible set. Also note that by [38, Section 6.3], for a generic point y in $\mathbb{C}^{i-1}$, $V(I_i(f)) \cap \pi_{i-1}^{-1}(y)$ is finite.
One obtains the first polynomial g_1 in the following way. We first compute generic points in $V(I_i(f)) \cap \pi_{i-1}^{-1}(y)$ where y is a generic point in $\mathbb{Q}^{i-1}$ using [17]; this is possible because Proposition 5 shows that assumptions required in [17] are satisfied. Next, we project those points on the X_i-coordinate (see e.g. [36, Lemma 10.5.5]) and repeat the process as many times as necessary to perform a multivariate interpolation. Since g_1 has degree D^{n-i+1}, we need $(D^{n-i+1} + 1)^i \leq (D + 1)^{i(n-i)}$ interpolation points. Due to the lack of space, we cannot enter into the details. Combining the complexity estimates in [17] and [36, Chap. 10], we get that this is done in time $\tilde{O}(n^5 D^{i(n-i)+3n-i+4})$.
We now show how to compute g_2. Note that, choosing $y \in \mathbb{Q}^{i-2}$ generically, $V(I_i(f)) \cap \pi_{i-2}^{-1}(y)$ has dimension 1. We start by computing generic points in this set, i.e. its intersection with a hyperplane H. This is done in two steps. We first obtain a generic point W and next use the lifting procedure [17, Lemma 3] to get a lifting curve that is finally intersected with the hypersurface defined by $f = 0$ [17,

Lemma 16]. Remark that repeating the computation with a different H allows us to select, from these generic points, those who actually lie in $V(\tilde{I}_2)$. Finally we project those points who lie in $V(\tilde{I}_2)$ on the X_{i-1}-coordinate using again [36, Lemma 10.5] and use again multivariate interpolation to finally reconstruct g_2. The cost of this step is done in the same complexity bound as above.

Other polynomials g_k are obtained similarly. All in all, there are $O(n)$ such steps to perform at most.

The total cost is $\tilde{O}(n^6 D^{i(n-i)+3n-i+4})$ and $\deg(g) \leq D^{n-i+1}$.

Step 2. We estimate the complexity to compute a set L of points meeting each connected component of the semi-algebraic set defined by $g \neq 0$ in $\mathbb{R}^i$ where $g \in \mathbb{Q}[X_1, \dots, X_i]$ has degree bounded by D^{n-i+1}. To compute such a set, we take the projection over $\mathbb{R}^i$ of a set of points meeting each connected component of the real algebraic set in $\mathbb{R}^{i+1}$ defined by $gY - 1 = 0$ (where Y is a new indeterminate). The degree of this set is bounded by $(D^{n-i+1}+1)^{i+1}$. It is straightforward to see that $gY-1$ is square-free and the algebraic set it defines is smooth and equidimensional. By [35, Theorem 3], this is done using $\tilde{O}(n^{16}(D^{n-i+1}+1)^{3i+5}) \subset \tilde{O}(n^{16}(1+D)^{3(n-i)i+5n-2i+5})$ arithmetic operations in $\mathbb{Q}$ at most.

Step 3. By [32, Theorem 4], deciding the emptiness of the real algebraic set defined by $f(x_1, \dots, x_i, X_{i+1}, \dots, X_n) = 0$ where $(x_1, \dots, x_i)$ is in L is done using $O((nD^{n-i}+n^4)n^4D^{2(n-i+1)})$ arithmetic operations in $\mathbb{Q}$ at most. By [35, Theorem 3], the number of points returned by Step 2 is bounded by $O((1+D^{n-i+1})^{i+1})$. We deduce that Step 3 uses $O(n^4(nD^n+n^4)(1+D)^{i(n-i)+3n-i+3})$ arithmetic operations in $\mathbb{Q}$ at most. Note that this step is negligible compared to the complexity of Step 2.

Finally, $\mathsf{HasEmptyInterior}(f, i, \mathrm{Id})$ runs in time

$$\tilde{O}(n^{16}(1+D)^{3i(n-i)+5n-2i+5})$$

Complexity of RealDimension**.** By [32, Theorem 4], one can decide the emptiness of $V \cap \mathbb{R}^n$ in Step 1 in probabilistic time $O((nD^n + n^4)n^4D^{2(n+1)})$.

Finally, Step 3 of RealDimension is a loop from 1 to $\lceil \frac{n}{2} \rceil$ calling HasEmptyInterior with inputs (f, i, Id) and $(f, n-i, \mathrm{Id})$. These calls require respectively at most $\tilde{O}(n^{16}(1+D)^{3i(n-i)+5n-2i+5})$ and $\tilde{O}(n^{16}(1+D)^{3i(n-i)+3n+2i+5})$ arithmetic operations. Since this loop stops when i or $n-i$ is equal to d, the complexity of Step 3 and then of RealDimension is bounded by $\tilde{O}(n^{16}(1+D)^{3d(n-d)+5n+5})$. □

4. PROOF OF PROPOSITION 5

Strategy of proof. In Subsection 4.1, we prove the existence of a non-empty Zariski open subset $\Gamma_3 \subset \mathrm{GL}_n(\mathbb{C})$ such that, for any $\mathbf{A} \in \Gamma_3 \cap \mathrm{GL}_n(\mathbb{Q})$, assertion (1) of Proposition 5 holds. In Subsection 4.2, we prove the existence of a non-empty Zariski open subset $\Gamma_4 \subset \mathrm{GL}_n(\mathbb{C})$ such that, for any $\mathbf{A} \in \Gamma_4 \cap \mathrm{GL}_n(\mathbb{Q})$, assertion (2) of Proposition 5 holds. Taking $\Gamma' = \Gamma_3 \cap \Gamma_4$ ends the proof.

4.1 Proof of assertion (1) of Proposition 5

This proof is widely inspired from [1, Prop. 3].

Consider the map Φ_i defined for every $(y, \mathbf{a}) = (y, (a_{k,l})) \in \mathbb{C}^n \times \mathbb{C}^{n(n-i)}$ by $\Phi_i(y, \mathbf{a}) = \left(\sum_{k=1}^n a_{k,j} \frac{\partial f}{\partial Y_k}(y)\right)_{i+1 \leq j \leq n} \in \mathbb{C}^{n-i}$ and, for $\mathbf{a} \in \mathbb{C}^{n(n-i)}$, its restriction $\Phi_{i,\mathbf{a}} : y \in \mathbb{C}^n \to \Phi_i(y, \mathbf{a}) \in \mathbb{C}^{n-i}$.

The Jacobian matrix of Φ_i with respect to $Y_1, \dots, Y_n, a_{1,i+1}, a_{2,i+1}, \dots, a_{n,n}$ at the point $\alpha = (y, (a_{k,l}))$ is the matrix

$$\begin{pmatrix} * & \cdots & * & \frac{\partial f}{\partial Y_1} & \cdots & \frac{\partial f}{\partial Y_n} & 0 & \cdots & \cdots & 0 \\ \vdots & & \vdots & 0 & \cdots & 0 & \ddots & 0 & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & \cdots & 0 & \frac{\partial f}{\partial Y_1} & \cdots & \frac{\partial f}{\partial Y_n} \end{pmatrix}.$$

Let $\mathcal{U}_i \subset \mathbb{C}^n$ be the Zariski open set defined as the set of points such that at least one of the first i partial derivatives of f does not vanish. Let α be in $(y, a) \in \mathcal{U}_i \times \mathbb{C}^{n(n-i)}$. The Jacobian matrix has maximal rank at α, since otherwise all the partial derivatives of f vanish at y and since y lies in $\mathcal{U}_i$, this is impossible. We deduce that α is a regular point of Φ_i, which implies that Φ_i is transversal to the origin $\mathbf{0}$.

By the Weak Transversality Theorem of Thom-Sard [36, Proposition 4.2.2], there exists a Zariski dense subset $\mathcal{O}_i \subset \mathbb{C}^{n(n-i)}$ such that for all $\mathbf{a} = (a_{k,l})$ in $\mathcal{O}_i \cap \mathbb{Q}^{n(n-i)}$, the map $\Phi_{i,\mathbf{a}} : \mathcal{U}_i \to \mathbb{C}^{n-i}$ is transversal to the origin.

Let Γ_3 be the non-empty Zariski subset of $\mathrm{GL}_n(\mathbb{C})$ defined as the set of matrices of $\mathrm{GL}_n(\mathbb{C})$ such that the $n-i$ last columns lie in $\mathcal{O}_i$. Let $\mathbf{A} \in \Gamma_3 \cap \mathrm{GL}_n(\mathbb{Q})$ and $\mathbf{a} \in \mathcal{O}_i$ be the $n-i$ last columns of $\mathbf{A}$.

We denote by $\mathcal{J}$ the Jacobian matrix of

$$\left(\sum_{k=1}^n a_{k,i+1} \frac{\partial f}{\partial X_k}, \dots, \sum_{k=1}^n a_{k,n} \frac{\partial f}{\partial X_k}\right).$$

Then, by the Jacobian Criterion, for all x in

$$V\left(\sum_{k=1}^n a_{k,i+1} \frac{\partial f}{\partial X_k}, \dots, \sum_{k=1}^n a_{k,i+1} \frac{\partial f}{\partial X_k}\right) - V\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_i}\right),$$

the matrix $\mathcal{J}$ has maximal rank at x. Since $\frac{\partial f^{\mathbf{A}}}{\partial X_j} = \sum_{k=1}^n a_{k,n} \frac{\partial f}{\partial X_k} \circ \mathbf{A}$, then the Jacobian of $\left(\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right)$ equals $\mathcal{J} \cdot \mathbf{A}$.

Let y be in $V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right) - V\left(\left(\frac{\partial f}{\partial X_1}\right)^{\mathbf{A}}, \dots, \left(\frac{\partial f}{\partial X_i}\right)^{\mathbf{A}}\right)$. Without loss of generality, we assume that $\frac{\partial f}{\partial X_1}(\mathbf{A}y) \neq 0$. We now prove that $y \notin V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_i}\right)$. Otherwise, since $\frac{\partial f^{\mathbf{A}}}{\partial X_j} = \sum_{k=1}^n a_{k,j} \frac{\partial f}{\partial X_k} \circ \mathbf{A}$, for every integer $j \leq n$, $\sum_{k=1}^n a_{k,1} \frac{\partial f}{\partial X_k}(\mathbf{A}y) = 0$ with $\mathbf{A} \in \mathrm{GL}_n(\mathbb{Q})$ and $\frac{\partial f}{\partial X_1}(\mathbf{A}y) \neq 0$ which is impossible.

Let $\mathbf{A} \in \Gamma_3 \cap \mathrm{GL}_n(\mathbb{Q})$ and let K_i be the set

$$\overline{V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_n}\right) - V\left(\frac{\partial f^{\mathbf{A}}}{\partial X_1}, \dots, \frac{\partial f^{\mathbf{A}}}{\partial X_i}\right)},$$

then by the Jacobian Criterion, K_i has dimension i. So, if $K_i \cap V^{\mathbf{A}}$ is non-empty then by Krull's theorem, it has either dimension i or dimension $i-1$. We prove below that $K_i \cap V^{\mathbf{A}}$ has dimension $i-1$ or is empty. In the sequel, we omit the superscript $\mathbf{A}$ to keep simple notations. Of course, we assume that $K_i \neq \emptyset$ in the sequel.

Let $z \in K_i$ be a regular point such that $y \notin V\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_i}\right)$. Without loss of generality, one can assume that z is the origin that we denote by $\mathbf{0}$. Since K_i has dimension i and $\mathbf{0}$ is a regular point of K_i, there exists $\{j_1, \dots, j_i\}$ such that the projection of $T_z K_i$ on the $(X_{j_1}, \dots, X_{j_i})$-space is full dimensional (hence the differential of the restriction of the projection to K_i at $\mathbf{0}$ is surjective). To keep notations as simple as possible, we assume without loss of generality that $\{j_1, \dots, j_i\} = \{1, \dots, i\}$.

By the Implicit Function Theorem, there exist two Zariski open sets $\mathcal{U} \subset \mathbb{C}^i$ and $\mathcal{V} \subset \mathbb{C}^{n-i}$ and there exists a function

$$\phi : \mathbf{x}_i \in \mathcal{U} \to (\phi_{i+1}(\mathbf{x}_i), \dots, \phi_n(\mathbf{x}_i)) \in \mathcal{V}$$

such that for every $(x_1, \dots, x_n) \in \mathcal{U} \times \mathcal{V}$, the following holds

$$(x_1, \dots, x_n) \in K_i \Leftrightarrow (x_{i+1}, \dots, x_n) = (\phi_{i+1}(\mathbf{x}_i), \dots, \phi_n(\mathbf{x}_i)).$$

We define now γ as the following map: $\mathbf{x}_i \in \mathcal{U} \to (\mathbf{x}_i, \phi(\mathbf{x}_i)) \in \mathcal{U} \times \mathcal{V}$. Remark that any point in the image of γ lies in K_i.

We deduce that for all $\mathbf{x}_i = (x_1, \dots, x_i)$,

$$f \circ \gamma(\mathbf{x}_i) = \sum_{j=1}^i \frac{\partial f \circ \gamma}{\partial X_j}(\mathbf{0}) x_j + o(\|\mathbf{x}_i\|),$$

so $f \circ \gamma(\mathbf{x}_i)$ equals to

$$\sum_{j=1}^i \left(\frac{\partial f}{\partial X_j}(\mathbf{0}) + \sum_{k=i+1}^n \frac{\partial f}{\partial X_k}(\mathbf{0}) \frac{\partial \phi_k}{\partial X_j}(\mathbf{0})\right) x_j + o(\|\mathbf{x}_i\|).$$

Now, recall that, by definition of K_i, $\frac{\partial f}{\partial X_j}(\mathbf{0}) = 0$ for every $j > i$, and that $\mathbf{0} \notin V\left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i}\right)$. Without loss of generality, assume that $\frac{\partial f}{\partial X_1}(\mathbf{0}) \neq 0$. Setting $x_2 = \cdots = x_i = 0$, we deduce that

$$f \circ \gamma(x_1, 0, \ldots, 0) = \frac{\partial f}{\partial X_1}(\mathbf{0})x_1 + o(\|\mathbf{x}_i\|)$$

which implies that f is not identically 0 along the curve defined by $\gamma(x_1, 0, \ldots, 0)$ with $(x_1, 0, \ldots, 0) \in \mathcal{U}$. This implies that $K_i \cap V$ is either empty or it has dimension $\dim(K_i) - 1 = i - 1$.

4.2 Proof of assertion (2) of Proposition 5

It relies on the two following lemmas.

LEMMA 8. *There exists a non-empty Zariski open subset $\Gamma_5 \subset \mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A} \in \Gamma_5 \cap \mathrm{GL}_n(\mathbb{Q})$ the following holds. Let $y \in \mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ and let $B_i \subset \mathbb{R}^i$ be a ball centered at y of radius $r > 0$. There exist $x \in V^{\mathbf{A}} \cap \mathbb{R}^n$ such that $\pi_i(x) = y$, a ball $B_n \subset \mathbb{R}^n$ centered at x of radius $r' > 0$ such that $\pi_i(B_n) \subset B_i$ and $y_\varepsilon \in \mathsf{Bd}(\pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}))$ such that $\lim_{\varepsilon\to 0}(y_\varepsilon) = y$.*

Recall from Section 2 that $V_\varepsilon \in \mathbb{C}\langle\varepsilon\rangle^n$ denotes the algebraic set defined by $f - \varepsilon = 0$ and that $W_{\varepsilon,i}^{\mathbf{A}} \subset \mathbb{C}\langle\varepsilon\rangle$ denotes the polar variety defined by $f^{\mathbf{A}} - \varepsilon = \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}} = \cdots = \frac{\partial f^{\mathbf{A}}}{\partial X_n} = 0$.

LEMMA 9. *There exists a non-empty Zariski open subset $\Gamma_6 \subset \mathrm{GL}_n(\mathbb{C})$ such that the following holds. For all $\mathbf{A} \in \Gamma_6 \cap \mathrm{GL}_n(\mathbb{Q})$, for any integer $1 \leq i \leq n-1$, the restriction of π_{i-1} to the polar variety $W_{\varepsilon,i}^{\mathbf{A}}$ is proper.*

PROOF OF ASSERTION (2) OF PROPOSITION 5. Let R_i and J_i be the ideals $\left\langle \frac{\partial f}{\partial X_{i+1}}, \ldots, \frac{\partial f}{\partial X_n} \right\rangle$ and $R_i : \left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i} \right\rangle^\infty$. Let $I_i(f)$ be the ideal $J_i + \langle f \rangle$. First we prove that $\lim_{\varepsilon\to 0}(W_{\varepsilon,i} \cap \mathbb{R}\langle\varepsilon\rangle^n) \subset V(I_i(f))$.
Let $(x, x_\varepsilon) \in \lim_{\varepsilon\to 0}(W_{\varepsilon,i} \cap \mathbb{R}\langle\varepsilon\rangle^n) \times W_{\varepsilon,i}$ such that $\lim_{\varepsilon\to 0}(x_\varepsilon) = x$. Since $\lim_{\varepsilon\to 0}$ is a ring homomorphism and $f \in \mathbb{Q}[X_1, \ldots, X_n]$, $\lim_{\varepsilon\to 0}(f(x_\varepsilon)) = 0 = f(\lim_{\varepsilon\to 0}(x_\varepsilon)) = f(x)$.
Let $h \in J_i$; by definition of J_i, there exists $m \in \mathbb{N}^*$ such that for all $g \in \left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i} \right\rangle$, the polynomial $g^m h$ lies in R_i. Since x_ε lies in $W_{\varepsilon,i}$, $g^m(x_\varepsilon)h(x_\varepsilon) = 0$ for all g in $\left\langle \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_i} \right\rangle$.
Since V_ε is smooth by Lemma 2, there exists $j \in \{1, \ldots, i\}$ such that $\frac{\partial f}{\partial X_j}(x_\varepsilon) \neq 0$. Let g be the polynomial $\frac{\partial f}{\partial X_j}$, then $g(x_\varepsilon) \neq 0$ and $g^m(x_\varepsilon)h(x_\varepsilon) = 0$ so $h(x_\varepsilon) = 0$. Finally, since h has rational coefficients and $\lim_{\varepsilon\to 0}$ is a ring homomorphism, we deduce that $h(\lim_{\varepsilon\to 0}(x_\varepsilon)) = h(x) = 0$. Then x lies in $V(J_i + \langle f \rangle) = V(I_i(f))$. We conclude that $\lim_{\varepsilon\to 0}(W_{\varepsilon,i} \cap \mathbb{R}\langle\varepsilon\rangle^n) \subset V(I_i(f))$.
We claim that there exists a non-empty Zariski open subset $\Gamma_4 \subset \mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A} \in \Gamma_4 \cap \mathrm{GL}_n(\mathbb{Q})$, and for $1 \leq i < n$, $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \subset \pi_i(\lim_{\varepsilon\to 0}(W_{\varepsilon,i}^{\mathbf{A}} \cap \mathbb{R}\langle\varepsilon\rangle^n))$.
Since we already proved $\lim_{\varepsilon\to 0}(W_{\varepsilon,i}^{\mathbf{A}} \cap \mathbb{R}\langle\varepsilon\rangle^n) \subset V(I_i^{\mathbf{A}})$, we deduce that $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \subset \pi_i(V(I_i^{\mathbf{A}}) \cap \mathbb{R}^n)$ which ends the proof of assertion (2) of Proposition 5.
Let Γ_5 be the non-empty Zariski open set of $\mathrm{GL}_n(\mathbb{C})$ defined in Lemma 8. Let Γ_6 be the non-empty Zariski open set of $\mathrm{GL}_n(\mathbb{C})$ defined in Lemma 9. We prove that taking the non-empty Zariski open set $\Gamma_4 = \Gamma_5 \cap \Gamma_6$ allows to prove our claim.
Let $\mathbf{A} \in \Gamma_4 \cap \mathrm{GL}_n(\mathbb{Q})$. Let $y \in \mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ and $B_i \subset \mathbb{R}^i$ be the ball centered at y of radius $r > 0$. Since $\mathbf{A} \in \Gamma_5 \cap \mathrm{GL}_n(\mathbb{Q})$, there exists x, B_n and y as in Lemma 8. In particular, we have $y_\varepsilon \in \mathsf{Bd}(\pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}))$ such that $\lim_{\varepsilon\to 0}(y_\varepsilon) = y$.
By Lemma 2, V_ε is either empty or smooth and equidimensional. By [35, Proposition 4], the set $\mathsf{Bd}(\pi_i(V_\varepsilon^{\mathbf{A}} \cap \mathbb{R}\langle\varepsilon\rangle^n))$ is a subset of $\pi_i(W_{\varepsilon,i}^{\mathbf{A}} \cap \mathbb{R}\langle\varepsilon\rangle^n)$, so $\mathsf{Bd}(\pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}))$ is a subset of $\pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap W_{\varepsilon,i}^{\mathbf{A}})$. By Lemma 9, since $\mathbf{A} \in \Gamma_6 \cap \mathrm{GL}_n(\mathbb{Q})$, the restriction of π_i to $W_{\varepsilon,i}^{\mathbf{A}}$ is proper. We deduce that $\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle)$ contains a neighborhood $\mathcal{O}$ of y_ε such that $\pi_i^{-1}(\overline{\mathcal{O}})$ is closed and bounded. In particular $\pi_i^{-1}(y_\varepsilon) \cap \mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap W_{\varepsilon,i}^{\mathbf{A}}$ is not empty. We let $x_\varepsilon \in \mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap \pi_i^{-1}(y_\varepsilon) \cap W_{\varepsilon,i}^{\mathbf{A}}$. Note that $x_\varepsilon \in \mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle)$ so it is bounded over $\mathbb{R}$; hence $x' = \lim_{\varepsilon\to 0}(x_\varepsilon)$ exists and $x' \in \lim_{\varepsilon\to 0}(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap W_{\varepsilon,i}^{\mathbf{A}}) = B_n \cap \lim_{\varepsilon\to 0}(W_{\varepsilon,i}^{\mathbf{A}})$.
We finish by proving that $\pi_i(x') = y$. We denote by $(x_{\varepsilon,1}, \ldots, x_{\varepsilon,n})$ the coordinates of x_ε. Then $\pi_i(x')$ is the point $\pi_i(\lim_{\varepsilon\to 0}(x_\varepsilon)) = (\lim_{\varepsilon\to 0}(x_{\varepsilon,1}), \ldots, \lim_{\varepsilon\to 0}(x_{\varepsilon,i}))$. So

$$\pi_i(x') = \lim_{\varepsilon\to 0}((x_{\varepsilon,1}, \ldots, x_{\varepsilon,i})) = \lim_{\varepsilon\to 0}(\pi_i(x_\varepsilon))$$

which is equal to $\lim_{\varepsilon\to 0}(y_\varepsilon) = y$.
Finally, we conclude that $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)) \subset \pi_i(\lim_{\varepsilon\to 0}(W_{\varepsilon,i}^{\mathbf{A}}) \cap B_n) \subset \pi_i(\lim_{\varepsilon\to 0}(W_{\varepsilon,i}^{\mathbf{A}} \cap \mathbb{R}\langle\varepsilon\rangle^n))$ as requested □

Proof of Lemma 8. By [22, Proposition 3], there exists a non-empty Zariski open set $\Gamma_5 \subset \mathrm{GL}_n(\mathbb{C})$ such that for all $\mathbf{A} \in \Gamma_5 \cap \mathrm{GL}_n(\mathbb{Q})$ the following holds. For $1 \leq i \leq n-1$ and for any connected component C of $V^{\mathbf{A}} \cap \mathbb{R}^n$, $\pi_i(C)$ is closed. We prove now that Γ_5 satisfies Lemma 8. Let $\mathbf{A}$ be in $\Gamma_5 \cap \mathrm{GL}_n(\mathbb{Q})$.
Assume for the moment the following assertions. (they are proved below).

1. There exists $x \in \pi_i^{-1}(y) \cap V^{\mathbf{A}} \cap \mathbb{R}^n$ such that the ball $B_n \subset \mathbb{R}^n$ centered at x of radius $r' > 0$ satisfies $\pi_i(B_n) \subset B_i$.
2. The ball $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ meets $\mathsf{Bd}\left(\pi_i(\mathrm{Ext}(B_n, \varepsilon) \cap V_\varepsilon^{\mathbf{A}})\right)$.

Let $x \in V^{\mathbf{A}} \cap \mathbb{R}^n$ and B_n be a ball of $\mathbb{R}^n$ as in assertion 1. For every radius $r > 0$, we denote by T_r the set $\pi_i\left(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}\right)$. Then by assertion 2, the set $\mathsf{Bd}(T_r) \cap \mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ is non-empty. It is a closed set as it is the intersections of closed sets. It is bounded over $\mathbb{R}$ since it is subset of $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$. We now consider the set T defined by $\bigcap_{r>0} \mathsf{Bd}(T_r) \cap \mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$. Again, the set T is closed and bounded over $\mathbb{R}$. When T is non-empty, there exists $y_\varepsilon \in T$, bounded over $\mathbb{R}$ that belongs to $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ for any $r > 0$. Then $\lim_{\varepsilon\to 0}(y_\varepsilon)$ exists and equals y.
Now, we prove by contradiction that T is non-empty. Fix $r > 0$ and let η be the distance between y and $\mathsf{Bd}(T_r)$ (defined in [8, Section 3.1]). There are two possible cases: either the distance η is 0 or infinitesimally small (i.e $\eta \in \mathbb{R}\langle\varepsilon\rangle$ such that $0 < \eta < s$, for all $s \in \mathbb{R}$) or there exists $s' > 0$, with $s' \in \mathbb{R}$ such that $\eta > s' > 0$.
In the first case, for all $s > 0$, with $s \in \mathbb{R}$, we have $0 \leq \eta < s$. Since η is the distance between y and $\mathsf{Bd}(T_r)$, there exists $y_\varepsilon \in T_r$ such that the distance η' between y and y_ε satisfies $\eta \leq \eta' \leq \eta + \varepsilon$. Since $\eta + \varepsilon$ is infinitesimal, then for all $s \in \mathbb{R}$ and $s > 0$, $0 < \eta + \varepsilon < s$ and y_ε lies in the ball $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ of radius $s > 0$ and $s \in \mathbb{R}$. We deduce that $y_\varepsilon \in T$ which contradicts our assumption.
Assume now we are in the second case. By assertion 2 there exists $y_\varepsilon \in \mathsf{Bd}(T_{s'}) \cap \mathrm{Ext}(B_i', \mathbb{R}\langle\varepsilon\rangle)$, where B_i' is the ball centered at y of radius s'. So the distance η' between y and y_ε satisfies $\eta \leq \eta' \leq s' < \eta$ and there is a contradiction.
Finally, the two cases contradict the fact that T is empty.

It remains the prove the above assertions.
1. Since y is in the boundary of $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$, the ball B_i meets $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$. We deduce that there exists a connected component C of $V^{\mathbf{A}} \cap \mathbb{R}^n$ such that $\pi_i(C)$ meets B_i. Since $\mathbf{A}$ is in $\Gamma_5 \cap \mathrm{GL}_n(\mathbb{Q})$, $\pi_i(C)$ is closed. So y is in the boundary of $\pi_i(C)$ which is a subset of $\pi_i(C)$. We consider a point $x \in \pi_i^{-1}(y) \cap C$. Let $r' > 0$ be such that the ball B_n of $\mathbb{R}^n$ centered at x of radius r', satisfies $\pi_i(B_n) \subset B_i$. Then $\pi_i^{-1}(y) \cap B_n$ is not empty.

Without loss of generality, we assume that for any r', there exists a point of B_n at which f is positive (else we change f to $-f$).
2. Assume for the moment the following

(a) There exists y_ε in $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ such that

$$\mathrm{Ext}\left(\pi_i^{-1}(y_\varepsilon) \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} \neq \emptyset.$$

(b) There exists y' in B_i such that

$$\mathrm{Ext}\left(\pi_i^{-1}(y') \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} = \emptyset.$$

On the one hand, since $\mathrm{Ext}\left(\pi_i^{-1}(y_\varepsilon) \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} \neq \emptyset$ by assertion (a) and $\pi_i(B_n) \subset B_i$ by assertion (1), we have

$$\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle) \cap \pi_i\left(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}\right) \neq \emptyset.$$

On the other hand, let U be the complementary of $\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}$ in $\mathbb{R}\langle\varepsilon\rangle^n$. Since $\mathrm{Ext}\left(\pi_i^{-1}(y') \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} = \emptyset$ by assertion (b) and since $\mathrm{Ext}(\pi_i^{-1}(y'), \mathbb{R}\langle\varepsilon\rangle) \neq \emptyset$ by assertion (1), we deduce that $\mathrm{Ext}\left(\pi_i^{-1}(y'), \mathbb{R}\langle\varepsilon\rangle\right) \cap U \neq \emptyset$. This implies that $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle) \cap \pi_i(U) \neq \emptyset$.
By [8, Prop 5.24], the set $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ is semi-algebraically connected. It is the disjoint union of $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle) \cap \pi_i(U)$ and

$$\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle) \cap \pi_i\left(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}\right)$$

which are semi-algebraic sets, closed in $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$. So the set $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ meets the boundary of $\pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}})$.

Finally, we prove (a) and (b).
(a). We prove that there exists a point x_ε in $\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}$. Since f is not non-positive over B_n, there exists x' in B_n such that $f^{\mathbf{A}}(x') > 0$. Since $f^{\mathbf{A}}(x') \in \mathbb{R}$, $x \in V^{\mathbf{A}} \cap \mathbb{R}^n$ and ε is an infinitesimal, we deduce that $f^{\mathbf{A}}(x) - \varepsilon < 0$ and $f^{\mathbf{A}}(x') - \varepsilon > 0$.
Let Ψ be the polynomial in $\mathbb{R}\langle\varepsilon\rangle[T]$ defined by $\Psi = f^{\mathbf{A}}(Tx + (1-T)x') - \varepsilon$. Then $\Psi(0) > 0$ and $\Psi(1) < 0$ so by [8, Thm 2.11], there exists $t_0 \in (0,1)$ such that $\Psi(t_0) = 0$. Let x_ε be the point $t_0 x + (1-t_0)x' \in \mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle)$, then $f^{\mathbf{A}}(x_\varepsilon) - \varepsilon = 0$. Then x_ε is in $\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}$.
Let y_ε be $\pi_i(x_\varepsilon)$, then $y_\varepsilon \in \pi_i(\mathrm{Ext}(B_n, \mathbb{R}\langle\varepsilon\rangle)) \subset \mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$. Then x_ε is in $\mathrm{Ext}(\pi_i^{-1}(y_\varepsilon) \cap B_n, \mathbb{R}\langle\varepsilon\rangle) \cap V_\varepsilon^{\mathbf{A}}$, so there exists y_ε in $\mathrm{Ext}(B_i, \mathbb{R}\langle\varepsilon\rangle)$ such that $\mathrm{Ext}\left(\pi_i^{-1}(y_\varepsilon) \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} \neq \emptyset$.

(b). Since y is in $\mathsf{Bd}(\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n))$ by assumption and since $\pi_i(B_n)$ is a neighborhood of y, there exists $y' \in \pi_i(B_n)$ such that y' is not in the Euclidean closure of $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$. We deduce that the distance between y' and $\pi_i(V^{\mathbf{A}} \cap \mathbb{R}^n)$ is positive. We also deduce that the set $F = \pi_i^{-1}(y') \cap B_n$ is not empty.
The distance Δ between F and $V \cap \mathbb{R}^n$ is also positive. Indeed, otherwise, for all $\beta > 0$ there is a point $z \in F$ such that the distance between z and $V \cap \mathbb{R}^n$ is less than β. Then $\Delta < \beta$ for all $\beta > 0$. This implies that $\Delta = 0$ and F meets the closure of $V \cap \mathbb{R}^n$ which contradicts the fact that y' is not in the closure of the projection.
Since $F \subset B_n$ is closed and bounded, then the polynomial function $x \mapsto f(x)$ reach its lower bound δ at $z_0 \in F$ and its upper bound δ' at $z_0' \in F$. Since $\Delta > 0$, either $\delta > 0$ or $\delta' < 0$.
We assume now $\delta > 0$. Then there exists $\eta > 0$, $\eta \in \mathbb{R}$ such that for all $z' \in F$, $f(z') > \eta$. If we denote the coordinates of y' by $y_1', \ldots, y_i'$, then the semi-algebraic set F can be defined by the equations $X_1 = y_1', \ldots, X_i = y_i'$, the polynomial inequality $d((X_1, \ldots, X_n), y) \leq r$ and both with or without the inequality $f(X_1, \ldots, X_n) > \eta$. Then by [8, Proposition 2.87], for all $z_\varepsilon \in \mathrm{Ext}(F, \mathbb{R}\langle\varepsilon\rangle)$, $f(z_\varepsilon) > \eta > 0$ with $\eta \in \mathbb{R}$. Thus f never equals to $\varepsilon < \eta$ on $\mathrm{Ext}(F, \mathbb{R}\langle\varepsilon\rangle) = \mathrm{Ext}(\pi_i^{-1}(y') \cap B_n, \mathbb{R}\langle\varepsilon\rangle)$. When $\delta' < 0$, the proof is similar. We prove that there exists $\eta' < 0$ such that f is never equals to $\varepsilon > 0 > \eta' > \delta'$ when $\delta' < 0$. In both cases, we deduce that $\mathrm{Ext}\left(\pi_i^{-1}(y') \cap B_n, \mathbb{R}\langle\varepsilon\rangle\right) \cap V_\varepsilon^{\mathbf{A}} = \emptyset$.
Proof of Lemma 9. This lemma is a generalization to $\mathbb{C}\langle\varepsilon\rangle$ of [35, Theorem 1].

PROOF. The proof of [35, Theorem 1] holds if the base field is $\mathbb{C}\langle\varepsilon\rangle$ instead of $\mathbb{C}$. This theorem can be restated as follows: *There exists a Zariski open set Γ in $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle)$ such that for $\mathbf{A}$ in Γ and $i \in \{1, \ldots, n-1\}$, the restriction of π_i to the i-th polar variety $W_{\varepsilon,i}^{\mathbf{A}}$ associated to $f - \varepsilon$ is proper.* We now prove that the previous property holds over a non-empty Zariski open subset of $\mathrm{GL}_n(\mathbb{C})$.
The proof of [35, Theorem 1] uses n^2 new indeterminates denoted by $\mathcal{A}_{1,1}, \ldots, \mathcal{A}_{n,n}$, and characterizes the matrices of $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle)$ which do not satisfy the properness property as the set of solutions of a polynomial system $G_1 = 0, \ldots, G_s = 0$ of $\mathbb{C}\langle\varepsilon\rangle[\mathcal{A}_{1,1}, \ldots, \mathcal{A}_{n,n}]$. Let G be the product of $G_1, \ldots, G_s$. The polynomials $G_1, \ldots, G_s$ are defined as generators of the prime components of the radical of an ideal generated by $f - \varepsilon \in \mathbb{Q}(\varepsilon)[X_1, \ldots, X_n]$ and some minors of the Jacobian matrix of $f - \varepsilon$ also in $\mathbb{Q}(\varepsilon)[X_1, \ldots, X_n]$ (see [35, Sections 2.3 and 2.4]) so their coefficients are in $\mathbb{Q}(\varepsilon)$ and the coefficients of G too. Let Ω be the non-empty Zariski open set of $\mathrm{GL}_n(\mathbb{C}\langle\varepsilon\rangle)$ defined as the complementary of this set of matrices.
If we multiply G by the least common multiple of the denominators of its coefficients, we obtain a polynomial with coefficients in $\mathbb{Q}[\varepsilon]$. Let P be the primitive part of this polynomial. Let P_0 be the polynomial with rational coefficients obtained by replacing ε by 0 in P, then $P - P_0$ can be factorized as $P - P_0 = \varepsilon^\nu P_\varepsilon$ with P_ε with coefficients in $\mathbb{Q}[\varepsilon]$ and $\nu > 0$ as large as possible. Hence, $P = P_0 + \varepsilon^\nu P_\varepsilon$.
Since the coefficients of P have non-trivial gcd, at least one of the two polynomials P_0 and P_ε is not identically 0. Indeed, if $P_0 = 0$, then since ν is maximal, $P = \varepsilon^\nu P_\varepsilon = \varepsilon^\nu(P_1 + \varepsilon^{\nu'} P_2)$ with P_1 with coefficients in $\mathbb{Q}$ and then $P_1 \neq 0$. Let $\Omega_0 \subset \mathrm{GL}_n(\mathbb{C})$ (resp. $\Omega_1 \subset \mathrm{GL}_n(\mathbb{C})$) be the non-empty Zariski open set defined by $P_0 \neq 0$ (resp. $P_1 \neq 0$).
Let $\mathbf{A}$ be in $(\Omega_0 \cup \Omega_1) \cap \mathrm{GL}_n(\mathbb{Q}) \subset \Omega$, then $P(\mathbf{A}) = P_0(\mathbf{A}) + \varepsilon^\nu P_\varepsilon(\mathbf{A}) \in \mathbb{Q}[\varepsilon]$. We now prove that $P(\mathbf{A}) \neq 0$. If $P_0(\mathbf{A}) = 0$, then $P_\varepsilon(\mathbf{A}) \neq 0$ and then $P(\mathbf{A}) = \varepsilon^\nu P_\varepsilon(\mathbf{A}) \neq 0$. If $P_0(\mathbf{A}) \neq 0$, then, since ε is transcendental, $P(\mathbf{A}) \neq 0$. Then $P(\mathbf{A}) \neq 0$ so the restriction of π_i to $W_{\varepsilon,i}^{\mathbf{A}}$ is proper. Let Γ be the set $\Omega_0 \cup \Omega_1$, then for all matrices $\mathbf{A}$ in $\Gamma \cap \mathrm{GL}_n(\mathbb{Q})$, the properness property holds. □

5. EXPERIMENTS

We report on timings obtained with a first implementation of our algorithm. This is a Maple implementation built-on the RAGlib Maple package [33] and the FGb library [15] written in C by J.-C. Faugère. RAGlib is used for deciding the emptiness of real algebraic sets and computing sample points in each of their connected components. It implements algorithms that essentially run in time $D^{O(n)}$. The library FGb is a state-of-the art library for Gröbner bases computations. We use it for all ideal-theoretic operations required by our algorithm. We also use Gröbner bases computations to check Noether position properties needed for the correctness of the algorithm. This allows us to try sparse linear changes of variables (or avoid them when unnecessary) which is crucial for practical performances.
We established that our algorithm runs in time $D^{O(d(n-d))}$ where d is the dimension of the real algebraic set under study. A first goal is to observe if the implementation has a practical behaviour that reflects this complexity. In other words computations should be "easier" when d or $n-d$ is "small" and harder when d is close to $n/2$.
Another goal is to identify if such an implementation can handle examples that are out of reach of the best implementations of Cylindrical Algebraic Decomposition such as QEPCAD [10], the implementation of CAD in Maple [11] or RedLog [39] among others. We report the timings obtained with the Maple implementation of CAD (other mentioned software behave similarly on our test-suite). While it is natural to compare with CAD since it is the unique other implemented technique, remember that CAD provides much more information than the dimension.
The choice of a test-suite is often subjective. With respect to our goals, we have chosen to run the software on sums of squares of random dense polynomials because this allows us to control the dimension of the real algebraic set and identifies if the implementation reflects the $D^{O(d(n-d))}$ complexity. We also have chosen discriminants of characteristic polynomials of linear symmetric matrices (entries are chosen random dense). These are known to be sums of squares

n	s	Degrees	d	CAD	Dim	Step 1	Step 2	Step 3	Fibers
4	1	2	3	0.8 s	4.5 s	/	/	/	/
4	2	2,1	2	∞	54 s	0%	99%	1%	2
4	3	2,1,1	1	∞	41 s	25%	5%	70%	2
4	4	2,1,1,1	0	∞	14 s	12%	1%	87%	3
5	1	2	4	0.6 s	4 s	/	/	/	/
5	2	2,1	3	∞	92 s	0%	99.4%	0.6%	4
5	3	2,1,1	2	∞	94 s	6%	88%	6%	1
5	4	2,1,1,1	1	∞	54 s	7%	13%	80%	4
5	5	2,1,1,1,1	0	∞	83 s	3%	0%	97%	3
6	1	2	5	7.5 s	4.8 s	/	/	/	/
6	2	2,1	4	∞	116 s	0%	99.8%	0.2%	1
6	3	2,1,1	3	∞	140 s	2%	95%	3%	2
6	4	2,1,1,1	2	∞	190 s	3%	95%	2%	1
6	5	2,1,1,1,1	1	∞	122 s	3%	19%	78%	4
6	6	2,1,1,1,1,1	0	∞	120 s	1%	0%	99%	3
7	1	2	6	3.1 s	4.7 s	/	/	/	/
7	2	2,1	5	∞	223 s	0%	99%	1%	6
7	3	2,1,1	4	∞	2278 s	1%	96%	3%	7
7	4	2,1,1,1	3	∞	103190 s	0%	100%	0%	3
7	5	2,1,1,1,1	2	∞	2202 s	2.5%	95%	2%	4
7	6	2,1,1,1,1,1	1	∞	507 s	2%	19%	79%	5
7	7	2,1,1,1,1,1,1	0	∞	355 s	1%	0%	99%	3
8	1	2	7	24.2 s	7.6 s	/	/	/	/
8	2	2,1	6	∞	2352 s	0%	99.5%	0.5%	8
8	3	2,1,1	5	∞	∞				
8	4	2,1,1,1	4	∞	∞				
8	5	2,1,1,1,1	3	∞	∞				
8	6	2,1,1,1,1,1	2	∞	35377 s	1%	99%	0%	1
8	7	2,1,1,1,1,1,1	1	∞	1230 s	2%	36%	62%	3
8	8	2,1,1,1,1,1,1,1	0	∞	1002 s	0.2%	0%	99.8%	3
4	2	2,2	2	∞	34 s	0.1%	97%	2.9%	8
4	3	2,2,1	1	∞	54 s	10%	74%	16%	6
4	4	2,2,1,1	0	∞	25 s	19%	2%	79%	3
5	2	2,2	3	∞	79 s	0%	99.7%	.3%	1
5	3	2,2,1	2	∞	8003 s	3.8%	96.1%	.1%	8
5	4	2,2,1,1	1	∞	4773 s	3.1%	96.6%	0.3%	6
5	5	2,2,1,1,1	0	∞	284 s	15%	0%	85%	3
6	2	2,2	4	∞	2477 s	0%	100%	0%	1
6	3	2,2,1	3	∞	∞				
6	4	2,2,1,1	2	∞	∞				
6	5	2,2,1,1,1	1	∞	152500 s	3.4%	96.6%	0%	6
6	6	2,2,1,1,1,1	0	∞	6376 s	11%	0%	89%	3

Table 1: Sum of square of s random polynomials in n variables.

[25]. We finally consider sparse polynomials from [20] and [30] that are known to be non-negative over the reals but which have real roots (this implies that the dimension of the real algebraic set they define is less than $n-1$).

The computations were performed on an Intel Xeon E7540 @ 2.00 Ghz and 250GB of RAM. All timings are given in seconds. The symbol ∞ means that the computation has been stopped after 48 hours.

Random dense polynomials. Timings are given in Table 1. We consider of sums-of-squares of s n-variate polynomials. The degrees of the polynomials are given and the computed dimension as well. The column CAD indicates the timing of the Maple CAD implementation. Those for our implementation are given in the column Dim. We also give the relative part of the three main steps of HasEmptyInterior: computation of $I_i^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_i]$, computing sample points outside the zero locus of the obtained polynomial and finally decide the emptiness of fibers; we indicate their number.

It appears that the practical behaviour of our implementation reflects the complexity of our algorithm: for fixed n, timings are longer when d approaches $n/2$ while they are smaller for $d = 1$ or $n-1$.

CAD implementations don't have this behaviour and seem to be essentially sensitive to the number of variables. Hence, *on this benchmark*, our implementation tackles more examples than CAD does.

Discriminant of characteristic polynomials of linear symmetric matrices. For linear matrices of size 2, the associated polynomials have degree 2. Both our implementation and CAD manage to compute the real dimension up to more than 45 variables, and in this specific case the CAD is faster than our implementation. But in Table 2 we see that when the size of the problem increases (through the size of the matrix and hence the input degree), our implementation outperforms CAD implementations when $n > 3$. In Table 2, k is the size of the matrix, n is the number of variables, D is the degree of the polynomial, and d is the real dimension of the real algebraic set.

Other examples. We finally considered the following polynomials.

1. $g_n := (\sum_{i=1}^n x_i^2)^2 - 4\sum_{i=1}^{n-1} x_i^2 x_{i+1}^2 - 4x_1^2 x_n^2$, non-negative for $n \geq 4$
2. $f_n := \prod_{i=1}^n (x_i^2 + n - 1) - n^{n-2}(\sum_{i=1}^n x_i)^2$.

Timings are given in Table 5. We observe a similar behaviour than the ones already observed.

k	n	D	d	CAD	Dim	Step 1	Step 2	Step 3	Fibers
3	3	6	1	∞	8.5 s	20%	52%	28%	6
3	4	6	2	∞	81 s	3%	95%	2%	11
3	5	6	3	∞	739 s	5%	90%	5%	29
3	6	6	4	∞	28867 s	12%	87%	1%	16
4	3	12	1	∞	1816 s	21%	79%	0%	15
4	4	12	2	∞	∞				

Table 2: Discriminant of characteristic polynomials.

	n	D	d	CAD	Dim	Step 1	Step 2	Step 3	Fibers
g_3	3	4	2	0.98 s	0.18 s	/	/	/	/
g_4	4	4	3	0.54 s	2.1 s	/	/	/	/
g_5	5	4	2	24.6 s	42 s	3%	2%	95%	14
g_6	6	4	2	∞	452 s	13%	43%	44%	49
g_7	7	4		∞	∞				
f_3	3	6	1	2.68 s	0.82 s	5%	30%	65%	2
f_4	4	8	1	65 s	10 s	34%	2%	64%	2
f_5	5	10	1	∞	650 s	11%	0%	89%	2
f_6	6	12	1	∞	∞				

Table 3: Series 1 and 2.

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