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Probabilistic Analysis of Directed Polymers in a Random Environment: a Review

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Abstract.

Directed polymers in random environment can be thought of as a model of statistical mechanics in which paths of stochastic processes interact with a quenched disorder (impurities), depending on both time and space. We review here main results which have been obtained during the last fifteen years, with proofs to most of the results. The material covers the diffusive behavior of the polymers in weak disorder phase studied by J. Imbrie, T. Spencer, E. Bolthausen, R. Song and X. Y. Zhou [11, 3, 25], and localization of the paths in strong disordered phase recently obtained by P. Carmona, Y. Hu, and the authors of the present article [4, 5].

CONTENTS

1. Introduction

1.1. Physical background

We start with an informal description of the situation we will discuss in these notes. Imagine a hydrophilic polymer chain wafting in water. Due to the thermal fluctuation, the shape of the polymer should be understood as a random object. We now suppose that the water contains randomly placed hydrophobic molecules as impurities, which repel the

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hydrophilic monomers which the polymer consists of. The question we address here is:

(1.1)
How do the impurities affect the global shape of the polymer chain?

We try to answer this question in a mathematical framework. However, as is everywhere else in mathematical physics, it is very difficult to do so without compromising with a rather simplified picture of the real world. Here, our simplification goes as follows. We first suppress entanglement and U-turns of the polymer; we shall represent the polymer chain as a graph $\{(j, \omega_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$, so that the polymer is supposed to live in $(1+d)$-dimensional discrete lattice and to stretch in the direction of the first coordinate. Each point $(j, \omega_j) \in \mathbb{N} \times \mathbb{Z}^d$ on the graph stands for the position of j-th monomer in this picture. Secondly, we assume that, the transversal motion $\{\omega_j\}_{j=1}^n$ performs a simple random walk in $\mathbb{Z}^d$, if the impurities are absent. We then define the energy of the path $\{(j, \omega_j)\}_{j=1}^n$ by

$$-\beta \sum_{j=1}^{n} \eta(j, \omega_j) \ , \tag{1.2}$$

where $\beta = 1/T > 0$ is the inverse temperature and $\{\eta(n,x) : n \geq 1, \ x \in \mathbb{Z}^d\}$ is a real i.i.d. random variables, with $\eta(n,x)$ describing the presence (or strength) of an impurity at site (n,x). The typical shape $\{(j, \omega_j)\}_{j=1}^n$ of the polymer is then given by the one that minimizes the energy (1.2). Let us suppose for example that $\eta(n,x)$ takes two different values $+1$ ("presence of a water molecule at (n,x)") and -1 ("presence of the hydrophobic impurity at (n,x)"). Then, the energy of the polymer is increased by $+\beta$ each time a monomer is in contact with the impurity ($\eta(j, \omega_j) = -1$). Therefore, the typical shape of the polymer for each given configuration of $\{\eta(j, \ x)\}$ is given by the one which tries to avoid the impurities as much as possible. The purpose of these notes is to introduce rigorous results which answer (1.1) roughly as follows.

- **(a):** If $d \geq 3$ and β small enough, the impurities do not affect the global shape of the polymer (*the weak disorder phase*).
- **(b):** If either (i): $d \leq 2$ and $\beta \neq 0$ or (ii): $d \geq 3$ and β large enough[1], then, the impurities change the global shape of the polymer drastically (*the strong disorder phase*).

[1]To be precise, there are some exceptions. See Remark 2.2.1 below.

1.2. Simple random walk model for directed polymers

We now put the informal description given in section 1.1 into a mathematical framework. As we mentioned before, the framework can be thought of as a model in statistical mechanics. However, no prior knowledge of statistical mechanics is needed in this paper. The model we consider here is defined as a random walk in a random environment. We first fix notation for the random walk and the random environment. Then, we introduce the polymer measure.

• *The random walk:* $(\{\omega_n\}_{n\geq 0}, P)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$. More precisely, we let Ω be the path space $\Omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0\}$, $\mathcal{F}$ be the cylindrical σ-field on Ω, and, for all $n \geq 0$, $\omega_n : \omega \mapsto \omega_n$ be the projection map. We consider the unique probability measure P on $(\Omega, \mathcal{F})$ such that $\omega_1 - \omega_0, \dots, \omega_n - \omega_{n-1}$ are independent and

$$P\{\omega_0 = 0\} = 1, \quad P\{\omega_n - \omega_{n-1} = \pm\delta_j\} = (2d)^{-1}, \quad j = 1, 2, \dots, d,$$

where $\delta_j = (\delta_{kj})_{k=1}^d$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, $P[X]$ denotes the P-expectation of a r.v.(random variable) X on $(\Omega, \mathcal{F}, P)$.

• *The random environment:* $\eta = \{\eta(n,x) : n \in \mathbb{N},\ x \in \mathbb{Z}^d\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.'s defined on a probability space $(H, \mathcal{G}, Q)$ such that

$$Q[\exp(\beta\eta(n,x))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.3}$$

Here, and in the sequel, $Q[Y]$ denotes the Q-expectation of a r.v. Y on $(H, \mathcal{G}, Q)$.

• *The polymer measure:* For any $n > 0$, define the probability measure μ_n on the path space $(\Omega, \mathcal{F})$ by

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right) P(d\omega), \tag{1.4}$$

where $\beta > 0$ is a parameter (the inverse temperature) and

$$Z_n = P\left[\exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right)\right] \tag{1.5}$$

is the normalizing constant (the partition function).

The polymer measure μ_n can be thought of as a Gibbs measure on the path space $(\Omega, \mathcal{F})$ with the Hamiltonian (1.2). We stress that the random environment η is contained in both Z_n and μ_n without being integrated out, so that they are r.v.'s on the probability space $(H, \mathcal{G}, Q)$. The polymer is attracted to sites where the random environment is positive, and repelled by sites where the environment is negative.

Remark 1.2.1. This model was originally introduced in physics literature [10] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [11, 3], where it was reformulated as above.

Here are two standard choices for the distribution of $\eta(n, x)$.

Example 1.2.1. *Bernoulli environment* ([3, 11, 25]); This is the case with

$$Q\{\eta(n,x) = -1\} = p > 0, \quad Q\{\eta(n,x) = +1\} = 1 - p > 0.$$

In the physical picture described in section 1.1, $\eta(n, x) = -1$ (resp. $\eta(n, x) = +1$) can be interpreted as the presence of a hydrophobic impurity (resp. a water molecule) at site (x, n).

Example 1.2.2. *Gaussian environment* ([4]); This is the case in which $\eta(n, x)$ is a standard normal random variable;

$$Q\{\eta(n,x) \in dt\} = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt.$$

2. Some typical results for the simple random walk model

In this section, we present some typical results for the simple random walk model. Here, we focus on the conceptual issues on these results and do not go into the proofs.

We now introduce an important quantity for this model, which appears in the assumptions of the results we present. Let $\lambda(\beta)$ be the logarithmic moment generating function of $\eta(n, x)$,

$$\lambda(\beta) = \ln Q[\exp(\beta\eta(n,x))], \quad \beta \in \mathbb{R}. \tag{2.1}$$

The function $\lambda(\beta)$ can be explicitly computed for some typical choice of the distribution of $\eta(n, x)$. For example, $\lambda(\beta) = \ln(pe^{-\beta} + (1-p)e^{\beta})$ for the Bernoulli environment (Example 1.2.2) and $\lambda(\beta) = \frac{1}{2}\beta^2$ for the Gaussian environment (Example 1.2.2).

2.1. The weak disorder phase

The results we present in this subsection show that the impurities do not change the transversal fluctuation of the polymer if $d \geq 3$ and β is small enough. We first recall the following fact about the return probability π_d for the simple random walk:

$$\pi_d \stackrel{\text{def.}}{=} P\{\omega_n = 0 \text{ for some } n \geq 1\} \begin{cases} = 1 & \text{if } d \leq 2, \\ < 1 & \text{if } d \geq 3. \end{cases} \tag{2.2}$$

More precisely, it is known that $\pi_{d+1} < \pi_d$ for all $d \geq 3$ [22, Lemma 1] and that $\pi_3 = 0.3405...$ [26, page 103]. In particular, $\pi_d \leq 0.3405...$ for all $d \geq 3$.

Theorem 2.1.1. **(The diffusive behavior;** [11, 3, 25]) *Suppose that $d \geq 3$ (hence $\pi_d < 1$) and that*

$$\gamma_1(\beta) \stackrel{\text{def.}}{=} \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d). \tag{2.3}$$

Then,

$$\lim_{n \nearrow \infty} \mu_n[|\omega_n|^2]/n = 1 \quad Q\text{-}a.s. \tag{2.4}$$

Note that $\gamma_1(\beta)$ is increasing on $[0, \infty)$ and $\gamma_1(0) = 0$ so that the condition in (2.3) does hold if β is small. Proof of Theorem 2.1.1 is given in section 3.2

Example 2.1.1. Consider the Bernoulli environment (Example 1.2.1). In this case, it is not difficult to see from direct computations that $\lim_{\beta \nearrow \infty} \gamma_1(\beta) = -\ln(1-p)$. This shows that (2.3) holds for all $\beta \geq 0$ if $p < 1 - \pi_d$.

Example 2.1.2. Consider the Gaussian environment (Example 1.2.2). Then, $\gamma_1(\beta) = \beta^2$ and hence (2.3) holds if $\beta < \sqrt{\ln(1/\pi_d)}$.

Remark 2.1.1. The first rigorous proof of Theorem 2.1.1 was obtained by J. Z. Imbrie and T. Spencer [11] in the case of Bernoulli environment. Soon afterwards, a more transparent proof based on the martingale analysis was given by E. Bolthausen [3]. The martingale proof was then extended to general environment under condition (2.3) by R. Song and X. Y. Zhou [25]. By the argument in [3, 25], it is possible to get a much more precise statement than (2.4). In fact, under the same assumption in Theorem 2.1.1, the following central limit theorem holds;

for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity,

(2.5)
$$\lim_{n \nearrow \infty} \mu_n \left[f\left(\omega_n/\sqrt{n}\right)\right] = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f\left(x/\sqrt{d}\right) \exp(-|x|^2/2)dx, \quad Q\text{-a.s.}$$

The diffusive behavior (2.4) follows from (2.5) by choosing $f(x) = |x|^2$. In [3], (2.5) is obtained for the Bernoulli environment only. However, with the help of the observation made in [25], it is not difficult to extend the central limit theorem to general environment under the assumption in Theorem 2.1.1. We will sketch the proof of (2.5) in Remark 3.2.4 below.

We now recall the following well known fact for the simple random walk, i.e., the case of $\beta = 0$;

$$\max_{x \in \mathbb{Z}^d} P\{\omega_n = x\} = \mathcal{O}(n^{-d/2}), \quad \text{as } n \nearrow \infty. \tag{2.6}$$

The decay rate $n^{-d/2}$ in (2.6) can be understood as a manifestation of the fact that the possible position of ω_n is spread over a ball in $\mathbb{Z}^d$ with radius const. $\times \sqrt{n}$.

For $\beta \neq 0$, we can still prove (2.6) in a weaker form as follows.

Theorem 2.1.2. (**Delocalization;**[4, 5]) *Suppose that $d \geq 3$ and that β is small enough so that (2.3) holds. Then,*

$$\sum_{n \geq 1} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\}^2 < \infty, \quad Q\text{-a.s.} \tag{2.7}$$

and thus,

$$\lim_{n \nearrow \infty} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} = 0, \quad Q\text{-a.s.} \tag{2.8}$$

Proof of Theorem 2.1.2 is given in section 3.3.

Remark 2.1.2. Theorem 2.1.2 was obtained for Gaussian environment by P. Carmona and Y. Hu [4] and then for general environment by F. Comets, T. Shiga and N. Yoshida [5].

2.2. The strong disorder phase

The result we present in this subsection shows that the behavior of the polymer is quite different from the usual random walk if either (i) $d = 1, 2$

and $\beta \neq 0$ or (ii) $d \geq 3$ and β is large[2] For this model, it is rather recent that the phenomena of this kind began to be studied rigorously.

We now present a result which is in sharp contrast with (2.6) and (2.8).

Theorem 2.2.1. **(Localization to the favorite sites** [4, 5]) *Suppose either that*

(i): $d = 1, 2$ *and* $\beta \neq 0$ *or*
(ii): $d \geq 1$ *and*

$$\gamma_2(\beta) \stackrel{\text{def.}}{=} \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d). \tag{2.9}$$

Then, there exists a constant $c = c(d, \beta) > 0$ *such that*

$$\varlimsup_{n \nearrow \infty} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} \geq c, \quad Q\text{-}a.s. \tag{2.10}$$

The bound (2.10) suggests that the position of ω_n, viewed under the polymer measure μ_{n-1}, is concentrated at a small region (the "favorite sites") with the size $\mathcal{O}(1)$ as $n \nearrow \infty$.

Note that γ_2 is increasing on $[0, \infty)$ and therefore that (2.9) holds for large enough β if

$$\lim_{\beta \nearrow \infty} \gamma_2(\beta) > \ln(2d). \tag{2.11}$$

We see from Theorem 2.1.2 and Theorem 2.2.1 that, if $d \geq 3$ and (2.11), then a phase transition occurs as β increases from the weak disorder phase to the strong disorder phase.

Theorem 2.2.1 under condition (ii) is proved in section 3.4. For the proof of this theorem under condition (i), we refer the reader to [4, 5].

Remark 2.2.1. For $d \geq 3$, there are exceptional choices of the distribution of $\eta(n, x)$ like the one discussed in Example 2.1.1, for which (2.10) does not hold even for large β (in fact, (2.8) holds for all β); to be on the safe side for this statement, one can consider unbounded environments, or bounded ones without mass on the top point of its support. In this case, one has (2.11), and hence (2.9) for large enough β. See Example 2.2.1 and Example 2.2.2 below.

Example 2.2.1. Consider the Bernoulli environment (Example 1.2.1). Then, it is not difficult to see from direct computations that $\lim_{\beta \nearrow \infty} \gamma_2(\beta) = \ln(1/(1-p))$. This shows that (2.9) holds for large enough β if $p > 1 - \frac{1}{2d}$.

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[2]This is again, up to some exceptions See Remark 2.2.1 below.

Example 2.2.2. Consider the Gaussian environment (Example 1.2.1). Then, $\gamma_2(\beta) = \beta^2/2$ and hence (2.9) holds if $\beta > \sqrt{2\ln(2d)}$.

Remark 2.2.2. Theorem 2.2.1 was obtained for Gaussian environment by P. Carmona and Y. Hu [4] and then for general environment by F. Comets, T. Shiga and N. Yoshida [5].

2.3. The normalized partition function and its positivity in the limit

We now introduce an important process on $(H, \mathcal{G}, Q)$ ((2.12) below), which is a martingale in fact. The large time behavior of this process characterizes the phase diagram of this model and for this reason, many of results on the model can be best understood from the viewpoint of this process.

Define the *normalized partition function* by

$$W_n = \exp(-\lambda(\beta)n)Z_n, \quad n \geq 1. \tag{2.12}$$

We then have

Lemma 2.3.1. *The limit*

$$W_\infty = \lim_{n \nearrow \infty} W_n \tag{2.13}$$

exists Q-a.s. Moreover, there are only two possibilities for the positivity of the limit;

$$Q\{W_\infty > 0\} = 1 \ , \tag{2.14}$$

or

$$Q\{W_\infty = 0\} = 1 \ . \tag{2.15}$$

The proof of this lemma is standard and is given in section 3.1.

The above contrasting situations (2.14) and (2.15) can be considered as the characterization of the weak disorder phase and the strong disorder phase, respectively. In fact, as are shown in Theorem 3.3.1 below, (2.14) implies (2.7), while (2.15) implies a weaker form of (2.10) that $\sum_{n\geq 1} \max_{x\in\mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} = \infty$, Q-a.s. It is even expected that (2.14) implies (2.5) and that (2.15) implies (2.10).

We close this subsection with the following result, which, in consistency with what we discuss above, describes the basic phase diagram of the model.

Theorem 2.3.2. **(a):** *For $d \geq 3$, (2.3) implies (2.14).*

(b): *Either (i) or (ii) in Theorem 2.2.1 implies (2.15).*

Proofs of Theorem 2.3.2 (a) and (b) are given in Sections 3.2 and 3.4, respectively.

Remark 2.3.1. For $d \geq 3$, it is an interesting question to find a characterization of (2.14) (or (2.15)) in terms of the distribution of $\eta(n,x)$. As is shown in section 3.1, $(W_n)_{n\geq 1}$ is a mean-one, positive martingale on $(H, \mathcal{G}, Q)$. In this respect, this question has somewhat similar flavor to some other topics in the probability theory such as Kakutani's dichotomy for infinite product measure (e.g.,[8, page 244]), nontriviality of the limit of the normalized Galton-Watson process [1] and of multiplicative chaos [14].

3. Martingale analysis on the simple random walk model

This section is devoted to the proofs of the results introduced in the previous one. We define an increasing sequence of sub σ-fields of $\mathcal{G}$ by

$$\mathcal{G}_n = \sigma[\eta(j,x) \; ; \; j \leq n, \; x \in \mathbb{Z}^d], \quad n \geq 1. \tag{3.1}$$

A major technical advantage of the model is that we can relate objects of interest such as

$$\mu_n[|\omega_n|^2] \text{ and } \max_{x\in\mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\}$$

to some martingale on $(H, \mathcal{G}, Q)$ with respect to the filtration $(\mathcal{G}_n)$. As is very easy to guess, what makes this possible is the independence of the environment $\{\eta(n,x)\}$, especially in the time parameter n. We will see from the arguments below, the martingale analysis plays a key role in everything we do.

3.1. Proof of Lemma 2.3.1

We first show that $(W_n)_{n\geq 1}$ is a mean-one, positive $(\mathcal{G}_n)$-martingale on $(H, \mathcal{G}, Q)$. Here and in what follows, we use the following notation.

$$e(n,x) = e(n,x,\eta) = \exp\left(\beta\eta(n,x) - \lambda(\beta)\right), \tag{3.2}$$

$$e_{1,n} = e_{1,n}(\omega,\eta) = \prod_{1\leq j\leq n} e(j,\omega_j). \tag{3.3}$$

Note that $W_n = P[e_{1,n}]$ in this notation. For any fixed $\omega \in \Omega$, $e_{1,n}$ is the product of mean-one i.i.d. random variables on $(H, \mathcal{G}, Q)$ and hence is a mean-one, positive $(\mathcal{G}_n)$-martingale. This implies the martingale property of W_n. By the martingale convergence theorem, the limit W_∞ exists Q-a.s. It is clear that the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{n\geq 1} \sigma[\eta(j,x)\ ;\ j \geq n,\ x \in \mathbb{Z}^d].$$

Therefore by Kolmogorov's zero-one law, only (2.14) and (2.15) are the possibilities. □

3.2. The second moment method

In this subsection, we give proofs to Theorem 2.1.1 and Theorem 2.3.2 (a). The proofs are based on the L^2 analysis of certain martingales on $(H, \mathcal{G}, Q)$. This approach was introduced by E. Bolthausen [3] and then investigated further by R. Song and X. Y. Zhou [25]. We summarize the main step in their analysis as Proposition 3.2.1 below. The proposition deals with a process $(M_n)_{n\geq 1}$ on $(H, \mathcal{G}, Q)$ of the form;

$$M_n = P[\varphi(n, \omega_n) e_{1,n}]. \tag{3.4}$$

Here, $e_{1,n}$ has been introduced by (3.3) and $\varphi : \mathbb{N} \times \mathbb{Z}^d \to \mathbb{R}$ is a function for which we assume the following properties:

(P1): There are constants $C_i, p \in [0, \infty)$, $i = 0, 1, 2$ such that

$$|\varphi(n,x)| \leq C_0 + C_1|x|^p + C_2 n^{p/2} \quad \text{for all } (n,x) \in \mathbb{N} \times \mathbb{Z}^d. \tag{3.5}$$

(P2): $\Phi_n \stackrel{\text{def.}}{=} \varphi(n, \omega_n)$, $n \geq 1$ is a martingale on $(\Omega, \mathcal{F}, P)$ with respect to the filtration

$$\mathcal{F}_n = \sigma[\omega_j\ ;\ j \leq n]. \tag{3.6}$$

It is easy to see from (P2) that $(M_n)_{n\geq 1}$ is a $(\mathcal{G}_n)$-martingale on $(H, \mathcal{G}, Q)$. The following proposition generalizes [3, Lemma 4] and [25, Theorem 2].

Proposition 3.2.1. *Consider the martingale $(M_n)_{n\geq 1}$ defined by (3.4). Suppose that $d \geq 3$ and that (2.3), (P1), (P2) are satisfied. Then, there exists $\kappa \in [0, p/2)$ such that*

$$\max_{0\leq j\leq n} |M_j| = \mathcal{O}(n^\kappa), \quad \textit{as } n \nearrow \infty,\ Q\textit{-a.s.} \tag{3.7}$$

If in addition, $p < \frac{1}{2}d - 1$, then

$$\lim_{n\nearrow\infty} M_n \ \textit{exists } Q\textit{-a.s. and in } L^2(Q). \tag{3.8}$$

Remark 3.2.1. As will be seen from the way (3.7) is used below, it is crucial that the divergence of the right-hand-side is strictly slower than $n^{p/2}$, and this is where the property (P2) is relevant. If we drop the property (P2) from the assumption of Proposition 3.2.1, we then have a larger bound:

$$M_n = \mathcal{O}(n^{p/2}), \quad \text{as } n \nearrow \infty, \ Q\text{-a.s.} \tag{3.9}$$

This larger bound from the weaker assumption can be obtained via Proposition 3.2.1 as in the proof of (2.1) in [3].

We will prove Proposition 3.2.1 later on. Before doing so, we explain how this proposition is used to derive the desired conclusions in Theorem 2.1.1 and in Theorem 2.3.2 (a).

• Theorem 2.3.2 (a) is proved by choosing $\varphi \equiv 1$ in Proposition 3.2.1. By (3.8), $M_n = W_n$ converges in $L^2(Q)$. In particular,

$$Q[W_\infty] = \lim_{n \nearrow \infty} Q[W_n] = 1.$$

This implies $Q\{W_\infty > 0\} > 0$ and hence that $Q\{W_\infty > 0\} = 1$ by the zero-one law.

• To prove (2.4), we take $\varphi(n, x) = |x|^2 - n$ (hence $p = 2$). Then, by Theorem 2.3.2 (a) and Proposition 3.2.1, there exists $\kappa \in [0, 1)$ such that

$$\mu_n[|\omega_n|^2] - n = P[\varphi(n, \omega_n) e_{1,n}]/W_n = \mathcal{O}(n^\kappa) \quad Q\text{-a.s.}$$

We now turn to the proof of Proposition 3.2.1. Here, we follow [25]. We present a key step in the proof as a lemma.

Lemma 3.2.2. *Suppose that $d \geq 3$ and that (2.3), (P1), (P2) are satisfied. Then,*

$$Q[M_n^2] = \mathcal{O}(b_n), \quad \text{as } n \nearrow \infty, \ Q\text{-a.s.} \tag{3.10}$$

where $b_n = 1$ if $p < \frac{d}{2} - 1$, $b_n = \ln n$ if $p = \frac{d}{2} - 1$, and $b_n = n^{p - \frac{d}{2} + 1}$ if $p > \frac{d}{2} - 1$.

Remark 3.2.2. The choice of b_n is made in order to have $\sum_{1 \leq j \leq n} j^{p - \frac{d}{2}} = \mathcal{O}(b_n)$. See (3.15) below for the reason of the power $p - \frac{d}{2}$.

Proof of Lemma 3.2.2: On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $P^{\otimes 2} = P^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega} = (\widetilde{\omega}_k)_{k\geq 0}$ an independent copy of $\omega = (\omega_k)_{k\geq 0}$. We write $\chi_{i_1,\ldots,i_k}$ for the indicator function of the event

$$\{\omega_{i_1} = \widetilde{\omega}_{i_1},\ \omega_{i_2} = \widetilde{\omega}_{i_2}\ ,\ldots,\ \omega_{i_k} = \widetilde{\omega}_{i_k}\}.$$

We first expand the second moment $Q[M_n^2]$ as follows:

$$Q[M_n^2] = \Phi_0^2 + \sum_{1\leq k\leq n} (e^{\gamma_1(\beta)} - 1)^k \sum_{1\leq i_1<\ldots<i_k\leq n} P^{\otimes 2}[\Phi_{i_k}(\omega)^2 \chi_{i_1,\ldots,i_k}]. \tag{3.11}$$

To see this, we write M_n^2 in terms of the independent copy:

$$\begin{aligned} M_n^2 &= P[\Phi_n e_{1,n}]^2 \\ &= P^{\otimes 2}[\Phi_n(\omega)\Phi_n(\widetilde{\omega}) e_{1,n}(\omega,\eta) e_{1,n}(\widetilde{\omega},\eta)]. \end{aligned} \tag{3.12}$$

It follows from (3.12) that

$$Q[M_n^2] = P^{\otimes 2}\left[\Phi_n(\omega)\Phi_n(\widetilde{\omega}) Q[e_{1,n}(\omega,\eta) e_{1,n}(\widetilde{\omega},\eta)]\right]. \tag{3.13}$$

On the other hand, with notation (3.2), we have that

$$Q[e(\omega_j,\eta)e(\widetilde{\omega}_j,\eta)] = 1 + (e^{\gamma_1(\beta)} - 1)\chi_j,$$

ane hence that

$$\begin{aligned} Q[e_{1,n}(\omega,\eta) e_{1,n}(\widetilde{\omega},\eta)] &= \prod_{1\leq j\leq n} \left(1 + (e^{\gamma_1(\beta)} - 1)\chi_j\right) \\ &= 1 + \sum_{1\leq k\leq n} (e^{\gamma_1(\beta)} - 1)^k \sum_{1\leq i_1<\ldots<i_k\leq n} \chi_{i_1,\ldots i_k}. \end{aligned} \tag{3.14}$$

The expansion (3.11) is now obtained by inserting (3.14) into (3.13) and by the martingale property of Φ_n.

Let us fix $i_1, \ldots, i_k$ for a moment. We then have by (3.5) that

$$P^{\otimes 2}[\Phi_{i_k}(\omega)^2 \chi_{i_1,\ldots,i_k}] \leq 3C_1^2 A_{i_1,\ldots,i_k} + 3(C_0^2 + C_2^2) B_{i_1,\ldots,i_k},$$

where

$$A_{i_1,\ldots,i_k} = P^{\otimes 2}[|\omega_{i_k}|^{2p} \chi_{i_1,\ldots,i_k}], \quad B_{i_1,\ldots,i_k} = i_k^p P^{\otimes 2}[\chi_{i_1,\ldots,i_k}].$$

We now bound $A_{i_1,\ldots,i_k}$ from above. As will be seen from the way it is done, the same bound (up to the multiplicative constant) is obtained for

$B_{i_1,\dots,i_k}$. Observe that

$$\begin{aligned} P^{\otimes 2}[|\omega_n|^{2p}\chi_n] &= \sum_{x\in\mathbb{Z}^d} P[|\omega_n|^{2p} : \omega_n = x]P[\omega_n = x] \\ &\le Cn^{-\frac{d}{2}}P[|\omega_n|^{2p}] \\ &\le Cn^{p-\frac{d}{2}}, \end{aligned} \tag{3.15}$$

where we have used (2.6) on the second line. We write $j_\ell = i_\ell - i_{\ell-1}$, $\ell = 1,2,..,k$ with $i_0 = 0$. We then see from the Markov property and (3.15) that

$$\begin{aligned} &A_{i_1,\dots,i_k} \\ &\le k^{2p-1}\sum_{1\le\ell\le k} P^{\otimes 2}[|\omega_{i_\ell} - \omega_{i_{\ell-1}}|^{2p}\chi_{i_1,\dots,i_k}] \\ &= k^{2p-1}\sum_{1\le\ell\le k}\left(\prod_{1\le m<\ell} P^{\otimes 2}[\chi_{j_m}]\right) P^{\otimes 2}[|\omega_{j_\ell}|^{2p}\chi_{j_\ell}]\left(\prod_{\ell<m\le k} P^{\otimes 2}[\chi_{j_m}]\right) \\ &\le Ck^{2p-1}\sum_{1\le\ell\le k} j_\ell^{p-\frac{d}{2}} \prod_{\substack{1\le m\le k\\ m\ne\ell}} P^{\otimes 2}[\chi_{j_m}] \end{aligned}$$

Note that $\sum_{1\le j\le n} j^{p-\frac{d}{2}} = \mathcal{O}(b_n)$ and that $\sum_{j\ge 1} P^{\otimes 2}[\chi_j] = \frac{\pi_d}{1-\pi_d}$. Therefore, we obtain from what we have seen above that

$$\begin{aligned} &\sum_{1\le i_1<\dots<i_k\le n} P^{\otimes 2}[\Phi_{i_k}(\omega)^2\chi_{i_1,\dots,i_k}] \\ &\quad\le Ck^{2p-1}\sum_{1\le\ell\le k}\sum_{1\le j_1\le n}\cdots\sum_{1\le j_k\le n} j_\ell^{p-\frac{d}{2}}\prod_{\substack{1\le m\le k\\ m\ne\ell}} P^{\otimes 2}[\chi_{j_m}] \\ &\quad\le \mathcal{O}(b_n)k^{2p}\left(\frac{\pi_d}{1-\pi_d}\right)^{k-1}. \end{aligned}$$

By this and (3.11), we now arrive at

$$Q[M_n^2] \le \Phi_0^2 + \mathcal{O}(b_n)\sum_{k\ge 1} k^{2p}(e^{\gamma_1(\beta)} - 1)^k\left(\frac{\pi_d}{1-\pi_d}\right)^{k-1}.$$

The summation in k converges, thanks to the assumptions $d \ge 3$ and (2.3). This finishes the proof of (3.10). □

Remark 3.2.3. We see from the proof of (3.10) that

$$\sup_{n\geq 1} Q[W_n^2] = 1 + \sum_{k\geq 1} (e^{\gamma_1(\beta)} - 1)^k \left(\frac{\pi_d}{1-\pi_d}\right)^{k-1}.$$

This shows that $\sup_{n\geq 1} Q[W_n^2] < \infty$ if and only if $d \geq 3$ and (2.3) holds.

It is now, easy to complete the proof of Proposition 3.2.1. We set $M_n^* = \max_{0\leq j\leq n} |M_j|$ to simplify the notation. For (3.10), it is sufficient to prove that for any $\delta > 0$,

$$M_n^* = \mathcal{O}(n^\delta \sqrt{b_n}) \quad \text{as } n \nearrow \infty,\ Q\text{-a.s.}, \tag{3.16}$$

where b_n is the L^2-bound in Lemma 3.2.2. Moreover, by the monotonicity of M_n^* and the polynomial growth of $n^\delta \sqrt{b_n}$, it is enough to prove (3.16) along a subsequence $\{n^k : n \geq 1\}$ for some power $k \geq 2$. Now, take $k > 1/\delta$. We then have by Chebychev's inequality, Doob's inequality and Lemma 3.2.2 that

$$\begin{aligned} Q\{M_{n^k}^* > n^{k\delta}\sqrt{b_{n^k}}\} &\leq Q\{M_{n^k}^* > n\sqrt{b_{n^k}}\} \\ &\leq Q[(M_{n^k}^*)^2]/(n^2 b_{n^k}) \\ &\leq 4Q[M_{n^k}^2]/(n^2 b_{n^k}) \\ &\leq Cn^{-2}. \end{aligned}$$

Then, it follows from the Borel-Cantelli lemma that

$$Q\{M_{n^k}^* \leq n^{k\delta}\sqrt{b_{n^k}} \text{ for large enough } n\text{'s}\} = 1.$$

This ends the proof of (3.7).

The second statement (3.8) in Proposition 3.2.1 follows from Lemma 3.2.2 and the martingale convergence theorem. This completes the proof of Proposition 3.2.1. □

Remark 3.2.4. With Proposition 3.2.1 in hand, we are no longer far away from the central limit theorem (2.5). Following [3], we now explain a route to (2.5).

We let $a = (a_j)_{j=1}^d$ and $b = (b_j)_{j=1}^d$ denote multi indices in what follows. We will use standard notation $|a|_1 = a_1 + ... + a_d$, $x^a = x_1^{a_1} \cdots x_d^{a_d}$ and $(\frac{\partial}{\partial x})^a = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{a_d}$ for $x \in \mathbb{R}^d$. It is enough to prove (2.5) for any monomial of the form $f(x) = x^a$. We will do this by induction on $|a|_1$. We introduce

$$\begin{aligned} \varphi(n,x) &= \left(\frac{\partial}{\partial\theta}\right)^a \exp(\theta\cdot x - n\rho(\theta))\Big|_{\theta=0}, \\ \psi(n,x) &= \left(\frac{\partial}{\partial\theta}\right)^a \exp\left(\theta\cdot x - n\frac{|\theta|^2}{2d}\right)\Big|_{\theta=0}, \end{aligned}$$

where $\rho(\theta) = \ln\left(\frac{1}{d}\sum_{1\le j\le d}\cosh(\theta_j)\right)$. Clearly, the function φ satisfies (P1) and (P2) with $p = |a|_1$. On the other hand, we see from the definition of ψ that

$$(3.17) \qquad (2\pi)^{-d/2}\int_{\mathbb{R}^d}\psi(1,x/\sqrt{d})e^{-|x|^2/2}dx = 0.$$

Moreover, it is not difficult to see [3, Lemma 3c] that $\varphi(n,x) = x^a + \varphi_0(n,x)$ and $\psi(n,x) = x^a + \psi_0(n,x)$ where

$$\varphi_0(n,x) = \sum_{\substack{|b|_1+2j\le|a|_1\\ j\ge1}} A_a(b,j)x^b n^j, \quad \psi_0(n,x) = \sum_{\substack{|b|_1+2j=|a|_1\\ j\ge1}} A_a(b,j)x^b n^j.$$

for some $A_a(b,j)\in\mathbb{R}$. In particular, φ_0 and ψ_0 have the same coefficients for $x^b n^j$ with $|b|_1+2j=|a|_1$. We now write $\mu_n[(\omega_n/\sqrt{n})^a]$ as

$$\begin{aligned} \mu_n[(\omega_n/\sqrt{n})^a] &= \frac{1}{W_n}P[\varphi(n,\omega_n)e_{1,n}]n^{-|a|_1/2} \\ &\quad -\frac{1}{W_n}P[\psi_0(1,\omega_n/\sqrt{n})e_{1,n}] \\ &\quad +\frac{1}{W_n}P[(\psi_0(n,\omega_n)-\varphi_0(n,\omega_n))\,e_{1,n}]n^{-|a|_1/2} \end{aligned}$$

As $n\nearrow\infty$, the second term converges to $(2\pi)^{-d/2}\int_{\mathbb{R}^d}(x/\sqrt{d})^a \times e^{-|x|^2/2}dx$ by the induction hypothesis and (3.17). The first and the third terms on the right-hand side vanish as $n\nearrow\infty$. In fact, we use Theorem 2.3.2 (a), Proposition 3.2.1 for the first term and Theorem 2.3.2 (a), (3.9) for the third term.

3.3. The replica overlap

In this subsection, we prove Theorem 2.1.2 as a consequence of Theorem 2.3.2 (a) and Theorem 3.3.1 below.

For $n\ge1$, we introduce the following random variables on $(H,\mathcal{G},Q)$;

$$I_n = \sum_{x\in\mathbb{Z}^d}\mu_{n-1}\{\omega_n = x\}^2, \quad J_n = \max_{x\in\mathbb{Z}^d}\mu_{n-1}\{\omega_n = x\}.$$

It is clear that

$$J_n^2 \le I_n \le J_n. \tag{3.18}$$

Both Theorem 2.1.2 and Theorem 2.2.1 deal with the large time behavior of J_n, $n \nearrow \infty$. As we will see below, I_n is better suited for the martingale analysis. For this reason, we will prove these theorems by studying I_n, rather than J_n itself.

We now mention to an interpretation of I_n. On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_n^{\otimes 2} = \mu_n^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega} = (\widetilde{\omega}_k)_{k\ge 0}$ an independent copy of $\omega = (\omega_k)_{k\ge 0}$ with law μ_n. We then have that

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \widetilde{\omega}_n)\ . \tag{3.19}$$

Hence, the summation

$$\sum_{1\le k\le n} I_k \tag{3.20}$$

is the expected amount of the overlap up to time n of two independent polymers in the same (fixed) environment. This can be viewed as an analogue to the so-called *replica overlap* often discussed in the context of disordered systems, e.g. mean field spin glass, and also of directed polymers on trees [7].

The large time behavior of (3.20) and the normalized partition function W_n are related as follows.

Theorem 3.3.1. *Let $\beta \neq 0$. Then,*

$$\{W_\infty = 0\} = \left\{ \sum_{n\ge 1} I_n = \infty \right\}, \quad Q\text{-}a.s. \tag{3.21}$$

Moreover, if $Q\{W_\infty = 0\} = 1$, there exist $c_1, c_2 \in (0,\infty)$ such that Q-a.s.,

$$c_1 \sum_{1\le k\le n} I_k \le -\ln W_n \le c_2 \sum_{1\le k\le n} I_k \qquad \text{for large enough } n\text{'s}. \tag{3.22}$$

We first note that Theorem 2.1.2 is now obtained as a consequence of Theorem 2.3.2 (a), (3.21) and (3.18).

Proof of Theorem 3.3.1: To conclude (3.21) and (3.22), it is enough to show the following (3.23) and (3.24):

$$\{W_\infty = 0\} \subset \left\{ \sum_{n\ge 1} I_n = \infty \right\}, \quad Q\text{-a.s.} \tag{3.23}$$

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{ \sum_{n \geq 1} I_n = \infty \right\} \subset \{(3.22) \text{ holds}\}, \quad Q\text{-a.s.} \tag{3.24}$$

The proof of (3.23) and (3.24) are based on Doob's decomposition for the process $-\ln W_n$. It is convenient to introduce some more notation. For a sequence $(a_n)_{n\geq 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geq 1$. Let us now recall Doob's decomposition in this context; any $(\mathcal{G}_n)$-adapted process $X = \{X_n\}_{n\geq 0} \subset L^1(Q)$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \geq 1,$$

where $M(X)$ is an $(\mathcal{G}_n)$-martingale and

$$A_0 = 0, \quad \Delta A_n = Q[\Delta X_n | \mathcal{G}_{n-1}], \quad n \geq 1.$$

$M_n(X)$ and $A_n(X)$ are called respectively, the martingale part and compensator of the process X. If X is a square integrable martingale, then the compensator $A_n(X^2)$ of the process $X^2 = \{(X_n)^2\}_{n\geq 0} \subset L^1(Q)$ is denoted by $\langle X \rangle_n$ and is given by the following formula:

$$\Delta\langle X \rangle_n = Q[(\Delta X_n)^2 | \mathcal{G}_{n-1}]$$

Here, we are interested in the Doob's decomposition of $X_n = -\ln W_n$, whose martingale part and the compensator will be henceforth denoted M_n and A_n respectively

$$-\ln W_n = M_n + A_n. \tag{3.25}$$

To compute M_n and A_n, we introduce $U_n = \mu_{n-1}[e(n, \omega_n)] - 1$ (Recall (3.2)). It is then clear that

$$W_n / W_{n-1} = 1 + U_n \tag{3.26}$$

and hence that

$$\begin{aligned} \Delta A_n &= -Q[\ln(1 + U_n) | \mathcal{G}_{n-1}], \\ \Delta M_n &= -\ln(1 + U_n) + Q[\ln(1 + U_n) | \mathcal{G}_{n-1}]. \end{aligned} \tag{3.27}$$

In particular,

$$\Delta\langle M \rangle_n \leq Q[\ln^2(1 + U_n) | \mathcal{G}_{n-1}]. \tag{3.28}$$

We now claim that there is a constant $c \in (0, \infty)$ such that

$$\frac{1}{c} I_n \leq \Delta A_n \leq c I_n, \quad \Delta\langle M \rangle_n \leq c I_n. \tag{3.29}$$

Indeed, both follow from (3.27), (3.28) and Lemma 3.3.2 below; $\{e_i\}$, $\{\alpha_i\}$ and Q in the lemma play the roles of $\{e(n,z)\}_{|z|_1 \leq n}$, $\{\mu_{n-1}(\omega_n = z)\}_{|z|_1 \leq n}$ and $Q[\,\cdot\,|\mathcal{G}_{n-1}]$.

We now conclude (3.23) from (3.29) as follows (the equalities and the inclusions here being understood as Q-a.s.):

$$\begin{aligned} \left\{ \sum_{n \geq 1} I_n < \infty \right\} &\subset \{A_\infty < \infty, \ \langle M \rangle_\infty < \infty\} \\ &\subset \{A_\infty < \infty, \ \lim_{n \nearrow \infty} M_n \text{ exists and is finite}\} \\ &\subset \{W_\infty > 0\}. \end{aligned}$$

Here, on the second line, we have used a well-known property for martingales, e.g. [8, page 255, (4.9)].

Finally we prove (3.24). By (3.29), it is enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{n \nearrow \infty} -\frac{\ln W_n}{A_n} = 1 \right\}, \quad Q\text{-a.s.} \tag{3.30}$$

Thus, let us suppose that $A_\infty = \infty$. If $\langle M \rangle_\infty < \infty$, then again by [8, page 255, (4.9)], $\lim_{n \nearrow \infty} M_n$ exists and is finite and therefore (3.30) holds. If, on the contrary, $\langle M \rangle_\infty = \infty$, then

$$-\frac{\ln W_n}{A_n} = \frac{M_n}{\langle M \rangle_n} \frac{\langle M \rangle_n}{A_n} + 1 \to 1 \quad Q\text{-a.s.}$$

by (3.29) and the law of large numbers for martingales, see [8, page 255, (4.10)]. This completes the proof of Theorem 3.3.1. □

Lemma 3.3.2. *Let e_i, $1 \leq i \leq m$ be positive, non-constant i.i.d. random variables on a probability space $(H, \mathcal{G}, Q)$ such that*

$$Q[e_1] = 1, \quad Q[e_1^3 + \ln^2 e_1] < \infty.$$

For $\{\alpha_i\}_{1 \leq i \leq m} \subset [0, \infty)$ such that $\sum_{1 \leq i \leq m} \alpha_i = 1$, define a centered random variable $U > -1$ by $U = \sum_{1 \leq i \leq m} \alpha_i e_i - 1$. Then, there exists a *constant* $c \in (0, \infty)$, *independent of* $\{\alpha_i\}_{1 \le i \le m}$ *such that*

$$\frac{1}{c} \sum_{1 \le i \le m} \alpha_i^2 \le Q\left[\frac{U^2}{2+U}\right], \tag{3.31}$$

$$\frac{1}{c} \sum_{1 \le i \le m} \alpha_i^2 \le -Q\left[\ln(1+U)\right] \le c \sum_{1 \le i \le m} \alpha_i^2, \tag{3.32}$$

$$Q\left[\ln^2(1+U)\right] \le c \sum_{1 \le i \le m} \alpha_i^2. \tag{3.33}$$

The readers are invited to try the proof of this lemma as an interesting exercise. A solution can be found in [5].

3.4. The fractional moment method

In this subsection, we prove Theorem 2.2.1(b) and Theorem 2.3.2(b). Both are obtained by dealing with the fractional moment $Q[W_n^\theta]$, $0 < \theta < 1$. To be more precise, we will prove that for some $\theta \in (0,1)$ and $a_n \nearrow \infty$,

$$\overline{\lim_{n \nearrow \infty}} \frac{1}{a_n} \ln Q[W_n^\theta] < 0. \tag{3.34}$$

Proof of Theorem 2.2.1 under condition (ii): We first assume (3.34) with $a_n = n$ for a moment to see that it implies (2.10). We then have by the Borel-Cantelli lemma that there is $c_3 \in (0, \infty)$ such that

$$\overline{\lim_{n \nearrow \infty}} \frac{1}{n} \ln W_n < -c_3, \quad Q\text{-a.s.} \tag{3.35}$$

Then, by (3.18) and (3.22) we conclude that

$$\begin{aligned} \underline{\lim_{n \nearrow \infty}} J_n &\ge \underline{\lim_{n \nearrow \infty}} \frac{1}{n} \sum_{1 \le k \le n} I_k \\ &\ge -\overline{\lim_{n \nearrow \infty}} \frac{1}{c_2 n} \ln W_n \\ &\ge c_3/c_2. \end{aligned}$$

We now turn to the proof of (3.34) with $a_n = n$. Recall the notation (3.2) and define

$$W_{n,m}^x = P\left[\prod_{1 \le j \le m} e(j+n, x+\omega_j)\right], \quad n, m \ge 1.$$

For $\theta \in (0,1)$, by the subadditive estimate $(u+v)^\theta \le u^\theta + v^\theta$, $u, v > 0$, we get

$$W_n^\theta \le (2d)^{-\theta} \sum_{x,|x|_1=1} e(1,x)^\theta (W_{1,n-1}^x)^\theta.$$

Since $W_{1,n-1}^x$ has the same law as W_{n-1},

$$Q[W_n^\theta] \le r(\theta) Q[W_{n-1}^\theta],$$

where $r(\theta) = (2d)^{1-\theta} Q[e(1,x)^\theta]$. Note that $\theta \mapsto \ln r(\theta)$ is convex, continuously differentiable, and that $\ln(2d) = \ln r(0) > \ln r(1) = 0$. Therefore $r(\theta) < 1$ for some $\theta \in (0,1)$ if and only if $0 < \left.\frac{d \ln r(\theta)}{d\theta}\right|_{\theta=1}$, which is equivalent to $\gamma_2(\beta) > \ln(2d)$. □

Proof of Theorem 2.3.2(b): We will check (3.34) where $a_n = n^{1/3}$ if $d = 1$ and $a_n = \sqrt{\ln n}$ if $d = 2$. In this respect, we first prove an auxiliary lemma.

Lemma 3.4.1. *For $\theta \in [0,1]$ and $\Lambda \subset \mathbb{Z}^d$,*

$$|\Lambda| Q\left[W_{n-1}^\theta I_n\right] \ge Q\left[W_{n-1}^\theta\right] - 2P(\omega_n \notin \Lambda)^\theta. \tag{3.36}$$

Proof: Repeating the argument in [19, page 453], we see that

$$\begin{aligned}
|\Lambda| I_n &\ge |\Lambda| \sum_{z \in \Lambda} \mu_{n-1}(\omega_n = z)^2 \\
&\ge \mu_{n-1}(\omega_n \in \Lambda)^2 \\
&= (1 - \mu_{n-1}(\omega_n \notin \Lambda))^2 \\
&\ge 1 - 2\mu_{n-1}(\omega_n \notin \Lambda) \\
&\ge 1 - 2\mu_{n-1}(\omega_n \notin \Lambda)^\theta.
\end{aligned}$$

Note also that

$$\begin{aligned}
Q\left[W_{n-1}^\theta \mu_{n-1}(\omega_n \notin \Lambda)^\theta\right] &\le Q\left[W_{n-1}\mu_{n-1}(\omega_n \notin \Lambda)\right]^\theta \\
&= P(\omega_n \notin \Lambda)^\theta.
\end{aligned}$$

We therefore see that

$$\begin{aligned}
|\Lambda| Q\left[W_{n-1}^\theta I_n\right] &\ge Q\left[W_{n-1}^\theta\right] - 2Q\left[W_{n-1}^\theta \mu_{n-1}(\omega_n \notin \Lambda)^\theta\right] \\
&\ge Q\left[W_{n-1}^\theta\right] - 2P(\omega_n \notin \Lambda)^\theta.
\end{aligned}$$

□

Assume now that $\theta \in (0,1)$, and define a function $f : (-1,\infty) \to [0,\infty)$ by

$$f(u) = 1 + \theta u - (1+u)^\theta.$$

It is then clear that there are constants $c_1, c_2 \in (0,\infty)$ such that

$$\frac{c_1 u^2}{2+u} \le f(u) \le c_2 u^2 \quad \text{for all } u \in (-1,\infty). \tag{3.37}$$

We see from (3.26), (3.37) and (3.31) that

$$\begin{aligned} Q[\Delta W_n^\theta | \mathcal{G}_{n-1}] &= W_{n-1}^\theta Q[(1+U_n)^\theta - 1 | \mathcal{G}_{n-1}] \\ &= -W_{n-1}^\theta Q[f(U_n)|\mathcal{G}_{n-1}] \\ &\le -c_3 W_{n-1}^\theta I_n. \end{aligned}$$

We therefore have by (3.36) that

$$QW_n^\theta \le \left(1 - \frac{c_3}{|\Lambda|}\right) Q\left[W_{n-1}^\theta\right] + \frac{2c_3}{|\Lambda|} P(\omega_n \notin \Lambda)^\theta. \tag{3.38}$$

For $d = 1$, set $\Lambda = (-n^{2/3}, n^{2/3}]$. Then,

$$P(\omega_n \notin \Lambda) = P\left(\left|\frac{\omega_n}{n^{1/2}}\right| \ge n^{1/6}\right) \le 2\exp(-\frac{n^{1/3}}{2}),$$

so that (3.38) reads,

$$QW_n^\theta \le \left(1 - \frac{c_3}{2n^{2/3}}\right) Q\left[W_{n-1}^\theta\right] + 4c_3 \exp(-\frac{\theta n^{1/3}}{2}).$$

It is not difficult to conclude (3.34) with $a_n = n^{1/3}$ from the above.

For $d = 2$, we set

$$\Lambda = (-n^{1/2} \ln^{1/4} n, n^{1/2} \ln^{1/4} n]^2$$

to get (3.34) with $a_n = \sqrt{\ln n}$ in a similar way as above. □

4. Some related models

The simple random walk model which we have discussed so far has a number of close relatives in the literature. We now mention some of them.

4.1. Gaussian random walk model

This model considered in M. Petermann [23] and by O. Mejane [20]. The polymer measure for this model is defined by the same expression (1.4). Here, however, the random walk $(\omega_n)_{n\geq 1}$ is the summation of independent Gaussian random variables in $\mathbb{R}^d$, i.e., Ω is replaced by $\Omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{R}^d, n \geq 0\}$ and P is the unique measure on $(\Omega, \mathcal{F})$ such that $\omega_1 - \omega_0, \dots, \omega_n - \omega_{n-1}$ are independent and

$$P\{\omega_0 = 0\} = 1, \quad P\{\omega_n - \omega_{n-1} \in dx\} = (2\pi)^{-d/2} \exp(-|x|^2/2)dx.$$

Moreover, as the random environment, one takes a random field

$$\{\eta(n,x) \ ; \ (n,x) \in \mathbb{N} \times \mathbb{R}^d\}$$

with a certain mild correlation in x variables. A major technical advantage in working with the Gaussian random walk rather than the simple random walk is the applicability of a Girsanov-type path transformation, which plays a key role in analyzing this model.

4.2. Brownian directed polymer

This model is introduced in [6] as a continuous model of directed polymers in random environment, defined in terms of Brownian motion and of a Poisson random measure. We first fix notation we use for the Brownian motion and Poisson random measure. Then, we introduce the polymer measure. We write $\mathbb{R}_+ = [0, \infty)$.

• *The Brownian motion:* Let $(\{\omega_t\}_{t\geq 0}, P)$ denote a d-dimensional standard Brownian motion. To be more specific, we let the measurable space $(\Omega, \mathcal{F})$ be $C(\mathbb{R}_+ \to \mathbb{R}^d)$ with the cylindrical σ-field, and P be the Wiener measure on $(\Omega, \mathcal{F})$ such that $P\{\omega_0 = 0\} = 1$.

• *The space-time Poisson random measure:* We let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with the unit intensity, defined on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. To make the definitions more precisely, we let $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ denote the class of Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$. Then, η is an integer valued random measure characterized by the following property: If $A_1, ..., A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ are disjoint and bounded, then

(4.1)
$$Q\left(\bigcap_{j=1}^{n} \{\eta(A_j) = k_j\}\right) = \prod_{j=1}^{n} \exp(-|A_j|) \frac{|A_j|^{k_j}}{k_j!} \quad \text{for } k_1, ..., k_n \in \mathbb{N}.$$

Here, $|A|$ denotes the Lebesgue measure of $\mathbb{R}^{d+1}$.

• *The polymer measure:* We let V_t denote a "tube" around the graph $\{(s,\omega_s)\}_{0<s\le t}$ of the Brownian path,

$$V_t = V_t(\omega) = \{(s,x)\ ;\ s \in (0,t],\ x \in U(\omega_s)\}, \tag{4.2}$$

where $U(x) \subset \mathbb{R}^d$ is the closed ball with the unit volume, centered at $x \in \mathbb{R}^d$. For any $t > 0$, define a probability measure μ_t on the path space $(\Omega, \mathcal{F})$ by

$$\mu_t(d\omega) = \frac{\exp(\beta\eta(V_t))}{Z_t}\, P(d\omega), \tag{4.3}$$

where $\beta \in \mathbb{R}$ is a parameter and

$$Z_t = P[\exp(\beta\eta(V_t))]\ . \tag{4.4}$$

The Brownian motion model defined above can be thought of as a natural transposition of the simple random walk model into continuum setting.

Analogous results of Theorem 2.1.2, Theorem 2.2.1, Theorem 2.3.2, and Theorem 3.3.1 as well as an almost sure large deviation principle for the polymer measure are obtained for this model in [6]. The model allows application of stochastic calculus, with respect to both Brownian motion and Poisson process, leading to qualitative properties of the quenched Lyapunov exponent and handy formulas for the fluctuation of the free energy.

Another strong motivation for the present model is its relation to some *stochastic partial differential equations.* To describe the connection, it is necessary to relativize the partition function, by specifying the ending point of the Brownian motion at time t. Let $P[\cdot|\omega_t = y]$ be the distribution of the Brownian bridge starting at the origin at time 0 and ending at y at time t. Define

$$Z_t(y) = P[\exp(\beta\eta(V_t))\,|\omega_t = y](2\pi t)^{-d/2}\exp\{-|y|^2/2t\}\ . \tag{4.5}$$

Then, by definition of the Brownian bridge,

$$Z_t = \int_{\mathbb{R}^d} Z_t(y)dy\ .$$

Similar to the Feynman-Kac formula, one obtains [6] the following stochastic heat equation (SHE) with multiplicative noise in a certain weak sense,

(4.6)

$$dZ_t(y) = \tfrac{1}{2}\Delta_y Z_t(y)dt + (e^\beta - 1)Z_{t-}(y)\eta(dt \times U(y))\ , \quad t \ge 0, y \in \mathbb{R}^d\ ,$$

where $dZ_t(y)$ denotes the time differential and $\Delta_y = (\frac{\partial}{\partial y^1})^2 + \ldots + (\frac{\partial}{\partial y^d})^2$.

In the literature, this equation has been extensively considered in the case of a Gaussian driving noise, instead of the Poisson process η here. Although we are able to prove (4.6) only in the weak sense, let us now pretend that (4.6) is true for all $y \in \mathbb{R}^d$. We would then see from Itô's formula that the function $h_t(y) = \ln Z_t(y)$ solves the Kardar-Parisi-Zhang equation (KPZ):

$$dh_t(y) = \tfrac{1}{2}\left(\Delta_y h_t(y) + |\nabla_y h_t(y)|^2\right) dt + \beta\eta(dt \times U(y)) .$$

We observe that, since h has jumps in the space variable y, the non-linearity makes the precise meaning of this equation somewhat knotty. This equation was introduced in [15] to describe the long scale behavior of growing interfaces. More precisely, the fluctuations in the KPZ equation –driven by a δ-correlated, gaussian noise–, are believed to be non standard, and universal, i.e., the same as in a large class of microscopic models. See [17] for a detailed review of kinetic roughening of growth models within the physics literature, in particular to Section 5 for the status of this equation.

4.3. Crossing Brownian motion in a soft Poissonian potential

This model is studied by M. Wüthrich [30, 31, 32], see also [28]. The model investigated there is described in terms of Brownian motion and of Poisson points. However, the Brownian motion there is "undirected", in other words, the d-dimensional Brownian motion travels through the Poisson points distributed in space $\mathbb{R}^d$, not in space-time as in the Brownian directed polymer.

4.4. First and last passage percolation

The first (resp. last) passage percolation can be thought of as an analogue of directed polymers at $\beta = -\infty$ (resp. $\beta = +\infty$). In fact, we have for example that

$$\lim_{\beta \nearrow +\infty} \frac{1}{\beta} \ln Z_n = T_n^* \stackrel{\text{def.}}{=} \max_{\omega \in \Omega : \omega_0 = 0} \sum_{1 \le j \le n} \eta(j, \omega_j),$$

i.e., the maximal passage time T_n^* in the context of the directed last passage percolation can be obtained as a limit of the free energy of the directed polymer. It is expected and even partly vindicated that the properties of the path with minimal/maximal passage time has similar feature to the typical paths under the polymer measure [16, 21, 18]. A few exactly solvable models of directed last passage percolation have recently been worked out in dimension $d = 1$ [2, 12, 13]. Johansson [12] treats the case of geometrically distributed η's, and Baik, Deift and Johansson analyze some continuous space Poissonian directed last passage percolation model in connection with the longest increasing sequence of the random permutation [2, 13]. For these exactly solvable models, it is proven that the maximal passage time T_n^* has the following asymptotic form in law as $n \nearrow \infty$:

$$c_0 n + c_1 n^{1/3} X, \tag{4.7}$$

where c_i, $i = 1, 2$ are positive constants and X is a random variable with the Tracy-Widom distribution. As is well known, the Tracy-Widom distribution appeared in the literatures in connection with the Gaussian Unitary Ensemble [29]. Since then, it has increasingly realized that this distribution is universal as the scaling limit of many other related models. For this reason, we are tempted to believe that for $d = 1$ and $\beta \neq 0$, the free energy $\ln Z_n$ of the directed polymer has the same large time behavior as (4.7) with c_i, $i = 1, 2$ depending on β and the choice of η [27].

4.5. Other models

Directed polymers in random environment, at positive or zero temperature, relate – even better, can sometimes be exactly mapped – to a number of interesting models of growing random surfaces (directed invasion percolation, ballistic deposition, polynuclear growth, low temperature Ising models), and non equilibrium dynamics (totally asymmetric simple exclusion, population dynamics in random environment); We refer to the survey paper [17] by Krug and Spohn for detailed account on these models and their relations.

5. Critical exponents

We write $\xi(d)$ for the "wandering exponent",i.e., the critical exponent for the transversal fluctuation of the path, and $\chi(d)$ for the the critical exponent for the longitudinal fluctuation of the free energy. Their definitions are roughly

$$\sup_{0 \leq j \leq n} |\omega_j| \approx n^{\xi(d)} \quad \text{and} \quad \ln Z_n - Q[\ln Z_n] \approx n^{\chi(d)} \quad \text{as } n \nearrow \infty. \tag{5.1}$$

There are various ways to define rigorously these exponents, e.g. (0.6) and (0.10-11) in [30], (2.4) and (2.6-7-8) in [24], and the equivalence between these specific definitions are often non trivial. Here, we do not go into such subtlety and take (5.1) as "definitions". The polymer is said to be *diffusive* if $\xi(d) = 1/2$ and *super-diffusive* if $\xi(d) > 1/2$.

These exponents are investigated in the context of various other models and in a large number of papers. In particular, it is conjectured in physics literature that the *scaling identity* holds in any dimension,

$$\chi(d) = 2\xi(d) - 1, \quad d \geq 1, \tag{5.2}$$

and that the polymer is super-diffusive in dimension one;

$$\chi(1) = 1/3, \quad \xi(1) = 2/3. \tag{5.3}$$

See, e.g., [10],[9, (3.4),(5.11),(5.12)], [17, (5.19),(5.28)]. For some models of directed first passage percolation, K. Johansson [12, 13] proves (5.3), cf. (4.7).

On the other hand, other rigorous results prove (or suggest) for example that

$$\chi(d) \geq 2\xi(d) - 1 \text{ for all } d \geq 1, \tag{5.4}$$

$$\xi(d) \leq 3/4 \text{ for all } d \geq 1, \tag{5.5}$$

$$\xi(1) > 1/2 \tag{5.6}$$

M. Piza [24] discusses (5.4) –(5.6) for the simple random walk model. For the Gaussian random walk model, M. Petermann [23] proves (5.6), while O. Mejane [20] shows (5.5). F. Comets and N. Yoshida [6] discuss (5.4)–(5.6) in the framework of Brownian directed polymer. Critical exponents similar to the above are also discussed for the crossing Brownian motion in a soft Poissonian potential by M. Wüthrich [30, 31, 32] and for the first passage percolation by C. Licea, M. Piza and C. Newman [21, 18].

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