Probabilistic Approach for Granular Media Equations in the Non Uniformly Convex Case

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Abstract

We use here a particle system to prove both a convergence result (with convergence rate) and a deviation inequality for solutions of granular media equation when the confinement potential and the interaction potential are no more uniformly convex. The proof of convergence is simpler than the one in Carrillo-McCann-Villani [CMCV03, CMCV06]. All the results complete former results of Malrieu [Mal03] in the uniformly convex case. The main tool is an uniform propagation of chaos property and a direct control in Wasserstein distance of solutions starting with different initial measures. The deviation inequality is obtained via a T_1 transportation cost inequality replacing the logarithmic Sobolev inequality which is no more clearly dimension free.

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1 Introduction

Our main goal will be to deal in a probabilistic way with the following nonlinear equation

$$\frac{\partial u}{\partial t} = \nabla \cdot [\nabla u + u\nabla V + u(\nabla W * u)] \tag{1}$$

where $u(t, x)$ is a time dependent probability measure, $*$ denotes the standard convolution operator and V and W are two convex (at infinity) potentials.

This equation arises (in dimension 1) in the modelling of granular media as follows: consider many infinitesimal particles colliding inelastically. With a correct renormalization between the frequency and the inelasticity of the collisions, $u(t, x)$ turns out to be the velocity of a representative particle (among an infinity). The potential V represents the friction and W the inelastic collisions between particles with different velocities. Note that the particular case $V = 0$ and $W(x) = |x|^3$ is of special interest and we refer to Benedetto-Caglioti-Pulvirenti [BCP97] for the physical issues (also see [BCCP98]).

Once the problem of existence and uniqueness is tackled, one major problem in this equation is the behavior at infinity: existence of a stationary measure and speed of convergence towards this stationary measure or even distance between two solutions starting from different points. It has been studied in parallel by Carrillo-McCann-Villani [CMCV03, CMCV06] and Malrieu [Mal01, Mal03], under various assumptions on the potentials V and W, using analytical and probabilistic approaches respectively (also see [BRV98, BRTV98] for one dimensional particles). We will consider here the probabilistic approach and will recover and generalize slightly results of [CMCV03, CMCV06] as

well as give a quantified probabilistic approximation of the stationary measure of the granular media equation. It is worthwhile noticing that the analytic methods in [CMCV03, CMCV06] cover a much larger spectrum of non linear p.d.e's (like the porous medium equation), for which the probabilistic approach remains to be written.
As physical interest (and in fact where the main mathematical difficulty resides) dictates the friction term to vanish, we will consider the following two sets of general assumptions:

Definition 1.1 • *We say that W satisfies the set of assumptions (A) if*

A1. the friction term $V = 0$;

A2. W is symmetric, i.e. $W(-x) = W(x)$;

A3. ∇W is locally Lipschitz with polynomial growth, i.e. for some m

$$\forall x, y \in \mathbb{R}^d, |\nabla W(x) - \nabla W(y)| \leq C(|x-y| \wedge 1)(1 + |x|^m + |y|^m).$$

We also assume that the second derivative of W has a (m) polynomial growth.

A4. W is the sum of a compactly supported $\mathcal{C}^2$ function and a $\mathcal{C}^2$ uniformly convex function. This last assumption entails some uniform convexity at infinity property, namely that there exist positive C, λ, such that

$$\forall x, y, \qquad (x-y).(\nabla W(x) - \nabla W(y)) \geq \lambda \|x-y\|^2 - C\,. \tag{2}$$

It also implies that for all x, $HessW(x) \geqslant -\beta Id$ (in the sense of quadratic forms) for some $\beta > 0$.

- *We say that V and W satisfy the set of assumptions (A') if V is uniformly convex at infinity (i.e. satisfies (A4)), W is convex at infinity (but not necessarily uniformly) and symmetric, and both satisfy (A3).*

As we will see later, these assumptions are not sufficient to get good properties for large time of the granular media equation and will be replaced by some "strict convexity except for a finite number of points" property.

Before further discussing the result, let us first present the probabilistic approach of this problem. The probabilistic interpretation is to consider a Markov process $(\bar{X}_t)_{t\geq 0}$, which law at time t is u. It is the solution of the nonlinear S.D.E.

$$\begin{cases} d\bar{X}_t = \sqrt{2}dB_t - \nabla V(\bar{X}_t)dt - \nabla W * u_t(\bar{X}_t)dt \\ \mathcal{L}(\bar{X}_t) = u_t\, dx. \end{cases} \tag{3}$$

By a solution, we here mean a process $X_.$ such that for all $t > 0$, the law of X_t is $u_t dx$, and which satisfies the (linear) SDE written above.
We wish here first to give sufficient conditions ensuring both existence and ergodicity of the solution of the nonlinear S.D.E. and in a second time to provide a way to simulate this law at each time t with some Gaussian confidence intervals independent of time t. These two goals will be carried out through the extensive use of some (linear) particle approximations, i.e. (X_t^N) solution of

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i - \nabla V(X_t^{i,N})dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N})dt & i = 1, ..., N \\ X_0^{i,N} = X_0^i & i = 1, ..., N. \end{cases} \tag{4}$$

If we suppose that $V = 0$, we may assume that the center of mass is fixed, and without loss of generality, set to 0, i.e. $\mathbb{E}\bar{X}_0 = 0$, indeed it is easy to remark that $\bar{X}_t - \mathbb{E}\bar{X}_0$ satisfies equation (3). This assumption obliges us to introduce the projected system onto the set $\sum_i x_i = 0$ (see Section 2), and introduces some intricacies.

Under the strong assumptions of (global) uniform convexity of one of the potentials (the other one being globally convex but non necessarily uniformly), Malrieu [Mal01, Mal03] successively fulfilled these two goals using extensively the so called Bakry-Émery criterion ensuring that a Logarithmic Sobolev inequality holds independently of N. He also provided in the same time asymptotic behavior in time and concentration inequality, enabling him to recover part of the results of Carrillo-McCann-Villani [CMCV03] (see also Bolley-Guillin-Villani [BGV05] for a strengthened deviation inequality for almost quadratic potentials).
The assumption of uniform convexity is however far too strong for applications (preventing for example to consider the case of cubic interaction potential). It can be removed (see [CMCV03]) for a one point degeneracy where the authors obtain various rates of convergence. Another approach is proposed by Carrillo-McCann-Villani in the subsequent [CMCV06] paper introducing tools of contractions in L^2-Wasserstein distance for length space enabling them to ensure a convergence in Wasserstein distance of the solution of the granular media equation.

We will see here that, with much simpler tools only using the particle system, we can recover their result and may provide also Gaussian confidence bounds for the approximations of the granular media equations. It is worthwhile noticing that, though the invariant measures of the particle system still satisfy some log-Sobolev inequality, it is difficult to obtain dimension-free estimates on the log-Sobolev constants (as for nearest neighbors interactions models in Statistical Mechanics). This prevents us to use this kind of approach.

Our main tool in order to get the Gaussian concentration inequality we need, will thus be a transportation cost-information inequality: let say that $\mu \in T_1(C)$ if for every probability measure ν

$$W_1(\nu, \mu) \leq \sqrt{C \operatorname{Ent}(\nu|\mu)} \tag{5}$$

where W_1 is the usual Wasserstein distance and Ent the Kullback information or relative entropy. In the final section we prove that, under one of the hypothesis (A) or (A'), the law of the particle system at time t satisfies a $T_1(CN)$ inequality, for some C that does not depend on N nor on t.

To complete the proof of the concentration inequality in the final section we need some "uniform (in time) propagation of chaos". It seems difficult to obtain such a result only assuming (A) or (A'). Indeed if the potentials are non attractive in some (bounded) region, the situation becomes the classical mean-field one where propagation of chaos is controlled on finite time intervals only. That is why we have to reinforce our assumptions.
Our condition below is inspired by the work of Carrillo-McCann-Villani [CMCV06]:
we say that condition $\mathbf{C}(\mathbf{A}, \alpha)$ holds if there exist $A, \alpha > 0$ such that for any $0 < \epsilon < 1$,

$$\forall x, y \in \mathbb{R}^d, \qquad (x - y).(\nabla W(x) - \nabla W(y)) \geq A\epsilon^\alpha(|x - y|^2 - \epsilon^2). \tag{6}$$

Remark that this condition implies convexity (if $|x - y| \leq 1/2$ choose $\epsilon = |x - y|$, otherwise choose $\epsilon = 1/4$) but is weaker than uniform convexity (which is true with $\alpha = 0$). Typical examples are polynomial potentials of the form $W(x) = |x|^{2+\alpha}$ which satisfy $\mathbf{C}(\mathbf{A}, \alpha)$ for some positive A. In fact this condition is related to the rate at which strict convexity is lost at points.
Condition $\mathbf{C}(\mathbf{A}, \alpha)$ allows us to prove the required "uniform (in time) propagation of chaos" and then to obtain the rate of convergence of u_t to the limit u_∞ in Wasserstein distance W_2. These results are obtained in sections 3 and 4.

To complete the description of the paper, the next section contains some useful estimates, and a complete proof of existence and uniqueness for all the equations we have considered ((1) and (3) in particular). The proof is quite natural (using the particle system). We did not find such a proof in the literature except in dimension 1, and as a matter of fact the set of hypotheses we need, slightly differs from the ones used in the analytical literature.

2 The particle system and solutions of the nonlinear SDE

In this section we study the particle system and the nonlinear SDE. The first discussion is about case (A) when V vanishes.
Consider once again the particle system (X_t^N) solution of

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i - \nabla V(X_t^{i,N})dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N})dt & i = 1, ..., N \\ X_0^{i,N} = X_0^i & i = 1, ..., N. \end{cases} \tag{7}$$

Here $X_0^{i,N}$ are i.i.d. variables, with common law μ_0.
As noted by Malrieu, when $V = 0$, the direction $\{(v, \cdot, , v)\}$ is quite singular for the particle system, and we then introduce its orthogonal hyperplane $\mathcal{M} = \left\{x \in (\mathbb{R}^d)^N; \sum_{i=1}^N x_i = 0\right\}$, and consider the projected particle system $Y_t^{i,N} = X_t^{i,N} - \frac{1}{N}\sum_{j=1}^N X_t^{j,N} \quad \forall i = 1, ..., N$. The process $(Y_t^N)_{t\geqslant 0}$ thus verifies the system of SDE's

$$\begin{cases} dY_t^{i,N} = \sqrt{2}dB_t^i - \frac{\sqrt{2}}{N}\sum_{j=1}^N dB_t^j - \frac{1}{N}\sum_{j=1}^N \nabla W(Y_t^{i,N} - Y_t^{j,N})dt & i = 1, ..., N \\ Y_0^{i,N} = X_0^i - \frac{1}{N}\sum_{j=1}^N X_0^{j,N} & i = 1, ..., N \end{cases} \tag{8}$$

hence is a diffusion on $\{\mathcal{M}\}$. Note that $\mathbb{E}(Y_t^{i,N}) = 0$.

We assume that W is satisfying the assumptions (A) in Definition 1.1. Let us first remark that this new particle system satisfies globally the "convexity at infinity" property. We have to verify, denoting $b(x) = (b^1(x), ..., b^N(x))$, b^i with values in $\mathbb{R}^d$ and

$$b^i(x) = -\frac{1}{N}\sum_{j=1}^N \nabla W(x_i - x_j),$$

that condition (2) holds true. Note that, with σ the (constant) matrix diffusion coefficient of $(Y_t^{i,N})_{1_lei\leq N}$, we have $\|\sigma\|_{HS} \leq c$ for some positive c, and, as W is symmetric and $x, y \in \mathcal{M}$

$$\begin{aligned} (x-y)\cdot(b(x)-b(y)) &= -\frac{1}{2N}\sum_{i,j=1}^N (x_i - x_j - (y_i - y_j))\cdot(\nabla W(x_i - x_j) - \nabla W(y_i - y_j)) \\ &\leq \frac{1}{2N}\sum_{i,j=1}^N \left(-\lambda|x_i - x_j - y_i + y_j|^2 + C\right) = -\lambda\|x-y\|^2 + \frac{CN}{2}, \end{aligned}$$

where the latter equality follows as $x, y \in \mathcal{M}$.
This remark allows us to prove existence, uniqueness and non-explosion for the solution of the SDE (8). Indeed recall that a sufficient condition for all this to hold is the following:
there exists some ψ such that $\psi(x) \to +\infty$ as $|x| \to +\infty$ and $\Delta\,\psi\, +\, b.\nabla\psi$ is bounded from above. ψ is a kind of Lyapounov function. Here we may choose $\psi(x) = |x|^2$ according to the previous remark applied with $y = 0$. In this case non-explosion holds as soon as $\mu_0(|x|^2) < +\infty$. The same argument can be used with (A') if W is everywhere convex. For the initial system (7) the only thing to show is non explosion. It will follow from Proposition 2.1 below.

2.1 Moment controls for the particle system

In order to prove tightness of the empirical measure of the particle system, and for later use, we first prove some controls on moments.

Proposition 2.1 *If W and V satisfy (A'), and $\mu_0(|x|^2) < +\infty$ the solution to (7) is non explosive. Furthermore there exists some $K > 0$ such that for all i,*

$$\sup_{t\geqslant 0} \mathbb{E}|X_t^{i,N}|^2 \leq \mu_0(|x|^2) + K\,.$$

If W satisfies (A) a similar result holds with $Y_t^{i,N}$ in place of $X_t^{i,N}$.

Proof

$\lhd$ Using Itô's formula up to the infimum between t and the exit time of a large ball (let T be this infimum) we have

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^N |X_T^{i,N}|^2 &= \mathbb{E}\sum_{i=1}^N |X_0^{i,N}|^2 + 2dN\mathbb{E}(T) - 2\mathbb{E}\sum_{i=1}^N \int_0^T X_s^{i,N}.\nabla V(X_s^{i,N})ds \\
&\quad - \frac{2}{N}\mathbb{E}\sum_{i,j=1}^N \int_0^T X_s^{i,N}.\nabla W(X_s^{i,N} - X_s^{j,N})ds \\
&= \mathbb{E}\sum_{i=1}^N |X_0^{i,N}|^2 + 2dN\mathbb{E}(T) - 2\mathbb{E}\sum_{i=1}^N \int_0^T X_s^{i,N}.\nabla V(X_s^{i,N})ds \\
&\quad - \frac{1}{N}\mathbb{E}\sum_{i,j=1}^N \int_0^T (X_s^{i,N} - X_s^{j,N}).\nabla W(X_s^{i,N} - X_s^{j,N})ds.
\end{aligned}$$

We may use condition (2), go to the limit with respect to the radius of the ball (hence replace T by t) and obtain the finiteness of the quantity. This implies non explosion.
Furthermore, denoting $v(t) = \mathbb{E}\sum_{i=1}^N |X_t^{i,N}|^2$, differentiating (see the proof of the next Proposition) and using our conditions (in particular $x.\nabla W(x) \geqslant \lambda_W |x|^2 - C$), we get

$$v'(t) \quad \leq \quad -2\lambda_V v(t) + 2dN + 3CN.$$

Gronwall's lemma and exchangeability conclude the proof in case (A').
The proof is similar for Y in case (A) using the convexity at infinity of the drift, as we previously remarked. $\rhd$

Proposition 2.2 *If W satisfies (A) or W and V satisfy (A'), for all $k \in \mathbb{N}$ there exists $C(k) > 0$ such that for all $1 \leq i, j \leq N$,*

$$\sup_{t\geqslant 0} \mathbb{E}\left(|X_t^{i,N} - X_t^{j,N}|^{2k}\right) \leq C(k)\left(1 + \mu_0 \otimes \mu_0\left(|x-y|^{2k}\right)\right).$$

Proof

$\lhd$ We write the proof in the case $V = 0$ (i.e (A)), the case (A') is similar. Recall that all particles are exchangeable. We may apply Itô's formula up to the exit time of a large ball (for the whole system) and then go to the limit in order to get,

$$\mathbb{E}\sum_{i,j=1}^N |X_t^{i,N} - X_t^{j,N}|^{2k} = \mathbb{E}\sum_{i,j=1}^N |X_0^{i,N} - X_0^{j,N}|^{2k} + 2k(2k-1)\int_0^t \mathbb{E}\sum_{i,j=1}^N |X_s^{i,N} - X_s^{j,N}|^{2k-2}ds$$

$$-\frac{2k}{N}\mathbb{E}\sum_{i,j,l=1}^N \int_0^t \left(\nabla W(X_s^{i,N} - X_s^{l,N}) - \nabla W(X_s^{i,N} - X_s^{l,N})\right).(X_s^{j,N} - X_s^{l,N})|X_s^{i,N} - X_s^{j,N}|^{2k-2}ds.$$

Denoting $A_k(t) = \mathbb{E}\sum_{i,j=1}^N |X_t^{i,N} - X_t^{j,N}|^{2k}$, and using (2) we obtain

$$A_k(t) \leq N\mu_0 \otimes \mu_0\left(|x-y|^{2k}\right) + 2k(2k-1+C)\int_0^t A_{k-1}(s)ds - 2k\lambda\int_0^t A_k(s)ds\,.$$

Applying Gronwall's lemma and an easy induction we thus have that $A_k(t)$ is finite for all t. Accordingly we may replace the pair of times $(0,t)$ by $(t,t+\varepsilon)$ and prove that $t \mapsto A_k(t)$ is differentiable. Differentiating at time t yields $A_1'(t) \leq -2\lambda A_1(t) + 2N(d+C)$. Gronwall's lemma yields the desired result for $k=1$. The proof follows, using this bound and an easy induction. $\triangleright$

We are able now to generalize Proposition 2.1 and get uniform moment estimates of every order (under assumptions on the initial condition).

Corollary 2.3 *If W satisfies (A) or if W and V satisfy (A') (where m is defined), then for all $t>0$ and all $k \geqslant 1$ there exists a constant $c(k)$ such that for all i*

$$\sup_{t\geqslant 0} \mathbb{E}|X_t^{i,N}|^{2k} \leq c(k)(1+\mu_0(|x|^{2mk}))\,.$$

If W satisfies (A) a similar result holds with $Y_t^{i,N}$ in place of $X_t^{i,N}$.

Proof
$\triangleleft$ We write the proof under (A'). Let $B_k(s) = \mathbb{E}(|X_s^{i,N}|^k)$. As in the previous propositions we shall use Itô's formula up to the stopping time T and then go to the limit. Using our hypotheses we get that for some nonnegative λ'

$$\begin{aligned} B_{2k}(t) &\leq B_{2k}(0) + 2k(C+2k-1)\mathbb{E}\int_0^t |X_s^{i,N}|^{2k-2}ds - 2k\lambda'\,\mathbb{E}\int_0^t |X_s^{i,N}|^{2k}ds \\ &\quad -\frac{2k}{N}\,\mathbb{E}\sum_{j=1}^N \int_0^t \left(\nabla W(X_s^{i,N} - X_s^{j,N})\right)\cdot X_s^{i,N}|X_s^{i,N}|^{2k-2}ds. \end{aligned}$$

To bound the last term above, we use (A3) i.e. $|\nabla W(X_s^{i,N} - X_s^{j,N})| \leq M(1+|X_s^{i,N} - X_s^{j,N}|^m)$, and Hölder inequality in order to obtain the following upper bound

$$2^{(2k-1)/2k}\,M \int_0^t \left[\mathbb{E}|X_s^{i,N}|^{2k}\right]^{\frac{2k-1}{2k}} \left[\mathbb{E}\Big(1+|X_s^{i,N} - X_s^{j,N}|^{2mk}\Big)\right]^{1/2k} ds\,.$$

Now we may use exchangeability and Proposition 2.2 to obtain

$$\begin{aligned} B_{2k}(t) &\leq B_{2k}(0) + 2k(C+2k-1)\,\mathbb{E}\int_0^t |X_s^{i,N}|^{2k-2}ds - 2k\lambda'\mathbb{E}\int_0^t |X_s^{i,N}|^{2k}ds \\ &\quad +c(k)(1+\mu_0(|x|^{2mk})^{1/2k}\int_0^t \left[\mathbb{E}\Big(|X_s^{i,N}|^{2k}\Big)\right]^{\frac{2k-1}{2k}} ds\,. \end{aligned}$$

As usual, one can get that B_{2k} is differentiable and satisfies

$$B_{2k}'(t) \leq c(k)B_{2k-2}(t) - 2kB_{2k}(t) + c(k)(1+\mu_0(|x|^{2mk}))B_{2k}(t)^{\frac{2k-1}{k}}\,.$$

Since, for every $\varepsilon>0$, it exists c such that $|x|^{2k-2} \leq c + \varepsilon|x|^{2k}$ and $a^{\frac{2k-1}{2k}} \leq 1+a$ we thus obtain

$$B_{2k}'(t) \leq c(k)(1+\mu_0(|x|^{2mk})) - \tilde\lambda B_{2k}(t),$$

for some $\tilde\lambda > 0$ and we can conclude using Gronwall one more time. $\triangleright$

Remark 2.4 *If V is identically 0 we may simply remark that*

$$|X_t^{i,N}|^k \leq 3^{k-1}\left(|X_0^{i,N}|^k + |\sqrt{2}B_t^i|^k + \left(\frac{1}{N}\sum_{j=1}^N \int_0^t C(1+|X_s^{i,N} - X_s^{j,N}|^m)ds\right)^k\right)$$

and directly conclude with the help of Proposition 2.2, thanks to the convexity of $x \to |x|^k$. One could mimic the preceding proof to get the uniform estimate of the moments of the projected particle system under assumptions (A).

2.2 Identification and existence of solutions of the nonlinear SDE

We prove here how a tightness criterion may ensure that the empirical law of the particles converges to the solution of the nonlinear SDE (3) proving thus the existence of such a solution.

Lemma 2.5 *If W satisfies (A) or W and V satisfy (A'), then for all s and t smaller than T, all $k \geqslant 2$ and all i*

$$\mathbb{E}|X_t^{i,N} - X_s^{i,N}|^{2k} \leq C(k,T)(1 + \mu_0(|x|^{m(m+2k-1)}))\, |t-s|^{\frac{3}{2}}\,.$$

Proof

$\lhd$ Let $0 \leq s \leq t \leq T$ and $k \geqslant 1$.

$$\begin{aligned}
\mathbb{E}|X_t^{i,N} - X_s^{i,N}|^{2k} &\leq 2k(2k-1)\mathbb{E}\int_s^t |X_u^{i,N} - X_s^{i,N}|^{2k-2} du \\
&\quad + (2k)\mathbb{E}\int_s^t |\nabla V(X_u^{i,N})||X_u^{i,N} - X_s^{i,N}|^{2k-1} du \\
&\quad + \frac{2k}{N}\,\mathbb{E}\sum_{j=1}^N \int_s^t \left(|\nabla W(X_u^{i,N} - X_u^{j,N})|\right) |X_u^{i,N} - X_s^{i,N}|^{2k-1} du.
\end{aligned}$$

Hence using (A3), Corollary 2.3 and Hölder (with the best choice of exponents) we get that for $k \geqslant 1$

$$\mathbb{E}|X_t^{i,N} - X_s^{i,N}|^{2k} \leq C(k,T)(1 + \mu_0(|x|^{m(m+2k-1)}))\, |t-s|\,.$$

Plugging this estimate into the previous inequality and using Cauchy-Schwarz , we obtain the desired result. $\rhd$

It is well known that Lemma 2.5 implies that the sequence of the laws of $(s \to X_s^{1,N})_N$ defined on $\mathcal{C}([0,T], dR^d)$ is tight. In order to build a solution to the nonlinear SDE (3) we may now follow some standard routine in mean-field particle systems. We here follow the one in [Mél95] Theorem 4.4. Thanks to Proposition 4.6 in [Mél95] and to Lemma 2.5, the empirical measures π_N defined on $\mathcal{P}(\mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d)))$ by $\pi_N = 1/N \sum_{i=1}^N \delta_{X^{i,N}_.}$ is tight too. According to the end of section 4.2 in [Mél95], for any limit point π_∞, any $Q \in \mathcal{P}(\mathcal{C}([0,T], \mathbb{R}^d))$ is π_∞ a.s. a solution to the nonlinear martingale problem (up to time T). Actually on one hand the proof here is simpler since we have no jumps, but on the other hand the drifts are unbounded (but with polynomial growth) so that to justify passage to the limit we have to use the (uniform in N) moment estimates in Corollary 2.3 and Proposition 2.2. Details are straightforward and left to the reader, who will check that a sufficient condition is $\mu_0(|x|^{m^2}) < +\infty$.

But consider two solutions $(\bar{X}_., \bar{Z}_.)$ of the nonlinear SDE (3), built with the same Brownian motion and the same initial condition, and introduce an independent copy $(\bar{X}'_., \bar{Z}'_.)$. Thanks to Proposition 2.7 below, we know that all processes have a finite second order moment. Hence, it holds

$$\begin{aligned}
A(t) = \mathbb{E}[|\bar{X}_t - \bar{Z}_t|^2] &= -\int_0^t 2\mathbb{E}[(\nabla V(\bar{X}_s) - \nabla V(\bar{Z}_s)).(\bar{X}_s - \bar{Z}_s)]ds \\
&\quad - \int_0^t 2\mathbb{E}[(\nabla W(\bar{X}_s - \bar{X}'_s) - \nabla W(\bar{Z}_s - \bar{Z}'_s)).(\bar{X}_s - \bar{Z}_s)]ds\,.
\end{aligned}$$

But again since $\nabla W(-a) = -\nabla W(a)$ the later integral can be rewritten

$$-\int_0^t \mathbb{E}[(\nabla W(\bar{X}_s - \bar{X}'_s) - \nabla W(\bar{Z}_s - \bar{Z}'_s)).\left((\bar{X}_s - \bar{X}'_s) - (\bar{Z}_s - \bar{Z}'_s)\right)]ds\,.$$

Using [A4] we have $-(\nabla V(x) - \nabla V(y)).(x-y) \leq \beta|x-y|^2$ a similar result holding for W. Using $(a+b)^2 \leq 2a^2 + 2b^2$ and the fact that $\bar{X} - \bar{Z}$ and $\bar{X}' - \bar{Z}'$ have the same law we get $A(t) \leq$

$3\beta \int_0^t A(s)ds$, so that $A(t) = 0$. Hence we have proved (strong) uniqueness for the nonlinear SDE. As for linear SDE, this notion of uniqueness implies uniqueness in law.

As a byproduct, we obtain that $\pi_\infty = \delta_Q$ for some Q which is the unique solution of (3) and that π_N goes to δ_Q.
What we have obtained is the following: there exists an unique probability measure Q defined on $C([0,T],\mathbb{R}^d)$ such that for all smooth f

$$(\omega, t) \mapsto f(\omega_t) - f(\omega_0) - \int_0^t \left(\Delta f(\omega_s) - \nabla V \cdot \nabla f(\omega_s) - \nabla W * Q_s(\omega_s) \cdot \nabla f(\omega_s)\right) ds$$

is a Q martingale, with

$$\nabla W * Q_s(\omega_s) = \int \nabla W(\omega_s - y) Q_s(dy).$$

Let $H(x) = \int \nabla W(x-y)Q_s(dy)$. Thanks to Corollary 2.3, $H(x) \leq C(1+|x|^m)$ (provided $\mu_0(|y|^{m^2})$ is finite) and is local Lipschitz (thanks to (A3)). Thus the SDE

$$dZ_t = \sqrt{2} dB_t - \nabla V(Z_t)\, dt \; - \; H(Z_t)\, dt$$

has an unique solution up to its explosion time. Since Q is a solution, it is the only one. The solution is thus non explosive, and Girsanov theory tells us that Q is absolutely continuous w.r.t. the Wiener measure, provided the drift is of finite energy (see e.g. [CL94] Proposition 2.3), i.e.

$$\int_0^T \int \left(|\nabla V|^2 + |H|^2\right)(x)\, Q_s(dx)\, ds \; < \; +\infty\,.$$

This condition is satisfied provided $\mu_0\left(|x|^{2m^2}\right) < +\infty$. Thus for all $s > 0$, Q_s is absolutely continuous w.r.t. Lebesgue measure, with a density u_s.
Let us summarize our results.

Theorem 2.6 *Assume that W satisfies (A) or that W and V satisfy (A'). Assume in addition that $\mu_0(|x|^a) < +\infty$ for $a = \max(m(m+3), 2m^2)$.*
Then the nonlinear SDE (3) has an unique solution Q.
Furthermore the density u_t satisfies for all $T > 0$,

$$\sup_{0<t\leq T} \int |x|^{2k} u_t(x)dx \leq C(T)(1 + \mu_0(|x|^{2mk}))\,,$$

and $t \to u_t$ is a solution of (1).
Finally $t \to u_t dx$ is the unique solution of (1) among the set of continuous flows of measures $t \to \nu_t$ satisfying for all $T > 0$,

$$\int_0^T \int |x|^{2m^2} d\nu_t dt < +\infty\,.$$

Proof
$\lhd$ The only thing it remains to prove is the last statement. Let $t \to \nu_t$ be a solution of (1) satisfying the (finite energy) condition above. Then, according to Theorem 4.18 in [CL94], a solution of the (linear time inhomogeneous) SDE

$$dZ_t = \sqrt{2} dB_t - \nabla V(Z_t)dt - (\nabla W * \nu_t)(Z_t)dt$$

with initial law μ_0 exists and furthermore its law at time t is given by ν_t. Hence $Z_.$ is a solution of the nonlinear SDE. Uniqueness of the later implies that $\nu_t = u_t dx$. $\rhd$

About the literature. If existence and uniqueness have been extensively discussed since the 80's in the framework of mean field interacting particle systems with Lipschitz interactions (i.e. $aId \le HessW \le bId$), the case of non-Lipschitz interactions was not studied in general. The above proof shows that convexity at infinity allows us to essentially mimic the bounded case, just changing the tightness criterion to be used, and using some recent aspects of stochastic calculus related to diffusion processes with $\mathbb{L}^2$ drifts.
If $d = 1$ another version of Theorem 2.6 is obtained in [BRTV98] Theorem 3.1 following a completely different way. One can notice that some moment condition on μ_0 similar to ours is also required therein. Our condition (as well as the one in [BRTV98]) is certainly non sharp. If a large part of the method in [BRTV98] can be extended to the d dimensional case, some aspects (strongly using monotonicity) require additional work.
For the nonlinear PDE, for $d = 1$ and $m = 2$ an existence and uniqueness result is stated in [BCCP98] p.983. However the initial measure has to be absolutely continuous with a $\mathcal{C}^2$ density satisfying $\int |x|^4 d\mu_0 < +\infty$ (while we need a 10 instead of a 4), and uniqueness holds for classical ($\mathcal{C}^2$) solutions.
The conclusion is that such results are certainly not useless. Since we shall need stronger moment assumptions in the sequel, we did not try to obtain the sharpest conditions in Theorem 2.6 (the interested reader may indeed remark that one need for the use of lemma 2.5 for tension only a dependence in time of the order $|t - s|^{1+\epsilon}$ so that the condition on the initial measure may be weakened).

We finally give the proof of the moment control for any solution of the nonlinear SDE.

Proposition 2.7 *Assume that W satisfies (A) or that W and V satisfy (A'). Then*

$$\sup_{t\geqslant 0} \mathbb{E}|\bar{X}_t|^2 \le \mu_0(|x|^2) + K \, .$$

Proof
$\lhd$ Let $s \to \bar{X}'_s$ an independent copy of $s \to \bar{X}_s$. Then

$$\begin{aligned}
\mathbb{E}(|\bar{X}_t|^2) &= \mathbb{E}(|\bar{X}_0|^2) - 2k\int_0^t \mathbb{E}\left(\bar{X}_s \cdot (\nabla V(\bar{X}_s) + \nabla W * u_s(\bar{X}_s))\right) ds + 2dt \\
&= \mathbb{E}(|\bar{X}_0|^2) - 2\int_0^t \mathbb{E}(\bar{X}_s \cdot \nabla V(\bar{X}_s)) ds - \int_0^t \mathbb{E}((\bar{X}_s - \bar{X}'_s) \cdot \nabla W(\bar{X}_s - \bar{X}'_s)) ds + 2dt
\end{aligned}$$

so that if we denote $v(t) = \mathbb{E}(|\bar{X}_t|^2)$, we may first differentiate w.r.t. time as we did before, and then use the hypotheses. Remark that, since W is convex at infinity and symmetric,

$$x \cdot \nabla W(x) \ge x \cdot \nabla W(0) + \lambda_W |x|^2 - C \geqslant -C.$$

It follows

$$v'(t) \le -2\lambda_V v(t) + (2d + 3C) - 2\lambda_W \left(v(t) - \mathbb{E}^2(\bar{X}_t)\right) \, .$$

If (A') holds, $\lambda_V > 0$ and Gronwall's lemma concludes the proof. If $V = 0$, $\lambda_W > 0$ and we have assumed that $E(\bar{X}_t) = 0$ so that Gronwall's lemma concludes the proof. $\rhd$

3 Propagation of chaos

This section is devoted to the comparison of the behavior, for a fixed number of particles, of the difference between one particle and the solution of the non-linear SDE. Ideally, it will be uniform on time and will decrease quickly to 0 as the number of particles increases. We will however see that to get such an estimate, we have to introduce a new convexity assumption, ensuring strict convexity except at some points.

All results in this section are written under assumption (A) i.e. with a vanishing confinement potential V. Replacing Y by X, the same results hold under assumption (A'), modifying the proofs in the same way as we did for various statements in the previous section.

We prove here a first uniform control on the mean square error between one particle and the solution of the nonlinear SDE. In the sequel we shall always make assumptions ensuring existence and uniqueness of strong solutions. Hence we may build solutions for (8) (or (7)) and (3) with the same Brownian motions and the same initial random variables (obtaining thus and i.i.d. sample $(\bar{X}^i_.)_{i=1,\dots,N}$ of $\bar{X}_.$).

Since V and W are the sum of an uniformly convex function and a Lipschitz compactly supported one, one may show the following bound

$$\sup_{0\le t\le T} \mathbb{E}\left(|Y^{i,N}_t - \bar{X}^i_t|^2\right) \le K \min\left(1, \frac{e^{aT}}{N}\right)$$

for some constants K and a only depending on V and W. This can be made by following [Mal01, Mal03] for the uniformly convex part and the standard estimates in mean-field systems with Lipschitz coefficients (see e.g. [Mél95]). This bound is quite bad, and it does not seem easy to improve it. Hence we need some additional assumption.
The condition (6) in the introduction and inspired by the work of Carrillo-McCann-Villani [CMCV06] is recalled below:
we say that condition $\mathbf{C}(\mathbf{A},\alpha)$ holds if there exist $A, \alpha > 0$ such that for any $0 < \epsilon < 1$,

$$\forall x, y \in \mathbb{R}^d, \qquad (x-y).(\nabla W(x) - \nabla W(y)) \ge A\epsilon^\alpha(|x-y|^2 - \epsilon^2).$$

We can now prove

Theorem 3.1 *Assume that W satisfies $\mathbf{C}(\mathbf{A},\alpha)$ and (A). Suppose that the law μ_0 has a large enough polynomial moment. Then there exists $K > 0$ such that*

$$\sup_{t\ge 0} \mathbb{E}\left(|Y^{i,N}_t - \bar{X}^i_t|^2\right) \le \frac{K}{N^{\frac{1}{1+\alpha}}}. \tag{9}$$

Proof
$\triangleleft$ Introduce $\bar{Y}^{i,N}_t = \bar{X}^i_t - \frac{1}{N}\sum_{j=1}^N \bar{X}^j_t$. Then

$$\begin{aligned}
\mathbb{E}\left(|Y^{i,N}_t - \bar{X}^i_t|^2\right) &\le 2\mathbb{E}\left(|Y^{i,N}_t - \bar{Y}^i_t|^2\right) + 2\mathbb{E}\left(|\bar{Y}^{i,N}_t - \bar{X}^i_t|^2\right)\\
&= 2\mathbb{E}\left(|Y^{i,N}_t - \bar{Y}^i_t|^2\right) + \frac{2}{N}\mathbb{E}(|\bar{X}^i_t|^2)
\end{aligned}$$

since $\mathbb{E}(\bar{X}^i_t) = 0$ and the $\bar{X}^j_t$'s are independent. The second term is of a better order thanks to Proposition 2.7 so that we focus on the first term. It holds

$$\begin{aligned}
Y^{i,N}_t - \bar{Y}^{i,N}_t &= -\frac{1}{N}\int_0^t \sum_{j=1}^N \left[\nabla W(Y^{i,N}_s - Y^{j,N}_s) - \nabla W * u_s(\bar{X}^i_s)\right] ds\\
&\quad -\frac{1}{N}\int_0^t \sum_{j=1}^N \nabla W * u_s(\bar{X}^j_s) ds
\end{aligned}$$

so that $\sum_{i=1}^N |Y^{i,N}_t - \bar{Y}^{i,N}_t|^2 = -\frac{2}{N}\sum_{i,j=1}^N \int_0^t (A_{ij}(s) + B_{ij}(s) + C_{ij}(s))ds$ with

$$\begin{aligned}
A_{ij}(s) &= (\nabla W(Y^{i,N}_s - Y^{j,N}_s) - \nabla W(\bar{Y}^{i,N}_s - \bar{Y}^{j,N}_s)) \cdot (Y^{i,N}_s - \bar{Y}^{i,N}_s),\\
B_{ij}(s) &= (\nabla W(\bar{X}^i_s - \bar{X}^j_s) - \nabla W * u_s(\bar{X}^i_s)) \cdot (Y^{i,N}_s - \bar{Y}^{i,N}_s),\\
C_{ij}(s) &= \nabla W * u_s(\bar{X}^i_s)) \cdot (Y^{i,N}_s - \bar{Y}^{i,N}_s).
\end{aligned}$$

Using (6) we obtain $\sum_{i,j=1}^N A_{ij}(s) \geq A\epsilon^\alpha \left(N\sum_{i=1}^N |Y_s^{i,N}-\bar{Y}_s^{i,N}|^2 - \epsilon^2 N^2/2\right)$. Remark now that

$$\begin{aligned} -\mathbb{E}\sum_{j=1}^N B_{ij}(s) &\leq \left(\mathbb{E}|Y_s^{i,N}-\bar{Y}_s^{i,N}|^2\right)^{1/2}\left(\mathbb{E}|\sum_{j=1}^N(\nabla W(\bar{X}_s^i-\bar{X}_s^j)-\nabla W * u_s(\bar{X}_s^i)|^2\right)^{1/2} \\ &\leq c\sqrt{N}\left(\mathbb{E}|Y_s^{i,N}-\bar{Y}_s^{i,N}|^2\right)^{1/2} \end{aligned}$$

by Cauchy-Schwarz inequality, the polynomial growth of ∇W, the controls of moments previously established and the key remark $\mathbb{E}(\nabla W(\bar{X}_s^i-\bar{X}_s^j)-\nabla W * u_s(\bar{X}_s^i))=0$ showing that the only terms with non zero expectation in $\mathbb{E}|\sum_{j=1}^N(\nabla W(\bar{X}_s^i-\bar{X}_s^j)-\nabla W * u_s(\bar{X}_s^i)|^2$ are the squared terms. The last term involving $C_{ij}(s)$ is controlled using the same tools. We finally get, defining $\alpha(s)=\mathbb{E}(|Y_s^{i,N}-\bar{Y}_s^{i,N}|^2)$ and reasoning as we did in the previous section, the differential inequality

$$\alpha'(s) \leq -2A\epsilon^\alpha(\alpha(s)-\epsilon^2)+\frac{c}{\sqrt{N}}\sqrt{\alpha(s)}.$$

For ε fixed (say 1) we thus have $\alpha'(s) \leq -2\lambda\alpha(s)+\frac{c}{\sqrt{N}}\sqrt{\alpha(s)}+2A$, with $\lambda = A\alpha$. Using $\sqrt{\alpha(s)}/\sqrt{N} \leq 1/2(\varepsilon^{-1}\frac{1}{N}+\varepsilon\alpha(s))$ and and choosing $\varepsilon = 2\lambda/c$, he previous inequality becomes a classical

$$\alpha'(s) \leq -\lambda\alpha(s)+(c/\varepsilon N)+2A$$

for which we can use Gronwall's lemma. This shows that $\alpha(s) \leq K$ for some $K>1$.
But now we may take $\epsilon = \sqrt{\alpha(s)}/2\sqrt{K} < 1$ and get

$$\alpha'(s) \leq -J\,\alpha(s)^{1+\alpha/2}+\frac{c}{\sqrt{N}}\sqrt{\alpha(s)}$$

with $J=\frac{2A}{(2\sqrt{K})^\alpha}(1-\frac{1}{4K})$. Define $\beta(s)=\sqrt{\alpha(s)}$. Then $\beta'(s)+(J/2)\beta^{1+\alpha}(s) \leq \frac{c}{2\sqrt{N}}$ so that

$$\beta(s) \leq C/N^{1/2(1+\alpha)}$$

for any s such that $\beta'(s) \geq 0$. Since $\beta(0)=0$ it easily follows that $\beta(s) \leq C/N^{1/2(1+\alpha)}$ everywhere, hence the result. $\triangleright$

Remark. If (A') holds we have to assume that V satisfies $\mathbf{C}(\mathbf{A},\alpha)$ and W is convex.

4 Ergodicity of the nonlinear SDE and Concentration around the stationary measure.

4.1 Existence of a stationary measure and control on $\mathbb{L}^2$ Wasserstein distance.

In fact, as long as we consider only the particle system, the condition (2) is sufficient to ensure the ergodicity, and even exponential ergodicity, of the particle system. Indeed, following [DMT95], we have

$$\|P_t^N(x,\cdot)-\pi^N(\cdot)\|_{TV} \leq c\,e^{-t\delta(N)}\,\Phi(x),$$

where P_t^N denotes the semigroup associated with the particle system, and Φ is a well chosen Lyapunov function.
As a byproduct, we get the existence of an invariant measure for the nonlinear SDE (hence an equilibrium for the nonlinear PDE), provided we have uniform (in time) propagation of chaos. Indeed the previous inequality together with uniform propagation of chaos show that the family $(u_t)_{t>0}$ is a Cauchy family in $\mathbf{L}^1(dx)$, which is a complete space.

Such a result is however not useful to control the convergence to equilibrium for the nonlinear SDE (or even the nonlinear PDE). Indeed there is no close form for $\delta(N)$ which heavily depends on N, so that even the uniform propagation of chaos cannot give good estimations for the rate of convergence of $(\bar{X}_t)$ to its invariant measure.

The method used in [Mal03], i.e. Bakry-Émery condition for the measure u_∞^N, cannot be used under assumption $\mathbf{C}(\mathbf{A}, \alpha)$ as the Hessian may degenerate. More precisely with some work one still can use log-Sobolev inequalities but obtain loose results.
The goal of this section is to show that even if the logarithmic Sobolev inequality cannot give useful result, one may use a direct approach to obtain a (uniform in N) control of the Wasserstein distance between two solutions of the particle system starting from different points. So, combining this with the uniform in time propagation of chaos, we recover in an elegant and simple way the results of Carrillo-McCann-Villani [CMCV06].

Theorem 4.1 *Assume that W satisfies (A) and the convexity condition $\mathbf{C}(A, \alpha)$. Let u_t and v_t be the unique solutions of the nonlinear PDE with initial conditions respectively u_0 and v_0. We assume for simplicity that both u_0 and v_0 have an exponential moment (or a large enough polynomial moment in order to ensure existence and uniqueness).*
Then $t \mapsto W_2^2(u_t, v_t)$ is non-increasing. Furthermore there exists $t_1 \le (2^{2+\alpha}/3)\,\log(W_2^2(u_0, v_0))/A$ such that

$$\begin{aligned} W_2^2(u_t, v_t) &\le e^{-(3A/2^{2+\alpha})t}\, W_2^2(u_0, v_0) \qquad \text{if } t < t_1\,, \\ &\le \left(1 + A\,\left(\alpha/(2+\alpha)\right)^{1+\alpha/2} (t - t_1)\right)^{-2/\alpha} \qquad \text{if } t \ge t_1\,. \end{aligned} \tag{10}$$

In particular, if $t > (1+\eta)(2^{2+\alpha}/3)\,\log(W_2^2(u_0, v_0))/A$ for some $\eta > 0$, one has

$$W_2^2(u_t, v_t) \le \left(1 + (A/(1+\eta))\,\left(\alpha/(2+\alpha)\right)^{1+\alpha/2}\, t\right)^{-2/\alpha}.$$

In addition for all $t \ge 0$ one has

$$W_2^2(u_t, v_t) \le \left(W_2^{-\alpha}(u_0, v_0) + A\,\left(\alpha/(2+\alpha)\right)^{1+\alpha/2}\, t\right)^{-2/\alpha}.$$

Under an uniform convexity condition for W (i.e. $\mathbf{C}(A, 0)$), the convergence is exponential i.e.

$$W_2(u_t, v_t) \le C e^{-At} W_2(u_0, v_0).$$

Note that the final assertion completes the result of Malrieu [Mal03].
Proof
$\lhd$ In what follows we shall always consider coupling consisting in picking the same Brownian motions (B^i) for the two particle systems, and only choose an ad-hoc coupling for the initial random variables. Denote $u_t^{1,N}$ the law of the first particle of the centered particle system (Y) starting with initial law u_0 and consider $v_t^{1,N}$ the law of the first particle of the centered particle system starting with initial law v_0. Let us first remark that for each coupling g_0 of u_0 and v_0

$$W_2^2(u_t^{1,N}, v_t^{1,N}) \le \frac{1}{N}\,\mathbb{E}_{g_0}\left(\sum_{i=1}^N |Y_t^{i,N} - Y_t'^{i,N}|^2\right)$$

where $(Y^{i,N})$ is given by the centered particle system where each particle starts with law u_0 , $(Y'^{i,N})$ with measure v_0 and the initial law of the pairs $(Y_0^{i,N}, Y_0'^{i,N})$ are given by independent copies of g_0. The subscript for the expectation is related to the initial law.

We then get by Itô's formula and symmetry (starting now from points y and y')

$$\begin{aligned}
&\mathbb{E}_{y,y'}\left(\sum_{i=1}^N |Y_t^{i,N}-Y_t^{\prime i,N}|^2\right)\\
&=\sum_{i=1}^N |y_i-y'_i|^2-\frac{1}{N}\sum_{i,j}\mathbb{E}_{y,y'}\int_0^t\langle Y_s^{i,N}-Y_s^{\prime i,N}\ ,\ \nabla W(Y_s^{i,N}-Y_s^{j,N})-\nabla W(Y_s^{\prime i,N}-Y_s^{\prime j,N})\rangle ds\\
&=\sum_{i=1}^N |y_i-y'_i|^2-\frac{1}{2N}\sum_{i,j}\mathbb{E}_{y,y'}\int_0^t \xi_{ij}(s)ds,
\end{aligned}$$

with $\xi_{ij}(s)=\langle (Y_s^{i,N}-Y_s^{j,N})-(Y_s^{\prime i,N}-Y_s^{\prime j,N})\ ,\ \nabla W(Y_s^{i,N}-Y_s^{j,N})-\nabla W(Y_s^{\prime i,N}-Y_s^{\prime j,N})\rangle$. We may now differentiate in time and then use condition $\mathbf{C}(A,\alpha)$ to get

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}_{y,y'}\left(\sum_{i=1}^N |Y_t^{i,N}-Y_t^{\prime i,N}|^2\right) &= -\frac{A\epsilon^\alpha}{2N}\sum_{i,j}\mathbb{E}_{y,y'}(|(Y_t^{i,N}-Y_t^{j,N})-(Y_t^{\prime i,N}-Y_t^{\prime j,N})|^2-\epsilon^2)\\
&= -A\epsilon^\alpha\,\mathbb{E}_{y,y'}\sum_i(|Y_t^{i,N}-Y_t^{\prime i,N}|^2-\epsilon^2)\,.
\end{aligned}$$

Define $\xi(t)=\mathbb{E}_{y,y'}\left(|Y_t^{i,N}-Y_t^{\prime i,N}|^2\right)$, we thus obtain using exchangeability that for all $\epsilon<1$

$$\xi'(t)\le -A\epsilon^\alpha(\xi(t)-\epsilon^2).$$

This inequality for ε going to 0, implies $\xi'(t)\le 0$, i.e. ξ is non-increasing. As a byproduct we get that $t\mapsto W_2(u_t^{1,N},v_t^{1,N})$ is also non-increasing. Indeed if we choose g_0 as the optimal coupling for the quadratic cost, it holds $W_2^2(u_0,v_0)=\xi(0)\ge\xi(t)\ge W_2^2(u_t^{1,N},v_t^{1,N})$, and we get the result just shifting the initial time.

Now we can separate two cases: either $\xi(t)>1$ or $\xi(t)\le 1$. Note that there exists some $t_1\ge 0$ such that the first case holds for $t<t_1$ and the second one for $t\ge t_1$.

If $t<t_1$ we may choose any ε, for instance here we choose $\varepsilon=1/2$ and obtain, since $\xi(t)>1$ and $\varepsilon^2=1/4<\xi(t)/4$, that $\xi'(t)\le -A(\alpha)\xi(t)$ with $A(\alpha)=(3A/4)\,(1/2)^\alpha$, which gives by Gronwall's lemma $\xi(t)\le e^{-A(\alpha)t}\,\xi(0)$. Choosing again the optimal coupling g_0 we obtain the first part of the result and the fact that $t_1\le\log(W_2^2(u_0,v_0))/A(\alpha)$.

For the second part, if $t\ge t_1$ choose $\epsilon^2=\alpha\xi(t)/(\alpha+2)$ to get

$$\xi'(t)\le -A\left(\frac{\alpha}{2+\alpha}\right)^{\alpha/2}\frac{2}{2+\alpha}\ \xi(t)^{1+\alpha/2}.$$

Integrating this differential inequality we get $\xi(t)\le(1+B(\alpha)(t-t_1))^{-2/\alpha}$, with $B(\alpha)=A\ (\alpha/(2+\alpha))^{1+\alpha/2}$. Our choice of ε gives the optimal constant (for this method).

Of course writing $\mathbf{C}(A,\alpha)$ as we did is a little bit artificial, and if we want a more homogeneous estimate we may remark that $\mathbf{C}(A,\alpha)$ implies

$$(x-y)\cdot(\nabla W(x)-\nabla W(y))\ge A(\epsilon/W_2(u_0,v_0))^\alpha(|x-y|^2-\epsilon^2)$$

for $\epsilon<W_2(u_0,v_0)$. So we may always choose $\epsilon^2=\alpha\xi(t)/(\alpha+2)$ and get for all t,

$$\xi(t)\le\left(W_2^{-\alpha}(u_0,v_0)+B(\alpha)t\right)^{-2/\alpha}.$$

We use then the uniform propagation of chaos property to transfer the inequality from the particles to the solutions of the nonlinear SDE:

$$W_2(u_t,v_t)\le W_2(u_t,u_t^{1,N})+W_2(u_t^{1,N},v_t^{1,N})+W_2(v_t,v_t^{1,N}),$$

and take the previous (uniform in N) estimation for the middle term and the uniform in time estimation for the first and third term and let N go to infinity. Of course if t_1 depends on N, its bound does not, so that we may find a converging subsequence and get the result. $\triangleright$

Remark. We have seen that there exists an invariant measure u_∞ for the nonlinear PDE. The bound for $W_2(u_t, v_t)$ obtained by introducing the particle system of order N, and then choosing choosing $v_0 = u_\infty^N$ allows us to prove this existence too (using completeness of the Wasserstein distance). In addition the decay of the Wasserstein distance trivially implies uniqueness of u_∞, at least in the set of measures having some large enough polynomial moment. Hence, we recover in the previous result the asymptotic rate of convergence to equilibrium obtained by Carrillo-McCann-Villani.

4.2 Concentration inequality

The main goal of this section is to complete the results on convergence of the particle system and of the nonlinear system by providing a (non asymptotic) deviation inequality. This inequality will be written for additive functionals of the particles and then allows us to estimate integrals with respect to the stationary measure of the nonlinear PDE. Once again it is interesting to get uniform in times estimation to be able to simulate at fixed time the particles and use them for the evaluation with no loss at each time of the constant in the concentration. Therefore we first prove in the general framework an uniform T_1 inequality for solution of SDE subject to some convexity at infinity condition. We then show how to use them on our example.

We first need an uniform control on transportation cost for diffusion processes, when the drift is confining only outside a compact set, and for non-constant diffusion coefficient, so that X is the solution of

$$dX_t = \sigma(t)dB_t + b(X_t)dt. \tag{11}$$

We use here the formalism of transportation cost inequalities in W_1 distance for which a practical criterion based on the integrability of the exponential of the square of the distance is sufficient [DGW04] (and Bolley-Villani [BV05] or Gozlan [Goz05] for a better evaluation of the constant) and which implies interesting deviation inequalities.

Proposition 4.2 *Suppose that there exist $\lambda, A > 0$ and C such that, for every $x, y \in \mathbb{R}^d$,*

$$(x-y)\cdot(b(x)-b(y)) \le -\lambda|x-y|^2 + C, \qquad \|\sigma\|_{HS} \le A \tag{12}$$

then for any $\delta < \lambda/2A$ and any $x \in \mathbb{R}^d$,

$$\sup_{t\geqslant 0} \mathbb{E}\left[e^{\delta|X_t^x - Y_t^x|^2}\right] \le 1 + (Ad + C + 1)e^{\delta(Ad+C+1)/(\lambda-2\delta A)},$$

where X^x and Y^x are two independent copies of (11) starting at x.
So the law of X_t^x satisfies a T_1 inequality with a constant $\mathfrak{C}$ independent of time and initial position. Recall that a T_1 inequality with constant $\mathfrak{C}$ for a measure μ reads as: $W_1(\nu,\mu) \le \sqrt{\mathfrak{C}\, Ent(\nu|\mu)}$ for any ν.

The proof of this proposition is an adaptation (using Gronwall's lemma to obtain uniform in time behavior) of Corollary 4.1 [DGW04] and is thus skipped.
An important consequence of a transportation inequality is that we easily obtain deviation inequality for Lipschitz functions. Indeed for all Lipschitz functions F with $\|F\|_{Lip} \le 1$ and all positive r

$$\mathbb{P}\left(F(X_t^x) - \mathbb{E}(F(X_t^x)) \ge r\right) \le e^{-r^2/\mathfrak{C}}.$$

Remark that the previous proof extends to the case when the initial law μ_0 satisfies

$$\int e^{\delta|x-y|^2}\,\mu_0(dx)\,\mu_0(dy)\;<\;+\infty$$

(that is μ_0 satisfies a T_1 inequality) just choosing two independent variables X_0 and Y_0 of law μ_0.

We may now recombine results inherited from the previous sections to get an useful inequality for the evaluation of $\int f du_\infty$ when f is a Lipschitz function. From the previous section, if (A) holds and μ_0 satisfies a T_1 inequality, the particle system satisfies a T_1 inequality with constant $\mathfrak{C}N$ (for some $\mathfrak{C}$ independent of time), which thus leads to the following:
for f Lipschitz (in $\mathbb{R}^d$) with $\|f\|_{Lip}\leq 1$, for all positive r and all t

$$\mathbb{P}\left(\frac{1}{N}\sum_{k=1}^N f(X_t^{k,N})-\mathbb{E}f(X_t^{1,N})\geq r\right)\leq e^{-Nr^2/\mathfrak{C}}.$$

Remark now that for a Lipschitz function satisfying $\|f\|_{Lip}\leq 1$, if the convexity condition $\mathbf{C}(\mathbf{A},\alpha)$ holds, the uniform propagation of chaos of Theorem 3.1 tells us

$$|\mathbb{E}f(X_t^{1,N})-\int f(y)u_t(y)dy|\leq W_1(u_t^{1,N},u_t)\leq W_2(u_t^{1,N},u_t)\leq\left(\frac{K}{N^{\frac{1}{1+\alpha}}}\right)^{1/2}$$

so that for all $r\geq\left(\frac{K}{N^{\frac{1}{1+\alpha}}}\right)^{1/2}$

$$\mathbb{P}\left(\frac{1}{N}\sum_{k=1}^N f(X_t^{k,N})-\int f(y)u_t(y)dy\geq r-\left(\frac{K}{N^{\frac{1}{1+\alpha}}}\right)^{1/2}\right)\leq e^{-Nr^2/\mathfrak{C}}.$$

We may now use convergence in W_2 distance of the solutions of the nonlinear SDE towards the stationary measure given in Theorem 4.1 to get

Proposition 4.3 *For all* $r\geq\left(\frac{K}{N^{\frac{1}{1+\alpha}}}\right)^{1/2}+\sqrt{\beta(t)}$,

$$\mathbb{P}\left(\frac{1}{N}\sum_{k=1}^N f(X_t^{k,N})-\int f(y)u_\infty(y)dy\geq r-\left(\frac{K}{N^{\frac{1}{1+\alpha}}}\right)^{1/2}-\sqrt{\beta(t)}\right)\leq e^{-Nr^2/\mathfrak{C}},\tag{13}$$

where $\beta(t)$ *is one of the functions governing the decay of* $W_2^2(u_t,u_\infty)$ *described in Theorem 4.1.*

Let us finally note than one can use an Euler-Maryama scheme preserving square exponential integrability and with good stability property [LMS06, Th1 and Th 4] to simulate the particle system leading to the same concentration inequality. Note that the recurrence property needed for the stability of this adaptive scheme in [LMS06] is exactly our condition of convexity at infinity.

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