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Probabilistic approximation and inviscid limits for one-dimensional fractional conservation laws

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We are interested in the one-dimensional scalar conservation law $\partial_t u(t, x) = \nu D^\alpha u(t, x) - \partial_x A(u(t, x))$ with fractional viscosity operator $D^\alpha v(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(v)(\xi))(x)$ when the initial condition $u(0, x)$ is the cumulative distribution function of a signed measure on $\mathbb{R}$. We associate a nonlinear martingale problem with the Fokker–Planck equation obtained by spatial differentiation of the conservation law. After checking uniqueness for both the conservation law and the martingale problem, we prove existence thanks to a propagation-of-chaos result for systems of interacting particles with fixed intensity of jumps related to ν. The empirical cumulative distribution functions of the particles converge to the solution of the conservation law. As a consequence, it is possible to approximate this solution numerically by simulating the stochastic differential equation which gives the evolution of particles. Finally, when the intensity of jumps vanishes ($\nu \to 0$) as the number of particles tends to $+\infty$, we obtain that the empirical cumulative distribution functions converge to the unique entropy solution of the inviscid ($\nu = 0$) conservation law.

Keywords: inviscid scalar conservation laws; nonlinear martingale problems; propagation-of-chaos; scalar conservation laws with fractional Laplacian; stable processes

1. Introduction

Let $\alpha \in (1, 2)$ and D^α denote the *symmetric fractional derivative (fractional Laplacian)* of order α on $\mathbb{R}$, that is, an operator defined either via the Fourier transform $\mathcal{F}$,

$$D^\alpha v(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(v)(\xi))(x), \tag{1.1}$$

or, equivalently, by its singular integral representation,

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$$D^\alpha v(x) = K\int_{\mathbb{R}} \big(v(x+y) - v(x) - \mathbf{1}_{\{|y|\leqslant 1\}} v'(x)y\big)\frac{\mathrm{d}y}{|y|^{1+\alpha}} \tag{1.2}$$

$$= K\int_{|y|>1} (v(x+y) - v(x))\frac{\mathrm{d}y}{|y|^{1+\alpha}} + K\int_{|y|\leqslant 1}\int_0^1 v''(x+zy)(1-z)\mathrm{d}z\frac{\mathrm{d}y}{|y|^{\alpha-1}}, \tag{1.3}$$

for a suitable positive constant K.

We are interested in the initial-value problem for the following *one-dimensional scalar conservation law with fractional viscosity*:

$$\partial_t u(t, x) = \nu D^\alpha u(t, x) - \partial_x A(u(t, x)), \tag{1.4}$$

$$u(0, x) = u_0(x), \tag{1.5}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $\nu > 0$, and $A : \mathbb{R} \to \mathbb{R}$ is a C^1-function.

The present work is motivated by the various physical applications of equations involving pseudo-differential terms like $D^\alpha u$ which model anomalous diffusion; see, for example, Klafter *et al.* (1995), Saichev and Zaslavsky (1997) and Piryatinska *et al.* (2005). Among those equations, the scalar conservation laws with fractional viscosity were introduced by Mann and Woyczynski (2001) in the context of growing interfaces in the presence of self-similar hopping surface diffusion. More precisely, the authors were interested in modelling at the mesoscopic level the chemical vapour deposition of thin semiconducting diamond films and were struck by the role that trapping may play in the growth mechanism. Kellog (1994) reported that the presence of impurity can act as a strong trap for an adatom migrating on a growing interface at room temperature. The conjecture in Mann and Woyczynski (2001) was that long jumps corresponding to Lévy flights are possible between trap sites – this in contrast to the nearest-neighbour hopping which leads to the Brownian motion in the continuum approximation for defect-free surfaces. Their paper developed mesoscopic equations for the growth of the interface based on the conservation of mass and introduced what was called the fractal KPZ-Burgers model, a special case of equation (1.4) (although in the two-dimensional situation). Biler *et al.* (2001) studied self-similar asymptotics and critical nonlinearities of conservation laws with fractional viscosity generalizing the fractal KPZ-Burgers model.

We will call (1.4)–(1.5) a *fractional conservation law*. The initial condition u_0 is assumed to be a non-constant function with bounded variation on $\mathbb{R}$. In other words, dx-almost everywhere on $\mathbb{R}$,

$$u_0(x) = c + \int_{-\infty}^{x} m(\mathrm{d}y) = c + H * m(x)$$

with $c \in \mathbb{R}$, m being a non-zero, bounded signed measure on $\mathbb{R}$, and $H(y)$ denoting the unit step function $\mathbf{1}_{\{y\geqslant 0\}}$.

More precisely, we consider bounded *weak solutions* u of equation (1.4) such that, for any $t \geqslant 0$ and for any C^∞-function ψ with compact support on $[0, t] \times \mathbb{R}$,

$$\int_{\mathbb{R}} \psi(t, x)u(t, x)\mathrm{d}x = \int_{\mathbb{R}} \psi(0, x)u_0(x)\mathrm{d}x + \int_0^t\int_{\mathbb{R}} (u(\partial_s\psi + \nu D^\alpha\psi) + A(u)\partial_x\psi)(s, x)\mathrm{d}x\,\mathrm{d}s. \tag{1.6}$$

Let $\|m\|$ denote the total mass of the measure m. Observe that $u(t, x)$ is a bounded weak solution of the conservation law (1.4) if and only if the function $(u(t, x) - c)/\|m\|$ is a bounded weak solution of the same conservation law with $A(\cdot)$ replaced by $A(c + \|m\|\cdot)/\|m\|$ and initial condition $v_0(x) = H * m(x)/\|m\|$. Therefore, without loss of generality, from now on will assume that

$$c = 0 \quad \text{and} \quad \|m\| = 1.$$

With this standing assumption, the total variation measure $|m|$ of m is a probability measure on $\mathbb{R}$. Denote by $h : \mathbb{R} \to \{-1, 1\}$ a density of m with respect to $|m|$.

To give a probabilistic interpretation to the fractional conservation law (1.4), we will use an approach introduced by Bossy and Talay (1996; 1997) for the viscous Burgers equation ($\alpha = 2$, $A(u) = u^2/2$) and generalized by Jourdain (2000) to any C^1-function A (but still for $\alpha = 2$). We deduce from (1.4) that the gradient $v(t, x) = \partial_x u(t, x)$ satisfies the evolution equation

$$\partial_t v = \nu D^\alpha v - \partial_x(A'(H * v)v), \qquad v(0, \cdot) = m. \tag{1.7}$$

If m is a probability measure on $\mathbb{R}$, this equation is a nonlinear Fokker–Planck equation. The case of a general signed measure can be dealt with using the approach developed by Jourdain (2000) which consists of associating a single nonlinear martingale problem with (1.7).

Let $\mathcal{P}$ and $(X_t)_{t\geqslant 0}$ denote, respectively, the space of probability measures and the canonical process on the space $D(\mathbb{R}_+, \mathbb{R})$ of cadlag functions from $\mathbb{R}_+$ to $\mathbb{R}$ endowed with the Skorokhod topology. We associate with each probability measure $P \in \mathcal{P}$ a signed measure $\tilde{P}$ with density $h(X_0)$ with respect to P and denote, respectively, by $(P_t)_{t\geqslant 0}$ and $(\tilde{P}_t)_{t\geqslant 0}$, the flows of time marginals of measures P and $\tilde{P}$. Thus, for any $B \in \mathcal{B}(\mathbb{R})$,

$$\tilde{P}_t(B) = \mathbb{E}^P\big(h(X_0)\mathbf{1}_B(X_t)\big).$$

Definition 1.1. *We say that $P \in \mathcal{P}$ solves the nonlinear martingale problem* (MP) *if the following conditions are satisfied:*

(i) $P_0 = |m|$.
(ii) *For any $\varphi(t, x)$ in the space $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ of functions which are continuously differentiable with respect to t and twice continuously differentiable with respect to x, and bounded together with their derivatives,*

$$M_t^\varphi \equiv \varphi(t, X_t) - \varphi(0, X_0) - \int_0^t \big(\partial_s\varphi + \nu D^\alpha\varphi + A'(H * \tilde{P}_s(X_s))\partial_x\varphi\big)(s, X_s)\mathrm{d}s$$

is a P-*martingale.*

When P solves this problem, $h(X_0)M_t^\varphi$ is a P-martingale. The constancy of the expectation of this martingale implies that $t \to \tilde{P}_t$ is a weak solution of (1.7).

The paper is organized as follows. In Section 2, we establish existence and uniqueness for martingale problems with linear generators belonging to a class which includes the generators of the particle systems that we study later in the paper.

In Section 3, we first check that the cumulative distribution functions of the signed marginals $\tilde{P}_t$ associated with any solution P of problem (MP) provide a bounded weak solution of the fractional conservation law (1.4). Since $t \to \tilde{P}_t$ is a weak solution of equation (1.7) obtained by spatial differentiation of (1.4), this result is not surprising. Then we prove uniqueness of bounded weak solutions of (1.4) and derive uniqueness for problem (MP).

Section 4 is devoted to the convergence of systems of particles with jumps as the number of particles tends to $+\infty$. We first suppose that the intensity of jumps is constant and obtain existence for problem (MP) and therefore for (1.4) by proving a propagation-of-chaos result. As a consequence, the weighted empirical cumulative distribution functions of the particles converge to the solution of the fractional conservation law (1.4). By discretizing the dynamics of the particles with respect to time, it is therefore possible to construct some Monte Carlo approximations of the solution of (1.4).

In Section 5 we assume that the intensity of jumps vanishes ($\nu \to 0$) as the number of particles tends to $+\infty$. We then prove that the empirical cumulative distribution functions converge to the unique entropy solution of the inviscid ($\nu = 0$) conservation law (1.4). This result can be related to the convergence of the solution of the fractional conservation law to the unique entropy solution of the inviscid conservation law as $\nu \to 0$ in arbitrary space dimension d, obtained by Droniou (2003): when the initial condition has bounded variation like the functions u_0 considered in the present paper, for any $T > 0$, the rate of convergence in $C([0, T], L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is proved to be $\mathcal{O}(\nu^{1/\alpha})$.

Jourdain *et al.* (2005) construct probabilistic approximations for evolution equations studied via analytic tools by Biler and Woyczynski (1998). These equations involve the fractional Laplacian and a singular nonlinear operator of order 1 similar to the term $-\partial_x((H * v)v)$ appearing in (1.7) in the case $A(u) = u^2/2$. The setting is d-dimensional; the Heaviside kernel H is replaced by a kernel $b : \mathbb{R}^d \to \mathbb{R}^d$ such that, for some $C > 0$ and $0 < \beta < +\infty$, and each $x \in \mathbb{R}^d$, $|b(x)| \leqslant C|x|^{\beta-d}$, and the initial measure m is assumed to be absolutely continuous with respect to the Lebesgue measure with a density belonging to $L^p(\mathbb{R}^d)$, where $p > 1$ is related to d, β and α. The study of the evolution equation of interest is based on the introduction of Lipschitz continuous and bounded cutoff versions of kernel b. In addition, the particles interact through these cutoff kernels. Here the approach is different: since the Heaviside kernel H is discontinuous but not singular at the origin, the cutoff procedure is not needed and we are able to deal directly with general signed measures m. In the proof of the vanishing viscosity limit result, it is important to consider particles interacting through the original kernel H.

We conclude this introduction by recalling some useful properties of the semigroup generated by the fractional Laplacian. Denote by p_t^α the convolution kernel of the Lévy semigroup $\exp(tD^\alpha)$ on $\mathbb{R}$. The kernel is self-similar, that is, for any positive t,

$$p_t^\alpha(x) = t^{-1/\alpha} p_1^\alpha(xt^{-1/\alpha}).$$

Moreover, there exists a constant $C_\alpha > 0$ (see, for example, Sato 1999, pp. 89 and 202) such that

$$0 \leqslant p_1^\alpha(x) \leqslant C_\alpha(1+|x|^{1+\alpha})^{-1}, \qquad |\partial_x p_1^\alpha(x)| \leqslant C_\alpha(1+|x|^{2+\alpha})^{-1}.$$

If, for $n \geqslant 1$ and $t \geqslant 0$, we introduce product kernels

$$G_t^{\alpha,n} : \mathbb{R}^n \ni y = (y_1, \ldots, y_n) \mapsto \prod_{i=1}^n p_t^\alpha(y_i),$$

then $G_t^{\alpha,1}(y) = p_t^\alpha(y)$, and the above properties of p_t^α immediately yield the following estimates for $G_t^{\alpha,n}$:

Lemma 1.1. *For any q, $1 \leqslant q \leqslant +\infty$, there is a constant $C > 0$ (depending on n, ν, α, and q) such that, for each $t > 0$, and $i = 1, \ldots, n$,*

$$\|G_{\nu t}^{\alpha,n}\|_q \leqslant Ct^{-n(q-1)/(\alpha q)} \quad \text{and} \quad \|\partial_i G_{\nu t}^{\alpha,n}\|_q \leqslant Ct^{-(n(q-1)+q)/(\alpha q)}.$$

Here ∂_i denotes the derivative with respect to the ith spatial coordinate and $\|\cdot\|_q$ stands for the norm in the usual Lebesgue space L^q.

2. Existence and uniqueness for a class of *n*-dimensional martingale problems

To construct particle systems whose empirical distributions approximate solutions of the fractional conservation law (1.4), we will initially prove the existence and uniqueness results for a class of martingale problems.

Let $\mathcal{P}_n$ and $(Y_t = (Y_t^1, \ldots, Y_t^n))_{t\geqslant 0}$ denote, respectively, the set of probability measures and the canonical process on $D(\mathbb{R}_+, \mathbb{R}^n)$.

Definition 2.1. *Let $b : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ be a bounded measurable function and $\eta \in \mathcal{P}(\mathbb{R}^n)$. We say that $Q \in \mathcal{P}_n$ solves the martingale problem with generator $\nu\sum_{i=1}^n D_i^\alpha + b\cdot\nabla$ starting from η if the initial marginal Q_0 of Q is equal to η and, for any $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$,*

$$M_t^\varphi = \varphi(t, Y_t) - \varphi(0, Y_0) - \int_0^t \left(\partial_s\varphi + \nu\sum_{i=1}^n D_i^\alpha\varphi + b\cdot\nabla\varphi\right)(s, Y_s)\mathrm{d}s$$

is a Q martingale. Here, ∇ and D_i^α denote, respectively, the gradient with respect to the n spatial coordinates, and the symmetric fractional derivative of order α acting on the ith spatial coordinate.

Proposition 2.1. *For any bounded measurable function $b = (b_1, \ldots, b_n) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and any probability measure η on $\mathbb{R}^n$, the martingale problem with generator $\nu\sum_{i=1}^n D_i^\alpha + b\cdot\nabla$ starting from η admits a unique solution $Q \in \mathcal{P}_n$. Additionally, for any*

$t > 0$, the marginal Q_t has a density ρ_t with respect to the Lebesgue measure on $\mathbb{R}^n$ satisfying,

$$\mathrm{d}y\text{-a.e. in } \mathbb{R}^n, \qquad \rho_t(y) = G^{\alpha,n}_{\nu t} * \eta(y) - \sum_{i=1}^n \int_0^t \partial_i G^{\alpha,n}_{\nu(t-s)} * (b_i(s, \cdot)\rho_s)(y)\mathrm{d}s. \tag{2.1}$$

Remark. Since we do not assume any regularity of the drift coefficient b in the spatial variable, existence and uniqueness for the martingale problem cannot be proved by checking existence and trajectorial uniqueness for the corresponding stochastic differential equation. Moreover, the Lévy measure

$$K \sum_{i=1}^n \frac{\mathrm{d}y_i}{|y_i|^{1+\alpha}} \delta_{(0,0,\ldots,0)}(\mathrm{d}y_1, \ldots, \mathrm{d}y_{i-1}, \mathrm{d}y_{i+1}, \ldots \mathrm{d}y_n)$$

corresponding to the operator $\sum_{i=1}^n D_i^\alpha$ is concentrated on the coordinate axes. Because of this singular feature, the general existence results obtained by Komatsu (1984) do not apply to the generator $\nu \sum_{i=1}^n D_i^\alpha \varphi + b \cdot \nabla$.

Proof of Proposition 2.1. To prove existence we regularize the drift by setting, for each $\epsilon \in (0, 1]$,

$$b^\epsilon(t, y) = \int_{\mathbb{R}^n} b(t, y - \epsilon z) \frac{\mathrm{e}^{-|z|^2/2}}{(2\pi)^{n/2}} \mathrm{d}z.$$

The function b^ϵ is bounded by a constant independent of ϵ, and Lipschitz continuous with respect to the spatial variables with constant C_ϵ. Now let $Z_0 = (Z_0^1, \ldots, Z_0^n)$ be a random variable with law η and $(S_t = (S_t^1, \ldots, S_t^n))_{t \geqslant 0}$ an independent process whose coordinates are independent one-dimensional symmetric stable processes with index α, both defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The properties of b^ϵ imply existence and trajectorial uniqueness for the stochastic differential equation

$$Z_t^\epsilon = Z_0 + \nu^{1/\alpha} S_t + \int_0^t b^\epsilon(s, Z_s^\epsilon)\mathrm{d}s. \tag{2.2}$$

Let $Q^\epsilon \in \mathcal{P}_n$ denote the law of the process Z^ϵ. Since b^ϵ is bounded, uniformly in ϵ, the family $\{Q^\epsilon, \epsilon \in (0, 1]\}$ is tight. Let $(\epsilon_k)_{k \in \mathbb{N}^*}$ be a sequence converging to 0 such that Q^{ϵ_k} converges weakly to some $Q \in \mathcal{P}_n$. Since, for any k, the initial marginal of Q^{ϵ_k} is equal to η, so is the initial marginal of Q. We will verify that, for $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, M_t^φ is a Q martingale. To accomplish this, we need estimates of the densities of the marginals Q_t^ϵ uniform in ϵ.

For $t > 0$, and ϕ a C^∞-function with compact support on $\mathbb{R}^n$, the function $\varphi(s, y) = G^{\alpha,n}_{\nu(t-s)} * \phi(y)$ solves the equation $\partial_s \varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi = 0$ on $[0, t] \times \mathbb{R}^n$. Consequently, computing $\varphi(s, Z_s^\epsilon)$ by Itô's formula, one verifies that

$$\left(\varphi(s, Y_s) - \varphi(0, Y_0) - \int_0^s b^\epsilon \cdot \nabla \varphi(r, Y_r)\mathrm{d}r \right)_{s \leqslant t}$$

is a Q^ϵ martingale vanishing at $s = 0$. Since the Q^ϵ expectation of this martingale at time $s = t$ is zero, we obtain

$$\int_{\mathbb{R}^n} \phi(y) Q_t^\epsilon(\mathrm{d}y) = \int_{\mathbb{R}^n} G_{\nu t}^{\alpha,n} * \phi(y)\eta(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^n} b^\epsilon(s, y) \cdot \nabla G_{\nu(t-s)}^{\alpha,n} * \phi(y) Q_s^\epsilon(\mathrm{d}y)\mathrm{d}s.$$

Since b is bounded, the estimates given in Lemma 1.1 with $q = 1$ justify the use of Fubini's theorem which yields

$$\int_{\mathbb{R}^n} \phi(y) Q_t^\epsilon(\mathrm{d}y) = \int_{\mathbb{R}^n} \phi(y) G_{\nu t}^{\alpha,n} * \eta(y)\mathrm{d}y - \int_{\mathbb{R}^n} \phi(y) \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i^\epsilon(s, \cdot) Q_s^\epsilon)(y) \mathrm{d}s\, \mathrm{d}y.$$

The minus sign appears because, for $s > 0$ and $1 \leqslant i \leqslant n$, the mapping $y \mapsto \partial_i G_s^{\alpha,n}(y)$ is an odd function. Since the equality holds for any test function ϕ we conclude that, for $t > 0$, Q_t^ϵ has a density ρ_t^ϵ with respect to the Lebesgue measure on $\mathbb{R}^n$ satisfying

$$\rho_t^\epsilon = G_{\nu t}^{\alpha,n} * \eta - \sum_{i=1}^n \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,n} * (b_i^\epsilon(s, \cdot)\rho_s^\epsilon)\mathrm{d}s.$$

By Lemma 1.1, and the fact that b^ϵ is bounded uniformly in ϵ, one obtains that, for $1 \leqslant q < n/(n + 1 - \alpha)$, and any $t > 0$,

$$\begin{aligned} \|\rho_t^\epsilon\|_q &\leqslant \|G_{\nu t}^{\alpha,n}\|_q + \sum_{i=1}^n \int_0^t \|\partial_i G_{\nu(t-s)}^{\alpha,n}\|_q \cdot \|b_i^\epsilon\|_\infty \cdot \|\rho_s^\epsilon\|_1 \,\mathrm{d}s \\ &\leqslant C\left(t^{-n(q-1)/\alpha q} + \int_0^t (t-s)^{-(n(q-1)+q)/\alpha q} \,\mathrm{d}s \right) \\ &\leqslant C\left(t^{-n(q-1)/\alpha q} + t^{(n-(n+1-\alpha)q)/\alpha q} \right) \end{aligned} \tag{2.3}$$

with a constant C independent of ϵ. Using the weak convergence of Q^{ϵ_k} to Q which implies the weak convergence of $Q_t^{\epsilon_k}$ to Q_t, for t outside of the at most countable set $D_Q = \{r \geqslant 0, Q(|Y_r - Y_{r^-}| > 0) > 0\}$, and the right-continuity of the mapping $t \mapsto Q_t$, one obtains that, for any positive t, Q_t has a density ρ_t which satisfies the estimates given above for ρ_t^ϵ.

Let $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$. To prove that M_t^φ is a Q-martingale it is sufficient to check that, for $l \in \mathbb{N}^*$, $g \in C_b(\mathbb{R}^{ln})$, and $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t \notin D_Q$, we have $\mathbb{E}^Q(F(Y)) = 0$, where

$$F(Y) = \left(\varphi(t, Y_t) - \varphi(s, Y_s) - \int_s^t \left(\partial_r\varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi + b \cdot \nabla\varphi \right)(r, Y_r)\mathrm{d}r \right) g(Y_{s_1}, \ldots, Y_{s_l}).$$

For $\epsilon \in (0, 1]$, let F^ϵ be defined like F but with b^ϵ replacing b. Since F^ϵ is bounded and Q gives full weight to the continuity points of this mapping, one has

$$\lim_{k\to+\infty} \mathbb{E}^{Q^{\epsilon_k}}(F^\epsilon(Y)) = \mathbb{E}^Q(F^\epsilon(Y)).$$

In addition, $\mathbb{E}^{Q^{\epsilon_k}}(F^{\epsilon_k}(Y)) = 0$. Hence

$$|\mathbb{E}^Q(F(Y))| \leqslant \limsup_{\epsilon\searrow 0} \mathbb{E}^Q(|F - F^\epsilon|(Y)) + \limsup_{\epsilon\searrow 0}\limsup_{k\to+\infty} \mathbb{E}^{Q^{\epsilon_k}}(|F^{\epsilon_k} - F^\epsilon|(Y)). \tag{2.4}$$

For $M > 0$, let $B(0, M)$ denote the open ball in $\mathbb{R}^n$ centred at the origin with radius M. Let $1 < q < n/(n+1-\alpha)$, which implies that $n(q-1)/(\alpha q) < 1 - 1/\alpha$. Then

$$\begin{aligned}
&\mathbb{E}^{Q^{\epsilon_k}}(|F^{\epsilon_k} - F^\epsilon|(Y))\\
&\leqslant C\mathbb{E}^{Q^{\epsilon_k}}\left(\int_s^t |b^{\epsilon_k}(r, Y_r) - b^\epsilon(r, Y_r)|\,\mathrm{d}r\right)\\
&\leqslant C\left(Q^{\epsilon_k}\left(\sup_{r\in[s,t]}|Y_r| \geqslant M\right) + \int_s^t \|b^{\epsilon_k}(r,\cdot) - b^\epsilon(r,\cdot)\|_{L^{q/(q-1)}(B(0,M))}\cdot\|\rho_r^{\epsilon_k}\|_q\,\mathrm{d}r\right).
\end{aligned}$$

Since, for any $r \geqslant 0$, by an easy adaptation of Brezis (1983, Theorem IV.22, p. 71), $b^\epsilon(r,\cdot)$ converges to $b(r,\cdot)$ in $L^{q/(q-1)}(B(0,M))$ as ϵ tends to 0, using (2.3) we obtain that

$$\begin{aligned}
&\limsup_{k\to+\infty}\int_s^t \|b^{\epsilon_k}(r,\cdot) - b^\epsilon(r,\cdot)\|_{L^{q/(q-1)}(B(0,M))}\cdot\|\rho_r^{\epsilon_k}\|_q\,\mathrm{d}r\\
&\quad\leqslant C\int_s^t \|b(r,\cdot) - b^\epsilon(r,\cdot)\|_{L^{q/(q-1)}(B(0,M))}\left(r^{-n(q-1)/\alpha q} + r^{(n-(n+1-\alpha)q)/\alpha q}\right)\mathrm{d}r.
\end{aligned}$$

Hence, for a fixed M,

$$\limsup_{\epsilon\to 0}\limsup_{k\to+\infty}\int_s^t \|b^{\epsilon_k}(r,\cdot) - b^\epsilon(r,\cdot)\|_{L^{q/(q-1)}(B(0,M))}\cdot\|\rho_r^{\epsilon_k}\|_q\,\mathrm{d}r = 0.$$

In addition, for any ϵ, b_ϵ is bounded by $\|b\|_\infty$, so that

$$Q^\epsilon\left(\sup_{r\in[s,t]}|Y_r| \geqslant M\right) \leqslant \mathbb{P}\left(\sup_{r\in[s,t]}|Z_0 + \nu^{1/\alpha}S_r| \geqslant M - \|b\|_\infty t\right),$$

which implies that $Q^{\epsilon_k}(\sup_{r\in[s,t]}|Y_r| \geqslant M)$ is arbitrarily small, uniformly in k, for M sufficiently large. Hence the second term on the right-hand side of (2.4) vanishes. Similar arguments give that the first term is zero as well.

We now turn to uniqueness. For $r \geqslant 0$ and $\chi \in \mathcal{P}(\mathbb{R}^n)$, let Q^r and $\bar{Q}^r$ be two solutions of the martingale problem with generator $\nu\sum_{i=1}^n D_i^\alpha + b(r+\cdot,\cdot)\cdot\nabla$ starting from χ. We will prove that Q^r and $\bar{Q}^r$ have the same time marginals. Then uniqueness for the martingale problem with generator $\nu\sum_{i=1}^n D_i^\alpha + b\cdot\nabla$ starting from η follows from an easy adaptation of Ethier and Kurtz (1986, Theorem 4.2, p. 184) to the case of time-dependent generators. By choosing test functions $\varphi(s,x) = G^{\alpha,n}_{\nu(t-s)} * \phi(x)$ as above, one obtains that, for $t > 0$, measure Q_t^r has a density ρ_t^r satisfying

$$\rho_t^r = G^{\alpha,n}_{\nu t} * \chi - \sum_{i=1}^n \int_0^t \partial_i G^{\alpha,n}_{\nu(t-s)} * (b_i(r+s,\cdot)\rho_s^r)\mathrm{d}s.$$

For the choice $r = 0$ and $\chi = \eta$, we recognize (2.1). Similarly, $\bar{Q}_t^r$ has a density $\bar{\rho}_t^r$ satisfying the same equation. For $t > 0$, let $g(t) = \|\rho^r_t - \bar\rho^r_t\|_1$. By the above evolution equation and Lemma 1.1,

$$\begin{aligned} g(t) &= \left\| \sum_{i=1}^n \int_0^t \partial_i G^{\alpha,n}_{\nu(t-s)} * (b_i(r+s,\cdot)(\rho^r_s - \bar\rho^r_s))\mathrm{d}s \right\|_1 \\ &\leqslant \sum_{i=1}^n \int_0^t \|\partial_i G^{\alpha,n}_{\nu(t-s)}\|_1 \cdot \|b_i(r+s,\cdot)\|_\infty\, g(s)\mathrm{d}s \\ &\leqslant C\int_0^t g(s)(t-s)^{-1/\alpha}\,\mathrm{d}s. \end{aligned}$$

Since $\alpha > 1$, we conclude that, for each $t > 0$, $g(t) = 0$ thanks to a version of Gronwall's lemma which is provided next. □

Lemma 2.2. *Let* $g : [0, T] \mapsto \mathbb{R}_+$ *be an integrable function on* $[0, T]$ *such that, for positive constants* A_0, C_0, *and* θ, *and each* $t \in [0, T]$,

$$g(t) \leqslant A_0 + C_0 \int_0^t g(s)(t-s)^{\theta-1}\,\mathrm{d}s.$$

Then there exists a positive constant C *independent of* A_0 *such that, for each* $t \in [0, T]$,

$$g(t) \leqslant CA_0.$$

Proof. Iterating the inequality satisfied by g and using Fubini's theorem, one obtains that, for each $t \in [0, T]$,

$$g(t) \leqslant A_0\left(1 + \frac{C_0 T^\theta}{\theta}\right) + C_0^2\left(\int_0^1 u^{\theta-1}(1-u)^{\theta-1}\,\mathrm{d}u\right)\int_0^t g(s)(t-s)^{2\theta-1}\,\mathrm{d}s.$$

Iterating inductively the successively obtained inequalities, one obtains after n steps that, for each $t \in [0, T]$, $g(t) \leqslant A_n + C_n \int_0^t g(s)(t-s)^{2^n\theta-1}\,\mathrm{d}s$. For sufficiently large n, $2^n\theta \geqslant 1$, and the standard version of Gronwall's lemma can be applied. □

We complete this section by proving an estimate for two-point densities which will be useful later on.

Proposition 2.3. *Let* $n \geqslant 2$ *and* Q *be a solution of the martingale problem given by Proposition 2.1. For* $1 \leqslant i < j \leqslant n$, *and* $t > 0$, *denote by* $\rho^{i,j}_t$ *the density of* $Q \circ (Y^i_t, Y^j_t)^{-1}$. *Then, for all* q, *such that* $1 \leqslant q < 2/(3-\alpha)$,

$$\|\rho^{i,j}_t\|_q \leqslant C\left(t^{-2(q-1)/\alpha q} + t^{(2-(3-\alpha)q)/\alpha q}\right),$$

where constant C *depends only on* ν, α, q, *and* $\|b_i\|_\infty + \|b_j\|_\infty$.

Proof. For simplicity's sake we assume that $i = 1$ and $j = 2$. Integrating (2.1) over $\mathbb{R}^{n-2}$ with respect to the $n-2$ last coordinates of y and setting $\eta^{1,2} = Q \circ (Y_0^1, Y_0^2)^{-1}$, we obtain

$$\rho_t^{1,2} = G_{\nu t}^{\alpha,2} * \eta^{1,2} - \sum_{i=1}^{2} \int_0^t \partial_i G_{\nu(t-s)}^{\alpha,2} * \bar{b}_i(s, \cdot) \mathrm{d}s,$$

where, for $1 \leqslant i \leqslant 2$ and $s > 0$,

$$\bar{b}_i(s, z_1, z_2) = \int_{\mathbb{R}^{n-2}} b_i(s, z_1, \ldots, z_n) \cdot \rho_s(z_1, \ldots, z_n) \mathrm{d}z_3 \ldots \mathrm{d}z_n.$$

Since $\|\bar{b}_i(s, \cdot)\|_1 \leqslant \|b_i\|_\infty$, for each $t > 0$,

$$\|\rho_t^{1,2}\|_q \leqslant \|G_{\nu t}^{\alpha,2}\|_q + \sum_{i=1}^{2} \int_0^t \|\partial_i G_{\nu t}^{\alpha,2}\|_q \cdot \|b_i\|_\infty \, \mathrm{d}s.$$

The proof can now be concluded by an application of estimates given in Lemma 1.1. □

3. Uniqueness for fractional conservation laws and the martingale problem (MP)

We begin this section by clarifying the connection between the martingale problem (MP) and the fractional conservation law (1.4). Since the martingale problem has been introduced by considering the equation obtained by spatial differentiation of (1.4) as a Fokker–Planck equation, the following result is not surprising:

Lemma 3.1. *If P solves the martingale problem* (MP) *then $u(t, x) = H * \tilde{P}_t(x)$ is a bounded weak solution of the fractional conservation law* (1.4).

Proof. First, observe that $H * \tilde{P}_t(x)$ is bounded because

$$|H * \tilde{P}_t(x)| = |\mathbb{E}^P(h(X_0)\mathbf{1}_{\{X_t \leqslant x\}})| \leqslant \mathbb{E}^P|h(X_0)| = 1.$$

Since $b(t, x) = A'(H * \tilde{P}_t(x))$ is a bounded function, by Proposition 2.1, for any $t > 0$, the measures P_t, and therefore $\tilde{P}_t$, are absolutely continuous with respect to the Lebesgue measure. Hence, for $t > 0$, the cumulative distribution function of the measure $A'(H * \tilde{P}_t(y))\tilde{P}_t(\mathrm{d}y)$ is $x \to A(H * \tilde{P}_t(x)) - A(0)$. Let $\psi(t, x)$ be a C^∞-function with compact support on $\mathbb{R}_+ \times \mathbb{R}$ and $\varphi(t, x) = \int_{-\infty}^{x} \psi(t, y)\mathrm{d}y$. The process $(h(X_0)M_t^\varphi)_{t \geqslant 0}$ is a P-martingale so that

$$\int_{\mathbb{R}} \varphi(t, x)\tilde{P}_t(\mathrm{d}x) = \int_{\mathbb{R}} \varphi(0, x)m(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}} \big(\partial_s\varphi + \nu D^\alpha \varphi + A'(H * \tilde{P}_s)\partial_x\varphi\big)(s, x)\tilde{P}_s(\mathrm{d}x)\mathrm{d}s. \tag{3.1}$$

By spatial integration by parts,

$$\int_{\mathbb{R}} D^\alpha \varphi(s, x)\tilde{P}_s(\mathrm{d}x) = K\int_{|y|>1}\int_{\mathbb{R}}\int_x^{x+y} \psi(s, z)\mathrm{d}z\tilde{P}_s(\mathrm{d}x)\frac{\mathrm{d}y}{|y|^{\alpha+1}}$$

$$+ K\int_{|y|\leqslant 1}\int_0^1\int_{\mathbb{R}} \partial_x\psi(s, x+zy)\tilde{P}_s(\mathrm{d}x)(1-z)\mathrm{d}z\frac{\mathrm{d}y}{|y|^{\alpha-1}}$$

$$= -K\int_{|y|>1}\int_{\mathbb{R}} (\psi(s, x+y) - \psi(s, x))H * \tilde{P}_s(x)\mathrm{d}x\frac{\mathrm{d}y}{|y|^{\alpha+1}}$$

$$- K\int_{|y|\leqslant 1}\int_0^1\int_{\mathbb{R}} \partial_{xx}\psi(s, x+zy)H * \tilde{P}_s(x)\mathrm{d}x(1-z)\mathrm{d}z\frac{\mathrm{d}y}{|y|^{\alpha-1}}$$

$$= -\int_{\mathbb{R}} D^\alpha\psi(s, x)H * \tilde{P}_s(x)\mathrm{d}x.$$

Treating the other terms in (3.1) in the same way, and using the fact that $H * \tilde{P}_t(+\infty) = \tilde{P}_t(\mathbb{R}) = \mathbb{E}^P(h(X_0))$ does not depend on t, we see that the weak equation (1.6) holds true for $u(t, x) = H * \tilde{P}_t(x)$. □

Proposition 3.2. *The fractional conservation law* (1.4) *has at most one bounded weak solution and the martingale problem* (MP) *has at most one solution.*

Proof. Let u be a weak solution of (1.4) bounded by M_u, and ϕ be a C^∞-function with compact support on $\mathbb{R}$. The function $\psi(s, x) = p^\alpha_{\nu(t-s)} * \phi(x)$ solves the equation $\partial_s\psi + \nu D^\alpha\psi = 0$ for $(s, x) \in [0, t] \times \mathbb{R}$. By spatial truncation of the function ψ, it is possible to exhibit a sequence $(\psi_n)_n$ of C^∞-functions with compact support in $[0, t] \times \mathbb{R}$ such that as n tends to infinity, ψ_n, $\partial_s\psi_n$, $\partial_x\psi_n$ and $\partial_{xx}\psi_n$ respectively converge in $L^1([0, t] \times \mathbb{R})$ to ψ, $\partial_s\psi$, $\partial_x\psi$ and $\partial_{xx}\psi$, and $\psi_n(0, \cdot)$ ($\psi_n(t, \cdot)$) converges in $L^1(\mathbb{R})$ to $\psi(0, \cdot)$ ($\psi(t, \cdot)$). Since

$$|D^\alpha\psi(s, x) - D^\alpha\psi^n(s, x)|$$

$$\leqslant K\int_{|y|>1} |\psi(s, x+y) - \psi^n(s, x+y)| + |\psi(s, x) - \psi^n(s, x)|\frac{\mathrm{d}y}{|y|^{1+\alpha}}$$

$$+ K\int_{|y|\leqslant 1}\int_0^1 |\partial_{xx}\psi(s, x+zy) - \partial_{xx}\psi^n(s, x+zy)|(1-z)\mathrm{d}z\,\frac{\mathrm{d}y}{|y|^{\alpha-1}},$$

$D^\alpha\psi^n$ also converges to $D^\alpha\psi$ in $L^1([0, t] \times \mathbb{R})$. Writing the weak equation (1.6) with the test function ψ^n and taking the limit $n \to +\infty$, in view of the boundedness of u one obtains that (1.6) holds true for the test function ψ. Using the partial differential equation satisfied by ψ, and then Fubini's theorem, one deduces that

$$\int_{\mathbb{R}} \phi(x)u(t, x)\mathrm{d}x = \int_{\mathbb{R}} \phi(x)p^\alpha_{\nu t} * u_0(x)\mathrm{d}x - \int_{\mathbb{R}} \phi(x)\int_0^t \partial_x p^\alpha_{\nu(t-s)} * A(u(s, \cdot))(x)\mathrm{d}s\,\mathrm{d}x.$$

Since ϕ is arbitrary, we deduce that the function u solves the mild equation

$$u(t, \cdot) = p^{\alpha}_{\nu t} * u_0 - \int_0^t \partial_x p^{\alpha}_{\nu(t-s)} * A(u(s, \cdot))\mathrm{d}s, \tag{3.2}$$

for all $t \geqslant 0$.

Let u' be another weak solution of (1.4) bounded by $M_{u'}$. One can estimate $g(t) = \|u(t, \cdot) - u'(t, \cdot)\|_\infty$ by substracting from (3.2) the same equation written for u' to obtain

$$g(t) \leqslant \max_{|x| \leqslant M_u \vee M_{u'}} |A'(x)| \int_0^t \|\partial_x p^{\alpha}_{\nu(t-s)}\|_1 \, g(s)\mathrm{d}s.$$

Therefore, by Lemma 1.1, there is a constant C such that, for all $t \geqslant 0$,

$$g(t) \leqslant C \int_0^t g(s)(t-s)^{-1/\alpha} \, \mathrm{d}s.$$

Since $\alpha > 1$, we have $1/\alpha < 1$ and, in view of Lemma 2.2, conclude that, for all $t \geqslant 0$, $g(t) = 0$. Hence $u = u'$.

If P and Q both solve the nonlinear martingale problem, combining Lemma 3.1 and the uniqueness result for the fractional conservation law (1.4), one obtains that, for all $t \geqslant 0$, dx-a.e., $H * \tilde{P}_t(x) = H * \tilde{Q}_t(x)$. This equality holds for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ since, for fixed t, both sides are right-continuous with respect to x. Hence both P and Q solve the martingale problem for the generator $\nu D^\alpha + b(t, x)\partial_x$ with $b(t, x) = A'(H * \tilde{P}_t(x))$, starting from $|m|$. Since b is a bounded measurable function, by Proposition 2.1, we conclude that $P = Q$. □

4. Interacting particle systems

In this section we develop a Monte Carlo method for simulating the fractional conservation law. More precisely, we find a sequence of interacting particle systems whose weighted cumulative empirical distribution functions converge, as the size of the system grows to infinity, to the solution of the conservation law. Results of this sort are also known as *propagation-of-chaos* results for the corresponding nonlinear, and in our case non-local, evolution equations.

4.1. Propagation of chaos for fixed fractional viscosity

Definition 4.1. *For $n \in \mathbb{N}^*$, we say that $Q \in \mathcal{P}_n$ solves the martingale problem* (Pn) *if:*

(i) $Q_0 = |m|^{\otimes n}$;
(ii) *for any* $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$,

$$\varphi(t, Y_t) - \varphi(0, Y_0) - \int_0^t \left(\partial_s \varphi + \nu \sum_{i=1}^n D_i^\alpha \varphi \right)(s, Y_s) + \sum_{i=1}^n A' \left(\frac{1}{n} \sum_{j=1}^n h(Y_0^j) \mathbf{1}_{\{Y_s^j \leq Y_s^i\}} \right) \partial_i \varphi(s, Y_s) \mathrm{d}s$$

is a Q-martingale.

In the drift coefficient of the nonlinear martingale problem (MP) the argument of the function A' is the cumulative distribution function of the weighted marginal $\tilde{P}_t$ of the solution. By comparison, in the martingale problem (Pn) the argument of the function A' (which gives the drift coefficient of each particle) is the weighted cumulative empirical distribution function of the particle system.

Remark 4.1. If m is a probability measure, $h(y) = 1$ for every $y \in \mathbb{R}$. Therefore, the existence and uniqueness for (Pn) are ensured by Proposition 2.1 for the time-homogeneous generator $\nu \sum_{i=1}^n D_i^\alpha + b^n \cdot \nabla$, where

$$b^n : (y_1, \ldots, y_n) \in \mathbb{R}^n \mapsto \left(A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{y_j \leq y_1\}} \right), \ldots, A' \left(\frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{y_j \leq y_n\}} \right) \right) \in \mathbb{R}^n. \tag{4.1}$$

However, in general, the drift coefficient at time $t > 0$ depends on the initial position Y_0 through the signed weights $h(Y_0^j)$, $1 \leq j \leq n$. Because of this dependence, the martingale problem (Pn) is non-standard. If D^α is replaced by the usual Laplacian on $\mathbb{R}$, the existence and uniqueness for the analogous non-standard martingale problem are an easy consequence of the Girsanov theorem, as in the standard case. But to obtain the existence result in our case we have to proceed more cautiously.

To deal with signed weights, we remark that the function $(y_1^0, \ldots, y_n^0) \in \mathbb{R}_n \mapsto (h(y_1^0), \ldots, h(y_n^0))$ takes its values in the finite set $\{-1, 1\}^n$. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \{-1, 1\}^n$, let us introduce the mappings

$$b^{n,\gamma} : (y_1, \ldots, y_n) \in \mathbb{R}^n \mapsto \left(A' \left(\frac{1}{n} \sum_{j=1}^n \gamma_j \mathbf{1}_{\{y_j \leq y_1\}} \right), \ldots, A' \left(\frac{1}{n} \sum_{j=1}^n \gamma_j \mathbf{1}_{\{y_j \leq y_n\}} \right) \right).$$

For $y^0 = (y_1^0, \ldots, y_n^0) \in \mathbb{R}^n$ let $Q^{y^0,\gamma}$ be the solution of the martingale problem with generator $\nu \sum_{i=1}^n D_i^\alpha + b^{n,\gamma} \cdot \nabla$ starting from δ_{y^0} and given by Proposition 2.1. By adapting the proof of Ethier and Kurtz (1985, Theorem 4.6, p. 188), we obtain measurability of $y^0 \to Q^{y^0,\gamma}$ for a fixed $\gamma \in \{-1, 1\}^n$.

Then

$$Q^n = \sum_{\gamma \in \{-1,1\}^n} \int_{\mathbb{R}^n} \mathbf{1}_{\{(h(y_1^0),\ldots,h(y_n^0))=\gamma\}} Q^{y^0,\gamma} |m|^{\otimes n}(\mathrm{d}y^0),$$

solves the martingale problem (Pn). Moreover, if σ denotes a permutation of $\{1, \ldots, n\}$, the uniqueness part of Proposition 2.1 ensures that, for $y^0 \in \mathbb{R}^n$ and $\gamma \in \{-1, 1\}^n$, under the probability measure $Q^{y^0,\gamma}$, $(Y^{\sigma(1)}, \ldots, Y^{\sigma(n)})$ is distributed according to $Q^{y^0_\sigma,\gamma_\sigma}$ where $y^0_\sigma = (y^0_{\sigma(1)}, \ldots, y^0_{\sigma(n)})$ and $\gamma_\sigma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$. With the above definition of Q^n, we deduce that the particles $Y^1, \ldots, Y^n$ are exchangeable under this probability measure.

Finally, since, for $n \geqslant 1$ and $1 \leqslant i \leqslant n$, the function $b_i^{n,\gamma}$ is bounded by the quantity $\max_{x\in[-1,1]}|A'(x)|$, we deduce from Proposition 2.3 that, for $n \geqslant 2$, $1 \leqslant i < j \leqslant n$ and $t > 0$, the measure $Q^n \circ (Y^i_t, Y^j_t)^{-1}$ has a density $\rho_t^{n,i,j}$ with respect to the Lebesgue measure on $\mathbb{R}^2$ such that, for each $1 \leqslant q < 2/(3-\alpha)$,

$$\|\rho_t^{n,i,j}\|_q \leqslant C\Big(t^{-2(q-1)/\alpha q} + t^{(2-(3-\alpha)q)/\alpha q}\Big), \tag{4.2}$$

where the constant C is independent of n and t.

Let $\pi_n = Q^n \circ (\mu^n)^{-1}$, where for $Y = (Y^1, \ldots, Y^n) \in D(\mathbb{R}_+, \mathbb{R}^n)$, $\mu^n(Y) = \frac{1}{n}\sum_{i=1}^n \delta_{Y^i} \in \mathcal{P}$ denotes the empirical measure. The following propagation-of-chaos result implies existence for the nonlinear martingale problem (MP):

Theorem 4.1. *The sequence $(\pi_n)_n$ converges weakly to δ_P, where P denotes the unique solution of the martingale problem* (MP).

Proof. The proof is similar to that given by Jourdain (2000, Theorem 2.1) where instead of the fractional Laplacian D^α there appears the classical Laplacian ∂_{xx}. Hence we only show its main steps.

Since the particles $(Y^1, \ldots, Y^n)$ are exchangeable under Q^n, the tightness of the sequence $(\pi_n)_n$ is equivalent to the tightness of the sequence $Q^n \circ (Y^1)^{-1}$ of the distributions of the first particle. The latter follows from the fact that for each $n \in \mathbb{N}^*$, and $y^0, y \in \mathbb{R}^n$,

$$\left|A'\left(\frac{1}{n}\sum_{i=1}^n h(y^0_j)\mathbf{1}_{\{y_j \leqslant y_1\}}\right)\right| \leqslant \max_{x\in[-1,1]}|A'(x)|.$$

Now let π_∞ denote the limit of a weakly convergent subsequence, for simplicity's sake also labelled $(\pi_n)_n$, and D_{π_∞} denote the at most countable set

$$\{t \geqslant 0, \pi_\infty(\{Q \in \mathcal{P}; Q(|Y_t - Y_{t-}| > 0) > 0\}) > 0\}.$$

Since, for any $n \in \mathbb{N}^*$, $Q^n_0 = |m|^{\otimes n}$, we have that π_∞-almost surely $Q_0 = |m|$. Hence, to prove that π_∞ gives full weight to solutions of the nonlinear martingale problem (MP) it is enough to check that, for any $\varphi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$, $l \in \mathbb{N}^*$, $g \in C_b(\mathbb{R}^l)$, and $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant s_l \leqslant s \leqslant t \notin D_{\pi_\infty}$, we have $\mathbb{E}^{\pi_\infty}|F(Q)| = 0$, where F associates with any $Q \in \mathcal{P}$

$$\left\langle Q, \left(\varphi(t, X_t) - \varphi(s, X_s) - \int_s^t (\partial_r\varphi + \nu D^\alpha\varphi + A'(H * \tilde{Q}_r(X_r))\partial_x\varphi)(r, X_r)\mathrm{d}r\right) \times g(X_{s_1}, \ldots, X_{s_l})\right\rangle.$$

According to Jourdain (2000, Lemma 2.2), for any $k \in \mathbb{N}^*$, there exists a Lipschitz continuous function h_k such that $|m|(\{x : h_k(x) \neq h(x)\}) \leqslant 1/k$. Let us also approximate the Heaviside function H by Lipschitz continuous functions $H_k(x) = (1 + kx)1_{[-1/k,0]}(x) + 1_{\{x>0\}}$, and define F_k like F but with $A'(H * \tilde{Q}_r(\cdot))$ replaced by $A'(\langle Q, H_k(\cdot - X_r)h_k(X_0)\rangle)$.

If $(P^j)_{j\geqslant 1}$ converges weakly to Q in $\mathcal{P}$, as $j \to +\infty$, for $r \geqslant 0$ outside the at most countable set $D_Q = \{r \geqslant 0 : Q(|X_r - X_{r^-}| > 0) > 0\}$, the measures $P^j \circ (X_0, X_r)^{-1}$ converge weakly to $Q \circ (X_0, X_r)^{-1}$. Therefore, by the continuity of A', the mapping $x \mapsto A'(\langle P^j, H_k(x - X_r)h_k(X_0)\rangle)$ converges uniformly to the mapping $x \mapsto A'(\langle Q, H_k(x - X_r)h_k(X_0)\rangle)$. We thus deduce that F_k is continuous at any Q such that $s_1, \ldots, s_l, s, t \notin D_Q$. Hence, π_∞ gives full weight to the continuity points of F_k. Now the boundedness of this mapping implies that

$$\mathbb{E}^{\pi_\infty}|F_k(Q)| = \lim_{n\to+\infty} \mathbb{E}^{\pi_n}|F_k(Q)| = \lim_{n\to+\infty} \mathbb{E}^{Q^n}|F_k(\mu^n)|.$$

Hence

$$\mathbb{E}^{\pi_\infty}|F(Q)| \leqslant \limsup_{k\to+\infty} \mathbb{E}^{\pi_\infty}|F - F_k|(Q) + \limsup_{n\to+\infty} \mathbb{E}^{Q^n}|F(\mu_n)| + \limsup_{k\to+\infty}\limsup_{n\to+\infty} \mathbb{E}^{Q^n}|F - F_k|(\mu^n).$$

The same arguments as those given by Jourdain (2000, proof of Theorem 2.1) imply that the sum of the two first terms of the right-hand side is zero and that the third term vanishes as long as

$$\limsup_{k\to+\infty}\limsup_{n\to+\infty} \mathbb{E}^{Q^n}\left(\int_s^t \mathbf{1}_{\{|Y_r^1 - Y_r^2| \leqslant 1/k\}} \mathrm{d}r\right) = 0. \tag{4.3}$$

To prove this equality we use the key estimate (4.2) of the two-particle density which replaces the one obtained via Girsanov's theorem by Jourdain (2000). Let $1 < q < 2/(3 - \alpha)$, which implies that $2(q - 1)/\alpha q < 1 - 1/\alpha$. By Hölder's inequality and (4.2),

$$\begin{aligned}
&\mathbb{E}^{Q^n}\left(\int_s^t \mathbf{1}_{\{|Y_r^1 - Y_r^2| \leqslant 1/k\}} \mathrm{d}r\right) \\
&\leqslant Q^n\left(\sup_{r\in[s,t]} |Y_r^1| \geqslant \sqrt{k}\right) + \int_s^t Q^n\left(|Y_r^1| \leqslant \sqrt{k},\, |Y_r^1 - Y_r^2| \leqslant \frac{1}{k}\right) \mathrm{d}r \\
&\leqslant Q^n\left(\sup_{r\in[s,t]} |Y_r^1| \geqslant \sqrt{k}\right) \\
&\quad + C\left(\int_{\mathbb{R}^2} \mathbf{1}_{\{|y_1| \leqslant \sqrt{k}, |y_1 - y_2| \leqslant 1/k\}} \mathrm{d}y_1\,\mathrm{d}y_2\right)^{(q-1)/q} \int_s^t r^{-2(q-1)/\alpha q} + r^{(2-(3-\alpha)q)/\alpha q}\,\mathrm{d}r.
\end{aligned}$$

The second term on the right-hand side does not depend on n and converges to 0, as $k \to +\infty$, because

$$\int_{\mathbb{R}^2} \mathbf{1}_{\{|y_1|\leqslant\sqrt{k}, |y_1-y_2|\leqslant 1/k\}} \, \mathrm{d}y_1 \, \mathrm{d}y_2 = \frac{4}{\sqrt{k}}.$$

Also, by the tightness of sequence $Q^n \circ (Y_1)^{-1}$, we have

$$\limsup_{k\to+\infty} \limsup_{n\to+\infty} Q^n \left(\sup_{r\in[s,t]} |Y_r^1| \geqslant \sqrt{k} \right) = 0.$$

Hence (4.3) holds true. □

Remark 4.2. Let $(S^i)_{i\geqslant 1}$ be a sequence of independent one-dimensional symmetric α-stable processes and $(Z_0^i)_{i\geqslant 1}$ be an independent sequence of initial variables which are independently and identically distributed (i.i.d.) with common distribution $|m|$. If the function A' is locally Lipschitz continuous then it is possible to define the n-particle system as the unique solution of the equations

$$Z_t^{i,n} = Z_0^i + \nu^{1/\alpha} S_t^i + \int_0^t A' \left(\frac{1}{n} \sum_{j=1}^n h(Z_0^j) H_{\epsilon_n}(Z_s^{i,n} - Z_s^{j,n}) \right) \mathrm{d}s, \qquad 1 \leqslant i \leqslant n,$$

where $\epsilon_n > 0$ and, for $\epsilon > 0$, $H_\epsilon(x) = (1 + x/\epsilon)\mathbf{1}_{[-\epsilon,0]}(x) + \mathbf{1}_{\{x>0\}}$ is a Lipschitz continuous regularization of the Heaviside function. If π_n denotes the law of the empirical measure $n^{-1}\sum_{i=1}^n \delta_{Z^{i,n}}$, then the propagation-of-chaos result stated in Theorem 4.1 holds as long as $\lim_{n\to+\infty} \epsilon_n = 0$.

The propagation-of-chaos result implies convergence of the weighted empirical cumulative distribution function of the system with n particles to the unique bounded weak solution of (1.4) as $n \to +\infty$.

Corollary 4.2. *Under Q^n, the approximate solution $n^{-1}\sum_{j=1}^n h(Y_0^j)H(x - Y_t^j)$ converges to the unique bounded weak solution $u(t, x) = H * \tilde{P}_t(x)$ of (1.4) in the following sense: for each $T > 0$,*

$$\lim_{n\to+\infty} \sup_{t\leqslant T} \int_{\mathbb{R}} \mathbb{E}^{Q^n} \left| \frac{1}{n} \sum_{j=1}^n h(Y_0^j) H(x - Y_t^j) - u(t, x) \right| \frac{\mathrm{d}x}{1 + x^2} = 0.$$

Remark 4.3. For $n \in \mathbb{N}^*$ and $\Delta t > 0$, it is possible to simulate the time-discretized system with n particles defined inductively by

$$Y_0^{i,n} = Y_0^i \quad \text{and} \quad Y_{(k+1)\Delta t}^{i,n} = Y_{k\Delta t}^{i,n} + (\nu\Delta t)^{1/\alpha} s_{k+1}^i + \Delta t A' \left(\frac{1}{n} \sum_{j=1}^n h(Y_0^j) 1_{\{Y_{k\Delta t}^{j,n} \leqslant Y_{k\Delta t}^{i,n}\}} \right), \quad i \leqslant n,$$

where the initial variables $(Y_0^i)_{i\geqslant 1}$ are independent and distributed according to $|m|$ and independent of the sequence $(s_k^i)_{i\geqslant 1, k\geqslant 1}$ of independent variables with common Fourier transform $\xi \to \mathrm{e}^{-|\xi|^\alpha}$. The previous result suggests that when n is large and Δt small, the step function $x \to n^{-1}\sum_{i=1}^n h(Y_0^i) 1_{\{Y_{k\Delta t}^{i,n} \leqslant x\}}$ provides a Monte Carlo approximation of the function $x \to u(k\Delta t, x)$ where u denotes the unique solution of the fractional conservation law (1.4), generalizing Bossy and Talay's approach (1996; 1997) to the Burgers equation ($\alpha = 2$, $A(u) = u^2/2$).

Proof. Let

$$N^Y(\mathrm{d}s, \mathrm{d}y) = \sum_t \mathbf{1}_{\{\Delta Y_t \neq 0\}} \delta_{(t, \Delta Y_t)}(\mathrm{d}s, \mathrm{d}y)$$

and

$$N^i(\mathrm{d}s, \mathrm{d}x) = \sum_t \mathbf{1}_{\{\Delta Y^i_t \neq 0\}} \delta_{(t, \nu^{-1/\alpha} \Delta Y^i_t)}(\mathrm{d}s, \mathrm{d}x)$$

respectively denote the jump measure on $\mathbb{R}_+ \times \mathbb{R}^n$ ($\mathbb{R}_+ \times \mathbb{R}$) associated with the canonical process $Y = (Y^1, \ldots, Y^n)$ on $D(\mathbb{R}_+, \mathbb{R}^n)$ (associated with $\nu^{-1/\alpha} Y^i$). According to Jacod and Shiryaev (1987, Theorem 2.42, p. 86), under Q^n, the predictable compensator of N^Y is

$$K\nu \sum_{i=1}^n \frac{\mathrm{d}y_i}{|y_i|^{1+\alpha}} \delta_{(0,0,\ldots,0)}(\mathrm{d}y_1, \ldots, \mathrm{d}y_{i-1}, \mathrm{d}y_{i+1}, \ldots \mathrm{d}y_n)\mathrm{d}s.$$

As a consequence, the measures N^i are independent Poisson point measures on $\mathbb{R}_+ \times \mathbb{R}$ with common intensity $K\,\mathrm{d}s\,\mathrm{d}x/|x|^{1+\alpha}$. Therefore, the processes

$$S^i_t = \int_{(0,t]\times\mathbb{R}} x\mathbf{1}_{\{|x|\leqslant 1\}} \left(N^i(\mathrm{d}s\,\mathrm{d}x) - \frac{K\,\mathrm{d}s\,\mathrm{d}x}{|x|^{1+\alpha}} \right) + \int_{(0,t]\times\mathbb{R}} x\mathbf{1}_{\{|x|>1\}} N^i(\mathrm{d}s\,\mathrm{d}x),$$

$1 \leqslant i \leqslant n$, are independent symmetric α-stable processes independent of the initial variables Y^i_0, $1 \leqslant i \leqslant n$, which are i.i.d. with common distribution $|m|$. Additionally, following Jacod and Shiryaev (1987, Theorem 2.42, p. 86, and Theorem 2.34, p. 84), we obtain that, for $1 \leqslant i \leqslant n$, and $t \geqslant 0$,

$$Y^i_t = Y^i_0 + \nu^{1/\alpha} S^i_t + \int_0^t A' \left(\frac{1}{n} \sum_{j=1}^n h(Y^j_0) \mathbf{1}_{\{Y^j_s \leqslant Y^i_s\}} \right) \mathrm{d}s.$$

Similarly, under the solution P of the nonlinear martingale problem (MP),

$$X_t = X_0 + \nu^{1/\alpha} S_t + \int_0^t A'(H * \tilde{P}_s(X_s))\mathrm{d}s,$$

where S_t is a symmetric α-stable process independent of the initial variable X_0 which is distributed according to $|m|$.

The scaling property of the α-stable process and the boundedness of the drift coefficients imply that, for $0 \leqslant s \leqslant t$ and $1 \leqslant i \leqslant n$,

$$\mathbb{E}^{Q^n}\big(|Y^i_t - Y^i_s|\big) \leqslant C\Big(\nu^{1/\alpha}(t-s)^{1/\alpha} + (t-s)\Big), \tag{4.4}$$

$$\mathbb{E}^{P}(|X_t - X_s|) \leqslant C\Big(\nu^{1/\alpha}(t-s)^{1/\alpha} + (t-s)\Big). \tag{4.5}$$

We set $k \in \mathbb{N}^*$. Let h_k and H_k be the Lipschitz continuous approximations of the functions h and H introduced in the proof of Theorem 4.1, and $\lfloor \cdot \rfloor$ denote the integer part. We have

$$\sup_{t \leq T} \int_{\mathbb{R}} \mathbb{E}^{Q^n} \left| \frac{1}{n} \sum_{j=1}^{n} h(Y_0^j) H(x - Y_t^j) - H * \tilde{P}_t(x) \right| \frac{\mathrm{d}x}{1 + x^2}$$

$$\leq \sup_{t \leq T} \int_{\mathbb{R}} \mathbb{E}^{Q^n} \left| h(Y_0^1) H(x - Y_t^1) - h_k(Y_0^1) H_k(x - Y^1_{T\lfloor k^2 t/T \rfloor / k^2}) \right| \frac{\mathrm{d}x}{1 + x^2} \tag{4.6}$$

$$+ \max_{0 \leq j \leq k^2 - 1} \mathbb{E}^{\pi_n} \left(\sup_{jT/k^2 \leq t \leq (j+1)T/k^2} \int_{\mathbb{R}} \left| \langle Q, \, h_k(X_0) H_k(x - X_{jT/k^2}) \rangle - H * \tilde{P}_t(x) \right| \frac{\mathrm{d}x}{1 + x^2} \right).$$

Since $|h(Y_0^1)H(x - Y_t^1) - h_k(Y_0^1)H_k(x - Y^1_{T\lfloor k^2 t/T \rfloor / k^2})|$ is smaller than

$$|h - h_k|(Y_0^1) + \mathbf{1}_{\{x - 1/k \leq Y_t^1 \leq x\}} + k|Y_t^1 - Y^1_{T\lfloor k^2 t/T \rfloor / k^2}|,$$

using (4.4) and the bound

$$\forall y \in \mathbb{R}, \int_{\mathbb{R}} \mathbf{1}_{\{x - 1/k \leq y \leq x\}} \frac{\mathrm{d}x}{1 + x^2} \leq 1/k$$

one obtains that the first term on the right-hand-side of (4.6) is smaller than $2\pi |m|(\{x : h_k(x) \neq h(x)\}) + 1/k + Ck^{1-2/\alpha}$ and vanishes as $k \to +\infty$.

Letting $s \to t^-$ in (4.5), one obtains that for any $t \geq 0$, $P(|X_t - X_{t^-}| > 0) = 0$. As a consequence, for $1 \leq j \leq k^2$, δ_P gives full weight to continuity points of the bounded mapping

$$Q \in \mathcal{P} \to \sup_{jT/k^2 \leq t \leq (j+1)T/k^2} \int_{\mathbb{R}} \left| \langle Q, \, h_k(X_0) H_k(x - X_{jT/k^2}) \rangle - H * \tilde{P}_t(x) \right| \frac{\mathrm{d}x}{1 + x^2}.$$

Hence for fixed k, Theorem 4.1 implies that when n tends to $+\infty$, the second term on the right-hand side of (4.6) converges to

$$\sup_{t \leq T} \int_{\mathbb{R}} \left| \langle P, \, h_k(X_0) H_k(x - X_{T\lfloor k^2 t/T \rfloor / k^2}) - h(X_0) H(x - X_t) \rangle \right| \frac{\mathrm{d}x}{1 + x^2}.$$

Reasoning as for the first term of the right-hand side of (4.6), with (4.5) replacing (4.4), one obtains that the previous limit converges to 0 as k tends to $+\infty$. □

4.2. The vanishing viscosity limit

In this subsection, we assume that m is a probability measure and introduce a sequence ν_n of positive numbers such that $\lim_{n \to +\infty} \nu_n = 0$. We will let the fractional viscosity vanish as the number n of particles tends to $+\infty$. We recall that uniqueness of bounded weak solutions fails to hold for the inviscid ($\nu = 0$) scalar conservation law (1.4):

$$\partial_t u(t, x) + \partial_x A(u(t, x)) = 0, \qquad u(0, x) = u_0(x). \tag{4.7}$$

However, in view of Kruzhkov's theorem (Kruzhkov 1970; Serre 1996), this equation admits a unique bounded entropy solution $u \in C(\mathbb{R}_+, L^1_{\text{loc}}(\mathbb{R}))$ characterized by the following entropy inequalities: for any $c \in \mathbb{R}$, and any non-negative C^∞-function ψ with compact support on $\mathbb{R}_+ \times \mathbb{R}$,

$$\int_{\mathbb{R}} |u_0(x) - c|\psi(0, x)\mathrm{d}x \tag{4.8}$$

$$+ \int_0^\infty \int_{\mathbb{R}} (|u - c|\partial_t\psi + \operatorname{sgn}(u - c)(A(u) - A(c))\partial_x\psi)(t, x)\mathrm{d}x\,\mathrm{d}t \geqslant 0.$$

For $n \geqslant 1$, let $Q^n \in \mathcal{P}_n$ be the solution of the martingale problem with generator $\nu_n \sum_{i=1}^n D_i^\alpha + b^n \cdot \nabla$ where b^n is defined in (4.1) starting from $m^{\otimes n}$ and given by Proposition 2.1, and $\pi_n = Q^n \circ (\mu^n)^{-1}$ where, for $Y = (Y^1, \ldots, Y^n) \in D(\mathbb{R}_+, \mathbb{R}^n)$, $\mu^n(Y) = n^{-1}\sum_{i=1}^n \delta_{Y^i} \in \mathcal{P}$. Since $\max_{x \in [0,1]}|A'(x)| < +\infty$, the sequence $(\pi_n)_n$ is tight.

Theorem 4.3. *Any weak limit of the sequence $(\pi_n)_n$ gives full weight to the set*

$$\{Q \in \mathcal{P},\ t \to H * Q_t(\cdot) \text{ is equal to } t \to u(t, \cdot)\}.$$

In addition, for each $T > 0$,

$$\lim_{n \to +\infty} \sup_{t \leqslant T} \int_{\mathbb{R}} \mathbb{E}^{Q^n} \left| \frac{1}{n} \sum_{j=1}^n H(x - Y_t^j) - u(t, x) \right| \frac{\mathrm{d}x}{1 + x^2} = 0.$$

Remark 4.4. If D^α is replaced by the Laplacian on $\mathbb{R}$ as the generator of the particle system, a similar result was proved by Jourdain (2002). In that case, the situation where the initial measure m is a signed measure can be handled by modifying the dynamics of the particle system by killing pairs of particles with opposite weights whenever they collide. Such a modification seems difficult to generalize for processes with jumps.

Proof. Let us first observe that the first assertion in the theorem implies the second. Indeed, from any subsequence of $(Q^n)_n$ one can extract a further subsequence $(Q^{n'})_{n'}$ such that $\pi_{n'}$ converges weakly to π_∞ giving full weight to $\{Q \in \mathcal{P},\ t \to H * Q_t(\cdot) \text{ is equal to } t \to u(t, \cdot)\}$. Since (4.4) holds with ν replaced by ν_n, first taking s and t outside $\{r \geqslant 0 : \pi_\infty(\{Q : Q(|Y_r - Y_{r^-}|) > 0\}) > 0\}$ and then using the right-continuity of sample paths, one obtains that, for $0 \leqslant s \leqslant t$,

$$\mathbb{E}^{\pi_\infty}(\langle Q, |X_t - X_s|\rangle) \leqslant C(t - s).$$

With this bound replacing (4.5), the arguments given in the proof of Corollary 4.2 imply that

$$\sup_{t \leqslant T} \int_{\mathbb{R}} \mathbb{E}^{Q^{n'}} \left| \frac{1}{n'} \sum_{j=1}^{n'} H(x - Y_t^j) - u(t, x) \right| \frac{\mathrm{d}x}{1 + x^2} \text{ converges to } 0.$$

Now let π_∞ be the limit of a converging subsequence of $(\pi_n)_n$, which we still index by n for the sake of notational simplicity, ψ be a non-negative C^∞-function with compact support on

$\mathbb{R}_+ \times \mathbb{R}$ and $c \in \mathbb{R}$. It is sufficient to prove that π_∞-a.s. the entropy inequality (4.8) holds true for $u(t, x) = H * Q_t(x)$, where Q denotes the canonical variable on $\mathcal{P}$. Indeed, we can then conclude by taking c and ψ in countably dense subsets.

Let us observe that since, for any $Q \in \mathcal{P}$, we have $(t, x) \to H * Q_t(x) \in [0, 1]$, the entropy inequality for $c = 1$ $(c = 0)$ implies the entropy inequality for any $c \geqslant 1$ $(c \leqslant 0)$. For this reason we assume that $c \in [0, 1]$.

As in the proof of Corollary 4.2, we obtain that, under the probability measure Q^n, for $1 \leqslant i \leqslant n$ and $t \geqslant 0$,

$$Y_t^i = Y_0^i + \nu_n^{1/\alpha} S_t^i + \int_0^t A'\left(\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{Y_s^j \leqslant Y_s^i\}}\right)\mathrm{d}s,$$

where S^i, $1 \leqslant i \leqslant n$, are independent α-stable processes independent of the initial variables Y_0^i, $1 \leqslant i \leqslant n$, which are i.i.d. with common distribution m.

For $n \geqslant 1$, we set $c_n = \lfloor nc \rfloor / n$, where $\lfloor \cdot \rfloor$ denotes the integer part. Our strategy, inspired by the proof of Lemma 3.1, is as follows: we want to integrate by parts in the spatial variable in order to evaluate $\int_{\mathbb{R}} \psi(t, x)|H * \mu_t^n(x) - c_n|\mathrm{d}x$, for $t > 0$. The distributional derivative of the step function with bounded variation

$$x \mapsto |H * \mu_t^n(x) - c_n| = |c_n| + \frac{1}{n}\sum_{i=1}^{\lfloor cn \rfloor} \mathbf{1}_{\{x < Y_t^{\sigma_t(i)}\}} + \frac{1}{n}\sum_{i=\lfloor cn \rfloor+1}^{n} \mathbf{1}_{\{Y_t^{\sigma_t(i)} \leqslant x\}}$$

is equal to $n^{-1}(\sum_{i=\lfloor cn \rfloor+1}^n - \sum_{i=1}^{\lfloor cn \rfloor})\delta_{Y_t^{\sigma_t(i)}}$, where σ_t denotes a permutation of $\{1, \ldots, n\}$ such that $Y_t^{\sigma_t(1)} \leqslant Y_t^{\sigma_t(2)} \leqslant \ldots \leqslant Y_t^{\sigma_t(n)}$. This justifies our interest in computing

$$\sum_{i=\lfloor cn \rfloor+1}^{n} \varphi(t, Y_t^{\sigma_t(i)}) - \sum_{i=1}^{\lfloor cn \rfloor} \varphi(t, Y_t^{\sigma_t(i)}),$$

where

$$\varphi(t, x) = \int_{-\infty}^{x} \psi(t, z)\mathrm{d}z, \qquad \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Because this calculation is delicate, we are going to approximate Y by a process with finite intensity of jumps by removing the small jumps of S^i, $1 \leqslant i \leqslant n$. More precisely, for $\epsilon > 0$, we set $S_t^{i,\epsilon} = \int_{(0,t]\times\mathbb{R}} y \mathbf{1}_{\{|x|>\epsilon\}} N^i(\mathrm{d}s\,\mathrm{d}x)$ and

$$Y_t^{i,\epsilon} = Y_0^i + \nu_n^{1/\alpha} S_t^{i,\epsilon} + \int_0^t A'\left(\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{\{Y_s^j \leqslant Y_s^i\}}\right)\mathrm{d}s.$$

Then introducing $Y_t^\epsilon = (Y_t^{1,\epsilon}, \ldots, Y_t^{n,\epsilon})$ and defininng σ_t^ϵ $(\sigma_{t^-}^\epsilon)$ to be a permutation such that $Y_t^{\sigma_t^\epsilon(1),\epsilon} \leqslant \ldots \leqslant Y_t^{\sigma_t^\epsilon(n),\epsilon}$ $(Y_{t^-}^{\sigma_{t^-}^\epsilon(1),\epsilon} \leqslant \ldots \leqslant Y_{t^-}^{\sigma_{t^-}^\epsilon(n),\epsilon})$, we have, for all $t \geqslant 0$,

$$\sum_{i=1}^{n}(Y_t^{\sigma_t^\epsilon(i),\epsilon} - Y_t^{\sigma_t(i)})^2 \leqslant |Y_t^\epsilon - Y_t|^2 \quad \text{and} \quad \lim_{\epsilon \to 0} \mathbb{E}^{Q^n}\left(\sup_{s \leqslant t}|Y_s^\epsilon - Y_s|^2\right) = 0. \tag{4.9}$$

Since, according to Proposition 2.1, for any $s > 0$, Q^n_s has a density with respect to the Lebesgue measure, Q^n-a.s. and ds-a.e. the positions $Y^1_s, \ldots, Y^n_s$ are distinct. Therefore Q^n_s-a.s., for all $t \geqslant 0$,

$$Y^{i,\epsilon}_t = Y^i_0 + \nu_n^{1/\alpha} S^{i,\epsilon}_t + \int_0^t A'\big(\sigma_s^{-1}(i)/n\big)\mathrm{d}s.$$

By considering successive jump times of the process Y^ϵ, one obtains

$$\begin{aligned}
&\sum_{i=1}^{\lfloor cn \rfloor} \varphi(t, Y_t^{\sigma^\epsilon_t(i),\epsilon}) \\
&\quad = \sum_{i=1}^{\lfloor cn \rfloor}\bigg(\varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t \big(\partial_s\varphi + A'(\sigma_s^{-1}(\sigma^\epsilon_s(i))/n)\partial_x\varphi\big)(s, Y_s^{\sigma^\epsilon_s(i),\epsilon})\mathrm{d}s\bigg) \qquad (4.10)\\
&\qquad + \int_0^t \sum_{j=1}^n \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)\geqslant \lfloor cn \rfloor\}}\bigg(\mathbf{1}_{\left\{\nu_n^{1/\alpha} y \geqslant Y_{s^-}^{\sigma^\epsilon_{s^-}(\lfloor cn \rfloor+1),\epsilon} - Y^{j,\epsilon}_{s^-}\right\}}\Big(\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha}y\Big) - \varphi(s, Y^{j,\epsilon}_{s^-})\Big). \\
&\qquad + \mathbf{1}_{\left\{\nu_n^{1/\alpha} y > Y_{s^-}^{\sigma^\epsilon_{s^-}(\lfloor cn \rfloor+1),\epsilon} - Y^{j,\epsilon}_{s^-}\right\}}\Big(\varphi(s, Y_{s^-}^{\sigma^\epsilon_{s^-}([cn]+1),\epsilon}) - \varphi(s, Y^{j,\epsilon}_{s^-})\Big)\bigg) \\
&\qquad + \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j) > \lfloor cn \rfloor\}}\bigg(\mathbf{1}_{\left\{\nu_n^{1/\alpha} y < Y_{s^-}^{\sigma^\epsilon_{s^-}(\lfloor cn \rfloor),\epsilon} - Y^{j,\epsilon}_{s^-}\right\}}\Big(\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha}y) - \varphi(s, Y_{s^-}^{\sigma^\epsilon_{s^-}(\lfloor cn \rfloor),\epsilon})\Big)\bigg) \\
&\quad N^j(\mathrm{d}s\,\mathrm{d}y).
\end{aligned}$$

Because ψ is non-negative, $x \mapsto \varphi(s, x) = \int_{-\infty}^x \psi(s, y)\mathrm{d}y$ is non-decreasing and

$$\begin{aligned}
&\sum_{i=1}^{\lfloor cn \rfloor}\bigg(\varphi(t, Y_t^{\sigma^\epsilon_t(i),\epsilon}) - \varphi(0, Y_0^{\sigma_0(i)}) - \int_0^t (\partial_s\varphi + A'(\sigma_s^{-1}(\sigma^\epsilon_s(i))/n)\partial_x\varphi)(s, Y_s^{\sigma^\epsilon_s(i),\epsilon})\mathrm{d}s\bigg) \\
&\qquad \leqslant \int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} \mathbf{1}_{\{(\sigma^\epsilon_{s-})^{-1}(j)\leqslant \lfloor cn \rfloor\}}\Big(\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha}y) - \varphi(s, Y^{j,\epsilon}_{s^-})\Big) N^j(\mathrm{d}s\,\mathrm{d}y).
\end{aligned}$$

By a similar but easier computation,

$$\begin{aligned}
&\sum_{i=1}^{n}\bigg(-\varphi(t, Y_t^{\sigma^\epsilon_t(i),\epsilon}) + \varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t (\partial_s\varphi + A'(\sigma_s^{-1}(\sigma^\epsilon_s(i))/n)\partial_x\varphi)(s, Y_s^{\sigma^\epsilon_s(i),\epsilon})\mathrm{d}s\bigg) \\
&\qquad = -\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\varphi(s, Y^{j,\epsilon}_{s^-} + \nu_n^{1/\alpha}y) - \varphi(s, Y^{j,\epsilon}_{s^-}))N^j(\mathrm{d}s\,\mathrm{d}y).
\end{aligned}$$

Adding this equality to the preceding inequality multiplied by 2, one obtains that $T_1^{n,\epsilon} \leqslant T_2^{n,\epsilon}$, where

$$T_1^{n,\epsilon} = \frac{1}{n}\left(\sum_{i=\lfloor cn\rfloor+1}^{n} - \sum_{i=1}^{\lfloor cn\rfloor}\right)\Big(\varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t (\partial_s\varphi + A'(\sigma_s^{-1}(\sigma_s^\epsilon(i))/n)\partial_x\varphi)(s, Y_s^{\sigma_s^\epsilon(i),\epsilon})\mathrm{d}s - \varphi(t, Y_t^{\sigma_t^\epsilon(i),\epsilon})\Big)$$

and

$$T_2^{n,\epsilon} = \frac{1}{n}\int_0^t \sum_{j=1}^{n}\int_{|y|>\epsilon}(\mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)\leqslant\lfloor cn\rfloor\}} - \mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)>\lfloor cn\rfloor\}})$$

$$(\varphi(s, Y_{s^-}^{j,\epsilon} + \nu_n^{1/\alpha}y) - \varphi(s, Y_{s^-}^{j,\epsilon}))N^j(\mathrm{d}s\,\mathrm{d}y).$$

According to Lemma 4.4 below,

$$\lim_{n\to+\infty}\sup_{\epsilon>0}\mathbb{E}^{Q^n}|T_2^{n,\epsilon}| = 0.$$

Hence

$$\lim_{n\to+\infty}\sup_{\epsilon>0}\mathbb{E}^{Q^n}((T_1^{n,\epsilon})^+) = 0.$$

According to (4.9), one can construct a sequence $(\epsilon_k)_k$ converging to 0 and such that Q^n-a.s., $\sup_{s\leqslant t}\sum_{i=1}^n (Y_s^{\sigma_s^{\epsilon_k}(i),\epsilon_k} - Y_s^{\sigma_s(i)})^2 \to 0$ as $k\to+\infty$. Moreover, since Q^n-a.s. and ds-a.e. the positions $Y_s^1, \ldots, Y_s^n$ are distinct, Q^n-a.s. and ds-a.e. $\sigma_s^{\epsilon_k}$ is equal to σ_s for k big enough. Hence Q^n-a.s., T_1^{n,ϵ_k} converges to

$$T^n = \frac{1}{n}\left(\sum_{i=\lfloor cn\rfloor+1}^{n} - \sum_{i=1}^{\lfloor cn\rfloor}\right)\left(\varphi(0, Y_0^{\sigma_0(i)}) + \int_0^t(\partial_s\varphi + A'(i/n)\partial_x\varphi)(s, Y_s^{\sigma_s(i)})\mathrm{d}s - \varphi(t, Y_t^{\sigma_t(i)})\right).$$

Since variables $T_1^{n,\epsilon}$ are uniformly bounded in ϵ, we have

$$\mathbb{E}^{Q^n}((T^n)^+) = \lim_{k\to+\infty}\mathbb{E}^{Q^n}((T_1^{n,\epsilon_k})^+),$$

so that we can conclude that

$$\lim_{n\to+\infty}\mathbb{E}^{Q^n}((T^n)^+) = 0. \tag{4.11}$$

We now choose t such that the support of ψ, and therefore of φ, is contained in $[0, t)\times\mathbb{R}$ which permits us to get rid of the terms involving $\varphi(t, \cdot)$ and perform spatial integration by parts as planned at the beginning of the proof. We thus obtain

$$T_n = (1 - c_n)\int_{\mathbb{R}} \psi(0, x)\mathrm{d}x - \int_{\mathbb{R}} \psi(0, x)|H * \mu_0^n(x) - c_n|\,\mathrm{d}x$$

$$+ (1 - c_n)\int_0^t\int_{\mathbb{R}} \partial_s\psi(s, x)\mathrm{d}x\,\mathrm{d}s$$

$$- \int_0^t\int_{\mathbb{R}} |H * \mu_s^n(x) - c_n|\partial_s\psi(s, x)$$

$$+ \left(A(c_n) - A(0) + \frac{1}{n}\sum_{i=1}^{nH*\mu_s^n(x)} (1_{\{i>\lfloor cn\rfloor\}} - 1_{\{i\leqslant\lfloor cn\rfloor\}})A'(i/n) \right)\partial_x\psi(s, x)\mathrm{d}x\,\mathrm{d}s.$$

As far as last term is concerned, observe that the cumulative distribution function of the signed measure

$$\frac{1}{n}\left(\sum_{i=\lfloor cn\rfloor+1}^{n} - \sum_{i=1}^{\lfloor cn\rfloor}\right) A'(i/n)\delta_{Y_s^{\sigma_s(i)}}$$

is a function

$$x \mapsto \frac{1}{n}\sum_{i=1}^{nH*\mu_s^n(x)} (1_{\{i>\lfloor cn\rfloor\}} - 1_{\{i\leqslant\lfloor cn\rfloor\}})A'(i/n),$$

where $nH * \mu_s^n(x)$ counts the number of particles with coordinates not greater than x at time s, and that no boundary term appears since $\lim_{x\to+\infty}\partial_x\varphi(s, x) = \lim_{x\to+\infty}\psi(s, x) = 0$.

The sum of the first and third terms on the right-hand side is zero. Moreover, for all $0 \leqslant k \leqslant n$,

$$\left|\mathrm{sgn}\left(\frac{k}{n} - c_n\right)\left(A\left(\frac{k}{n}\right) - A(c_n)\right) - A(c_n) + A(0) - \frac{1}{n}\sum_{i=1}^{k}(1_{\{i>[cn]\}} - 1_{\{i\leqslant[cn]\}})A'\left(\frac{i}{n}\right)\right|$$

$$= \left|\sum_{i=1}^{k}(1_{\{i>\lfloor cn\rfloor\}} - 1_{\{i\leqslant\lfloor cn\rfloor\}})\left(A\left(\frac{i}{n}\right) - A\left(\frac{i-1}{n}\right) - \frac{1}{n}A'\left(\frac{i}{n}\right)\right)\right|$$

$$\leqslant \sup_{x,y\in[0,1],|x-y|\leqslant\frac{1}{n}} |A'(x) - A'(y)|,$$

and, for each $u \in [0, 1]$,

$$|\mathrm{sgn}(u - c)(A(u) - A(c)) - \mathrm{sgn}(u - c_n)(A(u) - A(c_n))| \leqslant \sup_{u\in[c_n,c]} |A(c_n) + A(c) - 2A(u)|.$$

Hence the random variables

$$T_n + \int_{\mathbb{R}} \psi(0, x)|H * \mu_0^n(x) - c|\mathrm{d}x$$

$$+ \int_0^t \int_{\mathbb{R}} |H * \mu_s^n(x) - c|\partial_s \psi(s, x) + \mathrm{sgn}(H * \mu_s^n(x) - c)(A(H * \mu_s^n(x)) - A(c))\partial_x \psi(s, x)\mathrm{d}x\,\mathrm{d}s$$

converge uniformly to 0 as $n \to +\infty$. With the help of (4.11) we conclude that, for the continuous and bounded function G which associates with any $Q \in \mathcal{P}$

$$G(Q) = \int_{\mathbb{R}} \psi(0, x)|H * Q_0(x) - c|\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \big(|H * Q_s(x) - c|\partial_s \psi(s, x) + \mathrm{sgn}(H * Q_s(x) - c)(A(H * Q_s(x)) - A(c))\partial_x \psi(s, x)\big)\,\mathrm{d}x\,\mathrm{d}s, \tag{4.12}$$

we have

$$\mathbb{E}^{\pi_\infty}((G(Q))^-) = \lim_{n\to+\infty} \mathbb{E}^{\pi_n}((G(Q))^-) = 0.$$

We now can conclude the proof by observing that π_∞-a.s, $Q_0 = m$, and therefore $H * Q_0 = u_0$. □

Lemma 4.4. *In the notation introduced in the proof of Theorem 4.3, we have*

$$\lim_{n\to+\infty} \sup_{\epsilon>0} \mathbb{E}^{Q^n} |T_2^{n,\epsilon}| = 0.$$

Proof. For $1 \leqslant j \leqslant n$, let us denote by

$$\tilde{N}^j(\mathrm{d}s\,\mathrm{d}y) = N^j(\mathrm{d}s\,\mathrm{d}y) - \frac{K\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}$$

the compensated measure associated with N^j. We shall write $T_2^{n,\epsilon} = R_1 + R_2$, where

$$R_1 = \frac{1}{n}\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)\leqslant\lfloor cn\rfloor\}} - \mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)>\lfloor cn\rfloor\}})(\varphi(s, Y_{s-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s-}^{j,\epsilon}))\tilde{N}^j(\mathrm{d}s, \mathrm{d}y)$$

$$R_2 = \frac{1}{n}\int_0^t \sum_{j=1}^n \int_{|y|>\epsilon} (\mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)\leqslant\lfloor cn\rfloor\}} - \mathbf{1}_{\{(\sigma_{s-}^\epsilon)^{-1}(j)>\lfloor cn\rfloor\}})(\varphi(s, Y_{s-}^{j,\epsilon} + \nu_n^{1/\alpha} y) - \varphi(s, Y_{s-}^{j,\epsilon}) - \partial_x\varphi(s, Y_{s-}^{j,\epsilon})\nu_n^{1/\alpha} y 1_{\{\nu_n^{1/\alpha}|y|\leqslant 1\}})\frac{K\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}$$

In this notation we have

$$\mathbb{E}^{Q_n}((R_1)^2) = \frac{1}{n^2}\sum_{j=1}^{n}\int_0^t\int_{|z|>\nu_n^{1/\alpha}\epsilon}\mathbb{E}^{Q_n}\left((\varphi(s,\ Y_{s^-}^{j,\epsilon}+z)-\varphi(s,\ Y_{s^-}^{j,\epsilon}))^2\right)\frac{\nu_n K\,\mathrm{d}s\,\mathrm{d}z}{|z|^{1+\alpha}}$$

$$\leqslant \frac{C\nu_n}{n}\int_{\mathbb{R}}\frac{z^2\wedge 1}{|z|^{1+\alpha}}\mathrm{d}z$$

and

$$\mathbb{E}^{Q_n}|R_2| \leqslant \frac{1}{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}}\mathbb{E}^{Q_n}\left|\varphi(s,\ Y_{s^-}^{j,\epsilon}+z)-\varphi(s,\ Y_{s^-}^{j,\epsilon})-\partial_x\varphi(s,\ Y_{s^-}^{j,\epsilon})z\mathbf{1}_{\{|z|\leqslant 1\}}\right| \times \frac{\nu_n K\,\mathrm{d}s\,\mathrm{d}z}{|z|^{1+\alpha}}$$

$$\leqslant C\nu_n\int_{\mathbb{R}}\frac{z^2\wedge 1}{|z|^{1+\alpha}}\mathrm{d}z.$$

□

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