Probabilistic Methods in Combinatorics
Lecture 4

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Hamiltonian Paths

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Theorem [Alon, 1990]: $P(n) \leq cn^{3/2}\frac{n!}{2^{n-1}}$.

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$$F(T) = \mathrm{per}(A_T) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Here r_i is i-th row sum of A_T; $\sum_{i=1}^{n} r_i = \binom{n}{2}$.

A convex inequality

Lemma: For every two integers a, b satisfying $b \geq a + 2 > a \geq 1$, we have

$$(a!)^{1/a}(b!)^{1/b} < ((a+1)!)^{1/(a+1)}((b-1)!)^{1/(b-1)}.$$

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Proof: Let $f(x) = \frac{(x!)^{1/x}}{((x+1)!)^{1/(1+x)}}$. We need to show $f(a) < f(b-1)$. It suffices to show $f(x-1) < f(x)$.

$$((x-1)!)^{1/(x-1)}((x+1)!)^{1/(1+x)} < (x!)^{2/x}.$$

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It can be proved using $x! > (\frac{x+1}{2})^x$ for $x \geq 2$. $\square$

Proof of theorem

Observe that $\sum_{i=1}^{n}(r_i!)^{1/r_i}$ achieves the maximum when all r_i's are almost equal. We get

$$F(T) \leq (1+o(1))\frac{\sqrt{\pi}}{\sqrt{2}e}n^{3/2}\frac{(n-1)!}{2^n}.$$

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Construct a new tournament T' for T by adding a new vertex v, where the edges from v to T are oriented randomly and independently. Every Hamiltonian path in T can be extended to a Hamiltonian cycle in T' with probability $\frac{1}{4}$. We have

$$P(T) \leq \frac{1}{4} C(T') = O\left(n^{3/2} \frac{n!}{2^{n-1}} \right). \qquad \square$$

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Let X_v be the indicator random variable for $v \in I$.

$$\mathrm{E}(X_v) = \Pr(v \in I) = \frac{1}{d_v + 1}.$$

$$\alpha(G) \geq \mathrm{E}(|I|) = \sum_v \frac{1}{d_v + 1}.$$

Turán Theorem

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Turán Theorem: For $n = km + r$ $(0 \leq r < k)$,

$$t(n, K_{k+1}) = m^2 \binom{k}{2} + rm(k-1) + \binom{r}{2}.$$

The equality holds if and only if G is the complete k-partite graph with equitable partitions, denoted by $G_{n,k}$.

Dual version

For any $k \leq n$, let q, r satisfy $n = kq + r$, $0 \leq r < k$. Let $e = r\binom{q+1}{e} + (m - r)\binom{q}{2}$.

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$$\sum_v \frac{1}{d_v + 1} \geq k.$$

When the equality holds, I is a constant. G can not contain an induced P_2. Therefore $G = \bar{G}_{n,k}$.

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$t(n, K_{r,s}) < cs^{1/r}n^{2-1/r} + O(n)$.

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- **Erdős-Bondy-Simonovits (1963,1974):**
  $t(n, C_{2k}) \le ckn^{1+1/k}$.

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- **Conjecture ($250 for proof and $100 for disproof:)** Suppose H is a bipartite graph. Prove or disprove that $t(n, H) = O(n^{3/2})$ if and only if H does not contain a subgraph each vertex of which has degree > 2.

Alteration method

Suppose that the “random” structure does not have all desired properties but many have a few “blemishes”. With a small alteration we remove the blemishes, giving the desired structures.

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This gives $R(r,r) > (1+o(1))\frac{1}{e}r2^{r/2}$. □

Combinatorial geometry

- S: a set of n points in the unit square $[0,1]^2$.
- $T(S)$: the minimum area of a triangle whose vertices are three distinct points of S.

Koml'os, Pintz, Szemer'edi (1982): There exists a set S of n points in the unit square such that $T(S) = \Omega(\frac{\log n}{n^2})$.

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Proof: Select $2n$ random points uniformly and independently from $[0,1]^2$.

- P, Q, R: three random points.
- $\mu := \Delta PQR$: the area of PQR.

Proof

$$\Pr(x \leq |PQ| \leq x + \Delta x) \leq \pi(x + \Delta x)^2 - \pi x^2 \approx 2\pi x \Delta x.$$

If $\mu \leq \epsilon$, then R is in the region of a rectangle of width $\frac{4\epsilon}{x}$ and length at most $\sqrt{2}$.

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Delete one vertex from each small triangle and leave at least n vertices. Now no triangle has area less that $\frac{1}{100n^2}$. □

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Theorem: For any $0 < p < 1$, we have

$$R(k,t) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{t} (1-p)^{\binom{t}{2}}.$$

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Proof: Color each edge independently in red or blue; the probability of being red is p while the probability of being blue is $1-p$. Let X be the number of red K_k and Y be the number of blue K_t.

$$\begin{aligned} \mathrm{E}(X) &= \binom{n}{k} p^{\binom{k}{2}} \\ \mathrm{E}(Y) &= \binom{n}{t} (1-p)^{\binom{t}{2}}. \end{aligned}$$

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Best lower bound: **Kim (1995)** and best upper bound: **Shearer (1983)**.

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Before Shearer's result, **Ajtai-Komlós and Szemerédi (1980)** proved $R(3, t) \leq \frac{c't^2}{\ln t}$.