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Research Article

Probabilistic Modeling of Fatigue Damage Accumulation for Reliability Prediction

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A methodology for probabilistic modeling of fatigue damage accumulation for single stress level and multistress level loading is proposed in this paper. The methodology uses linear damage accumulation model of Palmgren-Miner, a probabilistic *S-N* curve, and an approach for a one-to-one transformation of probability density functions to achieve the objective. The damage accumulation is modeled as a nonstationary process as both the expected damage accumulation and its variability change with time. The proposed methodology is then used for reliability prediction under single stress level and multistress level loading, utilizing dynamic statistical model of cumulative fatigue damage. The reliability prediction under both types of loading is demonstrated with examples.

1. Introduction

Most of the mechanical components are subjected to fatigue due to random loading as well as constant amplitude loading during their usage. Fatigue is also recognized as one of the main reasons for failure of mechanical components [1]. This has led to a need for developing new approaches to predict the reliability and useful life of mechanical components, which are subjected to fatigue damage. This has been the primary focus of designers for many years, and the field still presents many challenges, even though extensive progress has been made in the past few decades [2].

Earlier models of fatigue damage accumulation reported in the literature focus on deterministic nature of the process whereas in practice, damage accumulation is of stochastic nature. This stochasticity results from the randomness in fatigue resistance of material as well as that of the loading process [3]. As a result of this, even under constant amplitude fatigue test at any given stress level, the fatigue life shows stochastic behavior with a specific distribution. The literature shows that fatigue life data follows either normal or lognormal distribution under constant amplitude or random loading [4–6]. Weibull distribution has also been reported to fit fatigue life data [7, 8], though there are no apparent physical or mathematical phenomenon explained for this [9]. Researchers have proposed different modeling approaches to the probabilistic damage accumulation paradigm. Shen et al. [3] developed a probabilistic distribution model of stochastic fatigue damage, wherein they have considered the randomness of loading process as well as the randomness of fatigue resistance of material by introducing a random variable of single cycle fatigue damage. Liu and Mahadevan [2] proposed a general methodology for stochastic fatigue life under variable amplitude loading by combining a nonlinear fatigue damage accumulation rule and stochastic *S-N* curve representation technique. Nagode and Fajdiga [10] have modeled a probability density function (PDF) of failure cycles at any stress level as a normal distribution based on the DeMoivre-Laplace principle to reliably predict endurance limit of a randomly loaded structural component. Liao et al. [11] proposed a new cumulative fatigue damage dynamic interference model assuming that cumulative damage follows normal or lognormal distribution. Wu and Huang [12] modeled fatigue damage and fatigue life of structural components subjected to variable loading as a Gaussian random process. Ben-Amoz [13] developed a cumulative damage theory based on the concept of bounds on residual fatigue life in two-stage cycling. Castillo et al. [14] developed a general model for predicting fatigue behavior for any stress level and range by generalizing the Weibull model. Sethuraman and Young [15] have developed a cumulative damage threshold crossing model. This model considers a multicomponent product which undergoes deterioration/damage at regular intervals of time and failure occurs as soon as the maximum damage to some component crosses a certain threshold. Time to failure data is used to estimate the model parameters. A comprehensive review of cumulative fatigue damage and life prediction theories can be found in Fatemi and Yang [16].

Two aspects are significantly important from the point of view of modeling probabilistic fatigue damage. First, an accurate physical damage accumulation model needs to be in place to predict expected or nominal fatigue damage. Second, an appropriate uncertainty modeling technique is required to account for stochasticity [2]. A review of literature has indicated that handling of stochasticity in modeling uncertainty involve complex mathematics. This fact is the primary motivation behind the development of a simpler approach for handling stochasticity in fatigue damage accumulation modeling in the proposed research work. This paper proposes a simpler approach to deduce the distribution of a fatigue damage accumulation from the fatigue life distribution using a one-to-one transformation methodology and to a certain extent attempts to minimize the mathematical complexity. It also proposes a simple and unique way to model the damage accumulation process treating it as a nonstationary probabilistic process to capture damage accumulation and its variability at any given point of time. The proposed methodology can be effectively used to predict reliability of mechanical components subjected to fatigue loading due to single stress level and multi-stress level.

An outline of this paper is as follows. Section 2 elaborates an approach for modeling probabilistic damage accumulation to capture expected values of damage and its variability. Section 3 presents a reliability prediction approach utilizing dynamic model of cumulative fatigue damage. The applicability of proposed methodology and its implications are illustrated with help of examples in Section 4. The conclusion is presented in Section 5.

2. Modeling Probabilistic Fatigue Damage Accumulation

Damage accumulation is a complex and irreversible phenomenon, wherein the damage of the product under consideration gradually accumulates and over a period of time leads to its failure. Therefore, damage accumulation can be treated as a measure of degradation in fatigue resistance of materials. Moreover, the damage accumulation is probabilistic in nature and it can be depicted graphically as shown in Figure 1. The Figure shows monotonically increasing degradation path where the degradation measure is increasing probabilistically with time.

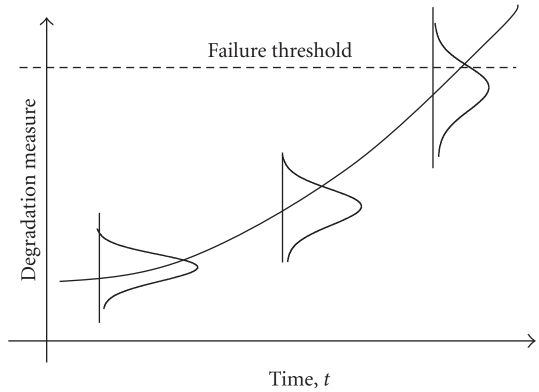

Figure 1: Degrading path example.

Wang and Coit [17] have explained that at any specified time, there exists a distribution of degradation measurements considering a population of similarly degrading components. They also pointed out that the variability in any given degradation measure increases with usage time. Since damage accumulation is also a measure of degradation, the reasoning given by Wang and Coit [17] can be applied in assuming that damage accumulation also follows certain probability distribution and that the expected value and variability of any damage accumulation measure will increase with usage time. Further, Wu et al. [6] have shown that under constant or random amplitude loading conditions normal or lognormal distribution provides good fit to fatigue failure data. Therefore, in the proposed work the damage accumulation is modeled as a nonstationary probabilistic process assuming that fatigue life follows normal distribution. Furthermore, since damage accumulation is a function of uses cycle, the probability distribution of damage accumulation can be treated as normal distribution as advocated in Benjamin and Cornell [18]. The nonstationary Guassian process of damage accumulation can be given as:

$$D(t) \approx N\{\mu_D(t), \sigma_D^2(t)\}, \quad (1)$$

where $D(t)$ is a damage accumulation measure that varies probabilistically with time t, and $\mu_D(t)$ and $\sigma_D^2(t)$ are its mean and variance. The proposed probabilistic modeling of damage accumulation is elaborated in subsequent sections.

2.1. Modeling Expected Value of Damage Accumulation. The two most widely used models for fatigue loading are *S-N* curve and Palmgren-Miner's damage accumulation models [2, 19]. The *S-N* curve model is used to express the relationship between fatigue life (N_f) and stress level (S) and is expressed by the well-known *S-N* curve equation as given below:

$$N_f S^m = A, \quad (2)$$

where A is a fatigue strength constant and m represents slope of the *S-N* curve. Figure 2 shows a probabilistic interpretation of the *S-N* curve, wherein PDFs (on normal

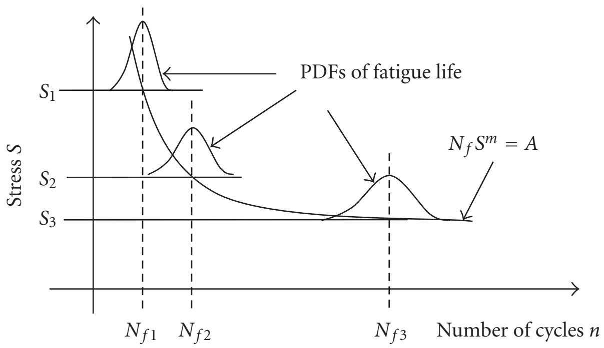

Figure 2: Probabilistic *S-N* curve.

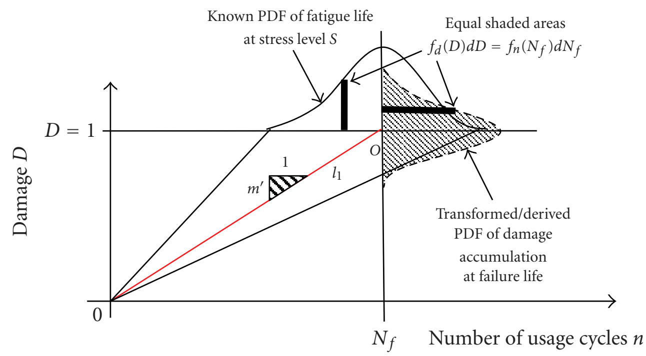

Figure 3: One-to-one transformation of PDF.

scales) of fatigue lives are depicted at different stress levels S_1, S_2 , and S_3.

The linear damage accumulation model, which is also known as Palmgren-Miner's rule, defines damage as the ratio of the number of cycles of operation to the number of cycles to failure at any given stress level [19]. Assuming no initial damage, the damage accumulation at single stress level is given as:

$$D = \frac{n}{N_f}. \tag{3}$$

Similarly, for multi-stress levels, damage accumulation can be expressed as:

$$D = \sum_{i=1}^{k} D_i = \sum_{i=1}^{k} \frac{n_i}{N_{fi}}, \tag{4}$$

where D is the total accumulated fatigue damage of the material, D_i is the damage accumulated when subjected to ith stress level, n_i is the number of usage cycles, and N_{fi} is the number of cycles to failure at the ith stress level. It is assumed that failure occurs when total damage accumulation reaches unity [6, 19]. The number of cycles to failure (N_{fi}) at any given stress level can be obtained from the *S-N* curve model. Therefore, by considering both *S-N* curve model and the linear damage accumulation model, a linear relationship model between damage accumulation and number of usage cycle at any given stress level can be derived by combining (2) and (3) as given below:

$$D = \frac{S^m}{A} n, \qquad D = CS^m n, \tag{5}$$

where C represents a reciprocal of fatigue strength constant, S denotes a constant amplitude stress level, m represents slope of the *S-N* curve, and n denotes number of usage cycles. Similarly, for multi-stress levels the linear damage accumulation model can be derived by modifying (5) as follows:

$$D = \sum_{i=1}^{k} D_i = \sum_{i=1}^{k} CS_i^m n_i. \tag{6}$$

Equations (5) and (6) provide expected value of damage accumulation at any given point of time (usage cycles) subjected to single stress level and multiple-stress levels, respectively. However, in order to come up with a realistic estimate of reliability of any given product, it is necessary to adopt a probabilistic approach. It is, therefore, important to treat the damage accumulation measure under consideration as a random variable and to establish an appropriate distribution of damage accumulation. In the following sections, a methodology is proposed to derive the PDF of the damage accumulation measure by considering the PDF of fatigue life (N_f).

2.2. Distribution of Fatigue Damage Accumulation. In order to establish the PDF of the damage accumulation measure (D) and to estimate distribution parameters, first the fatigue failure life is treated as random variable which follows certain distribution. Thereafter, the distribution of D is derived using the one-to-one PDF transformation methodology proposed by Benjamin and Cornell [18]. As per Benjamin and Cornell [18], the unknown PDF of a random variable can be derived using this transformation technique, if that variable is directly or functionally related to another random variable whose PDF is already known. Since cumulative damage accumulation is a function of usage life (or fatigue failure life), the PDF transformation methodology provides an effective means to establish distribution of damage accumulation measure. In Figure 3, line l_1 is the trend line of expected damage accumulation as given by (5) at a given stress level (S). The straight line depicts a linear relation between the damage accumulation measure and usage cycles (n). Now assuming that the PDF of usage cycles is known, Figure 3 provides the basic understanding of how this known PDF of fatigue life can be used to obtain the PDF of the damage accumulation measure. It must be noted, as shown in Figure 3, that initial variability of usage cycles is zero and it follows increasing trend in variability with increase in usage cycle. It is further assumed that this increase in variability also follows a linear trend as suggested by Coit et al. [20].

From the above discussion, it is clear that in order to derive the distribution of D using Benjamin and Cornell's [18] PDF transformation technique, there are two basic requirements that need to be fulfilled: (i) a clearly defined relation between damage accumulation and usage cycles and (ii) the knowledge of the distribution or PDF of the usage cycle.

In order to satisfy the first requirement, a linear relationship between the damage accumulation measure and usage time for a given stress level can be established by redefining (5) as follows:

$$D = m'n, \tag{7}$$

where m' represents the slope of the damage accumulation trend line. At fatigue failure life (i.e., at $n = N_f$), (7) can be written as:

$$D = m'N_f. \tag{8}$$

As mentioned earlier, under constant or random amplitude loading conditions, normal or lognormal distributions provide good fit to fatigue failure data than Weibull distribution [6]. Therefore, in the proposed work, it is assumed that the fatigue failure life N_f follows normal distribution. This fulfills the second requirement of known distribution of fatigue failure life too. The PDF of normally distributed fatigue life is given as:

$$f_n\left(N_f\right) = \frac{1}{\sigma_{N_f}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{N_f - \mu_{N_f}}{\sigma_{N_f}}\right)^2\right). \tag{9}$$

2.2.1. Application of the PDF Transformation Technique for Deriving the PDF of the Damage Accumulation Measure. The functional relationship between damage accumulation measure and fatigue life (N_f) can be generically expressed as given below:

$$D = g\left(N_f\right). \tag{10}$$

The inverse relation of (10) can be expressed as:

$$N_f = g^{-1}(D). \tag{11}$$

The cumulative distribution function (CDF) of the dependent variable D can be obtained from the CDF of N_f as:

$$F_d(D) = F_n\left(g^{-1}(D)\right). \tag{12}$$

Subsequently to obtain the PDF of the damage accumulation measure (D), take a derivative of its CDF as given below [18]:

$$\begin{aligned} f_d(D) &= \frac{d}{dD}F_d(D) \\ &= \frac{d}{dD}F_n(g^{-1}(D)) \\ &= \frac{d}{dD}\left[\int_{-\infty}^{g^{-1}(D)} f_n\left(N_f\right)dN_f\right] \\ &= \frac{dg^{-1}(D)}{dD} f_n(g^{-1}(D)). \end{aligned} \tag{13}$$

Using (11), (13) can be written in a more suggestive form as follows:

$$f_d(D) = \frac{dN_f}{dD} f_n\left(N_f\right) \tag{14}$$

or

$$f_d(D)dD = f_n\left(N_f\right)dN_f. \tag{15}$$

The graphical interpretation of (15) is shown in Figure 3 by equal shaded areas. As shown graphically, the interval widths dD and dN_f are not equal, but in case of the linear relationship between random variables, the ratio of dN_f/dD is constant [18].

Further, differentiating (8) with respect to D gives:

$$\frac{dN_f}{dD} = \frac{1}{m'}. \tag{16}$$

Substituting (16) and (9) in (14) gives the PDF of D as follows:

$$f_d(D) = \frac{1}{m'\sigma_{N_f}\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{(D/m') - \mu_{N_f}}{\sigma_{N_f}}\right)^2\right). \tag{17}$$

From (17), it is clear that the damage accumulation also follows similar distribution, and the relation between standard deviations before and after transformation is as follows:

$$\sigma_D = m'\sigma_{N_f}, \tag{18}$$

where σ_D represents standard deviation of damage accumulation, σ_{N_f} denotes standard deviation of failure life or usage cycle, and m' represents slope of damage accumulation trend line. This clearly indicates that variability in damage accumulation depends on slope of the damage accumulation trend line and variability in usage cycle.

Assuming that slope of the damage accumulation trend line is constant for any given stress level, the variability in damage accumulation can be treated as a function of the variability in usage cycle. Equations (17) and (18) derived using one-to-one transformation approach clearly explain that the PDF of damage accumulation will be the same as usage life distribution because of linear relationship between these two variables. However, the resulting PDF of damage accumulation is scaled down by factor m'.

2.3. Modeling Trend Line of Variance. As already discussed, the damage accumulation increases linearly with usage cycles at a given stress level. Further, Figure 4 shows increasing trend in variability or standard deviation of fatigue lives with decrease in stress levels as well. It must also be noted that the variability is zero at the initial stage (i.e., at $n = 0$) for all the stress levels. This indicates the presence of a trend of increasing variance in fatigue lives as stress level decreases, that is, there is low variability in fatigue life at higher stress level and higher variability in fatigue life when product is subjected to lower stress levels as shown in Figure 4. The challenge is to capture this increasing trend in variability with

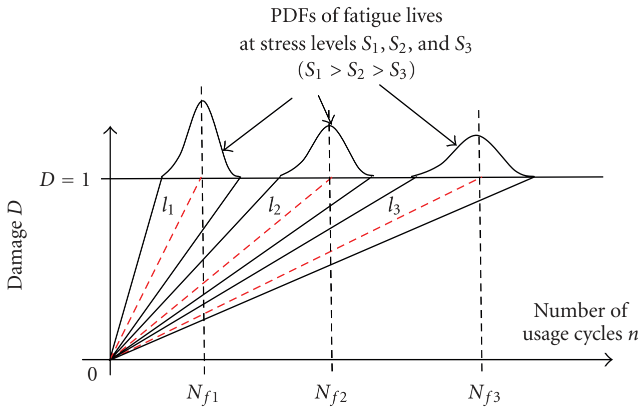

FIGURE 4: Damage accumulation for different stress.

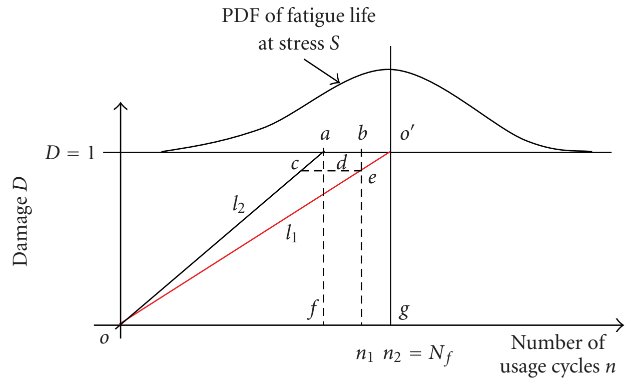

FIGURE 5: Geometric interpretation of rate of change of variability of usage cycles.

stress levels as well as usage cycle and model it into total variability of damage accumulation.

Based on the above discussion, there are two interesting aspects to be considered for modeling total variability in damage accumulation.

(1) At constant stress level, the variability in damage accumulation increases monotonically with increase in usage cycles as explained by Wang and Coit [17] and Coit et al. [20].

(2) The variability in fatigue life is lower at higher stress levels, whereas it monotonically increases with decrease in stress levels as shown in Figures 2 and 4. Pascual and Meeker [21] have also stated that in most fatigue experiments, the variance of the fatigue life increases as stress decreases, and therefore the standard deviation of fatigue life can be modeled as monotonic function of stress. In fact, the variability in fatigue life of a product subjected to multiple stress levels is another major source of variability in damage accumulation.

In order to model total variability in damage accumulation, these two sources of variability need to be considered. The following section elaborates the modeling of variability in damage accumulation considering these two aspects.

2.3.1. Deriving the Equation for the Variance Trend Line. Given the fact that the variability in usage cycles continuously increases from zero to a certain value at fatigue failure life, its rate of change can be interpreted using geometric reasoning. This is shown diagrammatically in Figure 5.

As shown in Figure 5, l_1 represents the mean damage accumulation trend line and line l_2 connects the origin with 1σ point (i.e., point a in Figure 5) of the fatigue life distribution. Therefore, ao' represents one standard deviation distance from center point of the fatigue life distribution, and hence, $\sigma_{N_f} = ao'$. Let us assume that slopes of these two lines l_1 and l_2 are represented as m' and m'', respectively.

From the geometric construction shown in Figure 5, the rate of change of standard deviation (r_σ) of usage cycles can be interpreted as:

$$r_\sigma = \frac{(ao' - ce)}{(n_2 - n_1)}, \tag{19}$$

where $ao' = ab + bo'$, $ce = cd + de$, and $bo' = n_2 - n_1$. Substituting these in (19) gives:

$$r_\sigma = \frac{(ab + bo' - cd - de)}{(bo')}. \tag{20}$$

Since $ab = de$, we have:

$$r_\sigma = \left(1 - \frac{(cd)}{(bo')}\right). \tag{21}$$

The slope of mean damage accumulation line l_1 can be obtained as:

$$m' = \frac{be}{bo'}. \tag{22}$$

Similarly, the slope of line l_2 can be derived as:

$$m'' = \frac{ad}{cd}. \tag{23}$$

By taking the ratio of two slopes and considering that $ad = be$, it gives:

$$\frac{cd}{bo'} = \left(\frac{m'}{m''}\right). \tag{24}$$

Substituting (24) in (21) gives:

$$r_\sigma = \left(1 - \frac{m'}{m''}\right). \tag{25}$$

Further, the slope of lines l_1 and l_2 can also be obtained as:

$$m' = \frac{o'g}{og} = \frac{1}{N_f}, \qquad m'' = \frac{af}{of} = \frac{1}{N_f - \sigma_{N_f}}. \tag{26}$$

Using (26) in (25) gives:

$$r_\sigma = \left(1 - \left(\frac{N_f - \sigma_{N_f}}{N_f}\right)\right),$$
$$r_\sigma = \frac{\sigma_{N_f}}{N_f}. \tag{27}$$

The variability (or standard deviation) in usage cycle (n) at constant amplitude stress level (S) can be given as:

$$\sigma_n = r_\sigma n. \tag{28}$$

Now by combining (27) and (28), the standard deviation of usage cycle is estimated as:

$$\sigma_n = \left(\frac{\sigma_{N_f}}{N_f}\right) n. \tag{29}$$

Considering (18) and (29), a relationship can be derived to estimate variability or standard deviation of damage accumulation measure (D) as follows:

$$\sigma_D = \left(\frac{\sigma_{N_f}}{N_f}\right) nm'. \tag{30}$$

Considering $m' = CS^m$, (30) can be modified in a more suggestive form as follows:

$$\sigma_D = CS^m n \left(\frac{\sigma_{N_f}}{N_f}\right). \tag{31}$$

Equation (31) indicates that variability in damage accumulation is a function of stress amplitude, usage cycles, fatigue failure life and variability in fatigue failure life. In order to estimate variability in damage accumulation, these parameter values can be obtained from the probabilistic S-N curve (refer to Figure 2).

The above equation represents a model that can be used to capture the variability in damage accumulation for a single stress level. The same model can be extended to capture the variability in damage accumulation if the product is subjected to multi-stress level as well. Figure 6 illustrates the concept of how variance of usage cycles changes in a multilevel stress loading scenario.

It has already been discussed (refer to Figure 4) that the rate of damage accumulation and its variability depend on stress levels and number of usage cycles. Moreover, the rate of change of variability is different for different stress levels. Therefore, considering multi-stress level scenario, the total variability or standard deviation in damage accumulation at the time of fatigue failure can be obtained using the following equation:

$$\sigma_D = \sqrt{\sum_{i=1}^{k} \left(CS_i^m n_i \left(\frac{\sigma_{N_{fi}}}{N_{fi}}\right)\right)^2}, \tag{32}$$

where suffix i represents the level of stress in the multilevel stress loading scenario.

In the next section, the models developed for capturing damage accumulation and its variability are used in conjunction with the dynamic statistical model for predicting the reliability of mechanical components.

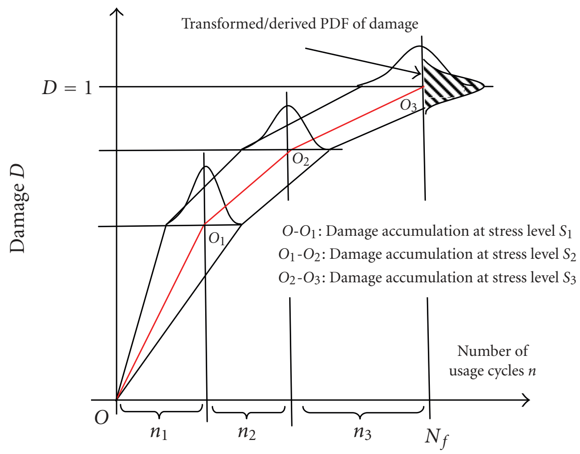

Figure 6: Damage accumulation for multi-stress loading and its distribution.

3. Reliability Prediction

The well-known stress-strength interference model considers product reliability from the probabilistic point of view. This concept has been used by many researchers for developing models to predict product reliability in the past [11, 22–24]. Liao et al. [11] have classified existing cumulative fatigue damage models for reliability prediction into two groups based on fundamental assumptions and hypothesis as: (i) static statistical models and (ii) dynamic statistical models. Unlike static models, dynamic models treat both expected value and variance of random variable as time dependent and their values continuously change with time. However, these dynamic statistical models are developed on existing classical stress-strength interference model considering random variable as dynamic random variable [11]. In the present work, the damage accumulation is treated as dynamic random variable whose distribution parameters (mean and variance) are dependent on usage life (time) as given by (5) and (31). This paper proposes a dynamic reliability prediction model considering probabilistic damage accumulation developed in the previous section of this paper. The following assumptions have been made while formulating a dynamic reliability prediction model.

(1) Fatigue failure occurs when damage accumulation (D) reaches the threshold damage (D_c), where $E(D_c) = 1$.

(2) The threshold damage or critical damage has the same distribution as the damage accumulation measure.

(3) When usage life is equal to the fatigue failure life ($n = N_f$), the variability of threshold damage accumulation ($\sigma_{D_c}^2$) is equal to the variability of damage accumulation measure (σ_D^2). The variability of damage accumulation measure continuously increases with usage life but when usage cycle reaches to fatigue failure level, the corresponding variability is assumed

to be the same as variability of threshold damage accumulation. However, it is statistically independent of (D).

Since the damage accumulation measure is treated as a normally distributed dynamic variate, the reliability of a product in terms of damage accumulation can be modeled as follows:

$$\begin{aligned} R &= \text{prob}(D < D_c) \\ &= 1 - \text{prob}(D_c - D \le 0) \\ &= 1 - \varnothing\left(-\frac{(\mu_{D_c} - \mu_D)}{\sqrt{\sigma_{D_c}^2 + \sigma_D^2}}\right). \end{aligned} \tag{33}$$

A diagrammatic representation of the above concept is shown in Figure 7. It is important to note that when usage cycle is equal to failure life ($n = N_f$), the variability of threshold damage accumulation will be equal to the variability of damage accumulation ($\sigma_{D_c}^2 = \sigma_D^2$).

Substituting (6) and (32) in (33), the reliability can be expressed in a more suggestive form as follows:

$$R = 1 - \varnothing\left(-\frac{\left(\mu_{D_c} - \sum_{i=1}^{k} CS_i^m n_i\right)}{\sqrt{\sigma_{D_c}^2 + \sum_{i=1}^{k}\left(CS_i^m n_i\left(\sigma_{N_{fi}}/N_{fi}\right)\right)^2}}\right). \tag{34}$$

The above model provides a dynamic reliability prediction considering dynamic behavior or continuous degradation phenomenon of the product with usage cycle. In essence, the proposed dynamic reliability prediction model captures product life cycle and assesses product reliability for a given time period or usage cycle. The proposed model can be used for predicting reliability of a product subjected to both single stress and multi-stress level scenarios. The applicability of the proposed model is demonstrated with the help of a helical spring (compression type) example.

4. Example

To demonstrate the applicability of the proposed reliability assessment approach, fatigue test data required to model *S-N* curve is adopted from Zaccone [7], which were obtained after conducting fatigue tests on helical compression springs used in compressors. Table 1 shows the fatigue test data at different amplitude stress levels and corresponding standard deviation considered.

Using the above data, the *S-N* curve model was fitted to obtain the model parameters as follows:

$$m = 8.46; \qquad A = 1.3544 \times 10^{27}. \tag{35}$$

The value of constant C is calculated by taking reciprocal of the fatigue strength constant and is obtained as:

$$C = 7.3829 \times 10^{-28}. \tag{36}$$

Using these parameters values and (6), the expected value of damage accumulation in the compression spring for single

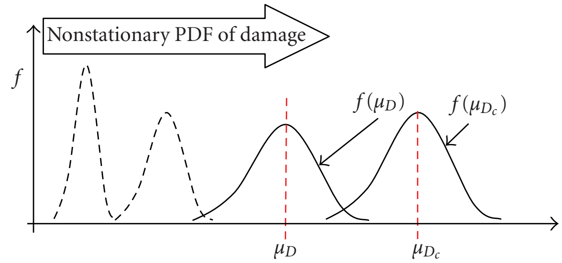

Figure 7: Dynamic stress-strength interference model for damage accumulation.

Table 1: Fatigue test data.

Constant amplitude stress (MPa) (S_i)	Mean number of cycle to failure (N_f)	Standard deviation (σ_{N_f})
470	33581	4785
435	64616	9692
395	146101	23291
360	320222	53947
320	867130	156303

stress or multi-stress level for given number of usage cycles can be obtained as follows:

$$\mu_D = \sum_{i=1}^{k} 7.3829 \times 10^{-28} \times S_i^{8.46} \times n_i. \tag{37}$$

4.1. Reliability Prediction for a Single Stress Level. First, the applicability of the proposed model is demonstrated by estimating the reliability for single stress level. A single stress level of 360 MPa is considered. For that purpose, one has to estimate the variability of the threshold damage (σ_{D_c}) at fatigue failure life and the variability of damage accumulation at any given usage cycle. The variability of threshold damage at fatigue failure life is calculated considering third assumption ($\sigma_{D_c} = \sigma_D$) at the fatigue failure life (N_f) and using (31) as follows:

$$\sigma_{D_c} = CS^m n\left(\frac{\sigma_{N_f}}{N_f}\right), \tag{38}$$

where $C = 7.3829 \times 10^{-28}$, $S = 360$ MPa, fatigue failure life $n = N_f = 320222$ cycles, and $\sigma_{N_f} = 53947$ cycles.

Using these values of variables and parameters, the variability at threshold damage is obtained as $(\sigma_{D_c}) = 0.16846$.

Similarly, one can calculate the variability in damage accumulation at any given time period or usage cycle (n). Once this element of variability is estimated, (34) can be used to estimate reliability of the compression spring subjected to single stress level for any given time period as given below:

$$R = 1 - \varnothing\left(-\frac{(1 - \mathscr{H})}{\sqrt{0.16846^2 + \left(\mathscr{H} \times \left(\sigma_{N_f}/N_f\right)\right)^2}}\right), \tag{39}$$

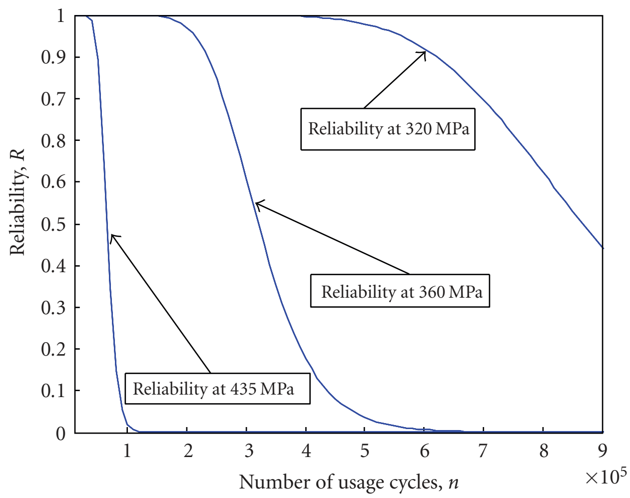

FIGURE 8: Reliability plot at single stress levels of 435, 360 and 320 MPa.

where ($\mathcal{H}$) denotes ($7.3829 \times 10^{-28} \times S^{8.46} \times n$). Equation (39) expresses the reliability as a function of usage cycles (n) at single stress level.

Considering reliability function equation (39), Figure 8 illustrates a reliability plot for single stress level at 435 MPa, 360 MPa, and 320 MPa. These reliability plots clearly reveal the trend of reliability loss with increase in usage cycles. The careful analysis of these reliability plots indicate that reliability remains higher (almost constant) for initial period and later on starts declining with usage cycle. This phenomenon explains existing understanding of crack initiation and crack propagation periods. The higher and stable reliability phase, although it varies with stress levels, represents crack initiation period, and reliability loss phase is indicative of crack propagation period. It is clear from these plots that crack initiation period is smaller for higher stress level and loss of reliability is faster during crack propagation indicating faster degradation or higher rate of damage accumulation. The total life of the product also varies with stress levels and therefore supports out argument that dynamic behavior or degradation phenomenon needs to be captured in design optimization models to provide more realistic solutions.

4.2. Reliability Prediction for Multi-Stress Levels. To demonstrate multi-stress loading scenarios, let's consider a helical compression spring subjected to three successive stress levels of 435 MPa, 360 MPa, and 320 MPa for 40000, 60000 and remaining number of usages cycles up to fatigue failure, respectively. To estimate reliability under multi-stress level loading, the fatigue life of spring needs to be predicted under multi-stress loading conditions.

4.2.1. Prediction of Fatigue Life. As mentioned earlier, the linear damage accumulation theory states that the damage fraction at any stress level is linearly proportional to the ratio of the number of usage cycles to the number of cycles to fatigue failure at that stress level [19]. In the case of multi-stress level loading, when these damage fractions equal unity, fatigue life can be predicted. Therefore, to predict fatigue life first, the remaining life (n_3) of the spring after first and second stress levels needs to be calculated. This can be obtained as follows:

$$n_3 = \frac{1}{CS_3^{8.46}}\left(1 - \sum_{i=1}^{2} CS_i^{8.46} n_i\right). \tag{40}$$

The remaining life is obtained as $n_3 = 167750$ number of usage cycles.

The estimated fatigue life (N_{fe}) of spring under multi-stress level loading condition is given as:

$$N_{fe} = (40000 + 60000 + 167750) = 267750, \tag{41}$$

number of usage cycles.

4.2.2. Prediction of Reliability. To estimate the reliability of spring subjected to given multilevel stress loading, the variability of the threshold damage at fatigue failure life needs to be estimated. The variability of the threshold damage at fatigue failure life is calculated as given in Section 4.1 (i.e., using third assumption) and(32)

$$\sigma_{D_c} = \sqrt{\sum_{i=1}^{3}\left(7.3829 \times 10^{-28} \times S_i^{8.46} \times n_i \times \left(\frac{\sigma_{N_{fi}}}{N_{fi}}\right)\right)^2}, \tag{42}$$

where usage cycles are $n_1 = 40000$, $n_2 = 60000$, and $n_3 = 167750$, and fatigue failure lives $N_{f1} = 64616$, $N_{f2} = 320222$, $N_{f3} = 867130$, and variability at each fatigue failure lives are $\sigma_{N_{f1}} = 9692$, $\sigma_{N_{f2}} = 53947$, and $\sigma_{N_{f3}} = 156303$.

Using the appropriate values of variables and parameters, the variability at the threshold damage (σ_{D_c}) is $\sigma_{D_c} = 0.16042$.

Similarly, the variability in damage accumulation at any given usage cycle can be obtained. After estimating the variability at threshold damage and variability at any given usage cycle, the reliability of the compression spring for any given time period subjected to given multilevel stress loading can be estimated using equation given below:

$$R = 1 - \varnothing\left(-\frac{\left(1 - \sum_{i=1}^{3} \mathcal{K}\right)}{\sqrt{0.16846^2 + \sum_{i=1}^{3}\left(\mathcal{K} \times \left(\sigma_{N_{fi}}/N_{fi}\right)\right)^2}}\right), \tag{43}$$

where ($\mathcal{K}_i$) denotes ($7.3829 \times 10^{-28} \times S_i^{8.46} \times n_i$). Using reliability function equation (43), a reliability plot for multi-stress level loading of three successive stress levels of 435 MPa, 360 MPa, and 320 MPa is illustrated as plot-A in Figure 9. A careful analysis of reliability plot-A indicates that initially the damage accumulation rate is higher because the spring is first subjected to higher stress level. As a result of this, the higher reliability phase (crack initiation period) is very short and crack propagation start at early stage of life.

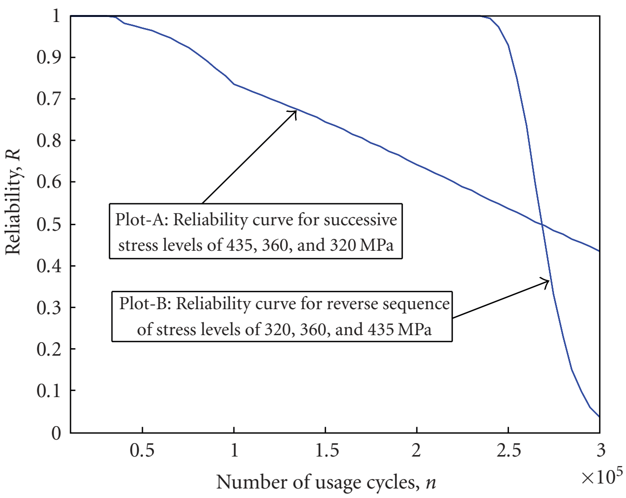

Figure 9: Reliability at multi-stress levels of 435, 360 and 320 MPa.

But as life progresses the rate of crack propagation is slightly reduced as compared to earlier rate at higher stress. In other words, when product is subjected to high-low sequence of stress, the rate of loss of reliability is higher and starts at early stage of life but slows down later when subjected to low stress levels. This happens because of shorter crack initiation period due to high stress levels initially and relatively longer crack propagation periods when subjected to lower stresses.

4.3. Reliability Prediction for Multi-Stress Levels by Changing Sequence of Loading. To study the effect of change in sequence of loading on reliability behavior of the helical compression spring, the sequence of multi-stress levels loading is reversed (i.e., 320 MPa, 360 MPa, and 435 MPa for successive usage cycles 167750, 60000, and 40000, resp.).

The reliability plot for this reversed multilevel stress loading is plotted as plot-B in Figure 9. In multi-stress loading scenario, reversal of loading sequence will not affect the fatigue life of spring much, as the fraction of life consumed at different stress level remains the same in light of linear damage accumulation theory. But reliability plot-B indicates considerable change in its shape. The initial higher reliability phase is considerably longer due to low-high stress sequence. This is an indication of longer crack initiation phase and slower rate of damage accumulation because of slow propagation of crack at lower stress levels. However, as soon as spring is subjected to higher stress level, the reliability starts declining sharply due to faster propagation of cracks and hence higher rate of damage accumulation at higher stress level. In other words, the implications of change in sequence of loading can be observed in the duration of crack initiation phase and the rate of loss of reliability with usage time (as shown in Figure 9). It can be observed that the rate of loss of reliability is generally higher for high-low sequence of loading compared to low-high sequence but it changes later.

Authors believe these two scenarios represent two extremes but ideal cases of loading phenomenon of multilevel stress loading. In real life applications, the loading pattern is not necessarily sequential but could be any combination of stress levels in any sequence of loading. Therefore, these two extreme scenarios provide a band of reliability behavior for multilevel stress loading, and actual reliability behavior, for any given real life loading pattern, could be somewhere within this band.

5. Conclusion

The proposed approach provides an easy methodology for modeling the probabilistic distribution of the damage accumulation measure and hence capturing real life behavior of the product. It proposes a simple and unique way to model the damage accumulation process treating it as a nonstationary process to capture damage accumulation and its variability at any given point of time. The proposed approach can be extended to consider other fatigue failure distributions such as lognormal and Weibull distributions to model probabilistic damage accumulation.

The major limitation of the proposed approach is that it treats damage accumulation as linear phenomenon whereas in actual practice damage accumulation could be a nonlinear phenomenon especially in multi-stress loading scenarios. Our future research efforts are directed towards developing an understanding to capture nonlinear damage accumulation phenomenon in multi-stress loading conditions. Another potential research area for future research can be towards capturing and incorporating degradation behavior (dynamic degradation phenomenon) into design optimization models to capture life-cycle issues and provide more realistic solutions.

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