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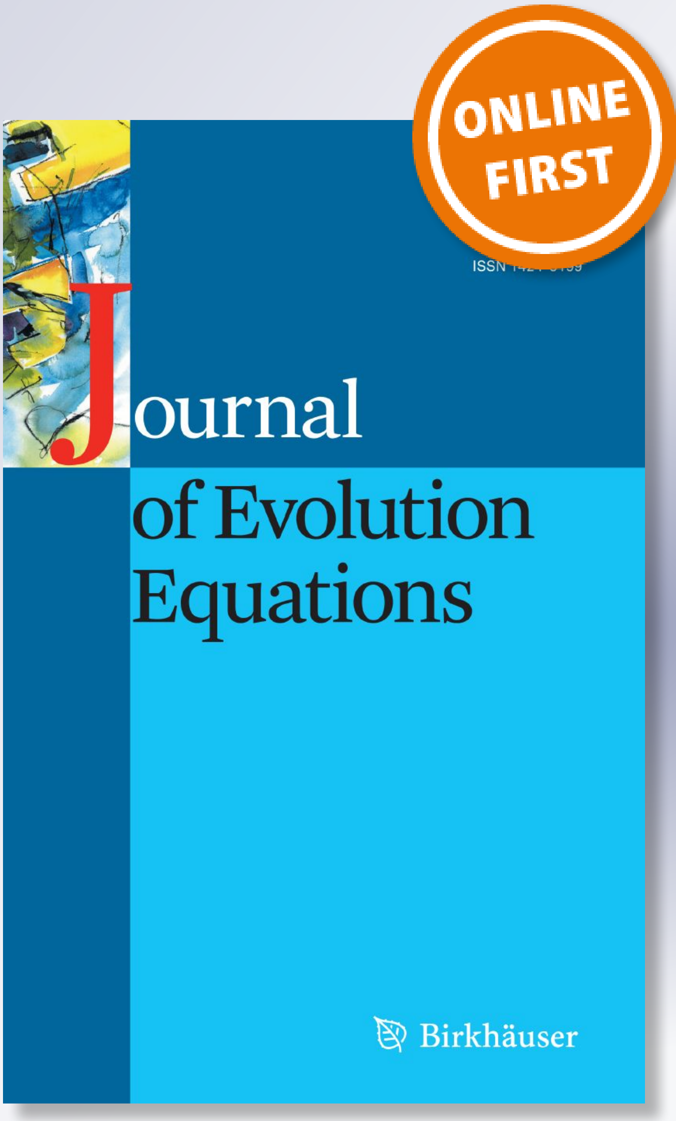

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Abstract. We propose a probabilistic construction for the solution of a general class of fractional high-order heat-type equations in the one-dimensional case, by using a sequence of random walks in the complex plane with a suitable scaling. A time change governed by a class of subordinated processes allows to handle the fractional part of the derivative in space. We first consider evolution equations with space fractional derivatives of any order, and later we show the extension to equations with time fractional derivative (in the sense of Caputo derivative) of order $\alpha \in (0, 1)$.

1. Introduction

The connection between partial differential equations and stochastic processes or, more generally, functional integration, is an extensively developed theory which covers one-dimensional, finite-dimensional and infinite-dimensional problems. Since this paper is mainly devoted to the one-dimensional case, we shall specialize this introduction to such problems. The first and main example is the Feynman–Kac formula [33,34] providing the solution of the Cauchy problem for the "heat equation with sink"

$$\partial_t u(t, x) = \frac{1}{2}\partial_x^2 u(t, x) + V(x)u(t, x), \qquad x \in \mathbb{R}, t \in \mathbb{R}^+, \tag{1}$$

in terms of a Wiener integral of the form

$$u(t, x) = \mathbb{E}\left[u(0, x + W(t))e^{\int_0^t V(x+W(s))\,\mathrm{d}s}\right],$$

where $W = (W(t))_{t\geq 0}$ denotes a one-dimensional Wiener process.

In fact, the connection between heat equation and Wiener process is just a particular case of a general theory connecting Markov processes with parabolic equations associated with second-order elliptic operators (see [21,25]). In the general, d-dimensional case, we are given a Lipschitz map $\sigma : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d)$ from $\mathbb{R}^d$ to the $d \times d$

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matrices, a Lipschitz vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ and a d-dimensional Wiener process $W = (W(t))_{t\geq 0}$. The solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}Tr[\sigma(x)\sigma^*(x)\nabla^2 u(t,x)] + \langle b(x), \nabla u(t,x)\rangle + V(x)u(t,x), \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

is related to the solution $X^x = \{X^x(t)\}_{t\geq 0}$ of the stochastic differential equation

$$\begin{cases} dX^x(t) = b(X^x(t))\,\mathrm{d}t + \sigma(X^x(t))\,\mathrm{d}W(t), \\ X^x(0) = x, \qquad x \in \mathbb{R}^d \end{cases} \tag{3}$$

by the probabilistic representation formula:

$$u(t,x) = \mathbb{E}\left[u(0, X^x(t))e^{\int_0^t V(X^x(s))\,\mathrm{d}s}\right], \quad t \geq 0, x \in \mathbb{R}^d. \tag{4}$$

One possible extension of the heat equation occurs if we replace the right-hand side of (1) by a spatial fractional derivative operator ∂_x^α, namely by a Fourier operator with symbol $\psi(y) = (-iy)^\alpha$, which leads to the equation

$$\partial_t u(t,x) = -\partial_x^\alpha u(t,x), \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,\ \alpha \in (0,1). \tag{5}$$

Fractional powers of operators have been introduced in [17,24] where the authors considered fractional powers of the Laplace operator. For a closed linear operator A, the fractional operator $(-A)^\alpha$ has been investigated by many researchers, and the reader can consult the works [8,30,39,40,66], for example. Equation (5) is associated with a Lévy process $\{H^\alpha(t)\}_{t\geq 0}$ called α-*stable subordinator* for $\alpha \in (0,1)$ (see [14] and Appendix A). In particular, the Laplace transform of $H^\alpha(t)$ has the form

$$\mathbb{E}[e^{-\lambda H^\alpha(t)}] = e^{-t\lambda^\alpha}, \qquad \lambda \in \mathbb{R}^+;$$

hence, the solution of the Cauchy problem associated with Eq. (5) is given by

$$u(t,x) = \mathbb{E}\left[u(0, x - H^\alpha(t))\right]. \tag{6}$$

The generalization of these results to different types of PDEs which do not satisfy the maximum principle is in general not possible [20]. In particular, a probabilistic representation of the form (4) or (6), giving the solution in terms of the expectation with respect to a *real* stochastic process with independent increments, cannot be proved in case the second-order elliptic operator on the right-hand side of Eq. (2) is replaced by a differential operator of order $N > 2$, obtaining a high-order heat-type equation of the form

$$\partial_t u(t,x) = \frac{\beta}{N!}\partial_x^N u(t,x), \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{7}$$

where β is a real constant satisfying some conditions on the sign, while the $\frac{1}{N!}$ term is the analog of the factor $\frac{1}{2}$ for the heat equation. In fact, this no-go result was stated originally by Krylov [42] in the case where $N = 4$ and is related to the non-positivity of the solution $g \equiv g(t, x)$ of the problem

$$\begin{cases} \partial_t g = \dfrac{1}{N!}\partial_x^N g \\ g(x, 0) = \delta(x), \quad x \in \mathbb{R} \end{cases} \tag{8}$$

as well as the rather restricting conditions for the generalization of Kolmogorov existence theorem to the limit of a projective system of either *signed* or *complex* measures (see [68] for this result and [1] for a discussion of its implication in the construction of a probabilistic representation for the solution of high-order PDEs).

The problem of a probabilistic representation for the Cauchy problem associated with Eq. (7), namely *a generalized Feynman–Kac formula*, is extensively studied and different approaches have been proposed, in particular in the case $N = 4$. One of the first approaches was introduced by Krylov [42] and continued by Hochberg [31], who introduced a stochastic pseudo-process whose transition probability function, defined as the solution of (8), is not positive definite. The generalized Feynman–Kac formula is constructed in terms of the expectation with respect to a signed measure on $\mathbb{R}^{[0,t]}$ with infinite total variation. For this reason, the integral on $\mathbb{R}^{[0,t]}$ is not defined in Lebesgue sense, but is meant as limit of finite-dimensional cylindrical approximations [10]. It is worth mentioning the work by Levin and Lyons [45] on rough paths, conjecturing that the signed measure (with infinite total variation) associated with the pseudo-process could exist on the quotient space of equivalence classes of paths corresponding to different parametrizations of the same path. Properties of the pseudo-process $X(t)$ associated with the signed measure $\mathbb{P}$, corresponding to the fundamental solution of (8) via the identity $\mathbb{P}_x(X(t) \in \mathrm{d}y) = p(t, x, \mathrm{d}y)$, were studied by several authors, see, e.g., [31,32,43,54–57]. It should be noticed that, in the case $N = 4$, paths of $X(t)$ are not continuous.

A different approach was proposed by Funaki [26] and continued by Burdzy and Mądrecki [18], based on the construction of a complex-valued stochastic process with dependent increments, obtained by a certain composition of two independent Brownian motions.

Recently, in [15] a new approach has been proposed. Starting from the weak convergence of the scaled random walk on the real line $S_n(t) := \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor}\xi_j$ to the Wiener process $B(t)$ (ξ_j for $j = 1, \dots, n$ being independent identically distributed Bernoulli random variables such that $\mathbb{P}(\xi_j = 1) = \mathbb{P}(\xi_j = -1) = 1/2$), the solution of the heat equation can be written as the limit $u(t, x) = \lim_{n\to+\infty}\mathbb{E}[u(0, x + S_n(t))]$. This formula can be generalized to the case of Eq.(7) with $N \in \mathbb{N}$ and $N > 2$, by constructing a sequence of complex random walks $\{W_n^{N,\beta}(t)\}_{n\in\mathbb{N}}$ as $W_n^{N,\beta}(t) := \frac{1}{n^{1/N}}\sum_{j=1}^{\lfloor nt\rfloor}\xi_j$, where ξ_j for $j = 1, \dots, n$ is a sequence of independent identically distributed complex random variables uniformly distributed on the set of Nth roots of β. In fact, if $N > 2$, the particular scaling exponent $1/N$ appearing in the definition of $W_n^{N,\beta}(t)$ does not allow the weak convergence of the sequence of random variables $W_n^{N,\beta}(t)$. However, for a suitable class of analytic initial data $u_0 : \mathbb{C} \to \mathbb{C}$, the limit of the expectation, namely $\lim_{n\to\infty} \mathbb{E}\left[u_0(x + W_n^{N,\beta}(t))\right]$, exists and provides a probabilistic representation for the solution of Eq. (7). Properties of the random walks $W^{N,\beta}$ are further studied in [16], where a kind of Itô calculus is introduced, by the construction of the Itô integral and an Itô formula for the limit of these processes. A similar approach to pseudo-processes was introduced by Lachal [44] for N even. In that paper, the ξ_j's take values in the discrete set $\{-N/2, -N/2+1, \dots, N/2-1, N/2\}$ with (positive or negative) real pseudo-probabilities $\mathbb{P}(\xi = k) = \delta_{k=0} + (-1)^{k-1}\binom{N}{k+N/2}$. He proves that with the same scaling exponent as ours, his sequence of pseudorandom walks converges to the pseudo-process associated with the signed measure $\mathbb{P}$ introduced before.

An extension of Eq. (8) to the case of higher-order space fractional derivatives, by replacing the order $N \in \mathbb{N}$ with the product $N\alpha$, with $N \in \mathbb{N}$ and $\alpha \in (0, 1)$, has been obtained in [60]. The authors define a sequence of pseudorandom walks, converging weakly to pseudo-processes stopped at stable subordinators. The fundamental solution of higher-order space fractional heat-type equations is obtained as the limit of the (signed) laws of the pseudorandom walks, which are signed measures.

A related problem is the study of time fractional equations of the form

$$\begin{aligned} \boldsymbol{D}_t^{\alpha} u(t,x) =& Au(t,x), \\ u(0,x) =& f(x), \end{aligned} \tag{9}$$

where $A = \frac{\beta}{N!}\partial_x^N$ and the time fractional derivative $\boldsymbol{D}_t^{\alpha}$ must be understood in the Caputo sense.

The fractional diffusion equations are related to the so-called fractional and anomalous diffusions, that is, diffusions in non-homogeneous media with random fractal structures; see, for instance, [47]. The term fractional is due to the replacement of standard derivatives with respect to time t with fractional derivatives and the corresponding equations describe delayed diffusions. However, we do not care about the geometrical structure of the medium, and therefore, our meaning of fractional diffusions is far from the definition of fractional diffusions introduced in [7]. Anomalous diffusion occurs when the mean square displacement (or time-dependent variance) is stretched by some index, in other words proportional to t^{α}, for instance. In the literature, equation (9) is used as a mathematical model of a wide range of important physical phenomena, usually named *sub-* or *super-diffusions*, for instance, in microelectronics (dielectrics and semiconductors), polymers, transport phenomena in complex systems and anomalous heat conduction in porous glasses and random media (see, for instance, [27,28,50,62]).

Fractional diffusion equations as (9) where $N = 2$ have been investigated by several researchers. In [37,53,67], the authors established the mathematical foundations of fractional diffusions. In [63,67] and later in [23,29], the authors studied the solutions to the heat-type fractional diffusion equation and the corresponding representation of the solutions in terms of Fox's functions. The explicit representation of the solutions by means of stable densities has been studied in [11,58] and, in the case $\alpha = 1/2^n$, Orsingher [58] proved the connection of this solution with the distribution of n-iterated Brownian motion.

For a general operator A acting in space, several results can also be listed. Nigmatullin [53] gave a physical interpretation when A is the generator of a Markov process. Zaslavsky [69] introduced the fractional kinetic equation for Hamiltonian chaos. The problem concerning an infinitely divisible generator A on a finite-dimensional space has been investigated in [4]. In general, a large class of fractional diffusion equations are solved by time-changed stochastic processes. We usually refer to such processes as stochastic solutions to the driving equations. Stochastic solutions to fractional diffusion equations can be realized through time change by inverse stable subordinators; see, for example, [46,58]. Indeed, for a guiding process $X(t)$ with generator A we have that $X(L^\alpha(t))$ is governed by $\partial_t^\alpha u = Au$ where the process $L^\alpha(t), t > 0$ is an inverse or hitting time process to an α-stable subordinator. The time fractional derivative comes from the fact that $X(L^\alpha(t))$ can be viewed as a scaling limit of continuous time random walks where the iid jumps are separated by iid power-law waiting times (see [48,49]). The interested reader can find a short survey on these results in [52]. Results on subordination principles for fractional evolution equations can be found in [9,17].

Besides the interest in studying fractional equations, many researchers have concentrated their efforts toward the study of the higher-order counterpart (13) of fractional equations; see, for example, [2,5,19,22,35,51]. When the underling operator generates a strongly continuous semigroup, the time-changed process can be considered in order to study the fractional diffusion equations and also, the higher-order equation with a non-homogeneous term involving higher-order powers of the driving operator. The reader can consult Keyantuo and Lizama [35] and the references therein.

The first aim of the present work is the generalization of the construction in [15] to the case of higher-order fractional derivatives of order $N\alpha$, with $N \in \mathbb{N}$ and $\alpha \in (0, 1)$. After a couple of sections where we introduce some preliminary results, mainly taken from [15] and [16], in Sect. 4 we provide a probabilistic representation of the solution to a family of equations of the form

$$\partial_t u = -Au, \tag{10}$$

where $A : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Fourier integral operator with symbol $\Psi : \mathbb{R} \to \mathbb{C}$ either of the form $\Psi(y) = c(iy)^{N\alpha}$ or $\Psi(y) = c'|y|^{N\alpha}$, with $c, c' \in \mathbb{C}$ suitable constants.

As opposite to [60], in our approach the solution of the equation is given in terms of the limit of expectations with respect to the probability laws of rather simple jump processes in the complex plane, without the need to introduce signed probabilities.

By subordinating the sequence of complex random walks $W_n^{N,\beta}(t)$ associated with the Nth-order Eq. (7) with a sequence of processes $\{S_m^\alpha(t)\}_{m\in\mathbb{N}}$ converging weakly as $m \to +\infty$ to the α-stable process $H^\alpha(t)$, a sequence $\{X_{n,m}(t)\}_{(n,m)\in\mathbb{N}^2}$ of jump processes on the complex plane is defined as $X_{n,m}(t) := W_n^{N,\beta}(S_m^\alpha(t))$. It converges formally to an $N\alpha$-stable process in the sense that

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[e^{iyX_{n,m}(t)}\right]=e^{-t\left((-1)^{N+1}i^N\frac{\beta}{N!}y^N\right)^\alpha},\qquad y\in\mathbb{R}.$$

This result allows the representation of the solution to (10) with $\widehat{Au}(y) := \left(\frac{(-1)^{N+1}i^N\beta y^N}{N!}\right)^\alpha \hat{u}(y)$ (the symbol $\hat{}$ denoting the Fourier transform) as the limit

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[u_0\left(x+W_n^{N,\beta}(S_m^\alpha(t))\right)\right],\qquad t\in\mathbb{R}^+, x\in\mathbb{R}, \tag{11}$$

for a suitable class of analytic initial data u_0. Moreover, we show that in the case the symbol of the operator A has the form $\Psi(y)=|y|^{N\alpha}$, the solution of (10) can still be given by a formula

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[u_0\left(x+X_{n,m}(t)+\tilde{X}_{n,m}(t)\right)\right],\qquad t\in\mathbb{R}^+, x\in\mathbb{R},$$

where $X_{n,m}(t)$ and $\tilde{X}_{n,m}(t)$ are two independent copies of the process $X_{n,m}(t) = W_n^{N,\beta}(S_m^\alpha(t))$ and $\tilde{X}_{n,m}(t)=W_n^{N,\beta'}(S_m^\alpha(t))$, constructed, respectively, by setting $\beta = N!$ and $\beta'=-N!$.

In Sect. 5, we also consider time fractional equations of the form (9) and we prove that the solution of the initial value problem, for a suitable class of initial data f, is given by

$$u(t,x)=\lim_{n\to+\infty}\mathbb{E}\left[f(x+W_n^{N,\beta}(L^\alpha(t)))\right], \tag{12}$$

where $L^\alpha(t)$ is the inverse of the subordinator $H^\alpha(t)$.

Furthermore, in the case where $\alpha=M^{-1}$ for some $M\in\mathbb{N}$, $M>1$, we prove that problem (9) is equivalent to the diffusion equation with non-local forcing term of the form

$$\begin{aligned}\partial_t u(t,x) &= A^{1/\alpha}u(t,x)+\sum_{k=1}^{1/\alpha-1}\frac{1}{\Gamma(\alpha k)}t^{\alpha k-1}A^k f(x),\\ u(0,x)&=f(x),\end{aligned} \tag{13}$$

in the sense that both problems share the same solution (12).

2. A sequence of random walks on the complex plane

The present section is devoted to the construction of a sequence of random walks in the complex plane whose limit can be interpreted in a very weak sense (see Theorem 1) as an N-stable stochastic process, with $N\in\mathbb{N}$.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Let β be a complex number and $N\geq 2$ a given integer.

Let $R(N) = \{e^{2i\pi k/N},\ k = 0, 1, \dots, N-1\}$ denote the set of the N-roots of the unity and let us consider the complex random variable $\xi_{N,\beta}$ that has uniform distribution on the set $\beta^{1/N} R(N)$, namely for any function $f : \mathbb{C} \to \mathbb{C}$:

$$\mathbb{E}[f(\xi_{N,\beta})] = \frac{1}{N} \sum_{k=0}^{N-1} f(\beta^{1/N} e^{2i\pi k/N}). \tag{14}$$

The random variable $\xi_{N,\beta}$ has some interesting properties (see [15] for detailed calculations). Indeed it admits (complex) moments of any order:

$$\mathbb{E}[\xi_{N,\beta}^m] = \begin{cases} \beta^{m/N}, & m = nN,\ n \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{15}$$

In particular, its characteristic function is

$$\psi_{\xi_{N,\beta}}(\lambda) = \frac{1}{N} \sum_{k=0}^{N-1} \exp(i\beta^{1/N} \lambda e^{2i\pi k/N}).$$

Further, we may compute the absolute moments of $\xi_{N,\beta}$ obtaining

$$\mathbb{E}[|\xi_{N,\beta}|^m] = |\beta|^{m/N}.$$

Equation (15) is the starting point for the construction of a particular sequence of random variables on the complex plane which converges (in a sense that will be explained soon) to a stable random variable of order $N \geq 2$.

Let $\{\xi_j,\ j \in \mathbb{N}\}$ be a sequence of iid random variables having uniform distribution on the set $\beta^{1/N} R(N)$ as in (14). Let $S(N, \beta)_n$ be the random walk defined by the $\{\xi_j\}$, i.e.,

$$S(N, \beta)_n = \sum_{j=1}^{n} \xi_j.$$

Interesting properties of the complex random walk $S(N, \beta)_n$ in the case $\beta = 1$ have been investigated in [15].

Consider the case where $N = 3$ and the walk $S(3, 1)_n$ occurs on the regular lattice generated by the vectors $\{(1, 0),\ (-\frac{1}{2}, \frac{\sqrt{3}}{2}),\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$, considered as a *directed* graph. Therefore, the motion is 3-periodic, and a return to the origin only happens if the same number of steps is made in every direction. Therefore, we compute

$$\mathbb{P}(S(3, 1)_{3m} = 0) = \frac{(3m)!}{(m!)^3} \frac{1}{3^{3m}},$$

and Stirling's formula implies $\mathbb{P}(S(3, 1)_{3m} = 0) \sim \frac{1}{2\pi m}$; hence, the expected number of returns to the origin is

$$\sum_{m=1}^{\infty} \mathbb{P}(S(3, 1)_{3m} = 0) \sim \sum_{m=1}^{\infty} \frac{1}{m} = +\infty,$$

and the process is recurrent.

The case $N = 4$ corresponds to the standard, two-dimensional random walk; hence, the motion is 2-periodic on the lattice $\mathbb{Z}^2$ (this time considered as an *undirected* graph). Finally, the motion is recurrent.

In the case where $N = 5$, the process is transient. Indeed, in this case, the motion is again 5-periodic and the only way to return to the origin is taking the same number of steps in each direction. Hence, again by an application of Stirling's formula,

$$\mathbb{P}(S(5, 1)_{5m} = 0) = \frac{(5m)!}{(m!)^5} \frac{1}{5^{5m}} \sim \frac{\sqrt{5}}{(2\pi m)^2},$$

and the expected number of returns is finite:

$$\sum_{m=1}^{\infty} \mathbb{P}(S(5, 1)_{5m} = 0) \sim \sum_{m=1}^{\infty} \frac{\sqrt{5}}{(2\pi m)^2} < \infty.$$

However, for $N > 5$ the following result holds [15].

Proposition 1. *Let $N \geq 5$. The process $\{S(N, 1)_n\}$ is neighborhood-recurrent, i.e., for every x in the lattice generated by the basis $\{\beta^{1/N} e^{2\pi i k/N},\ k = 0, 1, \ldots, N-1\}$ it holds*

$$\mathbb{P}(|S(N, 1)_n - x| \leq \varepsilon \ \ \textit{infinitely often}) = 1, \qquad \forall \epsilon > 0.$$

By the classical central limit theorem, the sequence $\frac{1}{n^{1/2}} S(N, \beta)_n$ converges to a Gaussian random variable for any $N \in \mathbb{N}$ and $\beta \in \mathbb{C}$. However, if we consider a normalized random walk $\tilde{S}(N, \beta)_n$ defined as

$$\tilde{S}(N, \beta)_n = \frac{1}{n^{1/N}} S(N, \beta)_n,$$

then we have convergence to a stable distribution of order N in the sense that (see [15, Theorem 2])

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda \tilde{S}(N, \beta)_n)] = \exp\left(\frac{i^N \beta}{N!} \lambda^N\right). \tag{16}$$

It is important to remark that for $N > 2$ the sequence $\tilde{S}(N, \beta)_n$ cannot converge in distribution because of the scaling exponent $1/N$. In fact, for $N > 2$, the function $\exp(cx^N)$ is not a well-defined characteristic function. Equation (16) states that even if the distributions of $\tilde{S}(N, \beta)_n$ cannot converge weakly to a measure, the integral of suitable functions (the exponentials) with respect to these measures has a well-defined limit as $n \to \infty$.

It is possible to extend the definition of $\tilde{S}(N, \beta)_n$ to a continuous time process and construct a sequence of random walks $W_n^{N,\beta}(t)$ on the complex plane such that $\tilde{S}(N, \beta)_n = W_n^{N,\beta}(1)$. Given a sequence $\{\xi_j\}$ of independent copies of the random variable $\xi_{N,\beta}$ defined in (14), let us consider for $t > 0$ the process $W_n^{N,\beta}(t)$ defined by

$$W_n^{N,\beta}(0) = 0;$$
$$W_n^{N,\beta}(t) = \frac{1}{n^{1/N}} \sum_{j=1}^{\lfloor nt \rfloor} \xi_j = \frac{1}{n^{1/N}} S(N,\beta)_{\lfloor nt \rfloor}. \qquad (17)$$

The process $W_n^{N,\beta}$ has some interesting properties. The following lemma shows the particular behavior of the complex moments.

Lemma 1. *For $k \in \mathbb{N}$ and $t \in \mathbb{R}^+$, the k-moment of $W_n^{N,\beta}(t)$ satisfies*

$$\mathbb{E}[(W_n^{N,\beta}(t))^k] = \begin{cases} \left(\frac{\beta t}{N!}\right)^{k/N} \frac{k!}{(k/N)!} \mathbf{1}_{[0,\lfloor nt \rfloor]}(k/N) + R(n,k;t), & k = hN,\ h \in \mathbb{N}, \\ 0, & \textit{otherwise}, \end{cases}$$

($\mathbf{1}_{[0,\lfloor nt \rfloor]}$ being the indicator function of the interval $[0, \lfloor nt \rfloor]$).

For $k = 0$ and $k = N$, i.e., for $h = 0$ and $h = 1$, the remainder term vanishes. For $k = hN$, $h \in \mathbb{N}$, $h \geq 2$, the remainder term $R(n,k;t)$ satisfies the following estimate:

$$|R(n,hN;t)| \leq \frac{|\beta|^h t^{h-1}(h^2+h)}{2n} \frac{(hN)!}{h!(N!)^h} + \frac{|\beta|^h t^{h-1}}{n} \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}.$$

Proof. Let $W_n^{N,\beta}(t) = \frac{1}{n^{1/N}} \sum_j^{\lfloor nt \rfloor} \xi_j$ and ψ_n be its characteristic function, namely:

$$\psi_n(\lambda) := \mathbb{E}[e^{i\lambda W_n^{N,\beta}(t)}].$$

We have that

$$\mathbb{E}[(W_n^{N,\beta}(t))^k] = (-i)^k \frac{d^k \psi_n}{d\lambda^k}(0),$$

where ψ_n is equal to

$$\psi_n(\lambda) = \left(\mathbb{E}\left[\exp\left(\frac{1}{n^{1/N}} i\lambda\xi\right)\right]\right)^{\lfloor nt \rfloor} = \left(\psi_\xi\left(\frac{\lambda}{n^{1/N}}\right)\right)^{\lfloor nt \rfloor},$$

with ψ_ξ being the characteristic function of ξ. By Faá di Bruno's formula [41]

$$\frac{d^k \psi_n}{d\lambda^k}(\lambda) = \sum_{\pi \in \Pi} C(|\pi|,\lambda) \prod_{B \in \pi} \left(\frac{\psi_\xi^{(|B|)}(\lambda/n^{1/N})}{n^{|B|/N}}\right), \qquad (18)$$

where π runs over the set Π of all partitions of the set $\{1,\ldots,k\}$, $B \in \pi$ means that the variable B runs through the list of blocks of the partition π, $|\pi|$ denotes the number of blocks of the partition π and $|B|$ is the cardinality of a set B, while the function $C : \mathbb{N} \times \mathbb{R} \to \mathbb{C}$ is equal to

$$C(j,\lambda) = \begin{cases} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - j)!} \left(\psi_\xi(\lambda/n^{1/N})\right)^{\lfloor nt \rfloor - j}, & \lfloor nt \rfloor \geq j \\ 0, & \text{otherwise}. \end{cases}$$

Formula (18) can be written in the equivalent form:

$$\frac{d^k\psi_n}{d\lambda^k}(\lambda) = \sum \frac{k!}{m_1!m_2!\cdots m_k!}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-(m_1+m_2+\cdots+m_k))!} \times \left(\psi_\xi(\lambda/n^{1/N})\right)^{\lfloor nt\rfloor-(m_1+m_2+\cdots+m_k)} \Pi_{j=1}^k\left(\frac{\psi_\xi^{(j)}(\lambda/n^{1/N})}{j!n^{j/N}}\right)^{m_j}, \tag{19}$$

where the sum is over the k-tuple of nonnegative integers $(m_1, m_2, \ldots, m_k)$ such that $m_1+2m_2+\cdots+km_k = k$ and $m_1+m_2+\cdots+m_k \le \lfloor nt\rfloor$. In particular, we have:

$$\frac{d^k\psi_n}{d\lambda^k}(0) = \sum_{\pi\in\Pi}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-|\pi|)!}\Pi_{B\in\pi}\left(\frac{\psi_\xi^{(|B|)}(0)}{n^{|B|/N}}\right), \tag{20}$$

where the first sum runs over the partitions π such that $|\pi|\le\lfloor nt\rfloor$ or equivalently

$$\frac{d^k\psi_n}{d\lambda^k}(0) = \sum \frac{k!}{m_1!m_2!\cdots m_k!}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-(m_1+m_2+\cdots+m_k))!}\Pi_{j=1}^k\left(\frac{\psi_\xi^{(j)}(0)}{j!n^{j/N}}\right)^{m_j}. \tag{21}$$

Since $\psi_\xi^{(j)}(0) = (i)^j\mathbb{E}[\xi^j]$, and $\mathbb{E}[\xi^j]\neq 0$ iff $j = mN$, with $m\in\mathbb{N}$, then product $\Pi_{j=1}^k\left(\frac{\psi_\xi^{(j)}(0)}{j!n^{j/N}}\right)^{m_j}$ is non-vanishing iff $m_j = 0$ for $j\neq lN$ and $k = Nm_N+2Nm_{2N}+\cdots$, i.e., if k is a multiple of N. Analogously in the sum appearing in formula (20), the only terms giving a non-vanishing contribution correspond to those partitions π having blocks B with a number of elements which is a multiple of N, giving, for $k = hN$:

$$\frac{d^{hN}\psi_n}{d\lambda^{hN}}(0) = i^{hN}\frac{\beta^h}{n^h}\sum_{\pi\in\Pi}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-|\pi|)!}, \tag{22}$$

where again the sum runs over the partitions π such that $|\pi|\le\lfloor nt\rfloor$. Equivalently:

$$\begin{aligned}\frac{d^{hN}\psi_n}{d\lambda^{hN}}(0) &= \sum\frac{(hN)!}{(m_N)!(m_{2N})!\cdots(m_{hN})!}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-(m_N+m_{2N}+\cdots+m_{hN}))!}\Pi_{l=1}^h \times\left(\frac{\psi_\xi^{(lN)}(0)}{(lN)!n^l}\right)^{m_{lN}},\\ &= \sum\frac{(hN)!}{(m_N)!(m_{2N})!\cdots(m_{hN})!}\frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-(m_N+m_{2N}+\cdots+m_{hN}))!}\Pi_{l=1}^h \times\left(\frac{i^{lN}\beta^l}{(lN)!n^l}\right)^{m_{lN}},\end{aligned}$$

$$= \frac{i^{hN}\beta^h}{n^h}\sum \frac{(hN)!}{(m_N)!(m_{2N})!\cdots(m_{hN})!} \times \frac{\lfloor nt\rfloor!}{(\lfloor nt\rfloor-(m_N+m_{2N}+\cdots+m_{hN}))!}\Pi_{l=1}^h \frac{1}{((lN)!)^{m_{lN}}},$$

where the sum is over the h-tuple of nonnegative integers $(m_N, m_{2N}, \dots, m_{hN})$ such that $m_N + 2m_{2N} + \cdots + hm_{hN} = h$ and $m_N + m_{2N} + \cdots + m_{hN} \le \lfloor nt\rfloor$. Hence, we have

$$\mathbb{E}[(W_n^{N,\beta}(t))^{hN}] = \beta^h \sum \frac{(hN)!}{(m_N)!(N!)^{m_N}(m_{2N})!(2N)!^{m_{2N}}\cdots(m_{hN})!(hN)!^{m_{hN}}} \times \frac{\lfloor nt\rfloor!}{n^h(\lfloor nt\rfloor-(m_N+m_{2N}+\cdots+m_{hN}))!}.$$

When $n\to\infty$, the leading term in the previous sum is the one corresponding to $m_N = h$ (hence $m_{2N} = \cdots = m_{hN} = 0$), which is equal to

$$\beta^h \frac{(hN)!}{(m_N)!(N!)^h}\frac{\lfloor nt\rfloor!}{n^h(\lfloor nt\rfloor-h)!} = \beta^h t^h \frac{(hN)!}{h!(N!)^h} + \beta^h\frac{(hN)!}{h!(N!)^h}\left(\frac{\lfloor nt\rfloor!}{n^h(\lfloor nt\rfloor-h)!} - t^h\right).$$

In the case where $\lfloor nt\rfloor < h$, this term does not appear in the sum and we can set it equal to 0. In the case where $\lfloor nt\rfloor \ge h$, we can estimate the quantity inside brackets as:

$$\begin{aligned}\left|\frac{\lfloor nt\rfloor!}{n^h(\lfloor nt\rfloor-h)!} - t^h\right| &= \frac{1}{n^h}\left|-(nt)^h + \Pi_{j=0}^{h-1}(\lfloor nt\rfloor - j)\right| \\ &= \frac{1}{n^h}\left|\Pi_{j=0}^{h-1}\big((\lfloor nt\rfloor - j) + (\{nt\}+j)\big) - \Pi_{j=0}^{h-1}(\lfloor nt\rfloor - j)\right| \\ &\le \frac{1}{n^h}\sum_{j=0}^{h-1}(\{nt\}+j)\Pi_{k\ne j} nt \\ &= \frac{(nt)^{h-1}}{n^h}\sum_{j=0}^{h-1}(\{nt\}+j) \le \frac{(nt)^{h-1}}{n^h}\sum_{j=0}^{h-1}(1+j) = \frac{t^{h-1}(h^2+h)}{2n},\end{aligned}$$

where in the second line we have used that if $a_j, b_j \in \mathbb{R}$, with $a_j, b_j \ge 0$ for all $j = 0, \dots, m$, then (see Appendix)

$$\Pi_{j=0}^m (a_j+b_j) - \Pi_{j=0}^m a_j \le \sum_{j=0}^m b_j \Pi_{k\ne j}(a_k+b_k).$$

Hence,

$$\begin{aligned}|R_1(n,h;t)| &= \left|\beta^h \frac{(hN)!}{(m_N)!(N!)^h}\frac{\lfloor nt\rfloor!}{n^h(\lfloor nt\rfloor-h)!} - \beta^h t^h\frac{(hN)!}{h!(N!)^h}\right| \\ &\le \frac{|\beta|^h t^{h-1}(h^2+h)}{2n}\frac{(hN)!}{h!(N!)^h}.\end{aligned}$$

By using formula (22), the remaining terms in the sum (corresponding to the h-tuple $(m_N, m_{2N}, \ldots, m_{hN})$ with $m_N < h$) are bounded by

$$R_2(n, h; t) = \frac{\beta^h}{n^h} \sum_{\pi \in \Pi} \frac{\lfloor nt \rfloor!}{(\lfloor nt \rfloor - |\pi|)!} - \beta^h \frac{(hN)!}{(m_N)!(N!)^h} \frac{\lfloor nt \rfloor!}{n^h(\lfloor nt \rfloor - h)!}$$
$$\leq \frac{\beta^h}{n^h} \sum_{\pi \in \Pi} \lfloor nt \rfloor^{h-1} = \frac{\beta^h}{n} t^{h-1} B_{hN},$$

where B_{hN} is the Bell number, i.e., the number of partitions of the set $\{1, \ldots, hN\}$. In particular, for any $h \in \mathbb{N}$ (see [13]) $B_{hN} < \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}$; hence,

$$|R_2(n, h; t)| \leq \frac{|\beta|^h t^{h-1}}{n} \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}.$$

□

A direct consequence of Lemma 1 is the following theorem, which generalizes formula (16) to the sequence of random walks $W_n^{N,\beta}$.

Theorem 1. *For any $\beta \in \mathbb{C}$ and $N \in \mathbb{N}$, $N \geq 2$, the sequence of random walks $W_n^{N,\beta}$ converges weakly to a N-stable process in the sense that for any $t \geq 0$ and $\lambda \in \mathbb{R}$ the following holds:*

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda W_n^{N,\beta}(t))] = \exp\left(\frac{i^N \beta t}{N!}\lambda^N\right). \tag{23}$$

Proof.

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda W_n^{N,\beta}(t))] = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} i^k \lambda^k \mathbb{E}[(W_n^{N,\beta}(t))^k]$$
$$= \lim_{n\to\infty} \sum_{h=0}^{\lfloor nt \rfloor} \frac{i^{hN}\lambda^{hN}}{(hN)!} \left(\frac{\beta t}{N!}\right)^h \frac{(hN)!}{h!}$$
$$+ \lim_{n\to\infty} \lim_{m\to\infty} \sum_{h=2}^{m} \frac{i^{hN}\lambda^{hN}}{(hN)!} R(n, hN; t)$$
$$= \exp\left(\frac{i^N \beta t}{N!}\lambda^N\right).$$

Indeed for $m, n \in \mathbb{N}$

$$\left|\sum_{h=2}^{m} \frac{i^{hN}\lambda^{hN}}{(hN)!} R(n, hN; t)\right| \leq \frac{C}{n}, \qquad C := \sum_{h=2}^{\infty} t^{h-1} \frac{|\beta|^h |\lambda|^{hN}}{(hN)!}$$
$$\left(\frac{(h^2+h)(hN)!}{2h!(N!)^h} + \left(\frac{0.792hN}{\log(hN+1)}\right)^{hN}\right) < \infty.$$

□

Theorem 2. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire analytic function with the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, such that the coefficients $\{a_k\}$ satisfy the following condition:*

$$\sum_{h=2}^{\infty} |a_{hN}| c^h \left(\frac{hN}{\log(hN+1)} \right)^{hN} < \infty \qquad \forall c \in \mathbb{R}^+. \tag{24}$$

Then

$$\lim_{n\to\infty} \mathbb{E}[f(W_n^{N,\beta}(t))] = \sum_{h=0}^{\infty} a_{hN} \frac{(hN)!}{h!} \left(\frac{\beta t}{N!} \right)^h = \sum_{h=0}^{\infty} \frac{f^{(hN)}(0)}{h!} \left(\frac{\beta t}{N!} \right)^h .$$

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[f(W_n^{N,\beta}(t))] &= \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k \mathbb{E}[(W_n^{N,\beta}(t))^k] \\
&= \lim_{n\to\infty} \sum_{h=0}^{\lfloor nt \rfloor} a_{hN} \left(\frac{\beta t}{N!} \right)^h \frac{(hN)!}{h!} + \lim_{n\to\infty} \sum_{h=2}^{\infty} a_{hN} R(n, hN; t) \\
&= \sum_{h=0}^{\infty} a_{hN} \left(\frac{\beta t}{N!} \right)^h \frac{(hN)!}{h!}.
\end{aligned}$$

Indeed, by assumption (24), we have

$$\left| \sum_{h=2}^{\infty} a_{hN} R(n, hN, t) \right| \le \frac{C}{n},$$

$$C := \sum_{h=2}^{\infty} |a_{hN}| \left(\frac{|\beta|^h t^{h-1} (h^2+h)}{2} \frac{(hN)!}{h!(N!)^h} + |\beta|^h t^{h-1} \left(\frac{0.792 hN}{\log(hN+1)} \right)^{hN} \right) < \infty,$$

where the series on the right-hand side is convergent thanks to condition (24). □

Remark 1. Let us discuss further the assumption (24).

1. First, we provide the following simple, yet widely applicable, condition about the coefficients $\{a_k\}$:

   there exist $C_1, C_2 \in \mathbb{R}$ such that for all k the coefficients a_k satisfy

   $$\text{the inequality } |a_k| \le \frac{C_1 C_2^k}{k!}. \tag{25}$$

   Then, if condition (25) holds, the sequence of coefficients $\{a_k\}$ satisfies assumption (24).
2. Recall that an analytic function f is said to be of *exponential type* c if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $(a_k k!)^{1/k} \to c$ as $k \to \infty$.
3. If f is of exponential type, then it satisfies assumption (24), since (25) holds.

4. If $f : \mathbb{C} \to \mathbb{C}$ is the Fourier transform of a complex bounded variation measure μ on $\mathbb{R}$ with compact support, then f is of exponential type; hence, in particular, it satisfies assumption (24).
5. If $f : \mathbb{C} \to \mathbb{C}$ is the Fourier transform of a complex bounded variation measure μ on $\mathbb{R}$ with compact support, i.e.,

$$f(x) = \int_{\mathbb{R}} e^{ixy}\, \mathrm{d}\mu(y),$$

then for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}$ it holds

$$\lim_{n\to\infty} \mathbb{E}[f(x + W_n^{N,\beta}(t))] = \int e^{iyx} e^{i^n \beta t \frac{y^N}{N!}}\, \mathrm{d}\mu(y). \tag{26}$$

3. A sequence of subordinated processes

Given a positive integer $N \in \mathbb{N}$ with $N \geq 2$ and a constant $\alpha \in (0, 1)$, in the present section we construct by means of Bochner's subordination a sequence of jump processes on the complex plane converging weakly (in the sense of Theorem 4) to an $N\alpha$-stable process. Our aim is the derivation of the limit of subordinated processes $W_n^{N,\beta}(H^\alpha(t))$, where $W_n^{N,\beta}$ is the sequence of complex random walks defined in Sect. 2 and $H^\alpha(t)$, $\alpha \in (0, 1)$ and $t \geq 0$, is the α-stable subordinator.

The first step is the construction of a sequence $\{S_m^\alpha(t)\}_{m\in\mathbb{N}}$ of compound Poisson processes (with finite moments of any order) approximating the α-stable subordinator $H^\alpha(t)$ (see Appendix A for the definition of α-stable processes).

Let $S_m^\alpha(t)$ be defined by

$$S_m^\alpha(t) = \frac{1}{m} \sum_{j=0}^{X} Y_j, \tag{27}$$

where $X \sim \mathrm{Po}(\lambda t m^{2\alpha})$ is a Poisson random variable of parameter $\lambda\, t\, m^{2\alpha}$, with $\lambda = (\Gamma(1-\alpha))^{-1}$, and Y_j are independent identically distributed[1] copies of the random variable $Y_{[m]}$, with density $f_m(y) = c_m\, y^{-\alpha-1}\, \mathbf{1}_{\left(\frac{1}{m}, m^2\right)}(y)$, $c_m = \frac{\alpha}{m^\alpha(1-m^{-3\alpha})}$ (see Appendix B), X and the $\{Y_j\}$ being independent as well.

Theorem 3. *The sequence of random variables $\{S_m^\alpha(t)\}_m$ converges weakly to the α-stable subordinator $H^\alpha(t)$:*

$$\lim_{m\to\infty} \mathbb{E}\left[e^{iyS_m^\alpha(t)}\right] = \exp\left(-t(-i\, y)^\alpha\right), \quad y \in \mathbb{R}. \tag{28}$$

[1] For notational simplicity, we do not write explicitly the dependence of Y_j and X on the index m

Proof.

$$\mathbb{E}\left[e^{iyS_m^\alpha(t)}\right] = \mathbb{E}\left[\left(\mathbb{E}[e^{i\frac{y}{m}Y_j}]\right)^X\right] = \exp\left(-\lambda\, t\, m^{2\alpha}\left(1-\mathbb{E}\left[e^{i\frac{y}{m}Y_{[m]}}\right]\right)\right)$$
$$= \exp\left(-\lambda\, t\, m^{2\alpha}\, c_m \int_{1/m}^{m^2}\left(1-e^{i\frac{y}{m}z}\right) z^{-\alpha-1}\,\mathrm{d}z\right),$$

by means of a change of variable $x = z/m$ we get

$$\mathbb{E}\left[e^{iyS_m^\alpha(t)}\right] = \exp\left(-\lambda\, t\, m^{2\alpha}\, c_m \int_{1/m^2}^{m}\left(1-e^{iyx}\right) x^{-\alpha-1} m^{-\alpha-1+1}\,\mathrm{d}x\right)$$
$$= \exp\left(-\lambda\, t\, m^{\alpha}\, c_m \int_{1/m^2}^{m}\left(1-e^{iyx}\right) x^{-\alpha-1}\,\mathrm{d}x\right).$$

Since $\lim_{m\to\infty} m^\alpha c_m = \alpha$, we eventually obtain

$$\lim_{m\to\infty}\mathbb{E}\left[e^{iyS_m^\alpha(t)}\right] = \exp\left(-t\,\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty\left(1-e^{iyx}\right) x^{-\alpha-1}\,\mathrm{d}x\right)$$
$$= \exp\left(-t(-i\,y)^\alpha\right).$$

□

Remark 2. Formula (28) remains valid by replacing iy, for $y \in \mathbb{R}$, with a complex variable $z \in \mathbb{C}$ with $\mathrm{Re}(z) \le 0$

$$\lim_{m\to\infty}\mathbb{E}\left[e^{zS_m^\alpha(t)}\right] = \exp\left(-t\,\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty\left(1-e^{zx}\right) x^{-\alpha-1}\,\mathrm{d}x\right)$$
$$= \exp\left(-t(-z)^\alpha\right). \tag{29}$$

Lemma 2. *There exists a constant $C(\alpha) \in \mathbb{R}^+$ such that the moments of the compound Poisson process $S_m^\alpha(t)$ given by (27) satisfy the following estimate*

$$\mathbb{E}[(S_m^\alpha(t))^k] \le C(\alpha)^k t^k m^{k+2\alpha k-3\alpha}\left(\frac{k+1}{\log(k+2)}\right)^{k+1}, \tag{30}$$

for every $k \ge 1$ and for every $m \ge (\Gamma(1-\alpha)/t)^{1/2\alpha}$

Proof.

$$\mathbb{E}[(S_m^\alpha(t))^k] = \mathbb{E}\left[\frac{1}{m^k}\sum_{j_1=0}^{X}\cdots\sum_{j_k=0}^{X} Y_{j_1}\ldots Y_{j_k}\right],$$

so we can rearrange the expectations and the sums as follows

$$\mathbb{E}[(S_m^\alpha(t))^k] = \frac{1}{m^k}\sum_{x=0}^{\infty}\mathbb{P}(X=x)\left(\sum_{j_1,\ldots,j_k=0}^{x}\mathbb{E}\left[Y_{j_1}\ldots Y_{j_k}\right]\right).$$

By the estimate (B.11) (see Appendix B), we obtain that for any choice of indexes $j_1, \dots, j_k = 0, \dots, x$ the following inequality holds:

$$\mathbb{E}\left[Y_{j_1} \dots Y_{j_k}\right] \le c(\alpha)^k m^{2k-3\alpha},$$

where $c(\alpha) = 1 \vee \frac{\alpha}{1-\alpha}$; hence, by estimate (B.10):

$$\begin{aligned}
\mathbb{E}[(S_m^\alpha(t))^k] &\le c(\alpha)^k m^{k-3\alpha} \sum_{x=0}^{\infty} \mathbb{P}(X = x)(x+1)^k \\
&= c(\alpha)^k m^{k-3\alpha} \frac{\Gamma(1-\alpha)}{tm^{2\alpha}} \sum_{x=1}^{\infty} \mathbb{P}(X = x) x^{k+1} \\
&\le c(\alpha)^k m^{k-3\alpha} \frac{\Gamma(1-\alpha)}{tm^{2\alpha}} \left(1 \vee \left(\frac{tm^{2\alpha}}{\Gamma(1-\alpha)}\right)^{k+1}\right) B_{k+1} \\
&\le c(\alpha)^k m^{k-3\alpha} \frac{\Gamma(1-\alpha)}{tm^{2\alpha}} \left(1 \vee \left(\frac{tm^{2\alpha}}{\Gamma(1-\alpha)}\right)^{k+1}\right) \left(\frac{0.792(k+1)}{\log(k+2)}\right)^{k+1}.
\end{aligned}$$

In particular, for m sufficiently large, i.e., for $m \ge (\Gamma(1-\alpha)/t)^{1/2\alpha}$, the following holds

$$\mathbb{E}[(S_m^\alpha(t))^k] \le C(\alpha)^k t^k m^{k+2\alpha k-3\alpha} \left(\frac{k+1}{\log(k+2)}\right)^{k+1},$$

where $C(\alpha) = c(\alpha) 0.792/\Gamma(1-\alpha)$.

□

Let us now consider the sequence of random walks $\{W_n^{N,\beta}(t)\}_n$ described in Sect. 2, and for any couple $(n, m) \in \mathbb{N}^2$, let us consider the subordinated process $X_{n,m}(t) = W_n^{N,\beta}(S_m^\alpha(t))$. We can think of $X_{n,m}$ as a jump process in the complex plane, where a random number of jumps, uniformly distributed on the set $\frac{\beta^{1/N}}{n^{1/N}} R(N)$, occur, namely:

$$X_{n,m}(t) = \frac{1}{n^{1/N}} \sum_{j=1}^{\lfloor n S_m^\alpha(t) \rfloor} \xi_j, \tag{31}$$

where ξ_j are iid uniformly distributed on the set $\beta^{1/N} R(N)$ [see Eq. (14)].

The following theorem is the analog of Theorem 1 for processes that are driven by subordinators, and shows that $X_{n,m}(t)$ converges in a suitable sense to an $N\alpha$-stable process.

Theorem 4. *Let the parameters $N \in \mathbb{N}$ and $\beta \in \mathbb{C}$ be chosen in such a way that the following inequality is satisfied*

$$\mathrm{Re}((-i)^N \beta y^N) \le 0 \quad \forall y \in \mathbb{R}. \tag{32}$$

Then the following holds

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}] = e^{-t\left((-1)^{N+1}i^N y^N \frac{\beta}{N!}\right)^\alpha}.$$

Before proving Theorem 4, we give an alternative estimate of the difference between $\mathbb{E}[e^{-iyW_n^{N,\beta}(t)}]$ and its limit for $n \to \infty$.

Lemma 3. *Under the assumption* (32)*, there exists a constant* $C(y)$ *depending continuously on the parameter* y *such that the following estimate holds*

$$|\mathbb{E}[e^{-iyW_n^{N,\beta}(t)}] - e^{(i)^N\beta t y^N/N!}| = \frac{C(y)t}{n}e^{tC(y)} + \frac{1}{n}\frac{|\beta||y|^N}{N!}. \tag{33}$$

Proof. By definition of $W_n^{N,\beta}(t)$, $\mathbb{E}[e^{-iyW_n^{N,\beta}(t)}] = (\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor}$, where $\psi_{\xi_{N,\beta}}$ is the characteristic function of the complex random variable $\xi_{N,\beta}$ defined in (14). Hence,

$$\begin{aligned}
&|\mathbb{E}[e^{-iyW_n^{N,\beta}(t)}] - e^{(i)^N\beta t y^N/N!}| = |(\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor} \\
&- e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N} + e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N} - e^{(i)^N\beta t y^N/N!}| \\
&\le |(\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor} - e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N}| + |e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N} - e^{(i)^N\beta t y^N/N!}|.
\end{aligned} \tag{34}$$

The first term can be estimated as

$$\begin{aligned}
&|(\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor} - e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N}| \le |(\psi_{\xi_{N,\beta}}(y/n^{1/N})) \\
&- e^{\frac{(i)^N\beta}{N!}\frac{y^N}{n}}| \sum_{j=0}^{\lfloor nt\rfloor-1} |\psi_{\xi_{N,\beta}}(y/n^{1/N}))|^j.
\end{aligned}$$

By setting $r(n,\beta,y) := \psi_{\xi_{N,\beta}}(y/n^{1/N}) - e^{\frac{(i)^N\beta}{N!}\frac{y^N}{n}}$, the latter estimate takes the following form:

$$\begin{aligned}
|(\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor} - e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N}| &\le |r(n,\beta,y)| \sum_{j=0}^{\lfloor nt\rfloor-1} |e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N} + r(n,\beta,y)|^j \\
&\le |r(n,\beta,y)| \sum_{j=0}^{\lfloor nt\rfloor-1} (1+|r(n,\beta,y)|)^j \\
&\le |r(n,\beta,y)|\lfloor nt\rfloor(1+|r(n,\beta,y)|)^{\lfloor nt\rfloor}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|r(n,\beta,y)| &= |\psi_{\xi_{N,\beta}}(y/n^{1/N}) - e^{\frac{(i)^N\beta}{N!}\frac{y^N}{n}}| \\
&= |\frac{\psi^{(N)}_{\xi_{N,\beta}}(0)}{N!}\frac{y^N}{n} + \frac{\psi^{(2N)}_{\xi_{N,\beta}}(\tilde{z})}{(2N)!}\frac{y^{2N}}{n^2} - \frac{i^N y^N\beta}{N!n} - \frac{1}{2}\left(\frac{i^N y^N\beta}{N!n}\right)^2 e^{\frac{(i)^N\beta}{N!}z}|,
\end{aligned}$$

where $z, \tilde{z} \in \mathbb{R}$, with $z \in [0, \frac{y^N}{n}]$ and $\tilde{z} \in [0, y/n^{1/N}]$. By formula (15) and the boundedness of the continuous map $\psi^{(2N)}_{\xi_{N,\beta}}$ over the interval $[0, y]$, i.e., $|\psi^{(2N)}_{\xi_{N,\beta}}(\tilde{z})| \le M$ $\forall \tilde{z} \in [0, y]$, we obtain:

$$|r(n, \beta, y)| \le \frac{|\beta|^2|y|^{2N}}{n^2}\left(\frac{1}{(2N)!}|\psi^{(2N)}_{\xi_{N,\beta}}(\tilde{z})| + \frac{1}{2(N!)^2}\right)$$
$$\le \frac{|\beta|^2|y|^{2N}}{n^2}\left(\frac{M}{(2N)!} + \frac{1}{2(N!)^2}\right);$$

hence, there exists a constant $C(y)$ depending continuously on the parameter y such that $|r(n, \beta, y)| \le \frac{C(y)}{n^2}$. We then obtain

$$|(\psi_{\xi_{N,\beta}}(y/n^{1/N}))^{\lfloor nt\rfloor} - e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N}| \le \frac{C(y)}{n^2}\lfloor nt\rfloor\left(1 + \frac{C(y)}{n^2}\right)^{\lfloor nt\rfloor} \le \frac{C(y)t}{n}e^{tC(y)}.$$

Further, the second term in (34) can be estimated as:

$$|e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor}{n}y^N} - e^{(i)^N\beta t y^N/N!}| = |e^{\frac{(i)^N\beta}{N!}\frac{\lfloor nt\rfloor - nt}{n}y^N} - 1| \le \frac{1}{n}\frac{|\beta||y|^N}{N!},$$

and we eventually obtain

$$|\mathbb{E}[e^{-iyW_n^{N,\beta}(t)}] - e^{(i)^N\beta t y^N/N!}| = \frac{C(y)t}{n}e^{tC(y)} + \frac{1}{n}\frac{|\beta||y|^N}{N!}.$$

□

Proof of Theorem 4.

By lemma 3,

$$\mathbb{E}\left[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}\right] = \mathbb{E}\left[e^{(i)^N\beta S_m^\alpha(t)y^N/N!}\right] + R(n, m, y, t),$$

where $|R(n, m, y, t)| \le \mathbb{E}\left[\frac{C(y)S_m^\alpha(t)}{n}e^{S_m^\alpha(t)C(y)} + \frac{1}{n}\frac{|\beta||y|^N}{N!}\right]$.

Since, by estimate (30), the expectation $\mathbb{E}\left[S_m^\alpha(t)e^{S_m^\alpha(t)C(y)}\right]$ is finite for any $m \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty}\mathbb{E}\left[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}\right] = \mathbb{E}\left[e^{(i)^N\beta S_m^\alpha(t)y^N/N!}\right].$$

Eventually, by using (29), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}\left[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}\right] = \lim_{m\to\infty}\mathbb{E}\left[\exp\left((-i)^N y^N\frac{\beta S_m^\alpha(t)}{N!}\right)\right]$$
$$= \exp\left(-t\left((-1)^{N+1}i^N y^N\frac{\beta}{N!}\right)^\alpha\right).$$

□

Theorem 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be the Fourier transform of a complex bounded measure μ on $\mathbb{R}$ with compact support, i.e., f be of the form $f(x) = \int e^{-iyx} d\mu(y)$. Then under the above assumptions on β and N*

$$\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}[f(W_n^{N,\beta}(S_m^\alpha(t)))] = \int_{\mathbb{R}} e^{-t\left((-1)^{N+1} i^N y^N \frac{\beta}{N!}\right)^\alpha} \mathrm{d}\mu(y). \tag{35}$$

Proof. Let $K \subset \mathbb{R}$ be the support of the measure μ. By lemma 3, there exists a positive constant $M \in \mathbb{R}^+$ such that for any $y \in K$ the following holds:

$$\mathbb{E}\left[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}\right] = \mathbb{E}\left[e^{(i)^N \beta S_m^\alpha(t) y^N/N!}\right] + R(n,m,t),$$

where $|R(n,m,t)| \le \mathbb{E}\left[\frac{MS_m^\alpha(t)}{n} e^{S_m^\alpha(t)M} + \frac{1}{n}\frac{|\beta||y|^N}{N!}\right] < +\infty$.

By Fubini theorem,

$$\begin{aligned}\mathbb{E}[f(W_n^{N,\beta}(S_m^\alpha(t))] &= \int_{\mathbb{R}} \mathbb{E}\left[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}\right] \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}} \mathbb{E}\left[\exp\left((-i)^N y^N \frac{\beta S_m^\alpha(t)}{N!}\right)\right] \mathrm{d}\mu(y) + |\mu| R(n,m,t),\end{aligned}$$

where $|\mu|$ denotes the total variation of the complex measure μ. By letting $n \to \infty$, we obtain:

$$\lim_{n\to\infty} \mathbb{E}[f(W_n^{N,\beta}(S_m^\alpha(t))] = \int_{\mathbb{R}} \mathbb{E}\left[\exp\left((-i)^N y^N \frac{\beta S_m^\alpha(t)}{N!}\right)\right] \mathrm{d}\mu(y).$$

Eventually, by dominated convergence theorem, the following holds

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}[f(W_n^{N,\beta}(S_m^\alpha(t))] &= \lim_{m\to\infty} \int_{\mathbb{R}} \mathbb{E}\left[\exp\left((-i)^N y^N \frac{\beta S_m^\alpha(t)}{N!}\right)\right] \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}} \lim_{m\to\infty} \mathbb{E}\left[\exp\left((-i)^N y^N \frac{\beta S_m^\alpha(t)}{N!}\right)\right] \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}} e^{-t\left((-1)^{N+1} i^N y^N \frac{\beta}{N!}\right)^\alpha} \mathrm{d}\mu(y).\end{aligned}$$

□

Remark 3. Theorem 4 allows to interpret *formally* the limit process of $X_{n,m}$ as an $N\alpha$-stable process. In fact, such a process cannot exist in the case $N\alpha > 2$ and, analogously to the random walk $W_n^{N,\beta}$ studied in Sect. 2, the sequence of complex random variables $X_{n,m}(t)$ does not converge in distribution. Theorems 4 and 5 have to be interpreted in a weaker sense; indeed, even if the distribution of $W_n^{N,\beta}(S_m^\alpha(t))$ does not converge to a well-defined probability measure on the complex plane, the integral of suitable functions (i.e., linear combinations of exponentials) converges and the limit is given by formula (35).

It is particularly interesting the study of the case N being an integer strictly greater than 2 and the product $N\alpha$ satisfies the inequality $N\alpha \leq 2$. In this case, an $N\alpha$-stable process $H^{N\alpha}(t)$ exists and its relation with the sequence of jump processes $\{W_n^{N,\beta}(S_m^\alpha(t))\}_{m,n}$ is worth of investigation. We can consider, for instance, the case where $N = 2M$ with $M \in \mathbb{N}$, $\alpha = 1/M$ and $\beta = (-1)^{M+1}\frac{(2M)!}{2^M}$. According to Theorem 4, we have the following convergence result

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}[e^{-iyW_n^{N,\beta}(S_m^\alpha(t))}] = e^{-t\frac{y^2}{2}}, \qquad y \in \mathbb{R}.$$

Nevertheless this is not sufficient to interpret the limit of $W_n^{N,\beta}(S_m^\alpha(t))$ as a real-valued Wiener process.

Indeed, for any $(n, m) \in \mathbb{N}^2$ the process $W_n^{N,\beta}(S_m^\alpha(t))$ has complex paths and we can prove that for any $t > 0$ given, the law of the random variable $W_n^{N,\beta}(S_m^\alpha(t))$ cannot converge to a Gaussian distribution on the real axis. Actually, a straightforward computation shows that

$$\lim_{n,m\to\infty}\mathbb{P}\left(W_n^{N,\beta}(S_m^\alpha(t)) \in B_R(0)\right) = 0,$$

for any given $R > 0$, with $B_R(0) \subset \mathbb{C}$. Therefore,

$$\lim_{n,m\to\infty}\mathbb{P}\left(\left\{\left|\mathrm{Re}[W_n^{N,\beta}(S_m^\alpha(t))]\right| \leq \sqrt{2}R\right\} \cap \left\{\left|\mathrm{Im}[W_n^{N,\beta}(S_m^\alpha(t))]\right| \leq \sqrt{2}R\right\}\right) = 0;$$

hence, even in the case where the imaginary part disappears, the real part cannot have a Gaussian distribution since it is concentrated outside the interval $(-\sqrt{2}R, \sqrt{2}R)$.

4. Probabilistic representation of evolution equations with fractional-order space derivative

Let $A_{N,\beta} : D(A_{N,\beta}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator defined by

$$\widehat{A_{N,\beta}f}(y) := \frac{(-i)^N\beta y^N}{N!}\hat{f}(y),$$

where $\hat{f}$ is the Fourier transform of $f \in L^2(\mathbb{R})$, i.e., $\hat{f}(y) = \int_{\mathbb{R}} e^{ixy}f(x)\mathrm{d}x$ for $f \in L^1(\mathbb{R})$, with

$$D(A_{N,\beta}) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^{2N}|\hat{f}(y)|^2\,\mathrm{d}y < \infty\}.$$

In other words, $A_{N,\beta}$ is the Fourier integral operator with symbol $\Psi(y) := \frac{(-i)^N\beta y^N}{N!}$. On smooth functions $f \in L^2(\mathbb{R}) \cap C^N(\mathbb{R})$, it is given by

$$A_{N,\beta}f(x) = \frac{\beta}{N!}\frac{\partial^N}{\partial x^N}f(x).$$

In the following, we shall always assume that $\beta \in \mathbb{R}$ is a real constant such that, whenever $N \in \mathbb{N}$ is even, the inequality $\mathrm{Re}((-1)^{N/2}\beta) \leq 0$ is satisfied. Under this assumption, the operator $A_{N,\beta}$ generates a strongly continuous contraction semigroup on $L^2(\mathbb{R})$. In addition, for $N \in \mathbb{N}$ odd, the operator $iA_{N,\beta}$ is self-adjoint and generates a strongly continuous unitary group on $L^2(\mathbb{R})$.

Let $B : D(B) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator defined by

$$\widehat{Bf}(y) := |y|\hat{f}(y).$$

with

$$D(B) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^2|\hat{f}(y)|^2\, \mathrm{d}y < \infty\}.$$

B is called *Riesz operator*, formally written as $B \equiv \partial_{|x|}$. It is given by

$$\partial_{|x|}f(x) = -k\int_0^\infty \frac{f(x-s) - 2f(x) + f(x+s)}{s^2}\,\mathrm{d}s, \qquad x \in \mathbb{R},$$

$$k = \left(2\int_0^\infty \frac{1-\cos s}{s^2}\,\mathrm{d}s\right)^{-1} = \frac{1}{\pi}.$$

Via functional calculus, it is straightforward to define the N-power of B, as the operator B^N with symbol $\Psi(y) = |y|^N$ and domain $D(B^N) = \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} y^{2N}|\hat{f}(y)|^2\,\mathrm{d}y < \infty\}$. For N even we have that $A_{N,\beta} = \frac{(-i)^N\beta}{N!}B^N$.

For a given real constant $\alpha \in (0,1)$, let us define the fractional power of $-A_{N,\beta}$ and B^N as the operators $(-A_{N,\beta})^\alpha$ and $B_{N,\alpha}$ with symbols, respectively

$$\widehat{(-A_{N,\beta})^\alpha f}(y) = \left(\frac{(-1)^{N+1}i^N\beta y^N}{N!}\right)^\alpha \hat{f}(y),$$

$$\widehat{B_{N,\alpha}f}(y) = |y|^{N\alpha}\hat{f}(y),$$

where, given a complex number $z \in \mathbb{C}$, with $z = |z|e^{i\theta}$, with $\theta \in (-\pi, \pi]$, the α-power of z is taken as

$$z^\alpha = |z|^\alpha e^{i\alpha\theta}.$$

Note that when N is odd and $i^N\beta$ is a purely imaginary number the symbol of $(-A_{N,\beta})^\alpha$ is explicitly given by $\left(\frac{(-1)^{N+1}i^N\beta y^N}{N!}\right)^\alpha := \frac{|\beta|^\alpha}{N!^\alpha}|y|^{N\alpha}e^{(-1)^{\frac{N-1}{2}}\frac{i\pi\alpha}{2}\frac{\beta y}{|\beta y|}}$.

The action of the operator $(-A_{N,\beta})^\alpha$ can be also represented by the following formula

$$(-A_{N,\beta})^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{f(x) - e^{sA_{N,\beta}}f(x)}{s^{\alpha+1}}\,\mathrm{d}s$$

$$= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \lim_{n\to\infty}\frac{f(x) - \mathbb{E}[f(x + W_n^{N,\beta}(s))]}{s^{\alpha+1}}\,\mathrm{d}s,$$

$e^{sA_{N,\beta}}$ being the semigroup generated by $A_{N,\beta}$ and $W_n^{N,\beta}$ the sequence of complex random walks defined in Sect. 2. For N an even integer, there is a trivial relation between the operators $(-A_{N,\beta})^\alpha$ and $B_{N,\alpha}$, namely

$$(-A_{N,\beta})^\alpha = \left(\frac{(-1)^{N+1} i^N \beta}{N!}\right)^\alpha B_{N,\alpha}.$$

For N odd, we have $B_{N,\alpha} = \frac{(A_{N,\beta})^\alpha + (-A_{N,\beta})^\alpha}{2(|\beta|/N!)^\alpha \cos(\pi\alpha/2)}$. The action of $B_{N,\alpha}$ can be also written in the following form:

$$\begin{aligned}
B_{N,\alpha} f(x) &= \left(2\int_0^\infty \frac{1-\cos s}{s^{\alpha+1}}\,\mathrm{d}s\right)^{-1} \\
&\quad\times \int_0^\infty \frac{e^{sA_{N,1}} f(x) - 2f(x) + e^{-sA_{N,1}} f(x)}{s^{\alpha+1}}\,\mathrm{d}s \\
&= \left(2\int_0^\infty \frac{1-\cos s}{s^{\alpha+1}}\,\mathrm{d}s\right)^{-1} \\
&\quad\times \int_0^\infty \lim_{n\to\infty} \frac{\mathbb{E}[f(x+W_n^{N,1}(s))] - 2f(x) + \mathbb{E}[f(x+W_n^{N,-1}(s))]}{s^{\alpha+1}}\,\mathrm{d}s.
\end{aligned}$$

Let $f \in L^2(\mathbb{R})$ be a function of the form

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixy} \hat{f}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}, \tag{36}$$

with $\hat{f} \in L^2(\mathbb{R})$ being a compactly supported function. It is straightforward to verify that f belongs to the domain of any of the operators above. Furthermore, the subset $D \subset L^2(\mathbb{R})$ of functions of the form (36) is an operator core. By considering the complex bounded Borel measure on the real line μ_f absolutely continuous with respect to the Lebesgue measure with density $\frac{\hat{f}}{2\pi}$, we can look at the function f defined by (36) as the Fourier transform of μ_f. Hence, f can be extended to an entire analytic function $f : \mathbb{C} \to \mathbb{C}$ of exponential type (see Remark 1).

For any of the following initial value problems

$$\begin{aligned}
\partial_t u(t) &= Au(t) \\
u(t_0) &= f, \qquad t \geq t_0,\ f \in D
\end{aligned} \tag{37}$$

with A being either one of the operators $A_{N,\beta}$ and $-B^N$ or one of their fractional powers $-(-A_{N,\beta})^\alpha$ and $-B_{N,\alpha}$ and with f of the form (36), we are going to construct a sequence of complex jump processes $\{X_{n,m}\}_{n,m\in\mathbb{N}}$ providing a probabilistic representation for the solution $u(t,x)$ of the form

$$u(t,x) = \lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}[f(x + X_{n,m}(t-t_0))].$$

The first result is taken from [15] and is a direct consequence of Remark 1.

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{C}$ be an $L^2(\mathbb{R})$ function of the form (36). Then the (classical [2]) solution of*

$$\begin{aligned} \partial_t u(t,x) &= A_{N,\beta} u(t,x) \\ u(t_0,x) &= f(x), \qquad t \geq t_0, x \in \mathbb{R} \end{aligned} \tag{38}$$

is given by

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[f(x + W_n^{N,\beta}(t-t_0))], \tag{39}$$

where $\{W_n^{N,\beta}(t)\}_{n\in\mathbb{N}}$ is the sequence of complex random walks defined in (17).

Proof. By formula (26), we have

$$\begin{aligned} u(t,x) &= \lim_{n\to\infty} \mathbb{E}[f(x + W_n^{N,\beta}(t-t_0))] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixy} \exp\left((-i)^N \beta(t-t_0)\frac{y^N}{N!}\right) \hat{f}(y)\, \mathrm{d}y. \end{aligned}$$

By the compactness of the support of the function $\hat{f}$, the function $u(t,x)$ is smooth and, by direct computation, a classical solution of the PDE

$$\partial_t u(t,x) = \frac{\beta}{N!} \partial_x^N u(t,x).$$

□

Let us consider now the fractional power $(-A_{N,\beta})^\alpha$ of the Nth-order differential operator $-A_{N,\beta}$ and construct a probabilistic representation of the associated C_0-contraction semigroup.

Theorem 7. *Let $f : \mathbb{R} \to \mathbb{C}$ be an $L^2(\mathbb{R})$ function of the form (36). Then the solution of*

$$\begin{aligned} \partial_t u(t,x) &= -(-A_{N,\beta})^\alpha u(t,x) \\ u(t_0,x) &= f(x), \qquad t \geq t_0, x \in \mathbb{R} \end{aligned} \tag{40}$$

is given by

$$u(t,x) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}[f(x + X_{n,m}(t-t_0))], \tag{41}$$

where $\{X_{n,m}(t)\}_{n,m\in\mathbb{N}}$ is the sequence of complex random walks defined in (31) as $X_{n,m}(t) = W_n^{N,\beta}(S_m^\alpha(t))$.

[2] For a classical solution of the initial value problem (38), we mean a function $u : [0,+\infty) \times \mathbb{R} \to \mathbb{C}$ which is of class C^1 in the time variable t, of class C^N in the space variable x and such that for any $(t,x) \in [0,+\infty) \times \mathbb{R}$ the equality $\partial_t u(t,x) = \frac{\beta}{N!}\frac{\partial^N}{\partial x^N} u(t,x)$ holds.

Equation (40) can be formally written as

$$\frac{\partial}{\partial t}u(t,x) = -\left(-\frac{\beta}{N!}\right)^{\alpha}\frac{\partial^{N\alpha}}{\partial x^{N\alpha}}u(t,x).$$

Proof. By Theorem 5, the function $u(t,x)$ defined by the r.h.s. of (41) is equal to

$$\begin{aligned}u(t,x) &= \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}[f(x+X_{n,m}(t-t_0))]\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixy}e^{-(t-t_0)\left((-1)^{N+1}i^N y^N \frac{\beta}{N!}\right)^{\alpha}}\hat{f}(y)\,\mathrm{d}y.\end{aligned}$$

The last line is exactly the action of the semigroup $e^{-(-A_{N,\beta})^{\alpha}(t-t_0)}$ on the vector $f\in L^2(\mathbb{R})$. Moreover, because of the assumptions (32) on the constants N, β and the compactness of the support of $\hat{f}\in L^2(\mathbb{R})$, the function $u(t,x)$ is smooth in both the time and space variables. □

Let us now consider the Riesz operator B and its powers. Given $N\in\mathbb{N}$, let us consider the initial value problem

$$\begin{aligned}&\partial_t u(t,x) = -B^N u(t,x)\\ &u(t_0,x) = f(x), \qquad t\ge t_0,\ x\in\mathbb{R}\end{aligned} \tag{42}$$

with f of the form (36) as above. For N even, the operator B^N coincides with $(-A_{N,\beta})$, with $\beta\equiv(-1)^{\frac{N}{2}+1}N!$. By Theorem 6, the solution of (42) is given by (39). In the case where N is odd, the construction of the associated process is neither simple, nor unique, as the following result shows.

Theorem 8. *Let us consider problem (42) with $N\in\mathbb{N}$ and $f\in D$. For any $M\in\mathbb{N}$, let us consider the sequence of processes $\{W_n^{2MN,\beta}(t)\}$ defined as in (17), with $\beta = (-1)^{MN+1}(2MN)!$. Then, choosing $\alpha=\frac{1}{2M}$, according to Theorem 7 the solution of the initial value problem (42) is given by*

$$u(t,x) = \lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}[f(x+W_n^{2MN,\beta}(S_m^{\frac{1}{2M}}(t-t_0)))], \tag{43}$$

where $S_m^{\alpha}(t)$ is the sequence of processes (27) approximating the α-stable subordinator.

Proof. By an application of Theorem 5, the function $u(t,x)$ defined by the r.h.s. of (43) is equal to

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixy}e^{-(t-t_0)|y|^N}\hat{f}(y)\,\mathrm{d}y.$$

□

More generally, let us consider the operator $B_{N,\alpha}$, and the corresponding associated initial value problem

$$\begin{aligned} \partial_t u(t,x) &= -B_{N,\alpha} u(t,x), \\ u(t_0,x) &= f(x), \end{aligned} \tag{44}$$

with $f \in L^2(\mathbb{R})$ of the form (36). For N even, we can take the process $W_n^{N,\beta}(t)$ associated with the operator $A_{N,\beta}$, with $\beta = (-1)^{N/2} N!$ and the solution of (44) is given by

$$u(t,x) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}[f(x + W_n^{N,\beta}(S_m^\alpha(t-t_0)))].$$

The following construction allows to handle the case where N is odd. Indeed, recall from [15] that the distribution of $-\xi_{N,\beta}$ is equal to the distribution of $\xi_{N,-\beta}$ and the same holds for the corresponding continuous time processes $W_n^{N,\beta}(t)$.

Theorem 9. *Let $W_n^{N,\beta}(t)$ and $\tilde{W}_n^{N,\beta'}(t)$ be two independent copies of the process (17), with $\beta = N!$ and $\beta' = -N!$, respectively, and let $S_m^\alpha(t)$ and $\tilde{S}_m^\alpha(t)$be two independent copies of the process (27). Taking a rescaled time variable $\tilde{t} := (2\cos\left(\frac{\alpha\pi}{2}\right))^{-1} t$, the solution of (44) with $f \in D$ is given by*

$$u(t,x) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}\left[f\left(x + W_n^{N,\beta}\left(S_m^\alpha(\tilde{t}-\tilde{t}_0)\right) + \tilde{W}_n^{N,\beta'}\left(\tilde{S}_m^\alpha(\tilde{t}-\tilde{t}_0)\right)\right)\right].$$

Proof. By applying Theorem 5, we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixy} e^{-(\tilde{t}-\tilde{t}_0)\left((iy^N)^\alpha + (-iy^N)^\alpha\right)} \hat{f}(y)\,\mathrm{d}y \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixy} e^{-(t-t_0)|y|^{N\alpha}} \hat{f}(y)\,\mathrm{d}y. \end{aligned}$$

□

Remark 4. In fact for any operator of the form B^N, with $N \in \mathbb{N}$, there exists a family of associated processes. For instance, if $N = 2$, the solution of the heat equation

$$\partial_t u(t,x) = \frac{1}{2} \partial_x^2 u(t,x) \tag{45}$$

can be represented by means of the Feynman–Kac formula, as the expectation with respect to the distribution of the Wiener process $W(t)$:

$$u(t,x) = \mathbb{E}[u(t_0, x + W(t-t_0))], \qquad t \geq t_0, x \in \mathbb{R},$$

but alternative constructions are possible. Let us consider, for instance, the sequence of processes $W_n^{N,\beta}(S_m^\alpha(t))$, with $N = 4$, $\alpha = 1/2$ and $\beta = -3!$. Then the solution of (45) can also be represented by

$$u(t,x) = \lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}[u(t_0, x + W_n^{N,\beta}(S_m^\alpha(t-t_0)))], \qquad t \geq t_0, \ x \in \mathbb{R}.$$

The same formula holds by taking a generic $M \in \mathbb{N}$ and setting $N \equiv 2M$, $\alpha \equiv 1/M$ and $\beta \equiv (-1)^{M+1} \frac{(2M)!}{2^M}$ It is important to remark that, even in these cases, the sequence of jump processes $\{W_n^{N,\beta}(S_m^\alpha(t))\}_{m,n}$ does not converge to the Wiener process W, as discussed in Remark 3.

5. Time fractional diffusion equations

In this section, we consider time fractional equations of the form

$$\begin{aligned} \boldsymbol{D}_t^\alpha u(t,x) =& A_{N,\beta} u(t,x) \\ u(0,x) =& f(x), \end{aligned} \tag{46}$$

where the time fractional derivative $\boldsymbol{D}_t^\alpha$ must be understood in the sense of Caputo derivative. Since throughout the paper $\alpha \in (0,1)$, we can define

$$\boldsymbol{D}_t^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial}{\partial s} v(s)\, \mathrm{d}s.$$

The Caputo derivative $\boldsymbol{D}_t^\alpha v(t)$, for $\alpha \in (0,1)$, can also be defined as the function with Laplace transform $\widetilde{\boldsymbol{D}^\alpha v}(\lambda) = \lambda^\alpha \tilde{v}(\lambda) - \lambda^{\alpha-1} v(0^+)$. The reader can consult the book by Samko et al. [62] for further details.

In our construction, equation (46) is solved with the aid of a random time change of the complex random walk $W_n^{N,\beta}(t)$. In order to explain our construction, we introduce the α-stable subordinator $H^\alpha(t)$, i.e., a subordinator with zero drift and Lévy measure

$$m_\alpha(\mathrm{d}x) = \frac{c}{x^{1+\alpha}} \mathbb{1}_{(0,\infty)}(x)\, \mathrm{d}x,$$

where $c > 0$ is a given constant. If we choose $c = \frac{1}{\Gamma(1-\alpha)}$, then the Laplace exponent becomes $\Phi(\lambda) = \lambda^\alpha$.

We denote by $L^\alpha(t)$ the inverse of the subordinator $H^\alpha(t)$ (see Appendix A), namely:

$$L^\alpha(t) = \inf\{s \geq 0 \,:\, H^\alpha(s) \geq t\}.$$

The moments of L_t^α are equal to

$$\mathbb{E}[(L^\alpha(t))^k] = \frac{k!\, t^{\alpha k}}{\Gamma(\alpha k + 1)}, \qquad k \in \mathbb{N},$$

while its Laplace transform is

$$\mathbb{E}[e^{-zL^\alpha(t)}] = \sum_{k \geq 0} \frac{(-z)^k t^{\alpha k}}{\Gamma(\alpha k + 1)}, \qquad z \in \mathbb{C}.$$

In the following result, we show that the sequence of subordinated processes $W_n^{N,\beta}(L_t^\alpha)$ can be associated with the PDE (46). Notice that, as opposite to the previous section, here we only need to take one limit.

Theorem 10. *Assume $\beta \in \mathbb{C}$ and $N \in \mathbb{N}$ satisfy assumption (32). If $f : \mathbb{R} \to \mathbb{R}$ is an analytic function that is the Fourier transform of a bounded complex Borel measure on $\mathbb{R}$ with compact support, i.e.,*

$$f(x) = \int_{\mathbb{R}} e^{-i\lambda x}\, \mathrm{d}\mu_f(\lambda),$$

then of the solution of the initial value problem

$$\begin{aligned} \boldsymbol{D}_t^\alpha u(t,x) =& A_{N,\beta} u(t,x) \\ u(0,x) =& f(x), \end{aligned} \tag{46}$$

is given by

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}\left[f(x + W_n^{N,\beta}(L^\alpha(t)))\right].$$

Proof. By explicit computation,

$$\begin{aligned} u(t,x) &= \lim_{n\to\infty} \mathbb{E}\left[f(x + W_n^{N,\beta}(L^\alpha(t)))\right] = \int_{\mathbb{R}} e^{-i\lambda x}\mathbb{E}[e^{\frac{\beta}{N!}(-i\lambda)^N L^\alpha(t)}]\,\mathrm{d}\mu_f(\lambda) \\ &= \int_{\mathbb{R}} e^{-i\lambda x} E_\alpha\left(\frac{\beta}{N!}(-i\lambda)^N t^\alpha\right) \mathrm{d}\mu_f(\lambda), \end{aligned} \tag{47}$$

where E_α denotes the Mittag-Leffler function, namely $E_\alpha(t) = \sum_{k\geq 0}\frac{t^k}{\Gamma(\alpha k+1)}$. Under the stated assumption on the constants $\beta \in \mathbb{C}$ and $N \in \mathbb{N}$, the argument $z \equiv \frac{\beta}{N!}(-i\lambda)^N t^\alpha$ is a complex number with non-positive real part for any $t \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}$; hence, the map $\lambda \mapsto E_\alpha\left(\frac{\beta}{N!}(-i\lambda)^N t^\alpha\right)$ is bounded and the integral (47) is absolutely convergent and defines a C^∞ function of the x variable. Furthermore, for any $t \in \mathbb{R}^+$ the function $u(t,x)$ is still the Fourier transform of a complex measure μ_t on $\mathbb{R}$ with compact support which is also absolutely continuous with respect to μ_f, namely :

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}\mu_f} := E_\alpha\left(\frac{\beta}{N!}(-i\lambda)^N t^\alpha\right).$$

In particular,

$$A_{N,\beta}u(t,x) = \int_{\mathbb{R}} e^{-i\lambda x}\frac{\beta}{N!}(-i\lambda)^N E_\alpha\left(\frac{\beta}{N!}(-i\lambda)^N t^\alpha\right) \mathrm{d}\mu_f(\lambda).$$

In order to prove that this is equal to $D_t^\alpha u(t,x)$, and hence that equation (46) holds, let us take the Laplace transform of both sides. Denoting $\tilde{u}(\rho,x)$ the Laplace transform of $u(t,x)$ and applying Fubini theorem as well as the properties of the Mittag-Leffler function and the condition $\mathrm{Re}\left(\frac{\beta}{N!}(-i\lambda)^N\right) \leq 0$, first of all we have

$$\begin{aligned} \tilde{u}(\rho,x) &= \int_0^\infty e^{-\rho t}u(t,x)\,\mathrm{d}t = \int_{\mathbb{R}} e^{-i\lambda x}\int_0^\infty e^{-\rho t}E_\alpha\left(\frac{\beta}{N!}(-i\lambda)^N t^\alpha\right)\mathrm{d}t\,\mathrm{d}\mu_f(\lambda) \\ &= \int_{\mathbb{R}} \frac{e^{-i\lambda x}}{\rho}\frac{1}{1-\left(\frac{\beta}{N!}(-i\lambda)^N\rho^{-\alpha}\right)}\mathrm{d}\mu_f(\lambda). \end{aligned} \tag{48}$$

On the other hand, by taking the Fourier–Laplace transform of both sides of (46) we obtain

$$\rho^\alpha\tilde{u}(\rho,\lambda) - \rho^{\alpha-1}\tilde{u}(0,\lambda) = \frac{\beta}{N!}(-i\lambda)^N\tilde{u}(\rho,\lambda),$$

which yields

$$\tilde{u}(\rho,\lambda) = \frac{\rho^{-1}}{1-\frac{\beta}{N!}(-i\lambda)^N\rho^{-\alpha}}\tilde{u}(0,\lambda). \tag{49}$$

By comparing (48) and (49), we obtain the final result. □

5.1. Time fractional equations and non-local space fractional equations

In this section, we discuss the relationship between equation (46) and the diffusion equation with non-local forcing term of the form

$$\begin{aligned}\partial_t u(t,x) &= (A_{N,\beta})^{1/\alpha}u(t,x) + \sum_{k=1}^{1/\alpha-1}\frac{1}{\Gamma(\alpha k)}t^{\alpha k-1}A_{N,\beta}^k f(x),\\ u(0,x) &= f(x).\end{aligned} \tag{50}$$

We shall require that $\alpha = M^{-1}$ for some $M \in \mathbb{N}$, $M > 1$, so in particular $\alpha \in (0,1)$.

The proof of the equivalence follows by taking Laplace (in time, parameter s) and Fourier (in space, parameter λ) transform of both equations, proving that the Laplace–Fourier transforms of the solutions coincide.

Let us first consider equation (46). By taking Laplace and Fourier transform on both sides, we get

$$s^\alpha\hat{\tilde{u}}(s,\lambda) - s^{\alpha-1}\widehat{f}(\lambda) = \frac{(-i)^N\beta\lambda^N}{N!}\hat{\tilde{u}}(s,\lambda);$$

hence,

$$\hat{\tilde{u}}(s,\lambda) = \frac{s^{\alpha-1}}{s^\alpha - \frac{(-i)^N\beta\lambda^N}{N!}}\widehat{f}(\lambda). \tag{51}$$

(It is worth noticing that the assumption on the sign of β: $\mathrm{Re}((-i)^N\beta) \le 0$ implies that the quantity we simplify on both sides is never zero.)

Next, we apply the same machinery to the solution of equation (50). We have

$$s\hat{\tilde{u}}(s,\lambda) - \widehat{f}(\lambda) = \left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^M\hat{\tilde{u}}(s,\lambda) + \sum_{k=1}^{M-1}s^{-\alpha k}\left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^k\widehat{f}(\lambda),$$

and rearranging both sides, we get

$$\begin{aligned}\left(s-\left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^M\right)\hat{\tilde{u}}(s,\lambda) &= \sum_{k=1}^{M-1}s^{-\alpha k}\left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^k\widehat{f}(\lambda)\\ &= \frac{1-\left(s^{-\alpha}\frac{(-i)^N\beta\lambda^N}{N!}\right)^M}{1-s^{-\alpha}\frac{(-i)^N\beta\lambda^N}{N!}}\widehat{f}(\lambda)\end{aligned}$$

$$= \frac{s - \left(\frac{(-i)^N \beta\lambda^N}{N!}\right)^M}{s - s^{1-\alpha}\frac{(-i)^N\beta\lambda^N}{N!}} \widehat{f}(\lambda),$$

so we simplify the quantity in the numerator

$$\hat{\tilde{u}}(s,\lambda) = \frac{1}{s - s^{1-\alpha}\frac{(-i)^N\beta\lambda^N}{N!}} \widehat{f}(\lambda),$$

which coincides with (51), as required.

Informally, we notice that a fractional time derivative of order α has become a fractional space derivative of order $1/\alpha$, but this transformation affects, in a rather complicated way, the initial condition. In the next and last example of this section, we shall see what happens if we start with an equation involving fractional derivatives of the same order in both time and space, and we compare it with an equation of integer-order derivatives. To be precise, the following equation takes place instead of (46)

$$\begin{aligned} \boldsymbol{D}_t^\alpha u(t,x) =& (A_{N,\beta})^\alpha u(t,x) \\ u(0,x) =& f(x), \end{aligned} \tag{52}$$

while the following non-local problem takes the place of (50):

$$\begin{aligned} \partial_t u(t,x) &= A_{N,\beta} u(t,x) + \sum_{k=1}^{1/\alpha-1} \frac{1}{\Gamma(\alpha k)} t^{-\alpha k} A_{N,\beta}^{\alpha k} f(x) \\ u(0,x) &= f(x). \end{aligned} \tag{53}$$

The proof is analogous to the previous one. Let us first consider equation (52); we have that

$$s^\alpha \hat{\tilde{u}}(s,\lambda) - s^{\alpha-1}\widehat{f}(\lambda) = \left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^\alpha \hat{\tilde{u}}(s,\lambda),$$

so that

$$\hat{\tilde{u}}(s,\lambda) = \frac{s^{\alpha-1}}{s^\alpha - \left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^\alpha} \widehat{f}(\lambda). \tag{54}$$

Consider now (53); we have

$$s\hat{\tilde{u}}(s,\lambda) - \widehat{f}(\lambda) = \left(\frac{(-i)^N\beta\lambda^N}{N!}\right) \hat{\tilde{u}}(s,\lambda) + \sum_{k=1}^{M-1} s^{-\alpha k} \left(\frac{(-i)^N\beta\lambda^N}{N!}\right)^{\alpha k} \widehat{f}(\lambda),$$

from which we obtain

$$\left(s - \left(\frac{(-i)^N \beta \lambda^N}{N!}\right)\right) \hat{\hat{u}}(s,\lambda) = \sum_{k=1}^{M-1} s^{-\alpha k} \left(\frac{(-i)^N \beta \lambda^N}{N!}\right)^{\alpha k} \widehat{f}(\lambda)$$
$$= \frac{1 - \frac{1}{s}\left(\frac{(-i)^N \beta \lambda^N}{N!}\right)}{1 - \frac{1}{s^\alpha}\left(\frac{(-i)^N \beta \lambda^N}{N!}\right)^\alpha} \widehat{f}(\lambda),$$

which, compared with (54), implies that the solution of (53) coincides with that of (52), as required.

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Appendix A. Fractional derivatives and Bochner's subordination

Fractional derivative

Let $\alpha \in (0, 1)$ be a real constant and let us consider the Lévy measure M on the positive half line defined by

$$M(\mathrm{d}s) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{\mathrm{d}s}{s^{\alpha+1}}. \tag{A.1}$$

It is well known that M is the Lévy measure of an α-stable subordinator ([14]) that is a totally (positively) skewed stable process for which the Lévy–Khinchin formula is written in terms of the Bernstein function

$$x^\alpha = \int_0^\infty (1 - e^{-xs})\, M(\mathrm{d}s). \tag{A.2}$$

Formula (A.2) is valid for any $x \in \mathbb{C}$, with $\mathrm{Re}(x) \geq 0$, in particular for $x = |x|e^{i\theta}$, $\theta \in [-\pi/2, \pi/2]$ it gives $|x|^\alpha e^{i\alpha\theta} = \int_0^\infty (1 - e^{-xs}) M(\mathrm{d}s)$. The representation (A.2) is therefore associated with the symbol of a positively skewed stable process, say $H^\alpha(t)$, $t \geq 0$. Indeed, for $\lambda > 0$, we have that $\mathbb{E}\left[e^{-\lambda H^\alpha(t)}\right] = e^{-t\lambda^\alpha}$ and we say that $H^\alpha(t)$ is a stable subordinator of order $\alpha \in (0, 1)$. It has nonnegative increments and therefore non-decreasing paths. Thus, $H^\alpha(t)$ can be considered as a time change and, given a stochastic process $X(t)$, one can consider the subordinated process $X(H^\alpha(t))$.

We recall that, for $\alpha \uparrow 1$, $H^\alpha(t)$ becomes the elementary subordinator t. The density law $h = h(t, x)$ of $H^\alpha(t)$ solves the problem

$$\begin{cases} \partial_t h = -\partial_x^\alpha h \\ h(0, x) = \delta(x), \quad x \in \mathbb{R}^+ \\ h(t, 0) = 0, \quad t \in \mathbb{R}^+ \end{cases} \tag{A.3}$$

where $\partial_x^\alpha = \partial^\alpha / \partial x^\alpha$ is the Riemann–Liouville fractional derivative with symbol $(-iy)^\alpha = |y|^\alpha e^{-i\frac{\pi\alpha}{2}\frac{y}{|y|}}$:

$$\frac{d^\alpha f}{dx^\alpha}(x) := \frac{1}{2\pi} \int e^{-ixy} (-iy)^\alpha \hat{f}(y)\, \mathrm{d}y, \qquad \hat{f}(y) = \int e^{ixy} f(x)\, \mathrm{d}x.$$

According to (A.2),

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(x) = \int_0^\infty (f(x) - f(x - s))\, M(\mathrm{d}s). \tag{A.4}$$

We also introduce the inverse to a stable subordinator that is the non-Markovian process

$$L^\alpha(t) = \inf\{s \geq 0 \,:\, H^\alpha(s) \geq t\}, \quad t \geq 0.$$

We have that $\mathbb{P}(L^\alpha(t) < x) = \mathbb{P}(H^\alpha(x) > t)$ and $\mathbb{E}\left[\exp -\lambda L^\alpha(t)\right] = E_\alpha(-\lambda t^\alpha)$ with $\lambda \geq 0$ where

$$E_\beta(z) := \frac{1}{2\pi i} \int_{Ha} \frac{\zeta^{\beta-1} e^\zeta}{\zeta^\beta - z}\, \mathrm{d}\zeta = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k + 1)}, \quad \mathrm{Re}\,(\beta) > 0,\ z \in \mathbb{C}, \tag{A.5}$$

(Ha is the Hankel path) is the Mittag-Leffler function. We observe that, for $\alpha \in (0, 1)$, $u(t) = E_\alpha(wt^\alpha)$ with $t > 0$, $w > 0$, is the fundamental solution to the fractional relaxation equation

$$\mathbf{D}_t^\alpha u(t) - w\, u(t) = 0, \tag{A.6}$$

where

$$\mathbf{D}_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u(s)\, \mathrm{d}s}{(t-s)^\alpha}, \tag{A.7}$$

is the so-called Caputo derivative or the Dzerbayshan–Caputo fractional derivative.

Fractional power of a generator

Given a strongly continuous contraction semigroup $T(t)$ on $(C^\infty(\mathbb{R}), \|\cdot\|_\infty)$ with infinitesimal generator $(A, D(A))$, we write $T(t) = e^{tA}$, $t > 0$. Let us consider the Markov process $X = (\{X(t)\}_{t\geq 0}; \mathbb{P}_x, x \in \mathbb{R})$ with

$$T(t) f(x) = \mathbb{E}_x f(X(t)), \qquad f \in D(A),$$

where $\mathbb{E}_x \mathbf{1}_{\mathcal{A}}(X(t)) = \mathbb{P}_x(X(t) \in \mathcal{A})$. We can define the subordinated semigroup given by the Bochner integral

$$T^\alpha(t) f(x) := \int_0^\infty T(s) f(x)\, h(t,s)\, \mathrm{d}s = \mathbb{E}_x f(X(H^\alpha(t))), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+ \tag{A.8}$$

where $h(t,s) = \mathbb{P}(H^\alpha(t) \in ds)/ds$ has been introduced before. According to the representation given by Phillips [61], for $f \in D(A)$, we also define

$$-(-A)^\alpha f(x) := \int_0^\infty (T(s) f(x) - f(x))\, M(\mathrm{d}s). \tag{A.9}$$

The formal representation e^{tA} of $T(t)$ shows that $A^\alpha = -(-A)^\alpha$ given in (A.9) is the generator of T^α given in (A.8) by considering a functional calculus which is referred to as Bochner–Phillips calculus (see, for example, [12]). The special case we introduced here can be extended to general time-changed processes with infinitesimal generators $-\phi(-A)$ where $\phi : (0, \infty) \mapsto [0, \infty)$ is a Bernstein function, which is the symbol of a time change (a non-decreasing process). Then, $-\phi(-A)$ is characterized via resolvents in terms of Dunford–Taylor integrals ([12,64]). For the generators we considered so far, we evidently have that the corresponding (Fourier) symbols are written in terms of Bernstein functions. If $-\Psi$ is the symbol of the generator A, i.e., for f belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decaying $C^\infty(\mathbb{R})$ functions,

$$Af(x) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x} \Psi(\lambda) \widehat{f}(\lambda)\, \mathrm{d}\lambda,$$

then the fractional power of $-A$ can be represented as:

$$(-A)^\alpha f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda x} (\Psi(\lambda))^\alpha \widehat{f}(\lambda)\, \mathrm{d}\lambda.$$

Appendix B. Some estimates

Appendix B.1: Moments of the Poisson distribution

Let X be a Poisson random variable with parameter λ. Then the moments are given by

$$m_k = \mathbb{E}[X^k] = \sum_{l=0}^{k} \lambda^l \left\{ {k \atop l} \right\},$$

where $\left\{ {k \atop l} \right\}$ denotes the Stirling numbers of the second kind, defined as

$$\left\{ {k \atop l} \right\} = \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} j^k.$$

In particular, $\left\{ n \atop 1 \right\} = 1$ and $\left\{ n \atop n \right\} = 1$ for any $n \geq 1$. Moreover, the sum over the first k values of the Stirling numbers of the second kind gives $\sum_{l=0}^{k} \left\{ k \atop l \right\} = B_k$, where B_k is the kth Bell number. Hence, if $\lambda > 1$, then $m_k \leq \lambda^k B_k$, while if $\lambda < 1$, then $m_k \leq B_k$. This eventually yields

$$m_k \leq (\lambda^k \vee 1) B_k \leq (\lambda^k \vee 1) \left(\frac{0.792k}{\log(k+1)} \right)^k . \tag{B.10}$$

Appendix B.2: Power-law distributions

Given $\alpha \in (0, 1)$, let us consider a sequence of random variables $Y_{[m]}$ with density

$$f_m(y) = c_m \, y^{-\alpha-1} \, \mathbf{1}_{\left(\frac{1}{m}, m^2\right)}(y),$$

where

$$c_m = \frac{\alpha}{m^\alpha (1 - m^{-3\alpha})}.$$

The moments of $Y_{[m]}$ are given by

$$\begin{aligned} \mathbb{E}[Y_{[m]}^k] =& c_m \int_{1/m}^{m^2} y^{k-\alpha-1} \, \mathrm{d}y = \frac{\alpha}{k-\alpha} \frac{1}{m^\alpha (1 - m^{-3\alpha})} \left(m^{2(k-\alpha)} - m^{-(k-\alpha)} \right) \\ =& \frac{\alpha}{k-\alpha} m^{2k-3\alpha} \frac{1 - m^{-3(k-\alpha)}}{1 - m^{-3\alpha}}, \end{aligned}$$

and for every $m > 1$ and every $k \geq 1$ we obtain the estimate

$$\mathbb{E}[Y_{[m]}^k] \leq c(\alpha) \, m^{2k-3\alpha}, \tag{B.11}$$

where $c(\alpha) := 1 \vee \frac{\alpha}{1-\alpha}$.

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