PROBABILISTIC TREATMENT OF THE BLOWING UP OF SOLUTIONS FOR A NONLINEAR INTEGRAL EQUATION

BY

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1. **Introduction.** The blowing up of the solutions of the following semilinear parabolic equation

$$(1.1)\qquad \begin{aligned} \partial u(t, x)/\partial t &= \boldsymbol{G}u(t, x)+c(x)u(t, x)^{\beta}, \qquad (\beta \geqq 2) \\ u(0, x) &= f(x), \qquad x \in R^d, \end{aligned}$$

depends on the dimension d and power β, where $\boldsymbol{G}$ is the infinitesimal generator of a linear nonnegative contraction semigroup on the space $\boldsymbol{B}(R^d)$ of bounded measurable functions on R^d and c is a bounded nonnegative measurable function on R^d. This fact was recently proved by Fujita [2] when $\boldsymbol{G}$ is the Laplacian operator. In this paper we will give upper and lower bounds for the solution of (1.1) constructed by a probabilistic method (cf. (3.4) and (4.7)). As a corollary we shall obtain Fujita's result when $\boldsymbol{G}$ is a fractional power $-(-\Delta)^{\alpha}$, $0<\alpha\leqq 2$, of the Laplacian operator.

Our method is based on probabilistic arguments relating to the branching Markov processes (cf. Ikeda-Nagasawa-Watanabe [3], Sirao [8] and Nagasawa [7]). The necessary facts of probabilistic arguments in this context will be summarized in §2, while in §3 and §4 we shall give upper and lower bounds of the probabilistic solution of (1.1) and some applications.

2. **Preliminaries.** Let D be a compact Hausdorff space with a countable open base, $\boldsymbol{B}(D)$ be the space of bounded Borel measurable functions on D. $\boldsymbol{B}^+(D)$ denotes the set of nonnegative elements of $\boldsymbol{B}(D)$. Let $\{T_t;\, t\geqq 0\}$ be a nonnegative contraction semigroup on $\boldsymbol{B}(D)$ defined through a kernel $T_t(x, dy)$ such that

(i) $T_t(x, \cdot)$ is a nonnegative Borel measure on D with $T_t(x, D)\leqq 1$;

(ii) $T_{\cdot}(\cdot, B)$ is measurable on $[0, \infty)\times D$ for any Borel subset B of D,

(iii) $T_{t+s}(x, B)=\int T_t(x, dy)T_s(y, B)$ for any $t, s\geqq 0$, $x\in D$ and Borel subset B, and

(iv) $T_tf(x)= \int T_t(x, dy)f(y)$ for $f\in \boldsymbol{B}(D)$.

We shall consider the following *nonlinear* integral equation with an initial data $f\in \boldsymbol{B}^+(D)$ instead of (1.1):

$$(2.1)\qquad v(t, x) = T_tf(x)+\int_0^t ds T_s(c\cdot v(t-s, \cdot)^{\beta})(x),$$

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where $c \in \boldsymbol{B}^+(D)$, which will be fixed throughout the paper, and $\beta = 2, 3, 4, \ldots$. One may apply the usual successive approximation method to obtain a solution of (2.1). This is, however, not appropriate for our present purpose. We shall treat the nonlinear integral equation in a different way, finding a *linear* integral equation which is a *linear dilatation* of the equation (2.1). This linear integral equation will be defined on an enlarged space

$$\text{(2.2)} \qquad \boldsymbol{S} = \bigcup_{n=1}^{\infty} D^n,$$

where D^n is the symmetric n-fold product(2) of D, $n \geqq 1$.

For $f \in \boldsymbol{B}^+(D)$, set

$$\text{(2.3)} \qquad \hat{f}(\boldsymbol{x}) = \prod_{j=1}^{n} f(x_j), \quad \text{when } \boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in D^n,$$

$\hat{f}$ is, then, a measurable function on $\boldsymbol{S}$ and $\hat{f} \in \boldsymbol{B}^+(\boldsymbol{S})$ when $f \leqq 1$.

We shall state some fundamental facts which will play an important role in the following discussion.

[a.1] There exist unique nonnegative kernels $\boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y})$ and $\Psi(\boldsymbol{x}, ds\, d\boldsymbol{y})$ defined on $[0, \infty) \times \boldsymbol{S} \times \boldsymbol{S}$ and $\boldsymbol{S} \times [0, \infty) \times \boldsymbol{S}$ respectively, such that when $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in D^n$

$$\text{(2.4)} \qquad \int_{\boldsymbol{S}} \boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y}) \hat{f}(\boldsymbol{y}) = \prod_{j=1}^{n} T_t f(x_j), \qquad f \in \boldsymbol{B}^+(D),$$

and(3)

$$\text{(2.5)} \quad \int_{\boldsymbol{S}} \Psi(\boldsymbol{x}, ds\, d\boldsymbol{y}) \hat{f}(s, \boldsymbol{y}) = ds \sum_{k=1}^{n} T_s(c \cdot f(s, \cdot)^{\beta})(x_k) \prod_{i \neq k;\, i=1}^{n} T_s(f(s, \cdot))(x_i).$$

Moreover the support of $\boldsymbol{T}_t(\boldsymbol{x}, \cdot)$ is concentrated on D^n and that of $\Psi(\boldsymbol{x}, ds \cdot)$ on $D^{n+\beta-1}$ (cf. Ikeda-Nagasawa-Watanabe [3, Lemma 0.3]).

Then we define a linear integral equation with an initial data $\hat{f}$

$$\text{(2.6)} \quad u(t, \boldsymbol{x}) = \boldsymbol{T}_t \hat{f}(\boldsymbol{x}) + \int_0^t \int_{\boldsymbol{S}} \Psi(\boldsymbol{x}, ds\, d\boldsymbol{y}) u(t-s, \boldsymbol{y}), \qquad \boldsymbol{x} \in \boldsymbol{S}, \qquad \hat{f} \in \boldsymbol{B}^+(\boldsymbol{S}),$$

where

$$\boldsymbol{T}_t \hat{f}(\boldsymbol{x}) = \int_{\boldsymbol{S}} \boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y}) \hat{f}(\boldsymbol{y}).$$

Now set

$$\text{(2.7)} \qquad \begin{aligned} u_0(t, \boldsymbol{x}) &= \boldsymbol{T}_t \hat{f}(\boldsymbol{x}), \\ u_k(t, \boldsymbol{x}) &= \int_0^t \int_{\boldsymbol{S}} \Psi(\boldsymbol{x}, ds\, d\boldsymbol{y}) u_{k-1}(t-s, \boldsymbol{y}), \qquad k \geqq 1. \end{aligned}$$

(2) That is, D^n is the quotient space of the n-fold product of D by the permutation of the coordinate.

(3) $\hat{f}(s, \boldsymbol{x})$ is obtained by applying (2.3) to $f(s, x)$ for fixed s.

[a.2] $u_k(t, x)$ is well defined and $\sum_{k=0}^{\infty} u_k(t, x)$ converges for sufficiently small $t>0$. If we put

$$u(t, x) = \sum_{k=0}^{\infty} u_k(t, x), \tag{2.8}$$

when the right-hand side converges, then it is the minimal (local) solution of (2.6) (cf. [3, Chapter IV]).

[a.3] The most important property of the $u(t, x)$ is the following *branching property*:

$$u(t, x) = \prod_{j=1}^{n} u(t, x_j), \quad \text{when } x = (x_1, x_2, \ldots, x_n) \in D^n, \tag{2.9}$$

(cf. [3, Chapter I]).

[a.4] Accordingly, by the branching property, (2.4), and (2.5), it is easy to see that the restriction of $u(t, x)$ on D is a solution of the nonlinear integral equation (2.1). Moreover, it is the *minimal solution* of (2.1), since if $v(t, x)$ is a solution of (2.1) then $v(t, x) = \prod_{j=1}^{n} v(t, x_j)$, $x \in D^n$ is a solution of (2.6) (cf. [3, Theorem 4.7]). We shall call this minimal solution $u(t, x)$, $x \in D$, of (2.1) obtained through (2.6) the *probabilistic solution* of (2.1)([4]).

[a.5] Let $f_{k_i}(s, x)$ $(i=1, 2, \ldots, m)$ be in $\boldsymbol{B}^+([0, \infty) \times D)$ and $a_{k_1 \cdots k_n}$ be certain constants which are symmetric with respect to $(k_1, k_2, \ldots, k_m)$. When $x = (x_1, x_2, \ldots, x_n) \in D^n$,

$$\begin{aligned} &\int_{D^m} \Psi(x, ds\, dy) \left\{ \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} \prod_{i=1}^{m} f_{k_i}(s, y_i) \right\} \\ &\quad = ds \sum_{l=1}^{n} \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} T_s\Big(c \cdot \prod^{\beta} f_{k_i}(s, \cdot)\Big)(x_l) \prod_{j \neq l;\, j=1}^{n} T_s(f_{k_j}(s, \cdot))(x_j), \end{aligned} \tag{2.10}$$

where $m = n+\beta-1$, $\sum^{(k,m)}$ denotes the sum over all $(k_1, k_2, \ldots, k_m)$ satisfying $\sum_{i=1}^{m} k_i = k$, and $\prod^{\beta}$ the product over $i = l, n+1, n+2, \ldots, m$. This representation of Ψ follows from the fact that the integrand of the left-hand side of (2.10) can be expressed by a linear combination of functions of the form $\hat{g}$, $g \in \boldsymbol{B}^+(D)$.

We will give upper and lower bounds of $u_k(t, x)$ in the following sections.

3. **Case 1. There exists a global solution.**

Now we give an upper bound of $u_k(t, x)$.

LEMMA 3.1. *For $f \in \boldsymbol{B}^+(D)$ and $x = (x_1, x_2, \ldots, x_n) \in D^n$, $u_k(t, x)$ which is defined by* (2.7) *has an upper bound*

([4]) When T_t is the semigroup of $\exp\left(-\int_0^t c(x_s)\, ds\right)$-subprocess of a conservative Markov process on D, $u(t, x)$ in (2.8) exists for all $t \geq 0$ and $U_t \hat{f}(x) = u(t, x)$, where U_t is the semigroup of a Markov process on S which has the branching property (branching Markov process). This remark is also true for any T_t, but we need some additional structure for branching Markov processes (cf. Sirao [8], Nagasawa [7]).

$$(3.1)\quad u_k(t,x) \leqq \|c\|^k \cdot \frac{\prod_{i=0}^{k-1}\{n+i(\beta-1)\}}{k!} \cdot \left\{\int_0^t \sup_{y\in D} h(s,y)^{\beta-1}\,ds\right\}^k \cdot \prod_{j=1}^{n} h(t,x_j),$$

where $k=1, 2, 3, \ldots$ *and*

$$(3.2)\qquad h(t,x) = T_t f(x).$$

Proof. We shall prove (3.1) by induction. $u_1(t, x)$ is estimated as follows: By (2.5) and (2.7)

$$(3.3)\qquad \begin{aligned} u_1(t,x) &= \int_0^t ds \sum_{l=1}^{n} T_s(c\cdot h_{t-s}^{\beta})(x_l)\cdot \prod_{i\neq l} T_s h_{t-s}(x_i) \\ &\leqq \|c\| \int_0^t ds\left\{\sup_{y\in D} h(t-s,y)^{\beta-1}\right\} \sum_{l=1}^{n} \prod_{i=1}^{n} T_s h_{t-s}(x_i) \\ &= n\|c\| \int_0^t ds\left\{\sup_{y\in D} h(s,y)^{\beta-1}\right\}\cdot \prod_{i=1}^{n} h(t,x_i), \end{aligned}$$

where we used that $T_s h_{t-s}(x) = h(t, x)$(5). Thus (3.1) is valid for $k=1$.

Suppose that (3.1) is valid for $k \geqq 1$. Then by (2.7) and the induction hypothesis, we have for $x=(x_1, x_2, \ldots, x_n)$ and $m=n+\beta-1$

$$\begin{aligned} u_{k+1}(t,x) &= \int_0^t \int_{D^m} \Psi(x, ds\,dy) u_k(t-s, y) \\ &\leqq \|c\|^k \int_0^t \frac{\prod_{i=0}^{k-1}\{n+\beta-1+i(\beta-1)\}}{k!} \cdot \left\{\int_0^{t-s} dr \sup_{y\in D} h(r,y)^{\beta-1}\right\}^k \\ &\qquad \int_{D^m} \Psi(x, ds\,dy)\cdot \prod_{j=1}^{m} h(t-s, y_j). \end{aligned}$$

By (2.5) this is equal to

$$\begin{aligned} &\|c\|^k \cdot \frac{\prod_{i=0}^{k-1}\{n+\beta-1+i(\beta-1)\}}{k!} \cdot \int_0^t ds\left\{\int_0^{t-s} dr \sup_{y\in D} h(r,y)^{\beta-1}\right\}^k \\ &\qquad\qquad \cdot \sum_{l=1}^{n} T_s(c\cdot h_{t-s}^{\beta})(x_l)\cdot \prod_{j\neq l} T_s h_{t-s}(x_j) \\ &\leqq n\|c\|^{k+1} \cdot \frac{\prod_{i=0}^{k-1}\{n+\beta-1+i(\beta-1)\}}{k!} \cdot \int_0^t ds \sup_{y\in D} h(s,y)^{\beta-1} \\ &\qquad\qquad \cdot \left\{\int_0^s dr \sup_{y\in D} h(r,y)^{\beta-1}\right\}^k \cdot \prod_{j=1}^{n} h(t,x_j) \\ &= \|c\|^{k+1} \cdot \frac{\prod_{i=0}^{k}\{n+i(\beta-1)\}}{(k+1)!} \cdot \left\{\int_0^t ds \sup_{y\in D} h(s,y)^{\beta-1}\right\}^{k+1} \cdot \prod_{j=1}^{n} h(t,x_i)(^6). \end{aligned}$$

This proves (3.1) for $k+1$, completing the proof.

(5) We write sometimes $h_t(x)$ for $h(t, x)$.

(6) Note: $\int_0^t dF(s)F(s)^k/k! = F(t)^{k+1}/(k+1)!$, $F(0)=0$.

COROLLARY 3.2. *Let $u(t, x)$ be the probabilistic solution of* (2.1), *then*

$$u(t, x) \leqq T_t f(x)\left\{1+\sum_{k=1}^{\infty} v_k(t)\right\}, \tag{3.4}$$

where

$$v_k(t) = \frac{\prod_{i=0}^{k-1}\{1+i(\beta-1)\}}{k!}\left\{\|c\| \int_0^t \sup_{y\in D} (T_s f(y))^{\beta-1}\, ds\right\}^k. \tag{3.5}$$

REMARK. When $T_t f(y) < 1$, (3.5) shows that larger β provides better converging factor $(T_t f(y))^{\beta-1}$. Therefore $\sum_{k=1}^{\infty} v_k(t)$ converges more easily for larger β.

THEOREM 3.3. *For $f \in \boldsymbol{B}^+(D)$ satisfying*

$$(\beta-1)\|c\| \int_0^{\infty} \sup_{y\in D} (T_t f(y))^{\beta-1}\, dt < 1, \tag{3.6}$$

there exists a global solution $u(t, x)$ of (2.1).

Moreover there exists a constant $M > 0$ such that

$$u(t, x) \leqq M T_t f(x)\text{(}^7\text{)}. \tag{3.7}$$

Proof. By (3.5) we have

$$\sup_t \frac{v_{k+1}(t)}{v_k(t)} \leqq \frac{1+k(\beta-1)}{k+1}\cdot \|c\| \int_0^{\infty} \sup_{y\in D} (T_t f(y))^{\beta-1}\, dt.$$

Therefore (3.6) implies

$$\sup_t \sum_{k=1}^{\infty} v_k(t) < \infty.$$

Thus the probabilistic solution actually provides a global solution. (3.7) follows from (3.4), completing the proof.

We shall give some applications of the preceding theorem.

COROLLARY 3.4. *Suppose that the semigroup T_t is transient in the following sense: For any open set $U \subset D$ with compact closure $\bar{U}$, ($\bar{U} \neq D$)*

$$\int_0^{\infty} \sup_x T_t(I_U)(x)\, dt < \infty\text{(}^8\text{)}. \tag{3.8}$$

If we assume $\beta \geqq 2$ and if $\delta > 0$ is sufficiently small, then there exists a global solution $u(t, x)$ of (2.1) *for $f = \delta I_U$, and it satisfies* (3.7).

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(7) In this case $u(t, x)$ is the unique bounded solution of (2.1), because u^β satisfies locally Lipschitz's condition.

(8) I_U is the indicator of U.

Proof. The assertion of this corollary is clear from

$$\int_0^\infty \sup_x (T_t f(x))^{\beta-1}\, dt \leqq \delta^{\beta-1} \int_0^\infty \sup_x T_t I_U(x)\, dt.$$

THEOREM 3.5. *Let T_t be the semigroup of the d-dimensional symmetric stable process of index α $(0<\alpha\leqq 2)$, i.e.*

$$\begin{aligned} T_t f(x) &= \int_{R^d} p(t, x-y) f(y)\, dy, \\ e^{-t|z|^\alpha} &= \int_{R^d} e^{i(z,x)} p(t, x)\, dx(^9). \end{aligned} \tag{3.9}$$

Let

$$d(\beta-1)/\alpha > 1, \tag{3.10}$$

and γ a positive number. Then there exists a positive number δ with the following property: If

$$0 \leqq f(x) \leqq \delta p(\gamma, x), \tag{3.11}$$

then there exists a global solution $u(t, x)$ of (2.1) *which satisfies*

$$0 \leqq u(t, x) \leqq M p(t+\gamma, x), \tag{3.12}$$

for some positive constant M.

When $\alpha=2$, i.e., T_t is the semigroup of the d-dimensional Brownian motion, this theorem was first proved by Fujita [2] by a different method.

Proof. If an initial data f satisfies (3.11), we have

$$T_t f(x) \leqq \delta p(t+\gamma, x).$$

Since

$$p(t+\gamma, x) = (t+\gamma)^{-d/\alpha} p(1, (t+\gamma)^{-1/\alpha} x),$$

and

$$p(1, y) \leqq p(1, 0), \quad \text{for } y \in R^d,$$

we have

$$\begin{aligned} \int_0^\infty \sup_x (T_t f(x))^{\beta-1}\, dt &\leqq \delta^{\beta-1} p(1, 0)^{\beta-1} \int_0^\infty (t+\gamma)^{-d(\beta-1)/\alpha}\, dt \\ &= \delta^{\beta-1} p(1, 0)^{\beta-1} \frac{\gamma^{1-d(\beta-1)/\alpha}}{d(\beta-1)/\alpha - 1}. \end{aligned}$$

Therefore if we take δ sufficiently small, (3.6) is satisfied. Hence the assertion of this theorem follows from Theorem 3.3.

(9) $|z|$ and (z, x) denote norm and inner product, respectively.

REMARK. Put

$$A = \sum_{i,j} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b^i(x) \frac{\partial}{\partial x^i},$$

where a^{ij} and b^i are sufficiently smooth and subject to $\sum_{i,j} a^{ij}(x)\lambda_i\lambda_j \geqq \sum_i \lambda_i^2$ for all $x \in R^d$. Then it is known that the elementary solution $p(t, x, y)$ of $\partial u/\partial t = Au$ has the following upper bound:

$$p(t, x, y) \leqq Kt^{-d/2} \exp(-a|x-y|^2/t),$$

where a and K are certain positive constants. Therefore Theorem 3.5 is true when we take $\int p(t, x, y)\, dy f(y)$ as $T_t f(x)$, where we put $\alpha = 2$.

4. **Case 2. There exists no global solution.** In order to obtain a criterion for existence of no global solution, we give a lower bound of $u_k(t, x)$.

LEMMA 4.1. *Assume*

$$\inf_{x \in D} c(x) = c_0 > 0. \tag{4.1}$$

Then, for nonnegative $f \in \boldsymbol{B}(D)$ *and* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $u_k(t, \boldsymbol{x})$, *which is defined by* (2.7), *has a lower bound*

$$\begin{aligned} u_k(t, \boldsymbol{x}) \geqq c_0^k &\left\{ \sum^{(k,n)} a_{k_1 k_2 \cdots k_n} h(t, x_1)^{k_1(\beta-1)} \cdot h(t, x_2)^{k_2(\beta-1)} \cdots h(t, x_n)^{k_n(\beta-1)} \right\} \\ &\cdot \prod_{j=1}^n h(t, x_j) \frac{t^k}{k!}, \end{aligned} \tag{4.2}$$

where $k = 1, 2, 3, \ldots$, $h(t, x) = T_t f(x)$, *and* $a_{k_1 k_2 \cdots k_n}$ *are certain symmetric constants satisfying*

$$\sum^{(k,n)} a_{k_1 k_2 \cdots k_n} = n(n+\beta-1)\cdots(n+(k-1)(\beta-1)), \tag{4.3}$$

where $\sum^{(k,n)}$ *denotes the sum over all* $(k_1, k_2, \ldots, k_n)$ *satisfying* $\sum_{i=1}^n k_i = k$.

Proof. We shall prove (4.2) by induction. Noting the following inequality which is justified by Jensen's inequality([10]),

$$T_s(h(t-s, \cdot)^\beta) \geqq \{T_s(h(t-s, \cdot))\}^\beta = h(t, x)^\beta, \tag{4.4}$$

we have by (2.7)

$$\begin{aligned} u_1(t, \boldsymbol{x}) &= \int_0^t ds \sum_{l=1}^n T_s(c \cdot h_{t-s}^\beta)(x_l) \cdot \prod_{i \neq l} T_s h_{t-s}(x_i)(^{11}), \\ &\geqq c_0 \int_0^t ds \sum_{l=1}^n h(t, x_l)^\beta \cdot \prod_{i \neq l} h(t, x_i), \end{aligned}$$

that is, (4.2) is verified for $k = 1$, with $a_{00 \cdots 010 \cdots 0} = 1$.

([10]) $\beta \geqq 2$.

([11]) We write $h_t(x) = h(t, x)$.

Suppose that (4.2) is valid for $k \geqq 1$. By (2.7), the assumption, and (2.10), we have for $x=(x_1, x_2, \ldots, x_n) \in D^n$ and $m=n-1+\beta$

$$
\begin{aligned}
u_{k+1}(t, x) &= \int_0^t \int_{D^m} \Psi(x, ds\, dy) u_k(t-s, y) \\
&\geqq c_0^k \int_0^t \int_{D^m} \Psi(x, ds\, dy) \Big\{ \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} \prod_{i=1}^m h(t-s, y_i)^{k_i(\beta-1)+1} \Big\} \cdot \frac{(t-s)^k}{k!} \\
&\geqq c_0^{k+1} \int_0^t ds \sum_{l=1}^n \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} T_s \Big(\prod^{\beta} h_{t-s}^{k_i(\beta-1)+1} \Big)(x_l) \\
&\qquad \prod_{j \neq l; j=1}^n T_s(h_{t-s}^{k_j(\beta-1)+1})(x_j) \cdot \frac{(t-s)^k}{k!} \text{ [12]} \\
&\geqq c_0^{k+1} \sum_{l=1}^n \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} \{ h(t, x_l)^{\Sigma^\beta (k_i(\beta-1)+1)} \} \\
&\qquad \prod_{j \neq l; j=1}^n h(t, x_j)^{k_j(\beta-1)+1} \cdot \frac{t^{k+1}}{(k+1)!} \text{ [13]},
\end{aligned}
$$

where we used (4.4) and performed the integration with respect to s in the last step. The last line is equal to

$$
\begin{aligned}
c_0^{k+1} \sum_{l=1}^n \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} h(t, x_l)^{(\Sigma^\beta k_i+1)(\beta-1)} \cdot \prod_{j \neq l; j=1}^n h(t, x_j)^{k_j(\beta-1)} \\
\cdot \prod_{j=1}^n h(t, x_j) \cdot \frac{t^{k+1}}{(k+1)!}.
\end{aligned} \tag{4.5}
$$

If we introduce $k_l' = \sum^\beta k_i + 1$, this can be written as

$$
c_0^{k+1} \sum_{l=1}^n \sum^{(k,m)} a_{k_1 k_2 \cdots k_m} h(t, x_l)^{k_l'(\beta-1)} \prod_{j \neq l} h(t, x_j)^{k_j(\beta-1)} \prod_{j=1}^n h(t, x_j) \cdot \frac{t^{k+1}}{(k+1)!}.
$$

Consequently we have

$$
\begin{aligned}
u_{k+1}(t, x) \geqq c_0^{k+1} \sum^{(k+1,n)} a'_{k_1 k_2 \cdots k_n} h(t, x_1)^{k_1(\beta-1)} \cdot h(t, x_2)^{k_2(\beta-1)} \cdots h(t, x_n)^{k_n(\beta-1)} \\
\cdot \prod_{j=1}^n h(t, x_j) \cdot \frac{t^{k+1}}{(k+1)!},
\end{aligned}
$$

where we put

$$
a'_{k_1 k_2 \cdots k_n} = \sum_{l=1}^n \sum_{p_l + k_{n+1} + \cdots + k_m = k_l - 1} a_{k_1 \cdots p_l \cdots k_m}. \tag{4.6}
$$

This proves (4.2) for $k+1$. $a'_{k_1 \cdots k_n}$ are symmetric because so are $a_{k_1 \cdots k}$. Moreover, since we have, by the induction hypothesis,

$$
\sum^{(k,m)} a_{k_1 k_2 \cdots k_m} = \{n+(\beta-1)\}\{n+2(\beta-1)\} \cdots \{n+k(\beta-1)\},
$$

[12] $\prod^\beta$ denotes the product over $i=l, n+1, n+2, \ldots, m$.

[13] $\sum^\beta$ denotes the sum over $i=l, n+1, n+2, \ldots, m$.

we have, noting (4.6),

$$\sum^{(k+1,n)} a'_{k_1k_2\cdots k_n} = n\{n+(\beta-1)\}\{n+2(\beta-1)\}\cdots\{n+k(\beta-1)\},$$

which proves (4.3) for $k+1$, completing the proof.

COROLLARY 4.2. *Let $u(t, x)$ be the probabilistic solution of* (2.1), *then*

$$u(t,x) \geqq T_tf(x)\left\{1+\sum_{k=1}^{\infty} v_k(t,x)\right\}, \qquad x \in D, \tag{4.7}$$

where

$$v_k(t,x) = \frac{1}{k!}\prod_{j=0}^{k-1}\{1+j(\beta-1)\}\{c_0t(T_tf(x))^{\beta-1}\}^k. \tag{4.8}$$

THEOREM 4.3. *For $f \in \boldsymbol{B}^+(D)$ satisfying, for some $x_0 \in D$ and $t_0>0$,*

$$(\beta-1)c_0t_0(T_{t_0}f(x_0))^{\beta-1} > 1\text{(14)}, \tag{4.9}$$

all solution $u(t, x)$ of the equation (2.1) *blows up at a point in a finite time interval* (*i.e. no global solution exists*).

Proof. By [a.3] the probabilistic solution $u(t, x)$ is the minimal solution of (2.1). Therefore it is sufficient to consider this solution $u(t, x)$. Assume that $u(t, x)$ does not blow up all $t>0$. Then $u(t, x)$ satisfies (4.7). We have, however, for sufficiently large k

$$\frac{v_{k+1}(t_0,x_0)}{v_k(t_0,x_0)} = \frac{1+k(\beta-1)}{k+1}c_0t_0(T_{t_0}f(x_0))^{\beta-1} > 1,$$

which contradicts the assumption.

We shall give some applications of the above theorem.

COROLLARY 4.4(15). *Let D be a bounded domain in R^d and let T_t be the semigroup of an A-diffusion on D with absorbing boundary*(16). *If the initial data $f \geqq 0$ takes sufficiently large values on an open set with positive Lebesgue measure, then the solution $u(t, x)$ of* (2.1) *blows up in a finite time interval*(17).

REMARK. In the above corollary, A-diffusion with absorbing boundary is a process on $\bar{D}=D\cup\{\delta\}$ (one-point compactification of D) with δ as the terminal point. We always assume $f(\delta)=0$ for $f \in \boldsymbol{B}(D)$.

(14) $c_0=\inf_{x\in D}\inf c(x)>0$.

(15) A different proof of this theorem is given in S. Ito [6].

(16) This is the process with transition probability $p(t, x, y)\,dy$, where $p(t, x, y)$ is the elementary solution of $\partial u/\partial t=Au$, $u|_{\partial D}=0$, $A=a^{ij}(x)(\partial^2/\partial x^i\,\partial x^j)+b^i(x)\partial/\partial x^i$.

(17) We assume $\inf_{x\in D} c(x)=c_0>0$.

THEOREM 4.5[18]. *Let T_t be the semigroup of the d-dimensional symmetric stable process with index* α $(0<\alpha\leqq 2)$. *Let*

$$0 < d(\beta-1)/\alpha < 1. \tag{4.10}$$

Then for any nonnegative measurable function f on R^d which has strictly positive values in an open set with positive Lebesgue measure, all solution $u(t, x)$ of (2.1)[19] *blows up in a finite time interval, i.e.*, (2.1) *has no global solution.*

Proof. First of all we note that we have, if $t\geqq 1$, $T_tf(x)\geqq t^{-d/\alpha}T_1f(x)$. On the other hand there exists $x_0\in R^d$ such that $0<T_1f(x_0)$ by the assumption. Therefore under the condition (4.10), we have, if t is sufficiently large,

$$(\beta-1)c_0t(T_tf(x_0))^{\beta-1} = (\beta-1)c_0t^{1-d(\beta-1)/\alpha}(T_1f(x_0))^{\beta-1} > 1.$$

Hence $u(t, x)$ blows up in a finite time interval by Theorem 4.3. This completes the proof.

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[18] This result was first proved by Fujita [2] in the case $\alpha=2$, i.e., for Brownian motion by a different method.

[19] We assume $\inf_{x\in R^d} c(x)=c_0>0$.