# Probability and Measure

# Second Edition

# Patrick Billingsley

The University of Chicago

JOHN WILEY & SONS

New York • Chichester • Brisbane • Toronto • Singapore

# Contents

## CHAPTER 1. PROBABILITY

1. Borel's Normal Number Theorem, 1

The Unit Interval—The Weak Law of Large Numbers—The Strong Law of Large Numbers—Strong Law Versus Weak—Extending the Probabilities

2. Probability Measures, 16

Spaces—Classes of Sets—Probability Measures—Lebesgue Measure on the Unit Interval—Constructing  $\sigma$ -Fields\*

3. Existence and Extension, 32

Construction of the Extension—Uniqueness and the  $\pi$ - $\lambda$ Theorem—Monotone Classes—Completeness—Lebesgue Measure on the Unit Interval—Nonmeasurable Sets

4. Denumerable Probabilities, 45

General Formulas—Limits Sets—Independent Events—Subfields— The Borel-Cantelli Lemmas—The Zero-One Law—Strong Laws Versus Weak

\*Stars indicate topics that may be omitted on a first reading.

5. Simple Random Variables, 63

Definition—Independence—Existence of Independent Sequences— Expected Value—Inequalities

6. The Law of Large Numbers, 80

The Strong Law—The Weak Law—Bernstein's Theorem—A Refinement of the Second Borel–Cantelli Lemma

7. Gambling Systems, 88

Gambler's Ruin—Selection Systems—Gambling Policies—Bold Play\* —Timid Play\*

8. Markov Chains, 107

Definitions—Higher-Order Transitions—An Existence Theorem—Transience and Persistence—Another Criterion for Persistence—Stationary Distributions—Exponential Convergence\*—Optimal Stopping\*

9. Large Deviations and the Law of the Iterated Logarithm,\* 142

Moment Generating Functions—Large Deviations—Chernoff's Theorem—The Law of the Iterated Logarithm

## CHAPTER 2. MEASURE

10. General Measures, 155

Classes of Sets—Conventions Involving  $\infty$ —Measures—Uniqueness

11. Outer Measure, 162

Outer Measure—Extension—An Approximation Theorem— Carathéodory's Condition\*

12. Measures in Euclidean Space, 171

Lebesgue Measure—Regularity—Specifying Measures on the Line—Specifying Measures in  $\mathbb{R}^k$ 

13. Measurable Functions and Mappings, 182

Measurable Mappings—Mappings into  $R^k$ —Limits and Measurability —Transformations of Measures

viii

14. Distribution Functions, 189

Distribution Functions—Exponential Distributions—Weak Convergence —Convergence of Types\*—Extremal Distributions\*

### **CHAPTER 3. INTEGRATION**

15. The Integral, 202

Definition—Nonnegative Functions—Uniqueness

16. Properties of the Integral, 209

Equalities and Inequalities—Integration to the Limit—Integration over Sets—Densities—Change of Variable—Uniform Integrability— Complex Functions

17. Integral with Respect to Lebesgue Measure, 224

The Lebesgue Integral on the Line—The Riemann Integral— The Fundamental Theorem of Calculus—Change of Variable— The Lebesgue Integral in  $R^k$ —Stieltjes Integrals

#### Product Measure and Fubini's Theorem, 234

Product Spaces—Product Measure—Fubini's Theorem—Integration by Parts—Products of Higher Order

Hausdorff Measure,\* 247

The Definition—The Normalizing Constant—Change of Variable—Calculations

# CHAPTER 4. RANDOM VARIABLES AND EXPECTED VALUES

259

20. Random Variables and Distributions, 259

Random Variables and Vectors—Subfields—Distributions— Multidimensional Distributions—Independence—Sequences of Random Variables—Convolution—Convergence in Probability—The Glivenko-Cantelli Theorem<sup>\*</sup>

#### CONTENTS

21. Expected Values, 280

Expected Value as Integral—Expected Values and Distributions—Moments—Inequalities—Joint Integrals—Independence and Expected Value—Moment Generating Functions

22. Sums of Independent Random Variables, 290

The Strong Law of Large Numbers—The Weak Law and Moment Generating Functions—Kolmogorov's Zero-One Law—Maximal Inequalities—Convergence of Random Series—Random Taylor Series\* —The Hewitt-Savage Zero-One Law\*

23. The Poisson Process, 307

Characterization of the Exponential Distribution—The Poisson Process —The Poisson Approximation—Other Characterizations of the Poisson Process—Stochastic Processes

24. Queues and Random Walk,\* 322

The Single-Server Queue—Random Walk and Ladder Indices—Exponential Right Tail—Exponential Left Tail—Queue Size

## CHAPTER 5. CONVERGENCE OF DISTRIBUTIONS

335

25. Weak Convergence, 335

Definitions—Uniform Distribution Modulo 1\*—Convergence in Distribution—Convergence in Probability—Fundamental Theorems—Helly's Theorem—Integration to the Limit

26. Characteristic Functions, 351

Definition—Moments and Derivatives—Independence—Inversion and the Uniqueness Theorem—The Continuity Theorem—Fourier Series\*

27. The Central Limit Theorem, 366

Identically Distributed Summands—The Lindeberg and Lyapounov Theorems—Feller's Theorem<sup>\*</sup>—Dependent Variables<sup>\*</sup>

- Infinitely Divisible Distributions,\* 382
  Vague Convergence—The Possible Limits—Characterizing the Limit
- 29. Limit Theorems in  $\mathbb{R}^k$ , 390

The Basic Theorems—Characteristic Functions—Normal Distributions in  $R^k$ —The Central Limit Theorem—Skorohod's Theorem in  $R^k \star$ 

Х

30. The Method of Moments,\* 405

The Moment Problem—Moment Generating Functions—Central Limit Theorem by Moments—Application to Sampling Theory—Application to Number Theory

# CHAPTER 6. DERIVATIVES AND CONDITIONAL PROBABILITY

31. Derivatives on the Line,\* 419

The Fundamental Theorem of Calculus—Derivatives of Integrals—Singular Functions—Integrals of Derivatives—Functions of Bounded Variation

32. The Radon-Nikodym Theorem, 440

Additive Set Functions—The Hahn Decomposition—Absolute Continuity and Singularity—The Main Theorem

## 33. Conditional Probability, 448

The Discrete Case—The General Case—Properties of Conditional Probability—Difficulties and Curiosities—Conditional Probability Distributions

34. Conditional Expectation, 466

Definition—Properties of Conditional Expectation—Conditional Distributions and Expectations—Sufficient Subfields\*—Minimum-Variance Estimation\*

35. Martingales, 480

Definition—Submartingales—Gambling—Functions of Martingales—Inequalities—Convergence Theorems—Reversed Martingales—Applications: Derivatives—Likelihood Ratios—Bayes Estimation—A Central Limit Theorem\*

## CHAPTER 7. STOCHASTIC PROCESSES

36. Kolmogorov's Existence Theorem, 506

Stochastic Processes—Finite-Dimensional Distributions—Product Spaces—Kolmogorov's Existence Theorem—The Inadequacy of  $\mathcal{R}^T$ 

419

37. Brownian Motion, 522

xii

Definition—Continuity of Paths—Measurable Processes—Irregularity of Brownian Motion Paths—The Strong Markov Property—Skorohod Embedding\*—Invariance\*

38. Separability,\* 551

Introduction—Definitions—Existence Theorems—Consequences of Separability—Separability in Product Space

APPENDIX	564
NOTES ON THE PROBLEMS	575
BIBLIOGRAPHY	610
LIST OF SYMBOLS	613
INDEX	615