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Abstract

The probability density function for wave propagating in a straight perfect electrical conductor (PEC) rough wall tunnel is deduced from the mathematical models of the random electromagnetic fields. The field propagating in caves or tunnels is a complex-valued Gaussian random processing by the Central Limit Theorem. The probability density function for single modal field amplitude in such structure is Ricean. Since both expected value and standard deviation of this field depend only on radial position, the probability density function, which gives what is the power distribution, is a radially dependent function.

The radio channel places fundamental limitations on the performance of wireless communication systems in tunnels and caves. The transmission path between the transmitter and receiver can vary from a simple direct line of sight to one that is severely obstructed by rough walls and corners. Unlike wired channels that are stationary and predictable, radio channels can be extremely random and difficult to analyze. In fact, modeling the radio channel has historically been one of the more challenging parts of any radio system design; this is often done using statistical methods. In this contribution, we present the most important statistic property, the field probability density function, of wave propagating in a straight PEC rough wall tunnel. This work only studies the simplest case – PEC boundary which is not the real world but the methods and conclusions developed herein are applicable to real world problems which the boundary is dielectric.

The mechanisms behind electromagnetic wave propagation in caves or tunnels are diverse, but can generally be attributed to reflection, diffraction, and scattering. Because of the multiple reflections from rough walls, the electromagnetic waves travel along different paths of varying lengths. The interactions between these waves cause multipath fading at any location, and the strengths of the waves decrease as the distance between the transmitter and receiver increases.

Since there exist multiple propagation paths, the received signal consists of many signals, each of which is described by a propagation delay and an attenuation factor, and they are statistical independent each other [1]. Both the propagation delays and the attenuation factors are spatially dependent, as a result of changes in the structure of the medium or boundaries. Let us assume that the received signal is made of *N* signals which when added produced a sum of phasors, and the random complex field has the form

$$\Phi(\rho,\phi,z,f) = \Phi_r + j\Phi_i, \tag{1}$$

where Φ_r and Φ_i are real and imaginary parts of the complex field, respectively.

$$\Phi_r - m_r = \sum_{n}^{N} \Phi_{rn}, \qquad \text{and} \qquad \Phi_i - m_i = \sum_{n}^{N} \Phi_{in}, \tag{2}$$

where m_r and m_i are expected values of the real and imaginary parts of the complex field, and Φ_{rn} and Φ_{in} are the individual field components of real and imaginary parts of the complex field, respectively. As a consequence of the central limit theorem, the received signals are approximately Gaussian random process since N is quite big. This means that the field propagating in a cave or tunnel is typically a complex-valued Gaussian random process. The electromagnetic fields $\Phi(\rho,\phi,z,t)$ we studied in time domain is the real quantities in the practice. It has been shown that Φ_r-m_r and Φ_i-m_i are uncorrelated [2]. This carries out that they are independent because of the Gaussian process. The joint probability density function of the complex field Φ is written as

$$p_2(\Phi_r,\Phi_i)=\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(\Phi_r-m_r)^2+(\Phi_i-m_i)^2}{2\sigma^2}\right], \quad (3)$$

where σ is the standard deviation. It is clear that in polar coordinates

$$\left.\begin{array}{l}\Phi_r=R\cos\phi\\ \Phi_i=R\sin\phi\end{array}\right\}. \quad (4)$$

The Jacobian is $|J|=R$. Then the joint probability density function of the complex field Φ in polar coordinates is

$$p_2(R,\phi)=|J|\,p_2(\Phi_r,\Phi_i)=\frac{R}{2\pi\sigma^2}\exp\left[-\frac{R^2-2Rm\cos\psi+m^2}{2\sigma^2}\right], \quad (5)$$

where $m_r\cos\phi+m_i\sin\phi=m\cos(\theta-\phi)$, $\theta-\phi=\psi$, $m=\sqrt{m_r^2+m_i^2}$, $\tan\theta=m_i/m_r$.

The probability density function for the random variable R is

$$p_1(R)=\int_{-\pi}^{\pi}p_2(R,\phi)d\phi=\frac{R}{\pi\sigma^2}\exp\left(-\frac{R^2+m^2}{2\sigma^2}\right)\int_0^{\pi}\exp\left(\frac{Rm\cos\psi}{\sigma^2}\right)d\psi. \quad (6)$$

By 9.6.16 of [3] we have

$$p_1(R) = \frac{R}{\sigma^2}\exp\left(-\frac{R^2+m^2}{2\sigma^2}\right)I_0\left(\frac{Rm}{\sigma^2}\right), \tag{7}$$

where $I_0(x)$ is the zeroth order modified Bessel function of the first kind. This probability density function for the random field amplitude R is Ricean. The parameter m denotes the specular field amplitude of the signal.

The electric and magnetic fields are expressed in terms of two scalar functions Φ and Ψ, [4]

$$\left.\begin{aligned}\vec{E} &= \frac{1}{\hat{y}}\nabla\times\nabla\times\Phi\hat{a}_z - \nabla\times\Psi\hat{a}_z\\ \vec{H} &= \frac{1}{\hat{z}}\nabla\times\nabla\times\Psi\hat{a}_z + \nabla\times\Phi\hat{a}_z\end{aligned}\right\}, \tag{8}$$

where $\hat{z} = j\omega\mu_0$ and $\hat{y} = j\omega\varepsilon_0$. Casey has solved these scalar functions [5]. For $Nlth$ order quasi-TM mode, the first-order potential is expressed in terms of Fourier-Stieltjes integral

$$\begin{aligned}\Phi_{Nl}(\rho,\phi,z) = e^{j(N\phi - k_{zNl}z)}\Bigg\{ & B_{Nl}J_N\left(\frac{p_{Nl}\rho}{a}\right) + \frac{A_{Nl}}{2\pi}\sum_{m=-\infty}^{\infty} e^{jm\phi}\,\frac{\hat{y}}{a}\\ & \int_{-\infty}^{\infty}\frac{J_{m+N}(\lambda_{Nl}\rho)Nk_z^{'}J_N(p_{Nl})}{\lambda_{Nl}^2[J_{m+N}(\lambda_{Nl}a) + \lambda_{Nl}^2\sigma_s^2 J^{"}_{m+N}(\lambda_{Nl}a)/2]}e^{-jk_z^{'}z}d\nu_m(k_z^{'})\\ & + \frac{B_{Nl}}{2\pi}\sum_{m=-\infty}^{\infty} e^{jm\phi}\int_{-\infty}^{\infty}\frac{p_{Nl}J_{m+N}(\lambda_{Nl}\rho)(k_{zNl}k_z^{'} - p_{Nl}^2/a^2)J_N^{'}(p_{Nl})}{\lambda_{Nl}^2[J_{m+N}(\lambda_{Nl}a) + \lambda_{Nl}^2\sigma_s^2 J^{"}_{m+N}(\lambda_{Nl}a)/2]a}e^{-jk_z^{'}z}d\nu_m(k_z^{'})\Bigg\},\end{aligned} \tag{9}$$

where $J_N(x)$ is Nth order first kind of Bessel function, k_{zNl} is the $Nlth$ mode cutoff wave number, $p_{Nl} = \sqrt{k_0^2 - k_{zNl}^2}a$, a is the average radius of the tunnel, $\lambda_{Nl} = \sqrt{k_0^2 - (k_{zNl} + k_z^{'})^2}$, σ_s is the standard deviation of the wall roughness, and

A_{Nl} and B_{Nl} are the *Nlth* constant coefficients for TE and TM modes, respectively.

$d\nu_m(k_z^{'})$ is Fourier-Stieltjes integral variable such that

$$\mathcal{E}\{d\nu_m(k_z)\} = 0, \tag{10}$$

and

$$\mathcal{E}\{d\nu_m(k_z)d\nu_n^*(k_z^{'})\} = 2\pi\delta_{mn}\delta(k_z - k_z^{'})dk_z dk_z^{'} S_m(k_z^{'}), \tag{11}$$

and $S_m(k_z)$ is roughness spectral densities. The first-order *Nlth* order quasi-TE mode potential is similar to the quasi-TM potential (9) but is much complicated function of m, N, ρ, $k_z^{'}$, and k_{zNl}. We shall not take the space to write it out. Using (8) the different electromagnetic field components are the combinations of the different derivatives associated with spatial variables ρ, ϕ, and z of Φ_{Nl} and Ψ_{Nl} [4]. The statistical properties (10) and (11) are dependent of $k_z^{'}$ only; therefore, the different electromagnetic fields possess the same statistical characteristics of which expressed by Φ_{Nl} and Ψ_{Nl}. Furthermore, the field is found to comprise a deterministic or coherent component, identical to that which would be found in a tunnel with a smooth wall, and a random or incoherent component whose expected value is zero and whose variance functions can be expressed in terms of integrals over the power spectral density of the wall roughness.

The real and imaginary parts of *Nlth* TE or TM mode of the expected values are

$$m_{rNl} = \begin{Bmatrix} \mathcal{E}\{\Re[\Psi_{Nl}(\rho,\phi,z)]\} \\ \mathcal{E}\{\Re[\Phi_{Nl}(\rho,\phi,z)]\} \end{Bmatrix} = \begin{Bmatrix} A_{Nl} \\ B_{Nl} \end{Bmatrix} J_N\left(\frac{p_{Nl}\rho}{a}\right)\cos(N\phi - k_{zNl}z), \tag{12}$$

$$m_{iNl} = \begin{Bmatrix} \mathcal{E}\{\Im[\Psi_{Nl}(\rho,\phi,z)]\} \\ \mathcal{E}\{\Im[\Phi_{Nl}(\rho,\phi,z)]\} \end{Bmatrix} = \begin{Bmatrix} A_{Nl} \\ B_{Nl} \end{Bmatrix} J_N\left(\frac{p_{Nl}\rho}{a}\right)\sin(N\phi - k_{zNl}z), \tag{13}$$

It is clear that the amplitude of the expected values are

$$m_{Nl} = \begin{Bmatrix} |A_{Nl}| \\ |B_{Nl}| \end{Bmatrix} \left| J_N\left(\frac{p_{Nl}\rho}{a}\right) \right|. \tag{14}$$

From (14) the amplitude of signal expected value m_{Nl} depends on radial variable ρ only, not the angular variable. The *Nlth* mode variances for the TE and TM cases are

$$\begin{aligned}(\sigma_{Nl}^{TE})^2 &\approx \mathcal{E}\{\Psi_{Nl}(\rho,\phi,z)\Psi_{Nl}^*(\rho,\phi,z)\} - \mathcal{E}^2\{\Psi_{Nl}(\rho,\phi,z)\} \\ &= \frac{|A_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{R_1(k_z^{'},\rho)}{a}[R_2(k_z^{'})P_1(k_z^{'}) - P_2(k_z^{'})]\right|^2 S_m(k_z^{'})dk_z^{'} \\ &+ \frac{|B_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|R_1(k_z^{'},\rho)[R_2(k_z^{'})P_3(k_z^{'}) - P_4(k_z^{'})]\frac{p_{Nl}}{\hat{y}a}J_N^{'}(p_{Nl})\right|^2 S_m(k_z^{'})dk_z^{'},\end{aligned} \tag{15}$$

$$\begin{aligned}(\sigma_{Nl}^{TM})^2 &\approx \mathcal{E}\{\Phi_{Nl}(\rho,\phi,z)\Phi_{Nl}^*(\rho,\phi,z)\} - \mathcal{E}^2\{\Phi_{Nl}(\rho,\phi,z)\} \\ &= \frac{|A_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\hat{y}}{a}R_3(k_z^{'},\rho)P_1(k_z^{'})\right|^2 S_m(k_z^{'})dk_z^{'} \\ &+ \frac{|B_{Nl}|^2}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{R_3(k_z^{'},\rho)P_3(k_z^{'})J_N(p_{Nl})}{a}\right|^2 S_m(k_z^{'})dk_z^{'},\end{aligned} \tag{16}$$

where

$$R_1(k_z^{'},\rho) = \frac{J_{m+N}(\lambda_{Nl}\rho)}{a\lambda_{Nl}J_{m+N}^{'}(\lambda_{Nl}a) + \lambda_{Nl}^2[a\lambda_{Nl}J_{m+N}^{'''}(\lambda_{Nl}a)/2 + J_{m+N}^{''}(\lambda_{Nl}a)]\sigma_s^2},$$

$$R_2(k_z^{'}) = -\frac{(m+N)(k_z^{'} + k_{zNl})[J_{m+N}(\lambda_{Nl}a) + \lambda_0^2\sigma_s^2 J_{m+N}^{'}(\lambda_{Nl}a)]}{\lambda_{Nl}^2[J_{m+N}(\lambda_{Nl}a) + \lambda_{Nl}^2\sigma_s^2 J_{m+N}^{'}(\lambda_{Nl}a)/2]},$$

$$R_3(k_z^{'},\rho) = \frac{J_{m+N}(\lambda_{Nl}\rho)}{\lambda_{Nl}^2[J_{m+N}(\lambda_{Nl}a) + \lambda_{Nl}^2\sigma_s^2 J_{m+N}^{'}(\lambda_{Nl}a)/2]},$$

$P_1(k_z^{'}) = Nk_z^{'}J_N(p_{Nl})$, $P_2(k_z^{'}) = p_{Nl}J_N^{'}(p_{Nl}) + p_{Nl}^2 J_N^{''}(p_{Nl}) + NmJ_N(p_{Nl})$,

$P_3(k_z^{'}) = k_z^{'}k_{zNl} - p_{Nl}^2/a^2$, and $P_4(k_z^{'}) = k_{zNl}(N+m)$.

We must choose $S_m(k_z^{'})$ carefully so that the numerical integral can be carrying out. The physics tells us that $S_m(k_z^{'})$ must remain real values. Let $S_m(k_z^{'})$ be a band-limited roughness spectral densities such that

$$\sigma_s^2 = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} S_m(k_z) dk_z. \tag{17}$$

We assume that $S_n(k_z) = S_{-n}(k_z)$ and $S_n(k_z) = S_n(-k_z)$, and set $S_n(k_z) = S_0$, $(N_0 \le n \le N_0 + \Delta N,\ k_{z0} \le k_z \le k_{z0} + \Delta k_z)$, and $S_n(k_z) = 0$ elsewhere in the region $n \geq 0,\ k_z \geq 0$. We evaluate the constant S_0 using equation (17):

$$\sigma_s^2 = \frac{2}{\pi} \sum_{n=N}^{N_0+\Delta N} \int_{k_{z0}}^{k_{z0}+\Delta k_z} S_n(k_z) dk_z = \frac{2}{\pi} S_0 \Delta k_z (\Delta N + 1), \tag{18}$$

so that

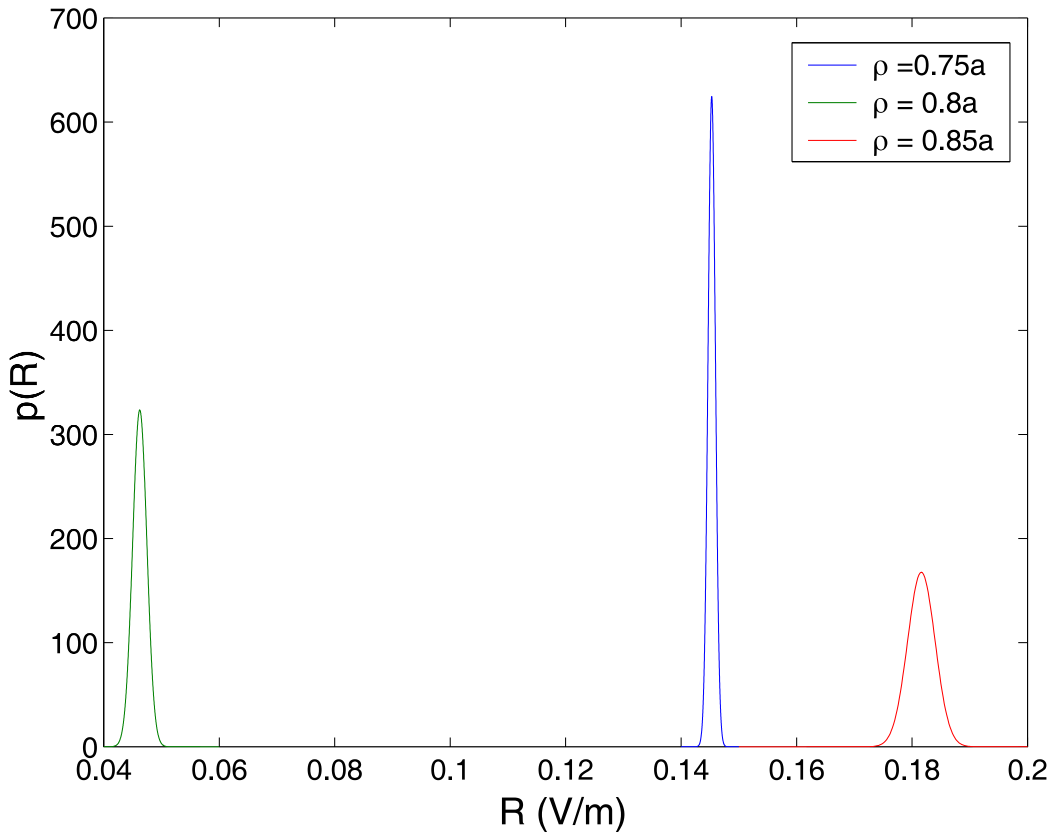
$$S_0 = \frac{\pi \sigma_s^2}{2\Delta k_z (\Delta N + 1)}. \tag{19}$$

The correlation function associated with this spectrum is easily shown to be

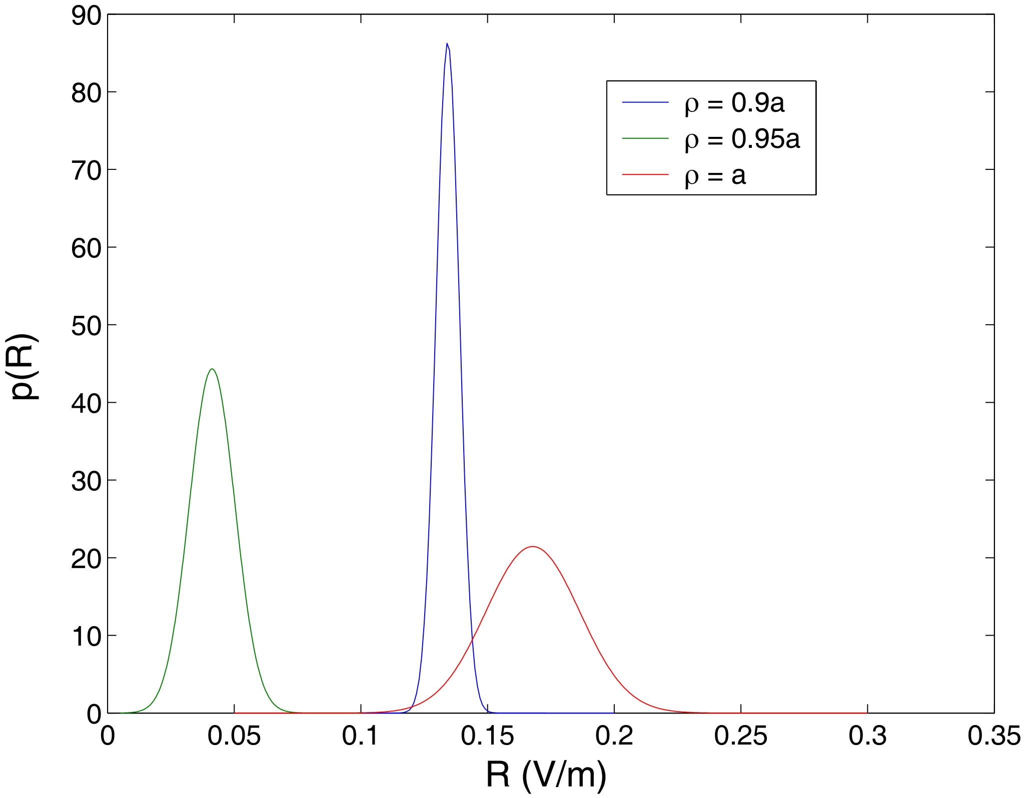
$$C(\phi, z) = \frac{2S_0}{\pi z} [\sin(k_{z0} + \Delta k_z) z - \sin k_{z0} z] \frac{\sin[(\Delta N + 1)\phi / 2]}{\sin(\phi / 2)} \cos[(N_0 + \Delta N / 2)\phi]. \tag{20}$$

It is obvious that the variance depends on the radial position as well.

Figure 1 (a) and (b) show the probability density function for TE_{11} mode at different ρ. It is clear that the width ΔR of the probability density function closest to the wall is largest compared to the ρ away from the wall. Figure 2 shows the probability density function for TE_{11} mode at different roughness σ_s. We find out that the larger of σ_s, the more field fluctuation is. Figure 3 illustrates the probability density function for the TE_{11} and TM_{01}modes at the same location. We found that TE_{11} mode, which has the lowest cutoff frequency, is affected more by wall roughness than TM_{01} mode. In addition, the shapes of all probability density functions are symmetrical like the Gaussian bell. We attribute this to the higher expected value to variance ratio.



(a)



(b)

Figure 1 Probability density function for TE_{11} mode at different radial location, σ $= 0.2a, f = 1GHz.$

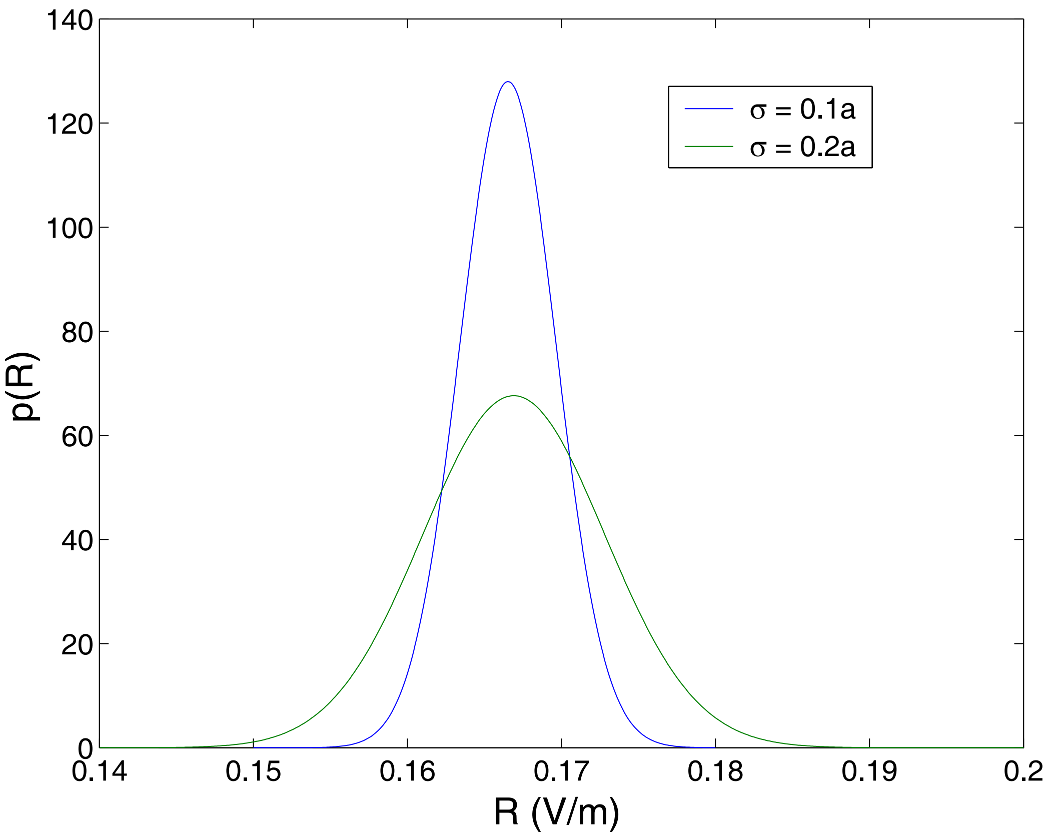


Figure 2 Probability density function for TE_{11} mode at different roughness situations. $\rho = a$, $f = 1GHz$.

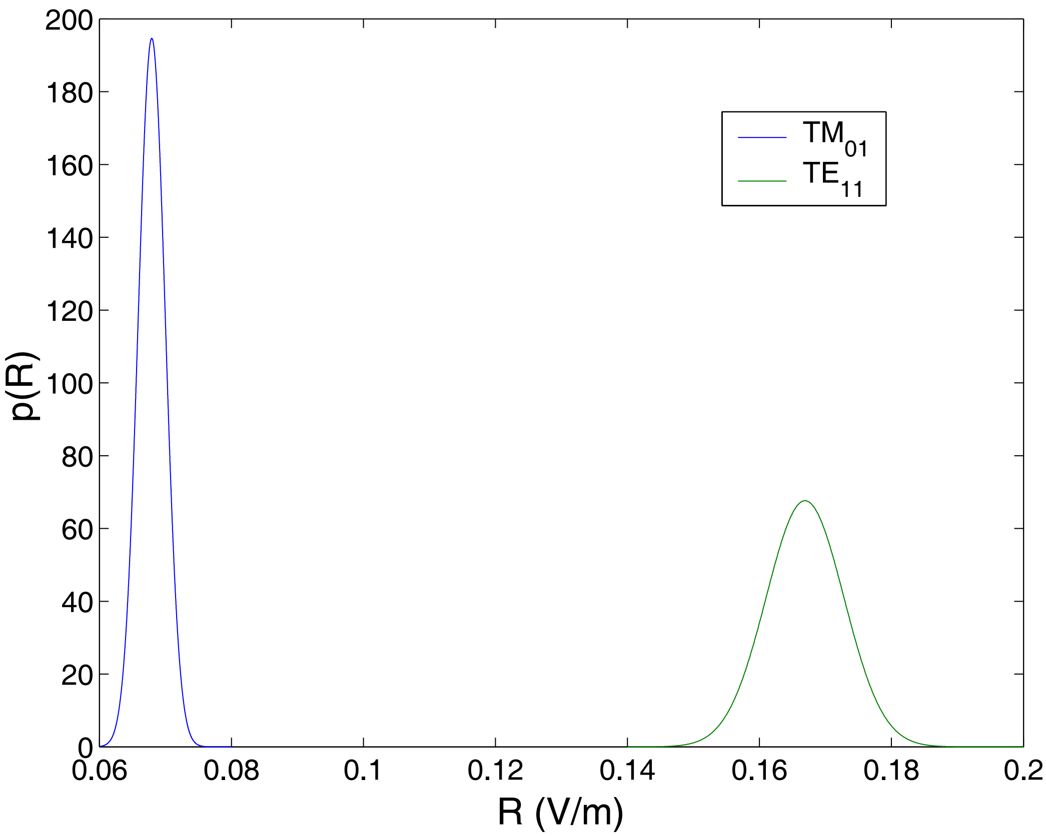


Figure 3 Probability density function for different modes. $\rho = a$, $\sigma = 0.2a, f = 1GHz.$

In conclusion, we deduce the single mode probability density function for waves propagating in a straight PEC rough wall cave or tunnel from the mathematical models of the random electromagnetic fields. The fields propagating in a cave or tunnel are complex-valued Gaussian random process, by using the Central Limit Theorem. The phase and amplitude of the field joint probability density function are independent because of the Gaussian process property. We have shown that the probability density function for single mode random field amplitude propagating in a straight rough wall tunnel or cave is Ricean. This tells us that there is a dominant signal component, such as a line-of-sight propagation path. In such a situation, random components arriving at different angles are superimposed towards a stationary signal. At the output of an envelope detector, it has the effect of adding a DC component to the random multi-path signal. Since both expected value and standard deviation depend only on radial position, the probability density function for random field amplitude propagating in a straight rough wall tunnel or cave is a radially dependent function. The lowest propagation mode is most affected by the rough wall than those higher order modes. The probability density functions are symmetric like the Gaussian bell because of the higher expected value to variance ratio.

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[1] S. O. Rice, Mathematical Analysis of Random Noise, Bell System Tech. J. **24,** (1945), 46-156.
[2] Athanasios Papoulis, Probability, Randow Variables, and Stochastic Process, McGraw-Hill, New York, 1991.
[3] Milton Abramowitz and Irene Stegun, Handbook of Mathematical Functions, Dover, New York, 1970.
[4] R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill, New York, 1961.
[5] Kendall F. Casey, “Modal Propagation in a Circular Waveguide with a Rough Wall”, North American Radio Science Meeting, Columbus, OH, June 2003.