Probability Density Functions of Large-Scale Turbulence in the Ocean

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Probability density functions (pdfs) of surface velocity and surface velocity gradients in the ocean are calculated using altimetric data from the Topex/Poseidon satellite. These provide information about turbulence in a high-Reynolds-number geophysical flow. Both velocity pdfs and velocity gradient pdfs calculated over small regions are Gaussian but have more exponential shapes as the size of the region increases. We develop a simple explanation for the non-Gaussianity of velocity pdfs based on the inhomogeneity of eddy kinetic energy in the ocean. [S0031-9007(98)07902-2]

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Two-dimensional turbulence is a natural paradigm for the high-Reynolds-number fluid flows that dominate ocean variability on scales of 50–80 km, the "mesoscale." Satellite altimeters offer a new means to study two-dimensional turbulent motions of the ocean. In this paper, we use altimeter data to calculate probability density functions (pdfs) for the ocean: pdfs are Gaussian locally but exponential over the global ocean.

Numerical and theoretical studies have shown that two-dimensional turbulence is characterized by coherent vortices separated by irrotational regions of straining motion [1]. This phenomenology provides a good conceptual model of the mesoscale and larger-scale oceanic circulation, which is dominated by two-dimensional motions associated with the constraints of strong stratification and the earth's rotation [2]. We therefore expect two-dimensional turbulence theory to illuminate processes such as eddy-induced transports of heat, chemical tracers, and biota, which are important to the earth's climate system [3]. In addition, observations can show us how notions of two-dimensional turbulence fail to describe the oceans and atmospheres. We therefore consider pdfs as measurable quantities that can be used to compare real-world turbulence with better understood physical analogs.

Pdfs are a standard statistical tool for analyzing three-dimensional turbulence [4]. They have been used less often to study two-dimensional turbulence [5], although there has been some work on vorticity pdfs [6]. Recently, however, the pdfs of the flow caused by an ensemble of identical point vortices have been calculated and compared to results from numerical two-dimensional turbulence simulations [7]. The theoretical velocity pdfs are Gaussian, and the velocity gradient pdfs have a truncated Cauchy distribution.

While the results from point vortex models provide a simple model for oceanic behavior, they are unlikely to describe fully the ocean mesoscale, which supports Rossby wave motions and has a finite Rossby radius. Recent studies of Lagrangian data from floats and numerical model output have revealed Lagrangian velocity pdfs that are significantly non-Gaussian [8]. Using satellite observations, we will show that Eulerian ocean velocity pdfs can be Gaussian if only a small region of the ocean is considered, but take on a more exponential structure if eddy kinetic energy (EKE) varies significantly through the region considered. Gradient pdfs follow a similar pattern, with Gaussian distributions for small regions and more exponential structure for regions large enough to have significant variability in root-mean-square (rms) velocity gradients. To our knowledge, this Letter represents the first calculation of Eulerian velocity and velocity gradient pdfs for a two-dimensional geophysical flow on the global scale.

The Topex/Poseidon satellite was launched in 1992 [9]. We will employ data from the Topex radar altimeter which measures the distance from the satellite to the ocean surface, providing $O(10^5)$ observations over the global ocean every ten days. From this can be calculated the sea-surface height anomaly relative to time–mean sea-surface height.

The geostrophic relation $v = -(g/f)\partial\eta/\partial l$ gives the velocity perpendicular to the satellite ground track; g is the local gravitational acceleration, f is the Coriolis parameter, η is the sea-surface height anomaly, and l is the distance along a ground track. Altimeter measurements are subject to errors due to atmospheric effects, solid earth tides, ocean tides, and responses due to surface pressure. We apply only the tidal corrections to avoid introducing high wave number noise into the velocity estimates. We also discard data within 10° of the equator where the geostrophic relation is more sensitive to measurement noise. "Transverse" velocity gradients are determined by computing the first derivative of v along satellite ground

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tracks, while "longitudinal" velocity gradients cannot be determined from altimeter measurements. Details of the data processing are discussed elsewhere [10,11].

We first calculate velocity and velocity gradient pdfs over 2.5° boxes in the ocean. Each box contains on average 5400 data points. For both velocity and velocity gradients, the resulting pdfs usually resemble Gaussian distributions, though in some cases the decay for large velocities is slower, and the pdfs are more exponential. Figure 1 shows examples of velocity pdfs for three 2.5° boxes. The first two, from the South Atlantic and South Pacific, show Gaussian distributions of varying widths, as is typical of most of the ocean. The third, from the Malvinas Current, a region of strongly varying eddy activity, is more exponential.

Figure 2 shows examples of velocity gradient pdfs. The first, from the South Atlantic location also depicted in Fig. 1, has a Gaussian distribution that is typical of most of the ocean. The second, from the midlatitude Indian Ocean, is an example of an exponential gradient pdf. Gradient pdf width varies strongly with latitude; low latitude pdfs are wider than high latitude pdfs. In no cases are gradient pdfs well represented by a Cauchy distribution.

Sophisticated statistical tests exist to measure departure from normality [12], but the presence of outliers in the data makes these tests too stringent for the present problem. The simple approach of comparing the goodness of fit of a Gaussian and of an exponential distribution shows that, for both the velocity pdf and the velocity gradient pdf, a Gaussian distribution fits the data better in about 80% of the cases. Velocity pdfs are typically exponential in regions of high eddy activity such as the Gulf Stream or the Kuroshio Extension. If we increase the size of the boxes, the velocity pdfs are increasingly likely to be more exponential than Gaussian: only 54% of 30° boxes are better represented by a Gaussian pdf.

Velocity gradient pdfs sometimes show exponential behavior at midlatitudes (typically around 40°) but boxes having exponential pdfs are not strongly associated with locations of energetic western boundary currents. Approximately 80% of boxes sized between 2.5° and 40° have Gaussian velocity gradient pdfs. In contrast, for 60° boxes, all of the velocity gradient pdfs are more exponential than Gaussian.

Why should the observed global pdfs fail to match the Gaussian and Cauchy distributions predicted by simple point vortex models? We begin by looking at velocity distributions. Our observations suggest that regions that encompass broad ranges of EKE have velocity pdfs that are more exponential.

In this paper, we will argue that velocity pdfs are in fact Gaussian over small regions of the ocean, as suggested by Fig. 1, so that locally the pdfs can be represented as

$$p_g(v) = \frac{1}{\sqrt{2\pi}s} \exp\left(-\frac{v^2}{2s^2}\right), \qquad (1)$$

but we allow the width of the distribution, s, to vary throughout the ocean. Pdf width is equivalent to rms velocity (or the square root of EKE). Thus this is the same as assuming that the energy dissipation is intermittent [13], which was the basis of K62 theory [14]. We shall draw s from a general set of pdfs $q(s)$. The pdf that we expect to measure is then given by

$$p(v) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{v^2}{2s^2}\right) \frac{q(s)}{s}\, ds\,. \qquad (2)$$

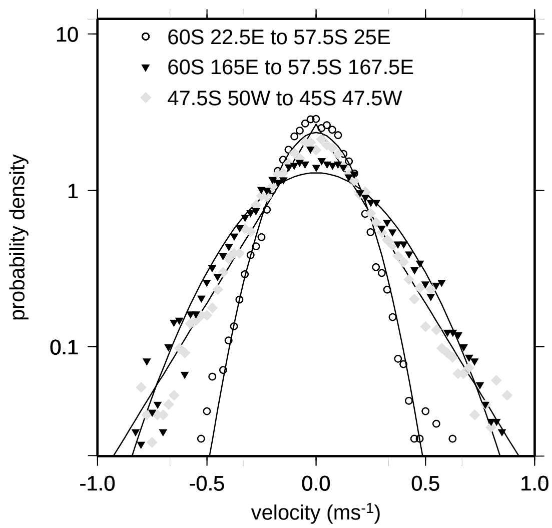

FIG. 1. Observed velocity pdfs for three 2.5° boxes. Open circles are from the South Atlantic, black triangles from the South Pacific Ocean, and gray diamonds from the energetically varying Malvinas Current in the South Atlantic, an exponential distribution. Solid lines show best fit Gaussian or exponential pdfs.

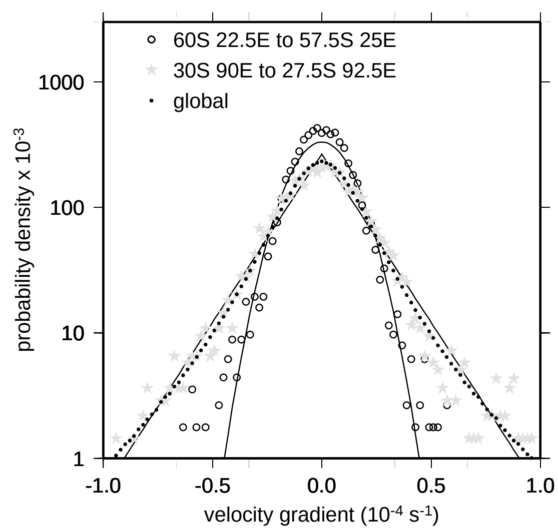

FIG. 2. Observed velocity gradient pdfs for two 2.5° boxes with fitted functional distributions (solid lines) and global velocity gradient pdf (heavy dots).

If we take $q(s) = \delta(s - s_0)$, the velocity pdf reduces to a Gaussian distribution with variance s_0. This corresponds to the observation that small boxes have Gaussian pdfs. In general, the pdf $p(v)$ will depend on the form of the function $q(s)$.

For the altimeter observations of the global ocean, Fig. 3 shows probability densities of the width of the fitted Gaussian pdf. Although the most frequent width is 0.2 ms^{-1}, much larger widths are observed in some of the 2.5° boxes. The distribution is noisy, but may be fitted approximately by a function that is parabolic near the origin, and exponentially decreasing for large s. Here, we shall consider the gamma probability distribution, with the origin shifted to s_o:

$$q(s) = \frac{H(s - s_0)}{a^{m+1}\Gamma(m + 1)} (s - s_0)^m \exp\left(-\frac{|s - s_0|}{a}\right), \tag{3}$$

where $H(s - s_0)$ is a Heaviside step function. Fitting (3) gives $a = 0.10$, $s_0 = 0.12$, and $m = 0.54$, shown in Fig. 3. The pdf calculated from (2) is shown in Fig. 4. (Alternate functional forms of $q(s)$ are discussed by [11].)

The gamma distribution prediction for the global velocity pdf based on the calculated functional fit agrees well with the observed velocity pdf, duplicating both the broad Gaussian pdf near the origin and the exponential tails at large v. In the range $-1.5 < v < 1.5$ ms^{-1} the rms difference between the log of the observed distribution and the log of the theoretical distribution is 0.21 for the gamma distribution, compared with 8.1 for the Gaussian distribution, and 0.24 for the exponential distribution.

One possible reason for the misfit between our predicted $p(v)$ and the observed velocity pdf is that $q(s)$ is not weighted by the number of measurements available in each box. In an alternate calculation of the observed $q(s)$, we weighted by the number of samples in each box, but the data were too noisy. Since s is related to the square root of EKE, in principle it would be possible to use observed EKE density to determine $q(s)$. However, such a calculation requires knowing both components of velocity, which are not simultaneously available from altimetry.

The exponential behavior of velocity pdfs in some of the small boxes might be explained using the same statistical framework that we used for the global pdfs, in this case allowing $q(s)$ to represent variations in pdf width over time in high-variability areas of the ocean.

Finally, since velocity gradients, like velocity, are observed to have Gaussian pdfs in small boxes, the same statistical framework as for the global velocity pdfs can be used to explain the global gradient pdf.

Our statistical model justifies why observed pdfs from the global ocean are not Gaussian, but what accounts for the fact that local velocity gradient pdfs have distributions that are nearly identical in shape to velocity pdfs? The Cauchy distribution of velocity gradients predicted by point vortex models [7] depends on the r^{-1} behavior of classical point vortices and also on the fact that all vortices are identical. In the ocean, vortices have a range of strengths, and velocity profiles differ from r^{-1} since the Rossby radius is finite. This allows us to use the central limit theorem to predict that both velocity and velocity gradient should be locally Gaussian, unlike the case of two-dimensional point vortices, truncated or not.

In addition, practical considerations suggest that observed gradient pdfs should resemble observed velocity pdfs. In the ocean, altimetric velocities cannot be sampled at very small spatial intervals and are subject to measurement noise. These two effects mean that adjacent velocity

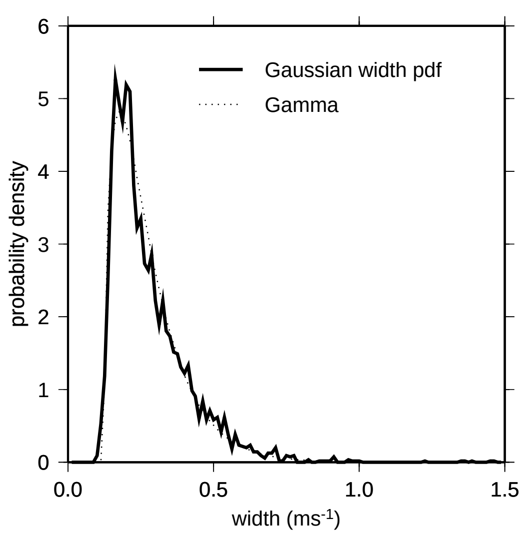

FIG. 3. Pdf of width of Gaussians fitted to observed velocity pdfs (heavy line) and gamma distribution fitted to the observed with pdf (dotted line).

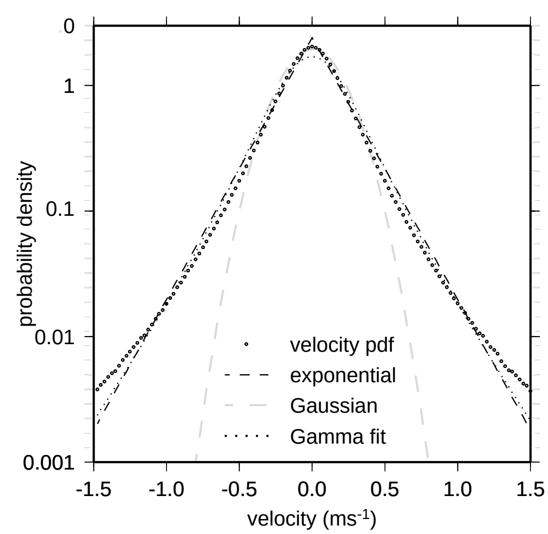

FIG. 4. Probability density distributions of all Topex velocities for the global ocean (large dots), along with (dashed line) an exponential fit to the pdf (dashed line), a Gaussian fit (gray dashed line), and predicted pdf (small dots) based on assuming that the pdf width is given by the distribution (3).

measurements are less than 20% correlated, so that $\langle v(l) v(l + \Delta l) \rangle$ is small. Therefore the observed velocity gradient pdf can be thought of as representing the difference between two data points randomly selected from a given pdf. If the velocities are initially drawn from a Gaussian distribution, the gradients will also be Gaussian. Because observed velocities are less subject to noise than velocity gradients, theoretical developments for velocity pdfs are likely to be more useful in interpreting observed geophysical turbulence.

In summary, we have used altimeter data to show that surface velocities and velocity gradients in the ocean have locally Gaussian distributions. Observed gradient pdfs do not match the Cauchy distributions of smoothed point vortex models. In contrast, observed velocity pdfs are as predicted by vortex models, but the reasons behind their Gaussian distributions may be different. Point vortex velocity pdfs converge to a Gaussian shape very slowly, while ocean eddies are smooth enough that we can apply the central limit theorem to predict Gaussian distributions under almost all circumstances for both velocity and velocity gradient. This suggests that the ocean mesoscale velocity field is not well represented by an ensemble of two-dimensional point vortices.

On the global scale, observed velocity and velocity gradient pdfs are exponential rather than Gaussian. We can explain the qualitative difference between global and local pdfs using a statistical framework that takes into account spatial variations in rms velocity or rms gradient. Spatial inhomogeneities in the ocean are due to mechanisms such as baroclinic instability that lead to regions of strong variability [11]. These results for the global pdfs suggest that spatial inhomogeneity is an essential part of any description of mesoscale turbulent flows in the global ocean.

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