PROBABILITY DISTRIBUTION OF THE FREE ENERGY OF THE CONTINUUM DIRECTED RANDOM POLYMER IN $1+1$ DIMENSIONS

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Abstract. We consider the solution of the stochastic heat equation

$$\partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} - \mathcal{Z}\dot{\mathscr{W}} \tag{1}$$

with delta function initial condition

$$\mathcal{Z}(T=0) = \delta_0 \tag{2}$$

whose logarithm, with appropriate normalizations, is the free energy of the continuum directed polymer, or the Hopf-Cole solution of the Kardar-Parisi-Zhang equation with narrow wedge initial conditions.

We obtain explicit formulas for the one-dimensional marginal distributions, the *crossover distributions*, which interpolate between a standard Gaussian distribution (small time) and the GUE Tracy-Widom distribution (large time).

The proof is via a rigorous steepest descent analysis of the Tracy-Widom formula for the asymmetric simple exclusion with anti-shock initial data, which is shown to converge to the continuum equations in an appropriate weakly asymmetric limit. The limit also describes the crossover behaviour between the symmetric and asymmetric exclusion processes.

1. Introduction

1.1. KPZ/Stochastic Heat Equation/Continuum Directed Random Polymer.

Despite its popularity as perhaps *the* default model of stochastic growth of a one dimensional interface, we are still far from a satisfactory theory of the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_T h = -\frac{1}{2}(\partial_X h)^2 + \frac{1}{2}\partial_X^2 h + \dot{\mathscr{W}} \tag{3}$$

where $\dot{\mathscr{W}}(T,X)$[1] is space-time white noise

$$E[\dot{\mathscr{W}}(T,X)\dot{\mathscr{W}}(S,Y)] = \delta(T-S)\delta(Y-X). \tag{4}$$

The reason is that even for nice initial data, the solution at a later time $T>0$ will look locally like a Brownian motion in X. Hence the nonlinear term is ill-defined. Naturally one expects that an appropriate Wick ordering of the non-linearity can lead to well defined solutions. However, numerous attempts have led to non-physical answers [14]. By a physical answer one means that for a large class of initial data, the solution $h(T,X)$ looks like

$$h(T,X) \sim C(T) + T^{1/3}\zeta(X) \tag{5}$$

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[1] We attempt to use capital letters for all variables (such as X, T) on the macroscopic level of the stochastic PDEs and polymers. Lower case letters (such as x, t) will denote WASEP variables, the microscopic discretization of these SPDEs.

where $C(T)$ is deterministic and where the statistics of ζ fits into various universality classes depending on the regime of initial data one is looking at. The correct interpretation appears to be that of [3] where $h(T,X)$ is simply *defined* by the Hopf-Cole transform:

$$h(T,X) = -\log \mathcal{Z}(T,X) \tag{6}$$

where $\mathcal{Z}(T,X)$ is the well-defined [35] solution of the stochastic heat equation,

$$\partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} - \mathcal{Z}\dot{\mathscr{W}}. \tag{7}$$

Recently [1] proved the $T^{1/3}$ scaling for this *Hopf-Cole solution h of KPZ* defined through (6) in the *equilibrium regime*, corresponding to starting (3) with a two sided Brownian motion. Strictly speaking, this is not an equilibrium solution for KPZ, but for the stochastic Burgers equation

$$\partial_T u = -\frac{1}{2}\partial_X u^2 + \frac{1}{2}\partial_X^2 u + \partial_X \dot{\mathscr{W}}, \tag{8}$$

formally satisfied by its derivative $u(T,X) = \partial_X h(T,X)$.

In this article, we will be interested in a very different regime, far from equilibrium. It is most convenient to state in terms of the stochastic heat equation (7) for which we will have as initial condition a delta function,

$$\mathcal{Z}(T=0) = \delta_0. \tag{9}$$

This initial condition is natural for the interpretation in terms of random polymers, where it corresponds to the point-to-point free energy. The free energy of the continuum directed random polymer in $1+1$ dimensions is

$$\mathcal{F}(T,X) = \log E_{T,X}\left[:\!\exp\!:\left\{-\int_0^T \dot{\mathscr{W}}(t,b(t))dt\right\}\right] \tag{10}$$

where $E_{T,X}$ denotes expectation over the Brownian bridge $b(t)$, $0 \le t \le T$ with $b(0) = 0$ and $b(T) = X$. The expectation of the Wick ordered exponential $:\!\exp\!:$ is defined using the n step probability densities $p_{t_1,\ldots,t_n}(x_1,\ldots,x_n)$ of the bridge in terms of a series of multiple Itô integrals;

$$\begin{aligned} &E_{T,X}\left[:\!\exp\!:\left\{-\int_0^T \dot{\mathscr{W}}(t,b(t))dt\right\}\right] \\ &= \sum_{n=0}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^n}(-1)^n p_{t_1,\ldots,t_n}(x_1,\ldots,x_n)\mathscr{W}(dt_1dx_1)\cdots\mathscr{W}(dt_ndx_n), \end{aligned} \tag{11}$$

where $\Delta_n(T) = \{(t_1,\ldots,t_n) : 0 \le t_1 \le \cdots \le t_n \le T\}$. Note that the series is convergent in $\mathscr{L}^2(\mathscr{W})$ as one can check that

$$\int_{\Delta_n(T)}\int_{\mathbb{R}^n} p^2_{t_1,\ldots,t_n}(x_1,\ldots,x_n)dt_1dx_1\cdots dt_ndx_n \le C(n!)^{-1/2} \tag{12}$$

and hence the square of the norm, $\sum_{n=0}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^n} p^2_{t_1,\ldots,t_n}(x_1,\ldots,x_n)dt_1dx_1\cdots dt_ndx_n$, is finite. Let

$$p(T,X) = \frac{1}{\sqrt{2\pi T}}e^{-X^2/2T} \tag{13}$$

denote the heat kernel. Then we have

$$\mathcal{Z}(T,X) = p(T,X)\exp\{\mathcal{F}(T,X)\} \tag{14}$$

as can be seen by writing the integral equation for $\mathcal{Z}(T,X)$;

$$\mathcal{Z}(T,X) = p(T,X) + \int_0^T\int_{-\infty}^{\infty} p(T-S,X-Y)\mathcal{Z}(S,Y)\mathscr{W}(dY,dS) \tag{15}$$

and iterating. The factor $p(T,X)$ in (14) represents the difference between conditioning on the bridge going to X, as in (11), and having a delta function initial condition, as in (9). The initial condition corresponds to

$$\mathcal{F}(0,X) = 0, \qquad X \in \mathbb{R}. \tag{16}$$

In terms of KPZ (3), there is no precise mathematical statement of the initial conditions; what one sees as $T \searrow 0$ is a narrowing parabola. In the physics literature this is referred as the *narrow wedge initial conditions.*

We can now state our main result which is an exact formula for the probability distribution for the free energy of the continuum directed random polymer in $1+1$ dimensions, or, equivalently, the one-point distributions of the stochastic heat equation with delta initial condition, or KPZ with narrow wedge initial conditions.

Theorem 1. *The* crossover distributions *defined by*

$$F_T(s) \stackrel{\text{def}}{=} P(\mathcal{F}(T,X) + \tfrac{T}{4!} \leq s) \tag{17}$$

are given explicitly by any of the following equivalent formulas where, for a function $\sigma(t)$, the operator K_σ is defined by its kernel

$$K_\sigma(x,y) = \int_{-\infty}^{\infty} \sigma(t)\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt, \tag{18}$$

where $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)\,dt$ is the Airy function, and where

$$a = a(s) = s - \log\sqrt{2\pi T}, \quad \text{and} \quad \kappa_T = 2^{-1/3}T^{1/3}. \tag{19}$$

(1) *The crossover Airy kernel formula,*

$$F_T(s) = \int_{\tilde{\mathcal{C}}} \frac{d\tilde{\mu}}{\tilde{\mu}} e^{-\tilde{\mu}} \det(I - K_{\sigma_{T,\tilde{\mu}}})_{L^2(\kappa_T^{-1}a,\infty)}, \tag{20}$$

where $\tilde{\mathcal{C}}$ is defined in Definition 9, and $K_{\sigma_{T,\tilde{\mu}}}$ is as above with

$$\sigma_{T,\tilde{\mu}}(t) = \frac{\tilde{\mu}}{\tilde{\mu} - e^{-\kappa_T t}}. \tag{21}$$

Alternatively

$$F_T(s) = \int_{\tilde{\mathcal{C}}} \frac{d\tilde{\mu}}{\tilde{\mu}} e^{-\tilde{\mu}} \det(I - \hat{K}_{\sigma_{T,\tilde{\mu}}})_{L^2(-\infty,\infty)} \tag{22}$$

$$\hat{K}_{\sigma_{T,\tilde{\mu}}}(x,y) = \sqrt{\sigma_{T,\tilde{\mu}}(x-s)}K_{\mathrm{Ai}}\sqrt{\sigma_{T,\tilde{\mu}}(y-s)} \tag{23}$$

where $K_{\mathrm{Ai}}(x,y)$ is the Airy kernel, ie. $K_{\mathrm{Ai}} = K_\sigma$ with $\sigma(t) = \mathbf{1}_{[0,\infty)}(t)$.

(2) *The Gumbel convolution formula,*

$$F_T(s) = 1 - \int_{-\infty}^{\infty} G(r)f(a-r)dr, \tag{24}$$

with $G(r)$ is given by $G(r) = e^{-e^{-r}}$ and where

$$f(r) = \kappa_T^{-1}\det(I - K_{\sigma_T})\mathrm{tr}\left((I - K_{\sigma_T})^{-1}\mathrm{P}_{\mathrm{Ai}}\right), \tag{25}$$

where the operators K_{σ_T} and P_{Ai} act on $L^2(\kappa_T^{-1}r,\infty)$ and are given by their kernels with

$$\mathrm{P}_{\mathrm{Ai}}(\mathrm{x},\mathrm{y}) = \mathrm{Ai}(x)\mathrm{Ai}(y), \tag{26}$$

$$\sigma_T(t) = \frac{1}{1 - e^{-\kappa_T t}}.$$

For σ_T above, the integral in (18) should be intepreted as a principal value integral. The operator K_{σ_T} contains a Hilbert transform of the product of Airy functions which can be partially computed showing that

$$K_{\sigma_T}(x,y) = \int_{-\infty}^{\infty} \tilde{\sigma}_T(t)\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt + \kappa_T^{-1}\pi G_{\frac{x-y}{2}}(\frac{x+y}{2}) \tag{27}$$

where

$$\begin{aligned} \tilde{\sigma}_T(t) &= \frac{1}{1-e^{-\kappa_T t}} - \frac{1}{\kappa_T t} \\ G_a(x) &= \frac{1}{2\pi^{3/2}} \int_0^\infty \frac{\sin(x\xi + \frac{\xi^3}{12} - \frac{a^2}{\xi} + \frac{\pi}{4})}{\sqrt{\xi}} d\xi. \end{aligned} \tag{28}$$

(3) *The cosecant kernel formula,*

$$F_T(s) = \int_{\tilde{\mathcal{C}}} e^{-\tilde{\mu}} \det(I - K_a^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)} \frac{d\tilde{\mu}}{\tilde{\mu}}, \tag{29}$$

where the contour $\tilde{\mathcal{C}}$, the contour $\tilde{\Gamma}_\eta$ and the operator K_a^{csc} is defined in Definition 9.

The proof of the theorem relies on the explicit limit calculation for the weakly asymmetric simple exclusion process (WASEP) contained in Theorem 8 as well as the relationship between WASEP and the stochastic heat equation stated in Theorem 10. Combining those two theorems proves the cosecant kernel formula. The other, alternative formulas are proved in Section 4

We also have the following representation for the Fredholm determinant involved in the above theorem. One should compare this result to the formula for the GUE Tracy-Widom distribution given in terms of the Painlevé II equation (see [28, 29] or the discussion of Section 5.2).

Proposition 2. *Let $\sigma_{T,\tilde{\mu}}$ be as in (21). Then*

$$\begin{aligned} \frac{d^2}{dr^2}\log\det(I - K_{\sigma_{T,\tilde{\mu}}})_{L^2(r,\infty)} &= -\int_{-\infty}^{\infty} \sigma'_{T,\tilde{\mu}}(t) q_t^2(r) dt \\ \det(I - K_{\sigma_{T,\tilde{\mu}}})_{L^2(r,\infty)} &= \exp\left(-\int_r^\infty (x-r)\int_{-\infty}^{\infty} \sigma'_{T,\tilde{\mu}}(t) q_t^2(x) dt dx\right) \end{aligned} \tag{30}$$

where

$$\frac{d^2}{dr^2} q_t(r) = \left(r + t + 2\int_{-\infty}^{\infty} \sigma'_{T,\tilde{\mu}}(t) q_t^2(r) dt\right) q_t(r) \tag{31}$$

with $q_t(r) \sim \mathrm{Ai}(t+r)$ as $r \to \infty$ and where $\sigma'_{T,\tilde{\mu}}(t)$ is the derivative of the function in (21).

This proposition is proved in Section 5.2 and follows from a more general theory developed in Section 5 about a class of generalized integrable integral operators.

It is not hard to show from the formulas in Theorem 1 that $\lim_{s\to\infty} F_T(s) = 1$, but we do not at the present time know how to show from the determinental formulas that $\lim_{s\to-\infty} F_T(s) = 0$, or even that F_T is non-decreasing in s. However, for each T, $\mathcal{F}(T,X)$ is an almost surely finite random variable, and hence we know from the definition (17) that F_T is indeed a non-degenerate distribution function.

The formulas in Theorem 1 suggest that in the limit as T goes to infinity, under $T^{1/3}$ scaling, we recover the celebrated F_{GUE} distribution (sometimes written as F_2) which is the GUE Tracy-Widom distribution, i.e., the limiting distribution of the scaled and centered largest eigenvalue in the Gaussian unitary ensemble.

Corollary 3. *As $T \nearrow \infty$,*

$$F_T\left(T^{1/3}s\right) \to F_{\mathrm{GUE}}(2^{1/3}s) \tag{32}$$

In particular, scaling X as $T^{2/3}X$,

$$\lim_{T\nearrow\infty} P\left(\frac{\mathcal{F}(T,T^{2/3}X)+\frac{T}{4!}}{T^{1/3}} \le s\right) = F_{\mathrm{GUE}}(2^{1/3}s) \tag{33}$$

This is most easily seen from the cosecant kernel formula for $F_T(s)$. Formally, as T goes to infinity the kernel K_a^{csc} behaves as $K_{T^{1/3}s}^{\mathrm{csc}}$ and making a change of variables to remove the T from the exponential argument of the kernel, this approaches the Airy kernel on a complex contour, as given in [32] equation (33). The full proof is given in Section 6.1.

It is elementary to add a temperature β^{-1} into the model. Let

$$\mathcal{F}_\beta(T,X) = \log E_{T,X}\left[:\!\exp\!:\left\{-\beta\int_0^T \dot{\mathscr{W}}(t,b(t))dt\right\}\right]. \tag{34}$$

The corresponding $\mathcal{Z}_\beta(T,X) = p(T,X)\exp\{\mathcal{F}_\beta(T,X)\}$ is the solution of $\partial_T\mathcal{Z}_\beta = \frac{1}{2}\partial_X^2\mathcal{Z}_\beta - \beta\dot{\mathscr{W}}\mathcal{Z}_\beta$ with $\mathcal{Z}_\beta(0,X) = \delta_0(X)$ and hence

$$\mathcal{Z}_\beta(T,X) \overset{\mathrm{distr.}}{=} \beta^2\mathcal{Z}(\beta^4T,\beta^2X) \tag{35}$$

giving the relationship

$$\beta \sim T^{1/4} \tag{36}$$

From this we see also that

$$\mathcal{F}_\beta(T,X) \overset{\mathrm{distr.}}{=} \mathcal{F}(\beta^4T,\beta^2X). \tag{37}$$

Hence the following result about the low temperature limit is, just like Corollary 3, a consequence of Theorem 1:

Corollary 4. *For each fixed $X\in\mathbb{R}$ and $T>0$,*

$$\lim_{\beta\to\infty} P\left(\frac{\mathcal{F}(\beta^4T,\beta^2X)+\frac{\beta^4T}{4!}}{\beta^{4/3}T^{1/3}} \le s\right) = F_{\mathrm{GUE}}(2^{1/3}s). \tag{38}$$

Now we turn to the behavior as T or $\beta \searrow 0$.

Proposition 5. *As $T\beta^4 \searrow 0$,*

$$2^{1/2}\pi^{-1/4}\beta^{-1}T^{-1/4}\mathcal{F}_\beta(T,X) \tag{39}$$

converges in distribution to a standard Gaussian.

This proposition is proved in Section 6.2.

For example with $\beta=1$ the above theorem shows that $\lim_{T\searrow 0} F_T(2^{-1/2}\pi^{1/4}T^{1/4}s) = \int_{-\infty}^s \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx$. Proposition 5 and Corollary 3 show that, under appropriate scalings, the family of distributions F_T

cross over from the Gaussian distribution for small T to the GUE Tracy-Widom distribution for large T.

An inspection the formula for F_T given above in Theorem 1 immediately reveals that there is no dependence on X in the formula. In fact, one can check directly from (11) that

Proposition 6. *For each $T \geq 0$, $\mathcal{F}(T, X)$ is stationary in X.*

This is simply because the Brownian bridge transition probabilities are affine transformations of each other. Performing the change of variables, the white noise remains invariant in distribution. The following conjecture is thus natural:

Conjecture 7. For each fixed $T > 0$, as $T \nearrow \infty$,

$$T^{-1/3}\left(\mathcal{F}(T, T^{2/3}X) + \frac{T}{4!}\right) \tag{40}$$

converges to the Airy$_2$ process in X.

Unfortunately, the very recent extensions of the Tracy-Widom formula for ASEP (74) to multipoint distributions [33] appear not to be conducive to the asymptotic analysis necessary to obtain this conjecture following the program of this article.

The main physical prediction (5) is based on the exact computation

$$\lim_{T\to\infty} T^{-1} \log E[\mathcal{Z}^n(T, 0)] = -\frac{1}{4!}n(n^2 - 1), \tag{41}$$

which can be performed rigorously [2] by expanding the Feynman-Kac formula (10) for $\mathcal{Z}(T, 0)$ into an expectation over n independent copies (replicas) of the Brownian bridge. In the physics literature, the computation is done by noting that the correlation functions

$$E[\mathcal{Z}(T, X_1) \cdots \mathcal{Z}(T, X_n)] \tag{42}$$

can be computed [17] using the Bethe ansatz [20] for a system of particles on the line interacting via an attractive delta function potential. (41) suggests the scaling (5) and is consistent with, but does not imply (3). Note the key point that the moments in (41) grow far too quickly to uniquely determine the underlying distribution. It is very interesting to note that the Tracy-Widom formula for ASEP (74), which is our main tool, is also based on the same idea that hard core interacting systems in one dimension can be rigorously solved via the Bethe ansatz, although, as H. Spohn has pointed out, the analogy is very unclear because the interaction is attractive in the case of the Bose gas.

The probability distribution for the free energy of the continuum directed random polymer, as well as for the solution to the stochastic heat equation and the KPZ equation has been a subject of interest for many years. The reason why we can now compute the distribution is because of the exact formula of Tracy and Widom for the asymmetric simple exclusion process (ASEP) with step initial condition. Once we observe that the weakly asymmetric simple exlusion process (WASEP) with these initial conditions converges to the solution of the stochastic heat equation with delta initial conditions, the calculation of the probability distribution boils down to a careful asymptotic analysis of the Tracy-Widom ASEP formula. This connection is made in Theorem 10 and the WASEP asymptotic analysis is recorded by Theorem 8.

1.1.1. *Outline.* There are three main results in this paper. The first pertains to the KPZ/ stochastic heat equation / continuum directed polymer and is contained in the theorems and corollaries above in Section 1.1. The proof of the equivalence of the formulas of Theorem 1 is given in Section 4. The Painlevé II like formula of Proposition 2 is proved in Section 5.2 along with the formulation of a general theory about a class of generalized integrable integral operators. The other results of the above section are proved in Section 6. The second result is about the WASEP. In Section 1.2 we introduce the fluctuation scaling theory of the ASEP and motivate the second main result which is contained in Section 1.3. The Tracy-Widom ASEP formula is reviewed in Section 1.5 and then a formal explanation of the result is given in Section 1.6. A full proof of this result is contained in Section 2 and its various subsections. The third result is about the connection between the first (stochastic heat equation, etc.) and second (WASEP). The result is stated in Section 1.4 and is proved in Section 3.

1.2. **ASEP scaling theory.** The simple exclusion process with parameters $p, q \geq 0$ (such that $p+q=1$) is a continuous time Markov process on the discrete lattice $\mathbb{Z}$ with state space $\{0,1\}^{\mathbb{Z}}$ (the 1's are thought of as particles and the 0's as holes). The dynamics for this process are given as follows: Each particle has an independent exponential alarmclock which rings at rate one. When the alarm goes off the particle flips a coin and with probability p attempts to jump one site to the right and with probability q attempts to jump one site to the left. If there is a particle at the destination, the jump is suppressed and the alarm is reset (see [21] for a rigorous construction of this process). If $q=1, p=0$ this process is the totally asymmetric simple exclusion process (TASEP); if $q>p$ it is the asymmetric simple exclusion process (ASEP); if $q=p$ it is the symmetric simple exclusion process (SSEP). Finally, if we introduce a parameter into the model, we can let $q-p$ go to zero with that parameter, and then this class of processes are known as the weakly asymmetric simple exclusion process (WASEP). It is the WASEP that is of central interest to us. ASEP is often thought of as a discretization of KPZ (for the height function) or stochastic Burgers (for the particle density). For WASEP the connection can be made precise (see Sections 1.4 and 3).

There are many ways to initialize these exclusion processes (such as stationary, flat, two-sided Bernoulli, etc.) analogous to the various initial conditions for KPZ/Stochastic Burgers. We consider a very simple initial condition known as *step initial condition* where every positive integer lattice site (i.e. $\{1,2,3,\ldots\}$) is initially occupied by a particle and every other site is empty. Associated to the ASEP are occupation variables $\eta(t,x)$ which equal 1 if there is a particle at position x at time t and 0 otherwise. From these we define $\hat{\eta}=2\eta-1$ which take values ± 1 and define the height function for WASEP with asymmetry $\gamma=q-p$ by

$$h_\gamma(t,x)=\begin{cases} 2N(t)+\sum_{0<y\leq x}\hat{\eta}(t,y), & x>0,\\ 2N(t), & x=0,\\ 2N(t)-\sum_{x<y\leq 0}\hat{\eta}(t,y), & x<0,\end{cases} \tag{43}$$

where $N(t)$ is equal to the net number of particles which crossed from the site 1 to the site 0 in time t. Since we are dealing with step initial conditions h_γ is initially given by (connecting the points with slope ± 1 lines) $h_\gamma(0,x)=|x|$. It is easy to show that because of step initial conditions, the following three events are equivalent:

$$\{h_\gamma(t,x)\geq 2m-x\}=\{J_\gamma(t,x)\geq m\}=\{\mathbf{x}_\gamma(t,m)\leq x\} \tag{44}$$

where $\mathbf{x}_\gamma(t,m)$ is the location at time t of the particle which started at $m>0$ and where $J_\gamma(t,x)$ is a random variable known as the current. $J_\gamma(t,x)$ is defined to be the number of particles which started to the right of the origin at time 0 and ended to the left or at x at time t. The γ emphasizes the strength of the asymmetry.

In the case of the ASEP ($q > p$, $\gamma \in (0,1)$) and the TASEP ($q = 1, p = 0$, $\gamma = 1$) there is a well developed fluctuation theory for the height function. We briefly review this now since it both motivates the time/space/fluctuation scale we will use throughout this paper, and also since we are ultimately interested in understanding the transition in behaviour from WASEP to ASEP.

The following result was proved for $\gamma = 1$ (TASEP) by Johansson [15] and for $0 < \gamma < 1$ (ASEP) by Tracy and Widom [32]:

$$\lim_{t\to\infty} P\left(\frac{h_\gamma(\frac{t}{\gamma},0) - \frac{1}{2}t}{t^{1/3}} \geq -s\right) = F_{\mathrm{GUE}}(2^{1/3}s). \tag{45}$$

In the case of TASEP, the one point distribution limit has been extended to a process level limit. Consider a time t, a space scale of order $t^{2/3}$ and a fluctuation scale of order $t^{1/3}$. Then, as t goes to infinity, the spatial fluctuation process, scaled by $t^{1/3}$ converges to the Airy$_2$ process (see [6, 9] for this result for TASEP, [16] for DTASEP and [22] for the closely related PNG model). Precisely, for $m \geq 1$ and real numbers $x_1, \ldots, x_m$ and $s_1, \ldots, s_m$:

$$\lim_{t\to\infty} P\left(h_\gamma(t, x_k t^{2/3}) \geq \frac{1}{2}t + (\frac{x_k^2}{2} - s_k)t^{1/3},\ k = 1,\ldots,m\right) = P\left(\mathcal{A}_2(x_k) \leq 2^{1/3}s_k,\ k = 1,\ldots,m\right) \tag{46}$$

where $\mathcal{A}_2$ is known as the Airy$_2$ process (see, for example,[6, 9]) and has one-point marginals F_{GUE}. Notice that in order to get this process limit, we needed to deal with the parabolic curvature of the height function above the origin by including $(\frac{x_k^2}{2} - s_k)$ rather than just $-s_k$. In fact, if one were to replace t by tT for some fixed T, then the parabola would become $\frac{x_k^2}{2T}$. We shall see that this parabola comes up again soon.

An important take away from the result above is the relationship between the exponents for time, space and fluctuations — their $3:2:1$ ratio. It is only with this ratio that we encounter a non-trivial limiting spatial process. For the purposes of this paper, it is more convenient for us to introduce a parameter ϵ which goes to zero, instead of the parameter t which goes to infinity.

Keeping in mind the $3:2:1$ ratio of time, space and fluctuations we define scaling variables

$$t = \epsilon^{-3/2}T, \qquad x = \epsilon^{-1}X, \tag{47}$$

where $T > 0$ and $X \in \mathbb{R}$. With these variables the height function fluctuations around the origin are written as

$$\epsilon^{1/2}\left(h_\gamma(\tfrac{t}{\gamma}, x) - \tfrac{1}{2}t\right). \tag{48}$$

Motivated by the relationship we will establish in Section 1.4, we are interested in studying the Hopf-Cole transformation of the height function fluctuations given by

$$\exp\left\{-\epsilon^{1/2}\left(h_\gamma(\tfrac{t}{\gamma}, x) - \tfrac{1}{2}t\right)\right\}. \tag{49}$$

When $T = 0$ we would like this transformed object to become, in some sense, a delta function at $X = 0$. Plugging in $T = 0$ we see that the height function is given by $|\epsilon^{-1}X|$ and so the exponential becomes $\exp\{-\epsilon^{-1/2}|X|\}$. If we introduce a factor of $\epsilon^{-1/2}/2$ in front of this, then the total integral in X is 1 and this does approach a delta function as ϵ goes to zero. Thus we consider

$$\frac{\epsilon^{-1/2}}{2}\exp\left\{-\epsilon^{1/2}\left(h_\gamma(\tfrac{t}{\gamma}, x) - \tfrac{1}{2}t\right)\right\}. \tag{50}$$

As we shall explain in Section 1.3, the correct scaling for γ to see different behavior than the ASEP or SSEP (i.e., the crossover behavior) is when $\gamma = b\epsilon^{1/2}$. We fix $b=1$, as scaling can give us other values of b. This corresponds with setting

$$\gamma = \epsilon^{1/2}, \qquad p = \frac{1}{2} - \frac{1}{2}\epsilon^{1/2}, \qquad q = \frac{1}{2} + \frac{1}{2}\epsilon^{1/2}. \tag{51}$$

Under this scaling the WASEP is related to the KPZ equation and stochastic heat equation. To help facilitate this connection define

$$\begin{aligned} \nu_\epsilon = &\quad p+q-2\sqrt{qp} &&= \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3), \\ \lambda_\epsilon = &\quad \tfrac{1}{2}\log(q/p) &&= \epsilon^{1/2} + \frac{1}{3}\epsilon^{3/2} + \mathcal{O}(\epsilon^{5/2}), \end{aligned} \tag{52}$$

and define

$$Z_\epsilon(T,X) = \tfrac{1}{2}\epsilon^{-1/2}\exp\left\{-\lambda_\epsilon h_\gamma(\tfrac{t}{\gamma},x) + \nu_\epsilon \epsilon^{-1/2} t\right\}. \tag{53}$$

Observe that this differs from the expression in (50) only to second order in ϵ. This second order difference, however, introduces a shift of $T/4!$ which we will see now. With the connection to the polymer free energy in mind write

$$Z_\epsilon(T,X) = p(T,X)\exp\{F_\epsilon(T,X)\}. \tag{54}$$

where $p(T,X)$ is the heat kernel defined in (13). This implies that the field is defined by

$$F_\epsilon(T,X) = \log(\epsilon^{-1/2}/2) - \lambda_\epsilon h_\gamma(\tfrac{t}{\gamma},x) + \nu_\epsilon\epsilon^{-1/2}t + \frac{X^2}{2T} + \log\sqrt{2\pi T}. \tag{55}$$

We are interested in understanding the behavior of $P(F_\epsilon(T,X) \leq s)$ as ϵ goes to zero. This probability can be translated into a probability for the height function, the current and finally the position of a tagged particle:

$$\begin{aligned} P(F_\epsilon(T,X) + \tfrac{T}{4!} \leq s) &= P\left(\log(\epsilon^{-1/2}/2) - \lambda_\epsilon h_\gamma(\tfrac{t}{\gamma},x) + \nu_\epsilon\epsilon^{-1/2}t + \frac{X^2}{2T} + \log\sqrt{2\pi T} \leq s\right) \\ &= P\left(h_\gamma(\tfrac{t}{\gamma},x) \geq \lambda_\epsilon^{-1}[-s + \log\sqrt{2\pi T} + \log(\epsilon^{-1/2}/2) + \frac{X^2}{2T} + \nu_\epsilon\epsilon^{-1/2}t]\right) \\ &= P\left(h_\gamma(\tfrac{t}{\gamma},x) \geq \epsilon^{-1/2}\left[-a + \log(\epsilon^{-1/2}/2) + \frac{X^2}{2T}\right] + \frac{t}{2}\right) \\ &= P(J_\gamma(\tfrac{t}{\gamma},x) \geq m) \\ &= P(\mathbf{x}_\gamma(\tfrac{t}{\gamma},m) \leq x), \end{aligned} \tag{56}$$

where m is defined as

$$\begin{aligned} m &= \frac{1}{2}\left[\epsilon^{-1/2}\left(-a + \log(\epsilon^{-1/2}/2) + \frac{X^2}{2T}\right) + \frac{1}{2}t + x\right] \\ a &= s - \log\sqrt{2\pi T}. \end{aligned} \tag{57}$$

The $\frac{T}{4!}$ added to $F_\epsilon(T,X)$ comes from taking into account the second order corrections to ν_ϵ and λ_ϵ. It is interesting to note that the same factor appears in [3].

1.3. **WASEP crossover regime.** We now turn to the question of how γ should vary with ϵ. The simplest heuristic argument is to use the KPZ equation

$$\partial_T h_\gamma = -\frac{\gamma}{2}(\partial_X h_\gamma)^2 + \frac{1}{2}\partial_X^2 h_\gamma + \dot{\mathscr{W}}. \tag{58}$$

as a proxy for its discretization ASEP, and rescale

$$h_{\epsilon,\gamma}(t,x) = \epsilon^{1/2} h_\gamma(t/\gamma, x) \tag{59}$$

to obtain

$$\partial_t h_{\epsilon,\gamma} = -\frac{1}{2}(\partial_x h_{\epsilon,\gamma})^2 + \frac{\epsilon^{1/2}\gamma^{-1}}{2}\partial_x^2 h_{\epsilon,\gamma} + \epsilon^{1/4}\gamma^{-1/2}\dot{\mathscr{W}} \tag{60}$$

from which we conclude that we want $\gamma = b\epsilon^{1/2}$ for some $b \in (0,\infty)$. We expect Gaussian behavior as $b \searrow 0$ and F_{GUE} behavior as $b \nearrow \infty$. On the other hand, a simple rescaling reduces everything to the case $b = 1$. Thus it suffices to consider

$$\gamma := \epsilon^{1/2}. \tag{61}$$

From now on we will assume that $\gamma = \epsilon^{1/2}$ unless we state explicitly otherwise. In particular, $F_\epsilon(T,X)$ now should be considered with respect to γ as defined above.

The following theorem is proved in Section 2 though an informative though non-rigorous derivation is given in Section 1.6.

Theorem 8. *For all $s \in \mathbb{R}$, $T > 0$ and $X \in \mathbb{R}$ we have the following convergence:*

$$F_T(s) := \lim_{\epsilon\to 0} P(F_\epsilon(T,X) + \tfrac{T}{4!} \le s) = \int_{\tilde{\mathcal{C}}} e^{-\tilde{\mu}} \det(I - K_a^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)} \frac{d\tilde{\mu}}{\tilde{\mu}}, \tag{62}$$

where $a = a(s)$ is given as in the statement of Theorem 1 and where the contour $\tilde{\mathcal{C}}$, the contour $\tilde{\Gamma}_\eta$ and the operator K_a^{csc} is defined below in Definition 9.

Definition 9. The contour $\tilde{\mathcal{C}}$ is defined as

$$\tilde{\mathcal{C}} = \{e^{i\theta}\}_{\pi/2 \le \theta \le 3\pi/2} \cup \{x \pm i\}_{x>0} \tag{63}$$

The contours $\tilde{\Gamma}_\eta$, $\tilde{\Gamma}_\zeta$ are defined as

$$\tilde{\Gamma}_\eta = \{\frac{c_3}{2} + c_3 i r : r \in (-\infty,\infty)\} \tag{64}$$

$$\tilde{\Gamma}_\zeta = \{-\frac{c_3}{2} + c_3 i r : r \in (-\infty,\infty)\}, \tag{65}$$

where the constant c_3 is defined henceforth as

$$c_3 = 2^{-4/3}. \tag{66}$$

The kernel K_a^{csc} acts on the function space $L^2(\tilde{\Gamma}_\eta)$ through its kernel:

$$K_a^{\mathrm{csc}}(\tilde{\eta},\tilde{\eta}') = \int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{T}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3}a(\tilde{\zeta} - \tilde{\eta}')\} 2^{1/3} \left(\int_{-\infty}^{\infty} \frac{\tilde{\mu} e^{-2^{1/3}t(\tilde{\zeta}-\tilde{\eta}')}}{e^t - \tilde{\mu}} dt\right) \frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}. \tag{67}$$

or, evaluating the inner integral, equivalently:

$$K_a^{\mathrm{csc}}(\tilde{\eta},\tilde{\eta}') = \int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{T}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3}a(\tilde{\zeta} - \tilde{\eta}')\} 2^{1/3} (-\tilde{\mu})^{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')} \pi \csc(\pi 2^{1/3}(\tilde{\zeta} - \tilde{\eta}')) \frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}. \tag{68}$$

1.4. **The connection between WASEP and the stochastic heat equation.** We now state the result about the convergence of the $Z_\epsilon(T,X)$ from (53) to the solution $\mathcal{Z}(T,X)$ of the stochastic heat equation (7) with delta initial data (9).

First we take the opportunity to state (7) precisely: $\mathscr{W}(T)$, $T \geq 0$ is the cylindrical Wiener process, i.e. the continuous Gaussian process taking values in $H^{-1/2-}_{\rm loc}(\mathbb{R}) = \cap_{\alpha<-1/2} H^{\alpha}_{\rm loc}(\mathbb{R})$ with

$$E[\langle \varphi, \mathscr{W}(T)\rangle \langle \psi, \mathscr{W}(S)\rangle] = \min(T,S)\langle \varphi, \psi\rangle \tag{69}$$

for any $\varphi, \psi \in C_c^\infty(\mathbb{R})$, the smooth functions with compact support in $\mathbb{R}$. Here $H^{\alpha}_{\rm loc}(\mathbb{R})$, $\alpha < 0$, consists of distributions f such that for any $\varphi \in C_c^\infty(\mathbb{R})$, φf is in the standard Sobolev space $H^{-\alpha}(\mathbb{R})$, i.e. the dual of $H^{\alpha}(\mathbb{R})$ under the L^2 pairing. $H^{-\alpha}(\mathbb{R})$ is the closure of $C_c^\infty(\mathbb{R})$ under the norm $\int_{-\infty}^{\infty}(1+|t|^{-2\alpha})|\hat{f}(t)|^2 dt$ where $\hat{f}$ denotes the Fourier transform. The distributional time derivative $\dot{\mathscr{W}}(T,X)$ is the space-time white noise

$$E[\dot{\mathscr{W}}(T,X)\dot{\mathscr{W}}(S,Y)] = \delta(T-S)\delta(Y-X). \tag{70}$$

Note the mild abuse of notation for the sake of clarity, as we write $\dot{\mathscr{W}}(T,X)$ even though it is a distribution on $(T,X) \in [0,\infty)\times\mathbb{R}$ as opposed to a classical function of T and X. Let $\mathscr{F}(T)$, $T \geq 0$, be the natural filtration, i.e. the smallest σ-field with respect to which $\mathscr{W}(S)$ are measurable for all $0 \leq S \leq T$.

The stochastic heat equation is then shorthand for its integrated version (15) where the stochastic integral is interpreted in the Itô sense [35], so that, in particular, if $f(T,X)$ is any non-anticipating integrand,

$$E[(\textstyle\int_0^T \int_{-\infty}^{\infty} f(S,Y)\mathscr{W}(dY,dS))^2] = E[(\int_0^T \int_{-\infty}^{\infty} f^2(S,Y) dY dS]. \tag{71}$$

The awkward notation is inherited from stochastic partial differential equations: $\mathscr{W}$ for (cylindrical) Wiener process, $\dot{\mathscr{W}}$ for white noise, and stochastic integrals are taken with respect to white noise $\mathscr{W}(dY,dS)$.

Note that the solution can be written explicitly as a series of multiple Wiener integrals;

$$\mathcal{Z}(T,X) = \sum_{n=0}^{\infty} \int_{\Delta'_n(T)} \int_{\mathbb{R}^n} (-1)^n \prod_{i=1}^{n} p(T_i - T_{i-1}, X_i - X_{i-1}) \mathscr{W}(dT_i dX_i) \tag{72}$$

where $\Delta'_n(T) = \{(t_0,\ldots,t_n) : 0 \leq t_0 \leq \cdots \leq t_n = T\}$.

The random functions $Z_\epsilon(T,X)$ from (53) have discontinuities both in space and in time. If desired, one can linearly interpolate in space so that they become a jump process taking values in the space of continuous functions. But it does not really make things easier. The key point is that the jumps are small, so we use instead the space $D_u([0,\infty); D_u(\mathbb{R}))$ where D refers to right continuous paths with left limits and $D_u(\mathbb{R})$ indicates that in space these functions are equipped with the topology of uniform convergence on compact sets. Let $\mathscr{P}_\epsilon$ denote the probability measure on $D_u([0,\infty); D_u(\mathbb{R}))$ corresponding to the process $Z_\epsilon(T,X)$.

Theorem 10. $\mathscr{P}_\epsilon$, $\epsilon \in (0, 1/4)$ *are a tight family of measures and the unique limit point is supported on* $C([0,\infty); C(\mathbb{R}))$ *and corresponds to the solution (72) of (7) with initial conditions (9).*

In particular, for each fixed X, T and s,

$$\lim_{\epsilon \searrow 0} P(F_\epsilon(T,X) \leq s) = P(\mathcal{F}(T,X) \leq s). \tag{73}$$

The result is motivated by, but does not follow directly from, the results of [3]. This is because of the delta function initial conditions, and the consequent difference in the scaling. It requires a certain amount of work to show that their basic computations are applicable to the present case. This is done in Section 3.

1.5. **The Tracy-Widom Step Initial Condition ASEP formula.** Due to the process level convergence of WASEP to the stochastic heat equation, exact information about WASEP can be, with care, translated into information about the stochastic heat equation. Until recently, very little exact information was known about ASEP and WASEP. The work of Tracy and Widom in the past few years, however, has changed that significantly. At the present their methods provide exact formulas for the one-point distribution of the height function for ASEP.

As such, the key tool in determining the limit as ϵ goes to zero of $P(F_\epsilon(T,X)+\frac{T}{4!}\leq s)$ is the following exact formula for the transition probability for a tagged particle in ASEP started from step initial conditions. This formula was stated in [32] in the form below, and was developed in the three papers [30, 31, 32]. We will apply it to the last line of (56) to give us an exact formula for $P(F_\epsilon(T,X)+\frac{T}{4!}\leq s)$.

Consider $q>p$ such that $q+p=1$ and let $\gamma=q-p$ and $\tau=p/q$. For $m>0$, $t\geq 0$ and $x\in\mathbb{Z}$, [32] gives the following exact formula

$$P(\mathbf{x}(\gamma^{-1}t,m)\leq x)=\int_{S_{\tau^+}}\frac{d\mu}{\mu}\prod_{k=0}^{\infty}(1-\mu\tau^k)\det(I+\mu J_{t,m,x,\mu})_{L^2(\Gamma_\eta)} \tag{74}$$

where S_{τ^+} is a circle centered at zero of radius strictly between τ and 1, and where the kernel of the determinant is given by

$$J_{t,m,x,\mu}(\eta,\eta')=\int_{\Gamma_\zeta}\exp\{\Psi_{t,m,x}(\zeta)-\Psi_{t,m,x}(\eta')\}\frac{f(\mu,\zeta/\eta')}{\eta'(\zeta-\eta)}d\zeta \tag{75}$$

where η and η' are on Γ_η, a circle centered at zero of radius ρ strictly between τ and 1, and the ζ integral is on Γ_ζ, a circle centered at zero of radius strictly between 1 and $\rho\tau^{-1}$ (so as to ensure that $|\zeta/\eta|\in(1,\tau^{-}1)$, and where

$$\begin{aligned} f(\mu,z) &= \sum_{k=-\infty}^{\infty}\frac{\tau^k}{1-\tau^k\mu}z^k \\ \Psi_{t,m,x}(\zeta) &= \Lambda_{t,m,x}(\zeta)-\Lambda_{t,m,x}(\xi) \\ \Lambda_{t,m,x}(\zeta) &= -x\log(1-\zeta)+\frac{t\zeta}{1-\zeta}+m\log\zeta. \end{aligned} \tag{76}$$

Remark 11. Throughout the rest of the paper we will only include the subscripts on J, Ψ and Λ when we want to emphasize the dependence of the kernel/functions on a given variable. Otherwise they will just be notated as J,Ψ and Λ.

1.6. **Weakly asymmetric limit of the Tracy-Widom ASEP formula.** The Tracy and Widom ASEP formula (74) provides an exact expression for the probability $P(F_\epsilon(T,X)+\frac{T}{4!}\leq s)$ by interpreting this, as in (56) in terms of a probability of the location of a tagged particle. It is of great interest to understand the limit of this probability as ϵ goes to zero, as it describes a number of interesting limiting objects. We called this limiting probability $F_T(s)$. Presently we will provide a purely formal explanation for the expression given in Theorem 8 (see Section 1.3) for this limiting function $F_T(s)$. After presenting this formal argument we will stress the point that there are a number of very important technical points which arise during this argument, many of which require serious work to resolve. In Section 2 we will provide a rigorous proof of Theorem 8 that $\lim_{\epsilon\to 0} P(F_\epsilon(T,X)+\frac{T}{4!}\le s) = F_T(s)$ in which we deal with all of these possible pitfalls.

Definition 12. Recall the definitions for the relevant quantities in this limit:

$$\begin{aligned} p &= \frac{1}{2}-\frac{1}{2}\epsilon^{1/2}, \qquad q = \frac{1}{2}+\frac{1}{2}\epsilon^{1/2} \\ \gamma &= \epsilon^{1/2}, \qquad \tau = \frac{1-\epsilon^{1/2}}{1+\epsilon^{1/2}} \\ x &= \epsilon^{-1}X, \qquad t = \epsilon^{-3/2}T \\ m &= \frac{1}{2}\left[\epsilon^{-1/2}\left(-a+\log(\epsilon^{-1/2}/2)+\frac{X^2}{2T}\right)+\frac{1}{2}t+x\right] \\ \{F_\epsilon(T,X)&+\tfrac{T}{4!}\le s\} = \{\mathbf{x}(\frac{t}{\gamma},m)\le x\}, \end{aligned} \tag{77}$$

where $a=a(s)$ is defined in the statement of Theorem 1. We also define the contours Γ_η and Γ_ζ to be

$$\Gamma_\eta = \{z:|z|=1-\tfrac{1}{2}\epsilon^{1/2}\} \qquad \text{and} \qquad \Gamma_\zeta=\{z:|z|=1+\tfrac{1}{2}\epsilon^{1/2}\} \tag{78}$$

The first term in the integrand of (74) is the infinite product $\prod_{k=0}^{\infty}(1-\mu\tau^k)$. Observe that $\tau\approx 1-2\epsilon^{1/2}$ and that S_{τ^+}, the contour on which μ lies, is a circle centered at zero of radius between τ and 1. The infinite product is not well behaved along most of this contour, so we will deform the contour to one along which the product is not highly oscillatory. Care must be taken, however, since the Fredholm determinant has poles at every $\mu=\tau^k$. The deformation must avoid passing through them. Observe now that

$$\prod_{k=0}^{\infty}(1-\mu\tau^k) = \exp\{\sum_{k=0}^{\infty}\log(1-\mu\tau^k)\} \tag{79}$$

and that

$$\sum_{k=0}^{\infty}\log(1-\mu(1-2\epsilon^{1/2})^k)\approx \epsilon^{-1/2}\int_0^\infty \log(1-\mu e^{-2r})dr \approx \epsilon^{-1/2}\mu\int_0^\infty e^{-2r}dr = -\frac{\epsilon^{-1/2}\mu}{2}. \tag{80}$$

With this in mind define

$$\tilde{\mu}=\epsilon^{-1/2}\mu \tag{81}$$

from which we see that if the Riemann sum approximation is reasonable then the infinite product converges to $e^{-\tilde{\mu}}$. We make the $\mu\mapsto\epsilon^{-1/2}\tilde{\mu}$ change of variables and find that if we consider a $\tilde{\mu}$ contour

$$\tilde{\mathcal{C}}_\epsilon = \{e^{i\theta}\}_{\pi/2\le\theta\le 3\pi/2}\cup\{x\pm i\}_{0<x<\epsilon^{-1/2}-1} \tag{82}$$

then the above approximations are reasonable. Thus the infinite product goes to $\exp\{-\tilde{\mu}/2\}$.

Now we turn to the Fredholm determinant. We determine a candidate for the pointwise limit of the kernel. That the combination of these two pointwise limits gives the actual limiting formula as ϵ goes to zero is, of course, completely unjustified at this point. Also, the pointwise limits here disregard the existence of a number of singularities encountered during the argument.

The kernel $J(\eta,\eta')$ is given by an integral and the integrand has three main components: An exponential term

$$\exp\{\Lambda(\zeta)-\Lambda(\eta')\}, \tag{83}$$

a rational function term (we include the differential with this term for scaling purposes)

$$\frac{d\zeta}{\eta'(\zeta-\eta)}, \tag{84}$$

and the term

$$\mu f(\mu,\zeta/\eta'). \tag{85}$$

We proceed by the method of steepest descent, so in order to determine the region along the ζ and η contours which affects the asymptotics we must consider the exponential term first. The argument of the exponential is given by $\Lambda(\zeta)-\Lambda(\eta')$ where

$$\Lambda(\zeta) = -x\log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m\log(\zeta), \tag{86}$$

and where, for the moment we take $m = \frac{1}{2}\left[\epsilon^{-1/2}(-a+\frac{X^2}{2T}) + \frac{1}{2}t + x\right]$. The real expression for m has a $\log(\epsilon^{-1/2}/2)$ term which we define in with the a for the moment (recall that a is defined in the statement of Theorem 1). Recall that x, t and m all depend on ϵ. For small ϵ, $\Lambda(\zeta)$ has a critical point in an $\epsilon^{1/2}$ neighborhood of -1. For purposes of having a nice ultimate answer we choose to center in on the point

$$\xi = -1 - 2\epsilon^{1/2}\frac{X}{T} \tag{87}$$

We can rewrite the argument of the exponential as $(\Lambda(\zeta)-\Lambda(\xi)) - (\Lambda(\eta')-\Lambda(\xi)) = \Psi(\zeta)-\Psi(\eta')$. The idea of extracting asymptotics for this term (which starts like those done in [32] but quickly becomes more involved due to the fact that τ tends to 1 as ϵ goes to zero) is then to deform the ζ and η contours to lie along curves such that outside the scale $\epsilon^{1/2}$ around ξ, $\Psi(\zeta)$ is very negative, and $\Psi(\eta')$ is very positive (in real part). This is so that we can completely forget about that part of the contours. Then, rescaling around ξ to blow up this $\epsilon^{1/2}$ scale, gives us the asymptotic exponential term. This final change of variables then sets the scale at which we should analyze the other two terms in the integrand for the J kernel.

Returning to $\Psi(\zeta)$, we make a Taylor expansion around ξ and find that in a neighborhood of ξ

$$\Psi(\zeta) \approx -\frac{T}{48}\epsilon^{-3/2}(\zeta-\xi)^3 + \frac{a}{2}\epsilon^{-1/2}(\zeta-\xi). \tag{88}$$

This suggests the following change of variables

$$\tilde\zeta = 2^{-4/3}\epsilon^{-1/2}(\zeta-\xi) \qquad \tilde\eta = 2^{-4/3}\epsilon^{-1/2}(\eta-\xi) \qquad \tilde\eta' = 2^{-4/3}\epsilon^{-1/2}(\eta'-\xi), \tag{89}$$

after which our Taylor expansion takes the form

$$\Psi(\tilde\zeta) \approx -\frac{T}{3}\tilde\zeta^3 + 2^{1/3}a\tilde\zeta. \tag{90}$$

In the spirit of steepest descent analysis we would like the ζ contour to leave ξ in a direction where this Taylor expansion is decreasing rapidly. This is accomplished by leaving at an angle $\pm 2\pi/3$. Likewise since $\Psi(\eta)$ should increase radidly, η should leave ξ at angle $\pm\pi/3$. The ζ contour was original centered at zero and of radius $1+\epsilon^{1/2}/2$ and the η contour of radius $1-\epsilon^{1/2}/2$. In order to deform these contours without changing the value of the determinant care must be taken since there are poles of f whenever $\zeta/\eta' = \tau^k$, $k\in\mathbb{Z}$. We ignore this issue for the formal calculation (and deal with it carefully in Section 2.)

Let us now assume that we can deform our contours to curves along which Ψ rapidly decays in ζ and increases in η, as we move along them away from ξ. If we apply the change of variables in (89) the straight part of our contours become infinite at angles $\pm 2\pi/3$ and $\pm\pi/3$ which we call $\tilde\Gamma_\zeta$ and

$\tilde{\Gamma}_\eta$. Note that this is *not* the actual definition of these contours which we use in the statement and proof of Theorem 1 because of the singularity problem mentioned above.

Applying this change of variables to the kernel of the Fredholm determinant changes the L^2 space and hence we must multiply the kernel by the Jacobian term $2^{4/3}\epsilon^{1/2}$. We will include this term with the $\mu f(\mu, z)$ term and take the $\epsilon \to 0$ limit of that product.

As noted before, the term $2^{1/3}a\tilde{\zeta}$ should actually have been $2^{1/3}(a - \log(\epsilon^{-1/2}/2))\tilde{\zeta}$ in the Taylor expansion above, giving

$$\Psi(\tilde{\zeta}) \approx -\frac{T}{3}\tilde{\zeta}^3 + 2^{1/3}(a - \log(\epsilon^{-1/2}/2))\tilde{\zeta}, \tag{91}$$

which would appear to blow up as ϵ goes to zero. We will now show how that the extra $\log \epsilon$ in the exponent can be absorbed into the $2^{4/3}\epsilon^{1/2}\mu f(\mu, \zeta/\eta')$ term.

Recall

$$\mu f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\mu\tau^k}{1 - \tau^k\mu} z^k. \tag{92}$$

If we let $n_0 = \lfloor \log(\epsilon^{-1/2})/\log(\tau) \rfloor$ then observe that

$$\mu f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\mu\tau^{k+n_0}}{1 - \tau^{k+n_0}\mu} z^{k+n_0} = z^{n_0}\tau^{n_0}\mu \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k\tau^{n_0}\mu} z^k. \tag{93}$$

By the choice of n_0, $\tau^{n_0} \approx \epsilon^{-1/2}$ so

$$\mu f(\mu, z) \approx z^{n_0}\tilde{\mu} f(\tilde{\mu}, z). \tag{94}$$

The discussion on the exponential term indicates that it suffices to understand the behavior of this function only in the region where ζ and η' are within a neighborhood of ξ of order $\epsilon^{1/2}$. Equivalently, letting $z = \zeta/\eta'$ it suffices to understand $\mu f(\mu, z) \approx z^{n_0}\tilde{\mu} f(\tilde{\mu}, z)$ for

$$z = \frac{\zeta}{\eta'} = \frac{\xi + 2^{4/3}\epsilon^{1/2}\tilde{\zeta}}{\xi + 2^{4/3}\epsilon^{1/2}\tilde{\eta}'} \approx 1 - \epsilon^{1/2}\tilde{z} \tag{95}$$

where we set $\tilde{z} = 2^{4/3}(\tilde{\zeta} - \tilde{\eta}')$.

Let us now consider z^{n_0} using the fact that $\log(\tau) \approx -2\epsilon^{1/2}$:

$$z^{n_0} \approx (1 - \epsilon^{1/2}\tilde{z})^{\epsilon^{-1/2}(\frac{1}{4}\log\epsilon)} \approx e^{-\frac{1}{4}\tilde{z}\log(\epsilon)}. \tag{96}$$

Plugging back in the value of $\tilde{z}$ in terms of $\tilde{\zeta}$ and $\tilde{\eta}'$ we see that this prefactor of z^{n_0} exactly cancels the $\log \epsilon$ term which accompanies a in the exponential.

What remains is to determine the limit of $2^{4/3}\epsilon^{1/2}\tilde{\mu} f(\tilde{\mu}, z)$ as ϵ goes to zero and for $z \approx 1 - \epsilon^{1/2}\tilde{z}$. This limit can be found by interpreting the infinite sum as a Reimann sum approximation for a certain integral. Define $t = k\epsilon^{1/2}$, then observe that

$$\epsilon^{1/2}\tilde{\mu} f(\tilde{\mu}, z) = \sum_{k=-\infty}^{\infty} \frac{\tilde{\mu}\tau^{t\epsilon^{-1/2}} z^{t\epsilon^{-1/2}}}{1 - \tilde{\mu}\tau^{t\epsilon^{-1/2}}} \epsilon^{1/2} \to \int_{-\infty}^{\infty} \frac{\tilde{\mu} e^{-2t} e^{-\tilde{z}t}}{1 - \tilde{\mu} e^{-2t}} dt. \tag{97}$$

This used the fact that $\tau^{t\epsilon^{-1/2}} \to e^{-2t}$ and that $z^{t\epsilon^{-1/2}} \to e^{-\tilde{z}t}$, which hold at least pointwise in t. If we change variables of t to $t/2$ and multiply the top and bottom by e^{-t} then we find that

$$2^{4/3}\epsilon^{1/2}\mu f(\mu, \zeta/\eta') \to 2^{1/3}\int_{-\infty}^{\infty} \frac{\tilde{\mu}e^{-\tilde{z}t/2}}{e^t - \tilde{\mu}} dt. \tag{98}$$

As far as the final term, the rational expression, under the change of variables and zooming in on ξ, the factor of $1/\eta'$ goes to -1 and the $\frac{d\zeta}{\zeta-\eta'}$ goes to $\frac{d\tilde{\zeta}}{\tilde{\zeta}-\tilde{\eta}'}$.

Therefore we formally find the following kernel: $-K_{a'}^{\mathrm{csc}}(\tilde{\eta}, \tilde{\eta}')$ acting on $L^2(\tilde{\Gamma}_\eta)$ where:

$$K_{a'}^{\mathrm{csc}}(\tilde{\eta}, \tilde{\eta}') = \int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{T}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3}a'(\tilde{\zeta} - \tilde{\eta}')\}2^{1/3}\left(\int_{-\infty}^{\infty} \frac{\tilde{\mu}e^{-2^{1/3}t(\tilde{\zeta}-\tilde{\eta}')}}{e^t - \tilde{\mu}} dt\right)\frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}, \tag{99}$$

where $a' = a + \log 2$. Recall that the $\log 2$ came from the $\log(\epsilon^{-1/2}/2)$ term.

We have the identity

$$\int_{-\infty}^{\infty} \frac{\tilde{\mu}e^{-\tilde{z}t/2}}{e^t - \tilde{\mu}} dt = (-\tilde{\mu})^{-\tilde{z}/2}\pi \csc(\pi\tilde{z}/2), \tag{100}$$

where the branch cut in $\tilde{\mu}$ is along the positive real axis, hence $(-\tilde{\mu})^{-\tilde{z}/2} = e^{-\log(-\tilde{\mu})\tilde{z}/2}$ where log is taken with the standard branch cut along the negative real axis. We may use the identity to rewrite the kernel as

$$K_{a'}^{\mathrm{csc}}(\tilde{\eta}, \tilde{\eta}') = \int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{T}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3}a'(\tilde{\zeta} - \tilde{\eta}')\}2^{1/3}(-\tilde{\mu})^{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')}\pi \csc(\pi 2^{1/3}(\tilde{\zeta} - \tilde{\eta}'))\frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}. \tag{101}$$

Therefore we have shown formally that

$$\lim_{\epsilon \to 0} P(F_\epsilon(T, X) + \tfrac{T}{4!} \leq s) := F_T(s) = \int_{\tilde{\mathcal{C}}} e^{-\tilde{\mu}/2}\frac{d\tilde{\mu}}{\tilde{\mu}} \det(I - K_{a'}^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)}, \tag{102}$$

where $a' = a + \log 2$. To make it cleaner we replace $\tilde{\mu}/2$ with $\tilde{\mu}$. This only affects the $\tilde{\mu}$ term inside of the kernel given now by $(-2\tilde{\mu})^{-\tilde{z}/2}{=}(-\tilde{\mu})^{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')}e^{-2^{1/3}\log 2(\tilde{\zeta}-\tilde{\eta}')}$. This can be absorbed and cancels the $\log 2$ in a' and thus we obtain,

$$\lim_{\epsilon \to 0} P(F_\epsilon(T, X) + \tfrac{T}{4!} \leq s) = F_T(s) = \int_{\tilde{\mathcal{C}}} e^{-\tilde{\mu}}\frac{d\tilde{\mu}}{\tilde{\mu}} \det(I - K_a^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)}, \tag{103}$$

which, up to the definitions of the contours $\tilde{\Gamma}_\eta$ and $\tilde{\Gamma}_\zeta$ is the desired limiting formula.

We now briefly note some of the potential pitfalls of the preceeding formal argument, all of which will be addressed in the real proof of Section 2.

Firstly, the pointwise convergence of both the prefactor infinite product and the Fredholm determinant is certainly not enough to prove convergence of the $\tilde{\mu}$ integral. Estimates must be made to control this convergence or to show that we can cut off the tails of the $\tilde{\mu}$ contour at negligible cost and then show uniform convergence on the trimmed contour.

Secondly, the deformations of the η and ζ contours to the steepest descent curves is *entirely* illegal, as it involves passing through many poles of the kernel (coming from the f term). In the case of [32] this problem could be dealt with rather simply by just slightly modifying the descent curves. However, in our case, since τ tends to 1 like $\epsilon^{1/2}$, such a patch is much harder and involves very fine estimates to show that there exists suitable contours which stay close enough together, yet along which Ψ displays the necessary descent and ascent required to make the argument work.

Finally, one must make precise tail estimates to show that the kernel convergence is in the sense of trace-class norm. The Reimann sum approximation argument can in fact be made rigorous (following the proof of Proposition 17). We choose, however, to give an alternative proof of the validity of that limit in which we identify and prove the limit of f via analysis of singularities and residues.

1.7. **Remark.** During the preparation of this article, we learned that T. Sasamoto and H. Spohn [23, 24, 25] independently obtained a formula equivalent to (62) for the distribution function F_T. They also use a steepest descent analysis on the Tracy-Widom ASEP formula. Note that their argument is at the level of formal asymptotics of operator kernels and they have not attempted a mathematical proof. Very recently two groups of physicists ([8], [11, 12]) have successfully employed the Bethe Ansatz for the attractive Lieb-Liniger model and the replica trick to rederive the distribution function F_T. While this is rather far from rigorous mathematics, it is suggestive of a deeper relationship between the work of Tracy and Widom for ASEP and the traditional study of the Bethe Ansatz.

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2. Proof of the weakly asymmetric limit of the Tracy-Widom ASEP formula

In this section we give a proof of the ϵ to zero limit for the properly scaled and normalized WASEP height function given in Section 1.3 as Theorem 8. In Section 1.6 we derived, at a formal level, the desired limiting formula for the one-point function. The purpose of this section is to rigorously prove this limiting formula. The heart of the argument is Proposition 16 which is proved in Section 2.1 and also relies on a number of technical lemmas. These lemmas as well as all of the other propositions are proved in Section 2.2.

2.0.1. *Proof of Theorem 8.* We will now present the proof of Theorem 8. The more technical computations and estimates are stated as lemmas and propositions and their proofs are relegated to a latter part of this section (Section 2.2).

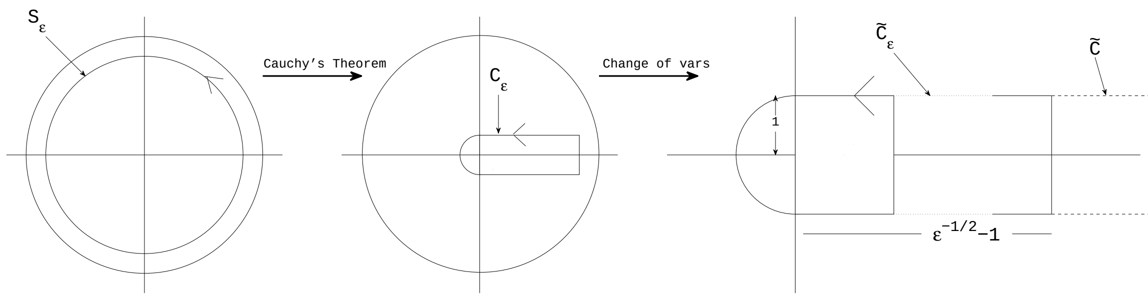

FIGURE 1. The S_ϵ contour is deformed to the C_ϵ contour via Cauchy's theorem and then a change of variables leads to $\tilde{C}_\epsilon$, with its infinite extension $\tilde{C}$.

The expression given in equation (74) for $P(F_\epsilon(T,X)+\frac{T}{4!}\leq s)$ contains an integral over a μ contour of a product of a prefactor infinite product and a Fredholm determinant. The first step towards taking the limit of this as ϵ goes to zero is to control the prefactor. Initially μ lies on a contour S_{τ^+} which is centered at zero and of radius between τ and 1. Recall the prefactor is given by $\prod_{k=0}^{\infty}(1-\mu\tau^k)$. Along this contour the partial products (i.e., product up to N) form a highly oscillatory sequence and hence it is hard to control the convergence of the sequence.

Therefore the first step in our proof is to deform the μ contour S_{τ^+} to the contour $\mathcal{C}_\epsilon$ (a long, skinny cigar shaped contour) where

$$\mathcal{C}_\epsilon=\{\epsilon^{1/2}e^{i\theta}\}\cup\{x\pm i\epsilon^{1/2}\}_{0<x\leq 1-\epsilon^{1/2}}\cup\{1-\epsilon^{1/2}+\epsilon^{1/2}iy\}_{-1<y<1} \tag{104}$$

(see figure 2.0.1.) We orient $\mathcal{C}_\epsilon$ counter-clockwise. Notice that this new contour still includes all of the poles at $\mu=\tau^k$ associated with the f function in the J kernel.

In order to justify replacing S_{τ^+} by $\mathcal{C}_\epsilon$ we need the following (for the proof see Section 2.2.2):

Lemma 13. *In equation (74) we can replace the contour S_ϵ with $\mathcal{C}_\epsilon$ as the contour of integration for μ without affecting the value of the integral.*

Having made this deformation of the μ contour we now observe that the natural scale for μ is on order $\epsilon^{1/2}$. With this in mind we make the following change of variables

$$\mu=\epsilon^{1/2}\tilde{\mu}. \tag{105}$$

Remark 14. Throughout the proof of this theorem and its lemmas and propositions, we will use the tilde to denote variables which are $\epsilon^{1/2}$ rescaled versions of the original, untilded variables.

The $\tilde{\mu}$ variable now lives on the contour

$$\tilde{\mathcal{C}}_\epsilon=\{e^{i\theta}\}\cup\{x\pm i\}_{0<x\leq\epsilon^{-1/2}-1}\cup\{\epsilon^{-1/2}-1+iy\}_{-1<y<1}. \tag{106}$$

Let us also define the increasing limit $\tilde{\mathcal{C}}$ of these contours

$$\tilde{\mathcal{C}}=\{e^{i\theta}\}\cup\{x\pm i\}_{x>0}. \tag{107}$$

The contour of integration for $\tilde{\mu}$ keeps growing and ultimately approaches $\tilde{\mathcal{C}}$. In order to show convergence of the integral as ϵ goes to zero, we must consider two things: the convergence of the integrand for $\tilde{\mu}$ in some compact region (near the origin) on $\tilde{\mathcal{C}}$; and the controlled decay of the integrand on $\tilde{\mathcal{C}}_\epsilon$ outside of that compact region. This second consideration will allow us to approximate the integral by a finite integral in $\tilde{\mu}$, while the first consideration will tell us what the limit of that integral is. When all is said and done, we will paste back in the remaining part of the $\tilde{\mu}$ integral (which we will show has small effect on the value) and have our answer.

With this in mind we give the following convergence / tail control lemma for the prefactor product. We define two regions (which depend on some parameter $r \geq 1$). The first region R_1 is compact, while the second region R_2 is infinite and contains the tail of the $\tilde{\mu}$ contour. Together these two regions cover the contour $\tilde{\mathcal{C}}_\epsilon$. The point of r is that increasing it amounts to cutting the $\tilde{\mu}$ contour further out.

Lemma 15. *Define two regions (which depend on a fixed parameter $r \geq 1$)*

$$R_1 \quad = \quad \{\tilde{\mu} : |\tilde{\mu}| \leq \frac{r}{\sin(\pi/10)}\} \tag{108}$$

$$R_2 \quad = \quad \{\tilde{\mu} : \mathrm{Re}(\tilde{\mu}) \in [\frac{r}{\tan(\pi/10)}, \epsilon^{-1/2}], \text{ and } \mathrm{Im}(\tilde{\mu}) \in [-2,2]\}. \tag{109}$$

Furthermore define the function (the infinite product after the change of variables)

$$g_\epsilon(\tilde{\mu}) = \prod_{k=0}^{\infty}(1 - \epsilon^{1/2}\tilde{\mu}\tau^k). \tag{110}$$

Then uniformly in $\tilde{\mu} \in R_1$,

$$g_\epsilon(\mu) \to e^{-\tilde{\mu}/2} \tag{111}$$

Also, for all $\epsilon < \epsilon_0$ (some positive constant) there exists a constant c such that for all $\tilde{\mu} \in R_2$ we have the following tail bound:

$$|g_\epsilon(\tilde{\mu})| \leq |e^{-\tilde{\mu}/2}||e^{-c\epsilon^{1/2}\tilde{\mu}^2}|. \tag{112}$$

(By the choice of R_2, for all $\tilde{\mu} \in R_2$, $\mathrm{Re}(\tilde{\mu}^2) > \delta > 0$ for some fixed δ. The constant c can be taken to be $1/8$.)

This lemma is proved in Section 2.2.2.

We now turn our attention to the Fredholm determinant term in the integrand. Just as we did for the prefactor infinite product in Lemma 15 we must establish uniform convergence of the determinant for $\tilde{\mu}$ in a fixed compact region (near the origin), and a suitable tail estimate valid outside that compact region. The tail estimate must be such that for each finite ϵ, we can combine the two tail estimates (from the prefactor and from the determinant) and show that their integral over the tail part of $\tilde{\mathcal{C}}_\epsilon$ is small and goes to zero as we enlarge the original compact region. For this we have the following two propositions (the first is the most substantial and is proved in Section 2.1, while the second is proved in Section 2.2.2).

Proposition 16. *Fix $s \in \mathbb{R}$, $T > 0$ and $X \in \mathbb{R}$. Then for all compact subsets of $\tilde{\mathcal{C}}$ we have that*

$$\det(I + \epsilon^{1/2}\tilde{\mu}J_{\epsilon^{1/2}\tilde{\mu}})_{L^2(\Gamma_\eta)} \to \det(I - K_{a'}^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)}, \tag{113}$$

uniformly over $\tilde{\mu}$ in the compact subset, where $a' = a + \log 2$ and where $K_{a'}^{\mathrm{csc}}$ is defined in Definition 67 and depends implicitly on $\tilde{\mu}$.

Proposition 17. *There exists a constant $c > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and all $\tilde{\mu}$ on $\tilde{\mathcal{C}}_\epsilon$,*

$$\left|g_\epsilon(\tilde{\mu}) \det(I + \epsilon^{1/2}\tilde{\mu}J_{\epsilon^{1/2}\tilde{\mu}})_{L^2(\Gamma_\eta)}\right| \leq e^{-c|\tilde{\mu}|}. \tag{114}$$

This exponential decay bound on the integrand shows that that, by choosing a suitably large (fixed) compact region around zero along the contour $\tilde{\mathcal{C}}_\epsilon$, it is possible to make the $\tilde{\mu}$ integral outside of this region arbitrarily small, uniformly in ϵ (smaller than some fixed ϵ_0). This means that we may henceforth assume that $\tilde{\mu}$ lies in a compact region along $\tilde{\mathcal{C}}$.

Now that we are on a fixed compact set of $\tilde{\mu}$, the first part of Lemma 15 and Proposition 16 combine to show that the integrand converges uniformly to

$$\frac{e^{-\tilde{\mu}/2}}{\tilde{\mu}}\det(I-K_{a'}^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)} \tag{115}$$

and hence the integral converges to the integral with this integrand.

To finish the proof of the limit in Theorem 8 it is necessary that for any δ we can find a suitably small ϵ_0 such that the difference between the two sides of the limit differ by less than δ for all $\epsilon<\epsilon_0$. Technically we are in the position of a $\delta/3$ argument. One portion of $\delta/3$ goes to the cost of cutting off the $\tilde{\mu}$ contour outside of some compact set. Another $\delta/3$ goes to the uniform convergence of the integrand. The final portion goes to repairing the $\tilde{\mu}$ contour. As δ gets smaller, the cut for the $\tilde{\mu}$ contour must occur further out. Therefore the limiting integral will be over the limit of the $\tilde{\mu}$ contours, which we called $\tilde{\mathcal{C}}$. The final $\delta/3$ is spent on the following:

Proposition 18. *There exists a constant $c>0$ such that for all $\tilde{\mu}$ with $|\tilde{\mu}|\geq 1$ along $\tilde{\mathcal{C}}$, we have*

$$\left|\frac{e^{-\tilde{\mu}/2}}{\tilde{\mu}}\det(I-K_{a}^{\mathrm{csc}})_{L^2(\tilde{\Gamma}_\eta)}\right|\leq |e^{-c\tilde{\mu}}|. \tag{116}$$

This proposition is proved in Section 2.2.2. One should note also that the argument used to prove this proposition immediately shows that K_a^{csc} is, in fact, a trace class operator on $L^2(\tilde{\Gamma}_\eta)$.

It is an immediate corollary of this exponential tail bound that for sufficiently large compact sets of $\tilde{\mu}$, the cost to include the rest of the $\tilde{\mu}$ contour is less than $\delta/3$. This, along with the change of variables in $\tilde{\mu}$ described at the end of Section 1.6 finishes the proof of Theorem 8.

2.1. **Proof of Proposition 16.** In this section we provide all of the steps necessary to prove Proposition 16. To ease understanding of the argument we relegate more technical points to lemmas whose proof we delay to Section 2.2.3.

For the entire proof of this proposition it is important that we keep in mind that at this point we may assume that $\tilde{\mu}$ lies on a fixed compact region of the curve $\tilde{\mathcal{C}}$. Recall that $\tilde{\mu}=\epsilon^{-1/2}\mu$. We proceed via the following strategy to find the limit of the Fredholm determinant as ϵ goes to zero. The first step is to deform the contours Γ_η and Γ_ζ to suitable curves along which there exists a small region outside of which the kernel of our operator is exponentially small. This justifies cutting the contours off outside of this small region. We may then rescale everything so this small region becomes order one in size. Then, for this compact region we must show uniform convergence of the kernel, as ϵ goes to zero, to our desired limit kernel. Finally we must show that we can complete the finite contour on which this limiting object is defined to an infinite contour without significantly changing the value of the determinant. This idea of cutting, taking the limit and then pasting back the remaining (limiting) contour is analogous to the idea behind the proof of Theorem 8.

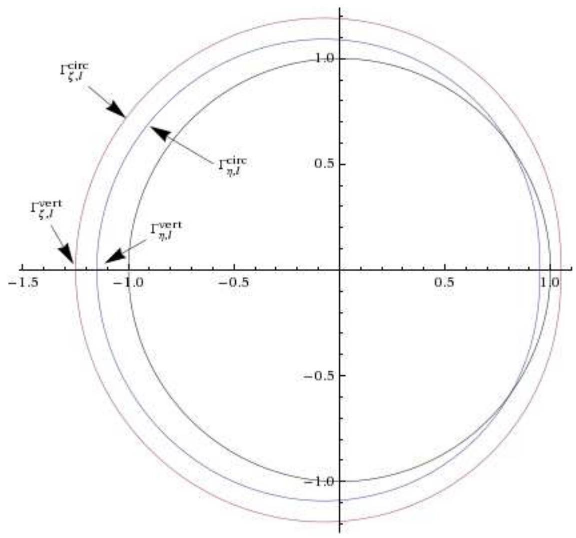

FIGURE 2. $\Gamma_{\zeta,l}$ (the outer most curve) is composed of a small verticle section near ξ labeled $\Gamma^{vert}_{\zeta,l}$ and a large almost circular (small modification due to the function $\kappa(\theta)$) section labeled $\Gamma^{circ}_{\zeta,l}$. Likewise $\Gamma_{\eta,l}$ is the middle curve, and the inner curve is the unit circle. These curves depend on ϵ in such a way that $|\zeta/\eta|$ is bounded between 1 and $\tau^{-1} \approx 1 + 2\epsilon^{1/2}$.

Recall now that Γ_ζ is defined to be a circle centered at zero of radius $1 + \epsilon^{1/2}/2$ and Γ_η is a circle centered at zero of radius $1 - \epsilon^{1/2}/2$ (this implies that $1 < |\zeta/\eta'| < \tau^{-1}$) and that

$$\xi = -1 - 2\epsilon^{1/2}\frac{X}{T}. \tag{117}$$

The function $f(\mu, \zeta/\eta')$ which shows up in the definition of the kernel for J has poles as every point $\zeta/\eta' = z = \tau^k$ for $k \in \mathbb{Z}$. As long as we simultaneously deform the Γ_ζ contour as we deform Γ_η so as to keep ζ/η' away from these poles, we may use Proposition 31 (Proposition 1 of [32]), to justify the fact that the determinant does not change under this deformation. In this way we may deform our contours to the following modified contours $\Gamma_{\eta,l}, \Gamma_{\zeta,l}$:

Definition 19. Let $\Gamma_{\eta,l}$ and $\Gamma_{\zeta,l}$ be two families (indexed by $l > 0$) of simple closed contours in $\mathbb{C}$ defined as follows. Let

$$\kappa(\theta) = \frac{2X}{T}\tan^2\left(\frac{\theta}{2}\right)\log\left(\frac{2}{1-\cos\theta}\right). \tag{118}$$

Both $\Gamma_{\eta,l}$ and $\Gamma_{\zeta,l}$ will be symmetric across the real axis, so we need only define them on the top half. $\Gamma_{\eta,l}$ begins at $\xi + \epsilon^{1/2}/2$ and moves along a straight vertical line for a distance $l\epsilon^{1/2}$ and then joins the curve (parametrized by the polar angle θ) given by

$$\left[1 + \epsilon^{1/2}(\kappa(\theta) + \alpha)\right] e^{i\theta} \tag{119}$$

where the value of θ ranges from $\theta \approx \pi - l\epsilon^{1/2} + \mathcal{O}(\epsilon)$ to $\theta = 0$ and where $\alpha = -1/2 + \mathcal{O}(\epsilon^{1/2})$ (see figure 2.1 for an illustration of these contours). The small errors are necessary to make sure that the curves join up at the end of the vertical section of the curve. As said before, we extend this to a closed contour by reflection through the real axis. The orientation to this contour is clockwise. We denote the first, vertical part, of the contour by $\Gamma^{vert}_{\eta,l}$ and the second, roughly circular part by $\Gamma^{circ}_{\eta,l}$. This means that $\Gamma_{\eta,l} = \Gamma^{vert}_{\eta,l} \cup \Gamma^{circ}_{\eta,l}$, and along this contour we can think of parametring η by $\theta \in [0, \pi]$.

We define $\Gamma_{\zeta,l}$ similarly except that it starts out at $\xi-\epsilon^{1/2}/2$ and joins the curve given by equation (119) where the value of θ ranges from $\theta\approx\pi-l\epsilon^{1/2}+\mathcal{O}(\epsilon)$ to $\theta=0$ and where $\alpha=1/2+\mathcal{O}(\epsilon^{1/2})$. We similarly denote this contour by the union of $\Gamma_{\zeta,l}^{vert}$ and $\Gamma_{\zeta,l}^{circ}$.

By virtue of the above definitions it is clear that $\epsilon^{-1/2}|\zeta/\eta'-\tau^k|$ stays bounded from zero for all k, and that $|\zeta/\eta'|$ is bounded in an open set contained in $(1,\tau^{-1})$ for all $\zeta\in\Gamma_{\zeta,l}$ and $\eta\in\Gamma_{\eta,l}$. Therefore, for any $l>0$ we may, by deforming both the η and ζ contours simultaneously, assume that our operator acts on $L^2(\Gamma_{\eta,l})$ and that its kernel is defined via an integral along $\Gamma_{\zeta,l}$. It is critical that we now show that, due to our choice of contours, we are able to forget about everything except for the vertical part of the contours. To formulate this we have the following:

Definition 20. Let χ_l^{vert} and χ_l^{circ} be projection operators acting on $L^2(\Gamma_{\eta,l})$ which project onto $L^2(\Gamma_{\eta,l}^{vert})$ and $L^2(\Gamma_{\eta,l}^{circ})$ respectively. Also define two operators J_l^{vert} and J_l^{circ} which act on $L^2(\Gamma_{\eta,l})$ and have kernels identical to J (see equation (75)) except the ζ integral is over $\Gamma_{\zeta,l}^{vert}$ and $\Gamma_{\zeta,l}^{circ}$ respectively. Thus we have a family of decompositions of our operator J as follows:

$$J=J_l^{vert}\chi_l^{vert}+J_l^{vert}\chi_l^{circ}+J_l^{circ}\chi_l^{vert}+J_l^{circ}\chi_l^{circ}. \tag{120}$$

We now show that it suffices to just consider the first part of this decomposition $(J_l^{vert}\chi_l^{vert})$.

Proposition 21. *Assume that $\tilde{\mu}$ is restricted to a bounded subset of the contour $\tilde{\mathcal{C}}$. For all $\delta>0$ there exists an $\epsilon_0>0$ and $l_0>0$ such that for all $\epsilon<\epsilon_0$ and all $l>l_0$*

$$|\det(I+\mu J)_{L^2(\Gamma_{\eta,l})}-\det(I+J_l^{vert})_{L^2(\Gamma_{\eta,l}^{vert})}|<\delta. \tag{121}$$

Proof. As was explained in the introduction, if we let

$$n_0=\lfloor\log(\epsilon^{-1/2})/\log(\tau)\rfloor \tag{122}$$

then it follows from the invariance of the doubly infinite sum for $f(\mu,z)$ that

$$\mu f(\mu,z)=z^{n_0}(\tilde{\mu}f(\tilde{\mu},z)+\mathcal{O}(\epsilon^{1/2})). \tag{123}$$

Note that the $\mathcal{O}(\epsilon^{1/2})$ above does not play a significant role in what follows so we will drop it off.

Using the above argument and the following two lemmas (which are proved in Section 2.2.3) we will be able to complete the proof of Proposition 21.

Lemma 22. *There exists an $l_0>0$ such that for all $l>l_0$ there exists $\epsilon_0>0$ and a constant $c>0$ such that for all $\epsilon<\epsilon_0$ and all $\eta\in\Gamma_{\eta,l}^{circ}$*

$$\mathrm{Re}(\Psi(\eta)+n_0\log(\eta))\geq c|\xi-\eta|\epsilon^{-1/2}, \tag{124}$$

where n_0 is defined in (122). Likewise, for all $\epsilon<\epsilon_0$ and $\zeta\in\Gamma_{\zeta,l}^{circ}$

$$\mathrm{Re}(\Psi(\zeta)+n_0\log(\zeta))\leq -c|\xi-\zeta|\epsilon^{-1/2}. \tag{125}$$

Lemma 23. *For all $l>0$ there exists $\epsilon_0>0$ and a constant $c>0$ such that for all $\epsilon<\epsilon_0$*

$$|\tilde{\mu}f(\tilde{\mu},\zeta/\eta')|\leq\frac{c}{|\zeta-\eta'|} \tag{126}$$

where $\eta'\in\Gamma_{\eta,l}$ and $\zeta\in\Gamma_{\zeta,l}$.

It now follows that for any $\delta > 0$, we can find l_0 large enough so that $||J_l^{vert}\chi_l^{circ}||_1, ||J_l^{circ}\chi_l^{vert}||_1$ and $||J_l^{circ}\chi_l^{circ}||_1$ are all bounded by $\delta/3$. This is because we may factor these various operators in the product of Hilbert-Schmidt operators and then use the exponential decay of Lemma 22 along with the polynomial control of Lemma 23 and the remaining term $1/(\zeta - \eta)$ to prove that each of the Hilbert-Schmidt norms goes to zero (see for instance the bottom of page 27 of [32]).

This estimate completes the proof of Proposition 21. □

We may now return to proving Proposition 16. We have successfully restricted ourselves to just considering J_l^{vert} acting on $L^2(\Gamma_{\eta,l}^{vert})$. Having focused in on the region of asymptotically non-trivial behavior, we can now rescale and show that the kernel uniformly converges on the compact contour to the limit kernel.

Definition 24. Fix $c_3 = 2^{-4/3}$ and let

$$\eta = \xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}, \qquad \eta' = \xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}', \qquad \zeta = \xi + c_3^{-1}\epsilon^{1/2}\tilde{\zeta}. \tag{127}$$

Under these change of variables the contours $\Gamma_{\eta,l}^{vert}$ and $\Gamma_{\zeta,l}^{vert}$ become

$$\tilde{\Gamma}_{\eta,l} = \{c_3/2 + c_3 ir : r \in (-l, l)\} \tag{128}$$

$$\tilde{\Gamma}_{\zeta,l} = \{-c_3/2 + c_3 ir : r \in (-l, l)\}. \tag{129}$$

As we increase l these contours approach the following infinite versions

$$\tilde{\Gamma}_{\eta} = \{c_3/2 + c_3 ir : r \in (-\infty, \infty)\} \tag{130}$$

$$\tilde{\Gamma}_{\zeta} = \{-c_3/2 + c_3 ir : r \in (-\infty, \infty)\}. \tag{131}$$

With respect to the change of variables define an operator $\tilde{J}$ acting on $L^2(\tilde{\Gamma}_\eta)$ via the kernel:

$$\mu\tilde{J}_l(\tilde{\eta}, \tilde{\eta}') = c_3^{-1}\epsilon^{1/2}\int_{\tilde{\Gamma}_{\zeta,l}} \exp\{\Psi(\xi + c_3^{-1}\epsilon^{1/2}\tilde{\zeta}) - \Psi(\xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}')\}\frac{\mu f(\mu, \frac{\xi + c_3^{-1}\epsilon^{1/2}\tilde{\zeta}}{\xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}'})}{(\xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}')(\tilde{\zeta} - \tilde{\eta})}d\tilde{\zeta}. \tag{132}$$

Lastly, define the projection operator $\tilde{\chi}_l$ which projects $L^2(\tilde{\Gamma}_\eta)$ onto $L^2(\tilde{\Gamma}_{\eta,l})$.

It is clear that under the change of variables the Fredholm determinant $\det(I + J_l^{vert})_{L^2(\Gamma_{\eta,l}^{vert})}$ becomes $\det(I + \tilde{\chi}_l\mu\tilde{J}_l\tilde{\chi}_l)_{L^2(\tilde{\Gamma}_{\eta,l})}$.

We now state a proposition which gives, with respect to these fixed contours $\tilde{\Gamma}_{\eta,l}$ and $\tilde{\Gamma}_{\zeta,l}$, the limit of the determinant in terms of the uniform limit of the kernel. Since all contours in question are finite, uniform convergence of the kernel suffices to show trace class convergence of the operators and hence convergence of the determinant.

Recall the definition of the operator K_a^{csc} given in Definition 9. For the purposes of this proposition, modify the kernel so that the integration in ζ occurs now only over $\tilde{\Gamma}_{\zeta,l}$ and not all of $\tilde{\Gamma}_\zeta$. Call this modified operator $K_{a',l}^{\mathrm{csc}}$.

Proposition 25. *For all $\delta > 0$ there exists $\epsilon_0 > 0$ and $l_0 > 0$ such that for all $\epsilon < \epsilon_0$ and $l > l_0$ we have (uniformly over the $\tilde{\mu}$ in our fixed compact subset of $\tilde{\mathcal{C}}$)*

$$\left|\det(I + \tilde{\chi}_l\mu\tilde{J}_l\tilde{\chi}_l)_{L^2(\tilde{\Gamma}_{\eta,l})} - \det(I - \tilde{\chi}_l K_{a',l}^{\mathrm{csc}}\tilde{\chi}_l)_{L^2(\tilde{\Gamma}_{\eta,l})}\right| < \delta, \tag{133}$$

where $a' = a + \log 2$.

Proof. The proof of this proposition relies on showing the uniform convergence of the kernel of $\mu\tilde{J}$ to the kernel of $K^{\mathrm{csc}}_{a',l}$, which suffices because of the compact contour. Furthermore, since the ζ integration is itself over a compact set, it suffices to show uniform convergence of this integrand. The two lemmas stated below will imply such uniform convergence and hence complete this proof.

First, however, recall that $\mu f(\mu,z) = z^{n_0}(\tilde{\mu}f(\tilde{\mu},z)+\mathcal{O}(\epsilon^{1/2}))$ where n_0 is defined in equation (122). We are interested in having $z=\zeta/\eta'$, which, under the change of variables can be written as

$$z = 1-\epsilon^{1/2}\tilde{z}+\mathcal{O}(\epsilon), \qquad \tilde{z} = c_3^{-1}(\tilde{\zeta}-\tilde{\eta}') = 2^{4/3}(\tilde{\zeta}-\tilde{\eta}'). \tag{134}$$

Therefore, since $n_0 = -\frac{1}{2}\log(\epsilon^{-1/2})\epsilon^{-1/2}+\mathcal{O}(1)$ it follows that

$$z^{n_0} = \exp\{-2^{1/3}(\tilde{\zeta}-\tilde{\eta}')\log(\epsilon^{-1/2})\}(1+o(1)). \tag{135}$$

This expansion still contains an ϵ and hence the argument blows up as ϵ goes to zero. However, this exactly counteracts the $\log(\epsilon^{-1/2})$ term in the definition of m which goes into the argument of the exponential of the integrand. We make use of this cancellation in the proof of this first lemma and hence include the $n_0\log(\zeta/\eta')$ term into the exponential argument.

Lemma 26. *For all $l>0$ and all $\delta>0$ there exists $\epsilon_0>0$ such that for all $\tilde{\eta}'\in\tilde{\Gamma}_{\eta,l}$ and $\tilde{\zeta}\in\tilde{\Gamma}_{\zeta,l}$ we have for $0<\epsilon\le\epsilon_0$,*

$$\left|\left(\Psi(\tilde{\zeta})-\Psi(\tilde{\eta}')+n_0\log(\zeta/\eta')\right)-\left(-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}'^3)+2^{1/3}a'(\tilde{\zeta}-\tilde{\eta})\right)\right|<\delta, \tag{136}$$

where $a=a'+\log 2$. Similarly we have

$$\left|\exp\left\{\Psi(\tilde{\zeta})-\Psi(\tilde{\eta}')+n_0\log(\zeta/\eta')\right\}-\exp\left\{-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}'^3)+2^{1/3}a'(\tilde{\zeta}-\tilde{\eta})\right\}\right|<\delta. \tag{137}$$

Lemma 27. *For all $l>0$ and all $\delta>0$ there exists $\epsilon_0>0$ such that for all $\tilde{\eta}'\in\tilde{\Gamma}_{\eta,l}$ and $\tilde{\zeta}\in\tilde{\Gamma}_{\zeta,l}$ we have for $0<\epsilon\le\epsilon_0$,*

$$\left|\epsilon^{1/2}\tilde{\mu}f\left(\tilde{\mu},\frac{\xi+c_3^{-1}\epsilon^{1/2}\tilde{\zeta}}{\xi+c_3^{-1}\epsilon^{1/2}\tilde{\eta}'}\right)-\int_{-\infty}^{\infty}\frac{\tilde{\mu}e^{-2^{1/3}t(\tilde{\zeta}-\tilde{\eta}')}}{e^t-\tilde{\mu}}dt\right|<\delta. \tag{138}$$

The above integral also has a representation (100) in terms of the csc function which in fact gives the analytic continuation for the integral to all $z\notin\mathbb{Z}$. Finally, the sign change in front of the kernel of the Fredholm determinant comes from the $1/\eta'$ term which, under the change of variables converges uniformly to -1. □

Having successfully taken the ϵ to zero limit, all that now remains is to paste the rest of the contours $\tilde{\Gamma}_\eta$ and $\tilde{\Gamma}_\zeta$ to their abbreviated versions $\tilde{\Gamma}_{\eta,l}$ and $\tilde{\Gamma}_{\zeta,l}$. To justify this we must show that the inclusion of the rest of these contours does not significantly affect the Fredholm determinant. Just as in the proof of Proposition 21 we have three operators which we must re-include at provably small cost. Each of these operators, however, can be factored into the product of Hilbert Schmidt operators and then an analysis similar to that done following Lemma 23 (see in particular page 27-28 of [32]) shows that because $\mathrm{Re}(\tilde{\zeta}^3)$ grows like $|\tilde{\zeta}|^2$ along $\tilde{\Gamma}_\zeta$ (and likewise but opposite for η') we have sufficiently strong exponential decay to assure us that the trace norms of these three additional kernels can be made arbitrarily small by taking l large enough.

This last estimate completes the proof of Proposition 16.

2.2. **Technical lemmas, propositions and proofs.**

2.2.1. *Preliminary lemmas and inequalities.* Before delving into the proofs of the propositions and lemmas, we state a few lemmas which will are useful in what follows. The first three lemmas are basic facts about Fredholm determinants. For a full treatment of the theory of Fredholm determinants, trace class operators and Hilbert-Schmidt operators see, for example, [26].

Lemma 28 (Pg. 40 of [7], from Theorem 2.20 from [26]). *The following conditions are equivalent:*

(1) $||K_n - K||_1 \to 0$*;*
(2) $\operatorname{tr} K_n \to \operatorname{tr} K$ *and* $K_n \to K$ *in the weak operator topology.*

Lemma 29 (Ch. 3 [26]). $A \mapsto \det(I+A)$ *is a continuous function on* J_1 *(the trace class operators). Explicitly,*

$$|\det(I+A) - \det(I+B)| \leq ||A-B||_1 \exp(||A||_1 + ||B||_1 + 1). \tag{139}$$

If $A \in J_1$ *and* $A = BC$ *with* $B, C \in J_2$ *(Hilbert-Schmidt operators) then*

$$||A||_1 \leq ||B||_2 ||C||_2. \tag{140}$$

For $A \in J_1$*,*

$$|\det(I+A)| \leq e^{||A||_1}. \tag{141}$$

If $A \in J_2$ *with kernel* $A(x,y)$ *then*

$$||A||_2 = \left(\int |A(x,y)|^2 dx dy \right)^{1/2}. \tag{142}$$

Lemma 30. *If* K *is an operator acting on a contour* Σ *and* χ *is a projection operator unto a subinterval of* Σ *then*

$$\det(I + K\chi)_{L^2(\Sigma,\mu)} = \det(I + \chi K \chi)_{L^2(\Sigma,\mu)}. \tag{143}$$

In performing steepest descent analysis on Fredholm determinants, the following proposition allows one to deform contours to descent curves.

Lemma 31 (Proposition 1 of [32]). *Suppose* $s \to \Gamma_s$ *is a deformation of closed curves and a kernel* $L(\eta, \eta')$ *is analytic in a neighborhood of* $\Gamma_s \times \Gamma_s \subset \mathbb{C}^2$ *for each* s*. Then the Fredholm determinant of* L *acting on* Γ_s *is independent of* s*.*

The following lemma, provided to us by Percy Deift, with proof provided in Appendix 7, allows us to use Cauchy's theorem when manipulating integrals which involve Fredholm determinants in the integrand.

Lemma 32. *Suppose* $T(z)$ *is an analytic map from a region* $D \in \mathbb{C}$ *into the trace-class operators on a (separable) Hilbert space* $\mathcal{H}$*. Then* $z \mapsto \det(I + T(z))$ *is analytic on* D*.*

The following is our key lemma on the meromorphic extension of $\mu f(\mu, z)$. Recall that $\mu f(\mu, z)$ has poles at $\mu = \tau^j$, $j \in \mathbb{Z}$.

Lemma 33. *For $\mu \neq \tau^j$ for $j \in \mathbb{Z}$, the function $\mu f(\mu, z)$ is analytic in z for $1 < |z| < \tau^{-1}$ and extends analytically to all $z \neq 0$ or τ^k for $k \in \mathbb{Z}$. This extension is given by first writing $\mu f(\mu, z) = g_+(z) + g_-(z)$ where*

$$g_+(z) = \sum_{k=0}^{\infty} \frac{\mu \tau^k z^k}{1 - \tau^k \mu} \qquad g_-(z) = \sum_{k=1}^{\infty} \frac{\mu \tau^{-k} z^{-k}}{1 - \tau^{-k} \mu}, \tag{144}$$

and where g_+ is now defined for $|z| < \tau^{-1}$ and g_- is defined for $|z| > 1$. These functions satisfy the following two functional equations which imply the analytic continuation:

$$g_+(z) = \frac{\mu}{1 - \tau z} + \mu g_+(\tau z), \qquad g_-(z) = \frac{1}{1 - z} + \frac{1}{\mu} g_-(z/\tau). \tag{145}$$

By repeating this functional equation we find that

$$g_+(z) = \sum_{k=1}^{N} \frac{\mu^k}{1 - \tau^k z} + \mu^N g_+(\tau^N z), \qquad g_-(z) = \sum_{k=0}^{N-1} \frac{\mu^{-k}}{1 - \tau^{-k} z} + \mu^{-N} g_-(z\tau^{-N}). \tag{146}$$

Proof. We'll prove the g_+ functional equation, since the g_- one follows similarly. Observe that

$$g_+(z) = \sum_{k=0}^{\infty} \mu (\tau z)^k (1 + \frac{1}{1 - \mu \tau^k} - 1) = \frac{\mu}{1 - \tau z} + \sum_{k=0}^{\infty} \frac{\mu^2 \tau^k}{1 - \mu \tau^k} (\tau z)^k = \frac{\mu}{1 - \tau z} + \mu g_+(\tau z), \tag{147}$$

which is the desired relation. □

2.2.2. *Proofs from Section 2.0.1.*

Proof of Lemma 13. The lemma follows from Cauchy's theorem once we show that for fixed ϵ, the integrand $\mu^{-1} \prod_{k=0}^{\infty} (1 - \mu \tau^k) \det(I + \mu J_\mu)$ is analytic in μ between S_ϵ and $\mathcal{C}_\epsilon$ (note that we now include a subscript μ on J to emphasize the dependence of the kernel on μ). It is clear that the infinite product and the μ^{-1} are analytic in this region. In order to show that $\det(I + \mu J_\mu)$ is analytic in the desired region we may appeal to Lemma 32. Therefore it suffices to show that the map $J(\mu)$ defined by $\mu \mapsto J_\mu$ is an analytic map from this region of μ between S_ϵ and $\mathcal{C}_\epsilon$ into the trace class operators (this suffices since the multiplication by μ is clearly analytic). The rest of this proof is devoted to the proof of this fact.

In order to prove this, we need to show that $J_\mu^h = \frac{J_{\mu+h} - J_\mu}{h}$ converges to some operator in the trace class operators as $h \in \mathbb{C}$ goes to zero. By the criteria of Lemma 28 it suffices to prove that the kernel associated to J_μ^h converges uniformly in $\eta, \eta' \in \Gamma_\eta$ to the kernel for some operator the operator J'_μ. This will prove both the convergence of traces as well as the weak convergence of operators necessary to prove trace norm convergence and complete this proof. The operator J'_μ acts on Γ_η, the circle centered at zero and of radius $1 - \frac{1}{2}\epsilon^{1/2}$, as

$$J'_\mu(\eta, \eta') = \int_{\Gamma_\zeta} \exp\{\Psi(\zeta) - \Psi(\eta')\} \frac{f'(\mu, \zeta/\eta')}{\eta'(\zeta - \eta)} d\zeta \tag{148}$$

where

$$f'(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^{2k}}{(1 - \tau^k \mu)^2} z^k. \tag{149}$$

Our desired convergence will follow if we can show that

$$\left| h^{-1} \left(f(\mu + h, \zeta/\eta') - f(\mu, \zeta/\eta') \right) - f'(\mu, \zeta/\eta') \right| \tag{150}$$

goes to zero uniformly in $\zeta \in \Gamma_\zeta$ and $\eta' \in \Gamma_\eta$ as $|h|$ goes to zero. Expanding this out and taking the absolute value inside of the infinite sum we have

$$\sum_{k=-\infty}^{\infty} \left| h^{-1}\left(\frac{\tau^k}{1-\tau^k(\mu+h)} - \frac{\tau^k}{1-\tau^k(\mu)}\right) - \frac{\tau^{2k}}{(1-\tau^k(\mu))^2}\right| z^k \tag{151}$$

where $z = |\zeta/\eta'| \in (1, \tau^{-1})$. For ϵ and μ fixed there is a $k = k^*$ at and above which

$$\left|\frac{\tau^k h}{1-\tau^k\mu}\right| < 1. \tag{152}$$

Furthermore, by choosing $|h|$ small enough we can make sure that k^* is negative. As a result we also see that for small enough $|h|$, for all $k < k^*$

$$\left|\frac{h}{\tau^{-1}-\mu}\right| < 1. \tag{153}$$

Therefore splitting our sum into these two sets of k values and using the fact that $1/(1-w) = 1 + w + \mathcal{O}(w)$ for $|w| < 1$ we can Taylor expand as follows: For $k \geq k^*$

$$\frac{\tau^k}{1-\tau^k(\mu+h)} = \frac{\tau^k}{1-\tau^k\mu}\frac{1}{1-\frac{\tau^k h}{1-\tau^k\mu}} = \frac{\tau^k\left(1+\frac{\tau^k h}{1-\tau^k\mu}+\left(\frac{\tau^k}{1-\tau^k\mu}\right)^2\mathcal{O}(h^2)\right)}{1-\tau^k\mu}. \tag{154}$$

Similarly expanding the second term inside the absolute value in equation (151) and canceling with the third term we are left with

$$\sum_{k=k^*}^{\infty} \frac{\tau^{3k}}{(1-\tau^k\mu)^3}\mathcal{O}(h)z^k. \tag{155}$$

The sum converges since $\tau^3 z < 1$ and thus behaves like $\mathcal{O}(h)$ as desired. Likewise for $k < k^*$ by multiplying the numerator and denominator by τ^{-k} the same type of expansion works and we find that the error is given by the same summand as above but over k from $-\infty$ to k^*-1. Again, however, the sum converges since the numerator and denominator cancel each other for k large negative, and z^k is a convegence series for k going to infinity. Thus this error series also behaves like $\mathcal{O}(h)$ as desired. This shows the necessary uniform convergence and completes the proof. □

Proof of Lemma 15. We prove this with the scaling parameter $r = 1$ as the general case follows in a similar way. Consider

$$\log(g_\epsilon(\tilde\mu)) = \sum_{k=0}^{\infty} \log(1-\epsilon^{1/2}\tilde\mu\tau_\epsilon^k). \tag{156}$$

For $\tilde\mu \in R_1$ we have

$$\begin{aligned}
|\log(g_\epsilon(\tilde\mu)) + \tilde\mu/2| &= \left|\sum_{k=0}^{\infty}\log(1-\epsilon^{1/2}\tilde\mu\tau^k) + \epsilon^{1/2}\tilde\mu\tau^k\right| && (157)\\
&\leq \sum_{k=0}^{\infty} |\log(1-\epsilon^{1/2}\tilde\mu\tau^k) + \epsilon^{1/2}\tilde\mu\tau^k| && (158)\\
&\leq \sum_{k=0}^{\infty} |\epsilon^{1/2}\tilde\mu\tau^k|^2 = \frac{\epsilon|\tilde\mu|^2}{1-\tau^2} = \frac{\epsilon^{1/2}|\tilde\mu|^2}{4-4\epsilon^{1/2}} \leq c\epsilon^{1/2}|\tilde\mu|^2 && (159)\\
&\leq c'\epsilon^{1/2}. && (160)
\end{aligned}$$

where the first line comes from the fact that $\sum_{k=0}^{\infty}\epsilon^{1/2}\tau^k = 1/2$. The second inequality uses the fact that for all $z \in \mathbb{C}$ such that $|z| \leq 1/2$, $|\log(1-z)+z| \leq |z|^2$. Since $\tilde\mu \in R_1$ it follows that

$|z| = \epsilon^{1/2}|\tilde{\mu}|$ is clearly bounded by 1/2 for small enough ϵ. The constants here are finite and do not depend on any of the parameters. This proves equation (111) and shows that the convergence is uniform in $\tilde{\mu}$ on R_1.

We now turn to the second inequality, equation (112). Consider a region $D \subset \mathbb{C}$

$$D = \{z : \arg(z) \in [-\frac{\pi}{10}, \frac{\pi}{10}]\} \cap \{z : \Im(z) \in (-\frac{1}{10}, \frac{1}{10})\} \cap \{z : \mathrm{Re}(z) \leq 1\}. \tag{161}$$

Then for all $z \in D$,

$$\mathrm{Re}(\log(1-z)) \leq \mathrm{Re}(-z - z^2/2). \tag{162}$$

For $\tilde{\mu} \in R_2$ it is clear that $\epsilon^{1/2}\tilde{\mu} \in D$. Therefore using (162),

$$\begin{aligned} \mathrm{Re}(\log(g_\epsilon(\tilde{\mu}))) &= \sum_{k=0}^{\infty} \mathrm{Re}[\log(1 - \epsilon^{1/2}\tilde{\mu}\tau^k)] && (163)\\ &\leq \sum_{k=0}^{\infty} \Big(-\mathrm{Re}[\epsilon^{1/2}\tilde{\mu}\tau^k] - \mathrm{Re}[(\epsilon^{1/2}\tilde{\mu}\tau^k)^2/2]\Big) && (164)\\ &\leq -\mathrm{Re}(\tilde{\mu}/2) - \frac{\epsilon^{1/2}}{8 - 8\epsilon^{1/2}}\mathrm{Re}(\tilde{\mu}^2) && (165)\\ &\leq -\mathrm{Re}(\tilde{\mu}/2) - \frac{1}{8}\epsilon^{1/2}\mathrm{Re}(\tilde{\mu}^2). && (166) \end{aligned}$$

This proves equation (112). Note that given the definition of region R_2 we can calculate the argument of $\tilde{\mu}$ and we see that $|\arg\tilde{\mu}| \leq \arctan(2\tan(\pi/10)) < \pi/4$ and $|\tilde{\mu}| \geq r \geq 1$. Therefore $\mathrm{Re}(\tilde{\mu}^2)$ is positive and bounded away from zero for all $\tilde{\mu} \in R_2$. □

Proof of Proposition 17. This proof proceeds in a similar manner to the proof of Proposition 18, however, since presently we have ϵ and changing contours, it is, by necessity, a little more complicated. For this reason we encourage readers to first study the simpler proof of Proposition 18.

In that proof we factor our operator into two pieces. Then, using the decay of the exponential term, and the control over the size of the csc term, we are able to show that the Hilbert-Schmidt norm of the first factor is finite and that for the second factor it is bounded by $|\tilde{\mu}|^\alpha$ for $\alpha < 1$ (we show it for $\alpha = 1/2$ though any $\alpha > 0$ works, just with constant getting large as $\alpha \searrow 0$). This gives an estimate on the trace norm of the operator, which, by exponentiating, gives an upper bound $\exp\{c|\tilde{\mu}|^\alpha\}$ on the size of the determinant. This upper bound is beat by the exponential decay in $\tilde{\mu}$ of the prefactor term g_ϵ.

For the proof of Proposition 17, we do the same sort of factorization of our operator into AB where here

$$A(\zeta, \eta) = \frac{e^{c[\Psi(\zeta) + n_0 \log(\zeta)]}}{\zeta - \eta} \tag{167}$$

with n_0 as explained before the statement of Lemma 22, and $0 < c < 1$ fixed, and

$$B(\eta, \zeta) = e^{-c[\Psi(\zeta) + n_0\log(\zeta)]} \exp\{\Psi(\zeta) - \Psi(\eta)\}\mu f(\mu, \zeta/\eta)\frac{1}{\eta}. \tag{168}$$

We must be careful in keeping track of the contours on which these operators act. As we have seen we may assume that the η variables are on $\Gamma_{\eta,l}$ and the ζ variables on $\Gamma_{\zeta,l}$ for any fixed choice of $l \geq 0$. Now using the estimates of Lemmas 22 and 26 we compute that $||A||_2 < \infty$ (uniformly in $\epsilon < \epsilon_0$ and trivial also in $\tilde{\mu}$). Here we calculate the Hilbert-Schmidt norm using Lemma 29. Intuitively this norm is uniformly bounded as ϵ goes to zero because while the denominator blows up as bad as $\epsilon^{-1/2}$, the numerator is roughly supported only on a region of measure $\epsilon^{1/2}$ (owing to the exponential decay of the exponential when ζ differs from ξ by more than order $\epsilon^{1/2}$).

We wish to control $||B||_2$ now. Using the discussion before Lemma 22 we may rewrite B as

$$B(\eta,\zeta) = e^{-c[\Psi(\zeta)+n_0\log(\zeta)]}\exp\{(\Psi(\zeta)+n_0\log(\zeta)) - (\Psi(\eta)-n_0\log(\eta))\}\tilde{\mu}f(\tilde{\mu},\zeta/\eta)\frac{1}{\eta} \tag{169}$$

Lemmas 22 and 26 apply and say that the exponentials decay as $\exp\{-\epsilon^{-1/2}c'|\zeta-\eta|\}$. Owing to that decay estimate the final ingredient in proving our proposition comes in the form of control over $|\tilde{\mu}f(\tilde{\mu},z)|$ for $z=\zeta/\eta'$. There are two regions of η',ζ we must consider: (1) when $|\eta'-\zeta|\leq c$ for a very small constant c and (2) $|\eta'-\zeta|>c$. We will compute $||B||_2$ as the squareroot of

$$\int_{\eta,\zeta\in\text{Case (1)}} |B(\eta,\zeta)|^2 d\eta d\zeta + \int_{\eta,\zeta\in\text{Case (2)}} |B(\eta,\zeta)|^2 d\eta d\zeta. \tag{170}$$

We will show that the first term can be bounded by $C|\tilde{\mu}|^{2\alpha}$ for any $\alpha<1$, while the second term can be bounded by a large constant. As a result $||B||_2\leq C|\tilde{\mu}|^{\alpha}$ which is exactly as desired since then $||AB||_1\leq\exp\{c|\tilde{\mu}|^{\alpha}\}$.

Consider case (1) where $|\eta'-\zeta|\leq c$ for a constant c which is positive but very small (depending on how small T is). One may easily check then that due to the choices of the contours we have that $\epsilon^{-1/2}(|\zeta/\eta|-1)$ is contained in a compact subset of $(0,2)$. In fact, ζ/η' almost exactly lies along the curve $|z|=1+\epsilon^{1/2}$ and in particular (by taking ϵ_0 small enough) we can assume that ζ/η never leaves the region bounded by $|z|=1+(1\pm r)\epsilon^{1/2}$ for any fixed $c<1$. Let us call this region $R_{\epsilon,r}$. The we have the following important lemma:

Lemma 34. *Fix ϵ_0 and $r<0$ then for all $\epsilon<\epsilon_0$ and for all $\tilde{\mu}$ on $\tilde{\mathcal{C}}_\epsilon$ and all $z\in R_{\epsilon,r}$,*

$$|\tilde{\mu}f(\tilde{\mu},z)|\leq\frac{c|\tilde{\mu}|^{\alpha}}{|1-z|} \tag{171}$$

for some $\alpha\in(0,1)$ fixed with $c=c_\alpha$ as constant independent of z, $\tilde{\mu}$ and ϵ.

Remark 35. It is worth noting that by changing the value of α in the definition of $\kappa(\theta)$ (which then goes into the definition of $\Gamma_{\eta,l}$ and $\Gamma_{\zeta,l}$) and also focusing the region $R_{\epsilon,r}$ around $|z|=1+2\alpha\epsilon^{1/2}$ we can take α arbitrarily small in the above lemma at cost of increasing the constant c_α (the same also applies for Proposition 18. The $|\tilde{\mu}|^{\alpha}$ comes out of the fact that $(1+2\alpha\epsilon^{1/2})^{\frac{1}{2}\epsilon^{-1/2}\log|\tilde{\mu}|}\approx|\tilde{\mu}|^{\alpha}$. Another important remark is that the proof below can be used to provide an alternative proof of Lemma 27 which is more direct and essentially just proves the convergence of the Riemann sum directly rather than by using functional equation properties of f and the analytic continuations.

Case (1) is now done since we can estimate the first integral in equation (170) using Lemma 34 and the exponential decay of the exponential term outside of $|\eta'-\zeta|=\epsilon^{\infty/\in}$. Therefore, just as with the A operator, the $\epsilon^{-1/2}$ blowup of $|\tilde{\mu}f(\tilde{\mu},\zeta/\eta')|$ is countered by the decay of the exponential and we are just left with a large constant time $|\tilde{\mu}|^{\alpha}$.

Turing to case (2) we need to show that the second integral in equation (170) is bounded uniformly in ϵ and $\tilde{\mu}\in\tilde{\mathbb{C}}_\epsilon$. This case corresponds to $|\eta'-\zeta|>c$ for some fixed but small constant c. Since $\epsilon^{-1/2}(|\zeta/\eta|-1)$ stays bounded in a compact set, using an argument almost identical to the proof of Lemma 23 we can show that $|\tilde{\mu}f(\tilde{\mu},\zeta/\eta)|$ can be bounded by $C|\tilde{\mu}|^{C'}$ for positive yet finite constants

C and C'. The important point is that there is only a finite power of $|\tilde{\mu}|$. Since $|\tilde{\mu}| < \epsilon^{-1/2}$ this means that this term can blowup at most polynomially in $\epsilon^{-1/2}$. On the other hand we know that the exponential term decays exponentially fast like $\exp\{-e^{-1/2}c\}$ and hence second integral in equation (170) in fact goes to zero.

Thus upon proving Lemma 34 we will have a complete proof of our desired result of Proposition 17.

Proof of Lemma 34. We will prove the desired estimate for $z : |z| = 1 + \epsilon^{1/2}$ and the proof for general $z \in R_{\epsilon,r}$ follows similarly.

Recall that

$$\tilde{\mu} f(\tilde{\mu}, z) = \sum_{k=-\infty}^{\infty} \frac{\tilde{\mu}\tau^k}{1 - \tilde{\mu}\tau^k} z^k. \tag{172}$$

Since $\tilde{\mu}$ has imaginary part 1, the denominator is smallest when $\tau^k = 1/|\tilde{\mu}|$, corresponding to

$$k = k^* = \lfloor \tfrac{1}{2}\epsilon^{-1/2} \log |\mu| \rfloor. \tag{173}$$

We start, therefore, by centering our doubly infinite sum at around this value,

$$\tilde{\mu} f(\tilde{\mu}, z) = \sum_{k=-\infty}^{\infty} \frac{\tilde{\mu}\tau^{k^*}\tau^k}{1 - \tilde{\mu}\tau^{k^*}\tau^k} z^{k^*} z^k. \tag{174}$$

By the definition of k^*,

$$|z|^{k^*} = |\tilde{\mu}|^{1/2}(1 + \mathcal{O}(\epsilon^{1/2})) \tag{175}$$

thus we find that

$$|\tilde{\mu} f(\tilde{\mu}, z)| = |\tilde{\mu}|^{1/2} \left| \sum_{k=-\infty}^{\infty} \frac{\omega\tau^k}{1 - \omega\tau^k} z^k \right| \tag{176}$$

where

$$\omega = \tilde{\mu}\tau^{k^*} \tag{177}$$

and is roughly on the unit circle except for a small dimple near 1. To be more precise, due to the rounding in the definition of k^* the ω is not exactly on the unit circle, however we do have the following two properties:

$$|1 - \omega| > \epsilon^{1/2}, \qquad |\omega| - 1 = \mathcal{O}(\epsilon^{1/2}). \tag{178}$$

The section of $\tilde{\mathcal{C}}_\epsilon$ in which $\tilde{\mu} = \epsilon^{-1/2} - 1 + iy$ for $y \in (-1, 1)$ corresponds to ω lieing along a small dimple around 1 (and still respects $|1 - \omega| > \epsilon^{1/2}$). We call the curve on which ω lies Ω.

We can bring the $|\tilde{\mu}|^{1/2}$ factor to the left and split the summation into three parts as

$$|\tilde{\mu}|^{-1/2} \tilde{\mu} f(\tilde{\mu}, z) = \left| \sum_{k=-\infty}^{-\epsilon^{-1/2}} \frac{\omega\tau^k}{1 - \omega\tau^k} z^k + \sum_{k=-\epsilon^{-1/2}}^{\epsilon^{-1/2}} \frac{\omega\tau^k}{1 - \omega\tau^k} z^k + \sum_{k=\epsilon^{-1/2}}^{\infty} \frac{\omega\tau^k}{1 - \omega\tau^k} z^k \right|. \tag{179}$$

We will control each of these term separately. The first and the third are easiest. Consider

$$\left| (z - 1) \sum_{k=-\infty}^{-\epsilon^{-1/2}} \frac{\omega\tau^k}{1 - \omega\tau^k} z^k \right|. \tag{180}$$

We wish to show this is bounded by a constant which is independent of $\tilde{\mu}$ and ϵ. Summing by parts the argument of the absolute value can be written as

$$\frac{\omega\tau^{-\epsilon^{-1/2}+1}}{1-\omega\tau^{-\epsilon^{-1/2}+1}}z^{-\epsilon^{-1/2}+1}+(1-\tau)\sum_{k=-\infty}^{-\epsilon^{-1/2}}\frac{\omega\tau^k}{(1-\omega\tau^k)(1-\omega\tau^{k+1})}z^k. \tag{181}$$

We have $\tau^{-\epsilon^{-1/2}+1}\approx e^2$ and $|z^{-\epsilon^{-1/2}+1}|\approx e^{-1}$ (where $e\sim 2.718$). The denominator of the first term is therefore bounded from zero. Thus the absolute value of this term is bounded by a constant. For the second term of (181) we can bring the absolute value inside of the summation to get

$$(1-\tau)\sum_{k=-\infty}^{-\epsilon^{-1/2}}\left|\frac{\omega\tau^k}{(1-\omega\tau^k)(1-\omega\tau^{k+1})}\right||z|^k. \tag{182}$$

The first term in absolute values stays bounded above by a constant times the value at $k=-\epsilon^{-1/2}$. Therefore, replacing this by a constant, we can sum in $|z|$ and we get $\frac{|z|^{-\epsilon^{-1/2}}}{1-1/|z|}$. The numerator, as noted before, is like e^{-1} but the denominator is like $\epsilon^{1/2}/2$. This is cancelled by the term $1-\tau=\mathcal{O}(\epsilon^{1/2})$ in front. Thus the absolute value is bounded.

The argument for the third term of equation (179) works in the same way, except rather than multiplying by $|1-z|$ and showing the result is constant, we multiply by $|1-\tau z|$. This is, however, sufficient since $|1-\tau z|$ and $|1-z|$ are effectively the same for z near 1 which is where our desired bound must be shown carefully.

We now turn to the middle term in equation (179) which is the more difficult term. We will show that

$$\left|(1-z)\sum_{k=-\epsilon^{-1/2}}^{\epsilon^{-1/2}}\frac{\omega\tau^k}{1-\omega\tau^k}z^k\right|=\mathcal{O}(\log|\tilde{\mu}|), \tag{183}$$

(recall the ω is defined in terms of $\tilde{\mu}$) This is of smaller order than $|\tilde{\mu}|$ raised to any positive real power and thus finishes the proof. For the sake of simplicity we will first show this with $z=1+\epsilon^{1/2}$. The general argument for points z of the same radius and non-zero angle is very similar as we will observe at the end of the proof. With the choice of z, observe that the $(1-z)$ prefactor is just $\epsilon^{1/2}$.

The method of proof we employ is to prove that this sum is well approximated by a Riemann sum and then that the Riemann sum is well approximated by a suitable integral. This idea was mentioned in the formal proof of the ϵ goes to zero limit. In fact, the argument below can be used to make that formal observation entirely rigorous and thus provides an alternative method to the complex analytic approach we take in the proof of Lemma 27. The sum we have is given by

$$\epsilon^{1/2}\sum_{k=-\epsilon^{-1/2}}^{\epsilon^{-1/2}}\frac{\omega\tau^k}{1-\omega\tau^k}z^k=\epsilon^{1/2}\sum_{k=-\epsilon^{-1/2}}^{\epsilon^{-1/2}}\frac{\omega(1-\epsilon^{1/2}+\mathcal{O}(\epsilon))^k}{1-\omega(1-2\epsilon^{1/2}+\mathcal{O}(\epsilon))^k} \tag{184}$$

where we have used the fact that $\tau=1-2\epsilon^{1/2}+\mathcal{O}(\epsilon)$ and that $\tau z=1-\epsilon^{1/2}+\mathcal{O}(\epsilon)$. Observe that if $k=t\epsilon^{-1/2}$ then this sum is close to a Riemann sum for

$$\int_{-1}^{1}\frac{\omega e^{-t}}{1-\omega e^{-2t}}dt. \tag{185}$$

We use this formal relationship to prove that the sum in equation (184) is $\mathcal{O}(\log|\tilde{\mu}|)$. We do this in a few steps. The first step is to consider the difference between each term in our sum and the analogous term in a Riemann sum for the integral. After estimating the difference we show that this can be summed over k and gives us a finite error. The second step is to estimate the error of this Riemann sum approximation to the actual integral. Finally, we estimate the size of the integral for ω on the dimpled curve Ω.

A single term in the Riemann sum for the integral looks like $\epsilon^{1/2}\frac{\omega e^{-k\epsilon^{1/2}}}{1-\omega e^{-2k\epsilon^{1/2}}}$. Thus we are interested in estimating

$$\epsilon^{1/2}\left|\frac{\omega(1-\epsilon^{1/2}+\mathcal{O}(\epsilon))^k}{1-\omega(1-2\epsilon^{1/2}+\mathcal{O}(\epsilon))^k}-\frac{\omega e^{-k\epsilon^{1/2}}}{1-\omega e^{-2k\epsilon^{1/2}}}\right|. \tag{186}$$

We claim that there exists $C<\infty$, independent of ϵ and k satisfying $k\epsilon^{1/2}\leq 1$, such that the previous line is bounded above by

$$\frac{Ck^2\epsilon^{3/2}}{(1-\omega+\omega 2k\epsilon^{1/2})}+\frac{Ck^3\epsilon^2}{(1-\omega+\omega 2k\epsilon^{1/2})^2}. \tag{187}$$

To prove that (186) $\leq$(187) we expand the powers of k and the exponentials. For the numerator and denominator of the first term inside of the absolute value in (186) we have $\omega(1-\epsilon^{1/2}+\mathcal{O}(\epsilon))^k=\omega-\omega k\epsilon^{1/2}+\mathcal{O}(k^2\epsilon)$ and

$$\begin{aligned}1-\omega(1-2\epsilon^{1/2}+\mathcal{O}(\epsilon))^k &= 1-\omega+\omega 2k\epsilon^{1/2}-\omega 2k^2\epsilon+\mathcal{O}(k\epsilon)+\mathcal{O}(k^3\epsilon^{3/2}) &(188)\\ &= (1-\omega+\omega 2k\epsilon^{1/2})(1-\frac{\omega 2k^2\epsilon+\mathcal{O}(k\epsilon)+\mathcal{O}(k^3\epsilon^{3/2})}{1-\omega+\omega 2k\epsilon^{1/2}}). &(189)\end{aligned}$$

Using $1/(1-z)=1+z+\mathcal{O}(z^2)$ for $|z|<1$ we see that

$$\frac{\omega(1-\epsilon^{1/2}+\mathcal{O}(\epsilon))^k}{1-\omega(1-2\epsilon^{1/2}+\mathcal{O}(\epsilon))^k}=\frac{\omega-\omega k\epsilon^{1/2}+\mathcal{O}(k^2\epsilon)}{1-\omega+\omega 2k\epsilon^{1/2}}\left(1+\frac{\omega 2k^2\epsilon+\mathcal{O}(k\epsilon)+\mathcal{O}(k^3\epsilon^{3/2})}{1-\omega+\omega 2k\epsilon^{1/2}}\right) \tag{190}$$

$$=\frac{\left(\omega-\omega k\epsilon^{1/2}+\mathcal{O}(k^2\epsilon)\right)\left(1-\omega+\omega 2k\epsilon^{1/2}+\omega 2k^2\epsilon+\mathcal{O}(k\epsilon)+\mathcal{O}(k^3\epsilon^{3/2})\right)}{(1-\omega+\omega 2k\epsilon^{1/2})^2} \tag{191}$$

Likewise, the second term from equation (186) can be similarly estimated and shown to be

$$\frac{\omega e^{-k\epsilon^{1/2}}}{1-\omega e^{-2k\epsilon^{1/2}}}=\frac{\left(\omega-\omega k\epsilon^{1/2}+\mathcal{O}(k^2\epsilon)\right)\left(1-\omega+\omega 2k\epsilon^{1/2}+\omega 2k^2\epsilon+\mathcal{O}(k^3\epsilon^{3/2})\right)}{(1-\omega+\omega 2k\epsilon^{1/2})^2}. \tag{192}$$

Taking the difference of these two terms, and noting the cancellation of a number of the terms in the numerator, gives (187).

To see that the error in (187) is bounded after the summation over $k\in\{-\epsilon^{-1/2},\ldots,\epsilon^{-1/2}\}$, note that this gives

$$\epsilon^{1/2}\sum_{-\epsilon^{-1/2}}^{\epsilon^{1/2}}\frac{(2k\epsilon^{1/2})^2}{1-\omega+\omega(2k\epsilon^{1/2})}+\frac{(2k\epsilon^{1/2})^3}{(1-\omega+\omega(2k\epsilon^{1/2}))^2}\sim\int_{-1}^{1}\frac{(2t)^2}{1-\omega+\omega 2t}+\frac{(2t)^3}{(1-\omega+\omega 2t)^2}dt. \tag{193}$$

The Riemann sums and integrals are easily shown to be convergent for our ω which one recalls lies on Ω which is roughly the unit circle, and avoids the point 1 by distance $\epsilon^{1/2}$.

Having completed this first step, we now must show that the Riemann sum for the integral in equation (185) converges to the integral. This involves the following estimate,

$$\sum_{k=-\epsilon^{-1/2}}^{\epsilon^{-1/2}}\epsilon^{1/2}\max_{(k-1/2)\epsilon^{1/2}\leq t\leq(k+1/2)\epsilon^{1/2}}\left|\frac{\omega e^{-k\epsilon^{1/2}}}{1-\omega e^{-2k\epsilon^{1/2}}}-\frac{\omega e^{-t}}{1-\omega e^{-2t}}\right|\leq C \tag{194}$$

To show this, observe that for $t \in \epsilon^{1/2}[k-1/2, k+1/2]$ we can expand the second fraction as

$$\frac{\omega e^{-k\epsilon^{1/2}}(1+\mathcal{O}(\epsilon^{1/2}))}{1-\omega e^{-2k\epsilon^{1/2}}(1-2l\epsilon^{1/2}+\mathcal{O}(\epsilon))} \tag{195}$$

where $l \in [-1/2, 1/2]$. Factoring the denominator as

$$(1-\omega e^{-2k\epsilon^{1/2}})(1+\frac{\omega e^{-2k\epsilon^{1/2}}(2l\epsilon^{1/2}+\mathcal{O}(\epsilon))}{1-\omega e^{-2k\epsilon^{1/2}}}) \tag{196}$$

we can use $1/(1+z) = 1 - z + \mathcal{O}(z^2)$ (valid since $|1-\omega e^{-2k\epsilon^{1/2}}| > \epsilon^{1/2}$ and $|l| \leq 1$) to rewrite equation (195) as

$$\frac{\omega e^{-k\epsilon^{1/2}}(1+\mathcal{O}(\epsilon^{1/2}))\left(1-\frac{\omega e^{-2k\epsilon^{1/2}}(2l\epsilon^{1/2}+\mathcal{O}(\epsilon))}{1-\omega e^{-2k\epsilon^{1/2}}}\right)}{1-\omega e^{-2k\epsilon^{1/2}}}. \tag{197}$$

Canceling terms in this expression with the terms in the first part of equation (194) we find that we are left with terms bounded by

$$\frac{\mathcal{O}(\epsilon^{1/2})}{1-\omega e^{-2k\epsilon^{1/2}}} + \frac{\mathcal{O}(\epsilon^{1/2})}{(1-\omega e^{-2k\epsilon^{1/2}})^2}. \tag{198}$$

These must be summed over k and multiplied by the prefactor $\epsilon^{1/2}$. Summing over k we find that these are approximated by the integrals

$$\epsilon^{1/2}\int_{-1}^{1}\frac{1}{1-\omega+\omega 2t}dt, \qquad \epsilon^{1/2}\int_{-1}^{1}\frac{1}{(1-\omega+\omega 2t)^2}dt \tag{199}$$

where $|1-\omega| > \epsilon^{1/2}$. The first integral has a logarithmic singularity at $t=0$ which gives $|\log(1-\omega)|$ which is clearly bounded by a constant time $|\log \epsilon^{1/2}|$ for $\omega \in \Omega$. When multiplied by $\epsilon^{1/2}$ this term is clearly bounded in ϵ. Likewise, the second integral diverges like $|1/(1-\omega)|$ which is bounded by $\epsilon^{-1/2}$ and again multiplying by the $\epsilon^{1/2}$ factor in front shows that this term is bounded. This proves the Riemann sum approximation.

The last steps is to control the behavior of

$$\int_{-1}^{1}\frac{\omega e^{-t}}{1-\omega e^{-2t}}dt \tag{200}$$

for $\omega \in \Omega$ (in particular where $|1-\omega| > \epsilon^{1/2}$). It is clear, however, that the divergence of this integral in t near zero gives a logarithmic divergence of the integral, and so this integral behaves like $|\log(1-\omega)|$ which behaves like $\log|\tilde{\mu}|$ and is hence smaller than any power of $|\tilde{\mu}|$.

This estimate completes the proof of the desired bound when $z = 1+\epsilon^{1/2}$. The general case of $|z| = 1+\epsilon^{1/2}$ is proved along a similar line by letting $z = 1+\rho\epsilon^{1/2}$ for ρ on a suitably defined contour such that z lies on the circle of radius $1+\epsilon^{1/2}$. The prefactor is no longer $\epsilon^{1/2}$ but rather now $\rho\epsilon^{1/2}$ and all estimates must take into account ρ. However, going through this carefully one finds that the same sort of estimates as above hold and hence the theorem is proved in general. □

This lemma completes the proof of Proposition 17 □

Proof of Proposition 18. We will focus on the growth of the absolute value of the determinant. Recall (see Lemma 29) that if K is trace class then $|\det(I+K)| \leq e^{||K||_1}$ where $||K||_1$ denotes the trace norm. Furthermore, if K can be factored into the product $K = AB$ where A and B are

Hilbert-Schmidt, then $||K||_1 \leq ||A||_2||B||_2$. We will demonstrate such a factorization and follow this approach to control the size of the determinant.

Define $A : L^2(\tilde{\Gamma}_\zeta) \to L^2(\tilde{\Gamma}_\eta)$ and $B : L^2(\tilde{\Gamma}_\eta) \to L^2(\tilde{\Gamma}_\zeta)$ via the kernels

$$A(\tilde{\zeta},\tilde{\eta}) = \frac{e^{-|\mathrm{Im}(\tilde{\zeta})|}}{\tilde{\zeta}-\tilde{\eta}}, \qquad B(\tilde{\eta},\tilde{\zeta}) = e^{|\mathrm{Im}(\tilde{\zeta})|}\exp\{-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}^3)+a\tilde{z}\}2^{1/3}(-\tilde{\mu})^{\tilde{z}}\pi\csc(\pi\tilde{z}), \tag{201}$$

where we let $\tilde{z} = 2^{1/3}(\tilde{\zeta}-\tilde{\eta})$. Notice that we have put the factor $e^{-|\mathrm{Im}(\tilde{\zeta})|}$ into the A kernel and removed it from the B contour. The point of this is to help control the A kernel, without significantly impacting the norm of the B kernel.

Consider first $||A||_2$ which is given by

$$||A||_2^2 = \int_{\tilde{\Gamma}_\zeta}\int_{\tilde{\Gamma}_\eta} d\tilde{\zeta}d\tilde{\eta}\frac{e^{-2|\mathrm{Im}(\tilde{\zeta})|}}{|\tilde{\zeta}-\tilde{\eta}|^2}. \tag{202}$$

The integral in $\tilde{\eta}$ converges and is independent of $\tilde{\zeta}$ (recall that $|\tilde{\zeta}-\tilde{\eta}|$ is bounded away from zero) while the remaining integral in $\tilde{\zeta}$ is clearly convergence (its exponentially small as $\tilde{\zeta}$ goes away from zero along $\tilde{\Gamma}_\zeta$. Thus $||A||_2 < c$ with no dependence on $\tilde{\mu}$ at all.

We now turn to computing $||B||_2$. First consider the cubic term $\tilde{\zeta}^3$. The contour $\tilde{\Gamma}_\zeta$ is parametrized by $-\frac{c_3}{2}+c_3 ir$ for $r \in (-\infty,\infty)$ — that is, a straight up and down line just to the left of the y axis. By plugging this parametrization in and cubing it, we see that, $\mathrm{Re}(\tilde{\zeta}^3)$ behaves like $|\mathrm{Im}(\tilde{\zeta})|^2$. This is a critical fact — even though our contours are parallel and only differ horizontally by a small distance, their relative location lead to very different behavior for the real part of their cube. For $\tilde{\eta}$ on the right of the y axis, the real part still grows quadratically, however with a negative sign. This is important because this implies that $|\exp\{-\frac{T}{3}(\tilde{\zeta}^3-\tilde{\eta}^3)\}|$ behaves like the exponential of the real part of the argument, which is to say, like

$$\exp\{-\frac{T}{3}(|\mathrm{Im}(\tilde{\zeta})|^2+|\mathrm{Im}(\tilde{\eta})|^2)\}. \tag{203}$$

Turning to the $\tilde{\mu}$ term, observe that

$$\begin{aligned} |(-\tilde{\mu})^{-\tilde{z}}| &= \exp(\mathrm{Re}\,[(\log|\tilde{\mu}|+i\arg(-\tilde{\mu}))(-\mathrm{Re}(\tilde{z})-i\mathrm{Im}(\tilde{z}))]) &(204)\\ &= \exp(-\log|\tilde{\mu}|\mathrm{Re}(\tilde{z})+\arg(-\tilde{\mu})\mathrm{Im}(\tilde{z})). &(205)\end{aligned}$$

The csc term behaves, for large $\mathrm{Im}(\tilde{z})$ like $\exp(-\pi|\mathrm{Im}(\tilde{z})|)$, and putting all these estimates together gives that for $\tilde{\zeta}$ and $\tilde{\eta}$ far from the origin on their respective contours, $|B(\tilde{\eta},\tilde{\zeta})|$ behaves like the following product of exponentials:

$$\exp\{|\mathrm{Im}(\tilde{\zeta})|\}\exp\{-\frac{T}{3}(|\mathrm{Im}(\tilde{\zeta})|^2+|\mathrm{Im}(\tilde{\eta})|^2)\}\exp\{-\log|\tilde{\mu}|\mathrm{Re}(\tilde{z})+\arg(-\tilde{\mu})\mathrm{Im}(\tilde{z})-\pi|\mathrm{Im}(\tilde{z})|\}. \tag{206}$$

Now observe that due to the location of the contours, $-\mathrm{Re}(\tilde{z})$ is constant and less than one (in fact equal to $1/2$ by our choice of contours). Therefore we may factor out the term $\exp\{-\log|\tilde{\mu}|\mathrm{Re}(\tilde{z})\} = |\tilde{\mu}|^\alpha$ for $\alpha = 1/2 < 1$.

The Hilbert-Schmidt norm of what remains is clearly finite and independent of $\tilde{\mu}$ (this is just due to the strong exponential decay from the quadratic terms $-\mathrm{Im}(\zeta)^2$ and $-\mathrm{Im}(\eta)^2$ in the exponential. Therefore we find that $||B||_2 \leq c|\tilde{\mu}|^\alpha$ for some constant c.

This shows that $||K_a^{\rm csc}||_1$ behaves like $|\tilde{\mu}|^\alpha$ for $\alpha < 1$. Using the bound $|\det(I + K_a^{\rm csc})| \leq e^{||K_a^{\rm csc}||}$ we find that $|\det(I + K_a^{\rm csc})| \leq e^{|\tilde{\mu}|^\alpha}$. Comparing this to $e^{-\tilde{\mu}}$ we have our desired result. Note that this proof also shows that $K_a^{\rm csc}$ is trace class. □

2.2.3. *Proofs from Section 2.1.*

Proof of Lemma 22. Before starting this proof, we remark that the choice (118) of $\kappa(\theta)$ function was specifically to make the calculations in this proof more tractable. Certainly other choices of contours would do, however, the estimates would likely be harder in that case. As it is, we used Mathematica as a preliminary tool to assist us in computing the series expansions and simplifying the resulting expressions.

Now define the function $g(\eta) = \Psi(\eta) + n_0 \log(\eta)$. We wish to control the real part of this function for both the η contour and the ζ contour. Combining these estimates proves the lemma.

We may expand $g(\eta)$ into powers of ϵ with the expression for η in terms of $\kappa(\theta)$ from (118) with $\alpha = -1/2$ (similarly $1/2$ for the ζ expansion). Doing this we see that the $n_0 \log(\eta)$ term plays an important role in canceling the $\log(\epsilon)$ term in the Ψ and we are left with

$$\mathrm{Re}(g(\eta)) = -\frac{\epsilon^{-1}}{4} T\alpha \cot^2\left(\frac{\theta}{2}\right) + \frac{\epsilon^{-1/2}}{8} T\left[\alpha + \kappa(\theta)\right]^2 \cot^2\left(\frac{\theta}{2}\right) + \mathcal{O}(1). \tag{207}$$

Plugging in the expression for $\kappa(\theta)$ and factoring out an $\epsilon^{-1/2}$ we find that

$$\mathrm{Re}(g(\eta)) = \epsilon^{-1/2}\left(-\frac{\epsilon^{-1/2}}{4} T\alpha \cot^2\left(\frac{\theta}{2}\right) + \frac{1}{8} T\left[\alpha + \kappa(\theta)\right]^2 \cot^2\left(\frac{\theta}{2}\right)\right) + \mathcal{O}(1). \tag{208}$$

We must show that everything in the parenthesis above is bounded below by a positive constant times $|\eta - \xi|$ for all η which start at roughly angle $l\epsilon^{1/2}$. Equivalently we can show that the terms in the parenthesis behave bounded below by a positive constant times $|\pi - \theta|$, where θ is the polar angle of η.

The second part of this expression is clearly positive regardless of the value of α. What this suggests is that we must show (in order to also be able to deal with $\alpha = 1/2$ corresponding to the ζ estimate) that for η starting at angle $l\epsilon^{1/2}$ and going to zero, the first term dominates (if l is large enough).

To see this we first note that since $\alpha = -1/2$, the first term is clearly positive and dominates for θ bounded away from π. This proves the inequality for any range of η with θ bounded from π. Now observe the following asymptotic behavior of the following three functions of θ as θ goes to π:

$$\cot\left(\frac{\theta}{2}\right)^2 \approx \frac{1}{4}(\pi - \theta)^2 \tag{209}$$

$$\tan\left(\frac{\theta}{2}\right)^2 \approx \frac{4}{(\pi - \theta)^2} \tag{210}$$

$$\log\left(\frac{2}{1 - \cos(\theta)}\right)^2 \approx \frac{1}{16}(\pi - \theta)^4. \tag{211}$$

The behavior expressed above is dominant for θ close to π. We may expand the square in the second term in (208) and use the above expressions to find that for some suitable constant $C > 0$

(which depends on X and T only), we have

$$\mathrm{Re}(g(\eta)) = \epsilon^{-1/2}\left(-\frac{\epsilon^{-1/2}}{16}T\alpha(\pi-\theta)^2 + C(\pi-\theta)^2\right) + \mathcal{O}(1). \tag{212}$$

Now use the fact that $\pi - \theta \geq l\epsilon^{1/2}$ to give

$$\mathrm{Re}(g(\eta)) = \epsilon^{-1/2}\left(-\frac{l}{16}T\alpha(\pi-\theta) + \frac{X^2}{8T}(\pi-\theta)^2\right) + \mathcal{O}(1). \tag{213}$$

Since $\pi - \theta$ is bounded by π, we see that taking l large enough, the first term always dominates for the entire range of $\theta \in [0, \pi - l\epsilon^{1/2}]$. Therefore since $\alpha = -1/2$, we find that we have have the desired lower bound in $\epsilon^{-1/2}$ and $|\pi - \theta|$.

Turn now to the bound for $\mathrm{Re}(g(\zeta))$. In the case of the η contour we took $\alpha = -1/2$, however since we now are dealing with the ζ contour we must take $\alpha = 1/2$. This change in the sign of α and the argument above shows that equation (213) implies the desired bound for $\mathrm{Re}(g(\zeta))$ (for l large enough). □

Proof of Lemma 23. Recall that $\tilde{\mu}$ lies on a compact set along $\tilde{\mathcal{C}}$ and hence that $|1-\tilde{\mu}\tau^k|$ stays bounded from below as k varies. Also observe that due to our choices of contours for η' and ζ, $\epsilon^{-1/2}(|z|-1)$ stays bounded in a compact set.

In Lemma 33 we write $\tilde{\mu}f(\tilde{\mu}, z) = g_+(z) + g_-(z)$ and give functional equations for g_+ and g_-. Here we set $z = \zeta/\eta'$. Lets just focus on $g_+(z)$ as the bound for $g_-(z)$ follows similarly. The function $g_+(z)$ is initially only defined for $|z| < \tau^{-1}$. Due to the boundedness of $\epsilon^{-1/2}(|z|-1)$ it follows that there is a finite N such that $|z| < \tau^{-N}$. Thus if we apply the functional equation for g_+ N times we find that

$$g_+(z) = \sum_{k=1}^{N} \frac{\tilde{\mu}^k}{1-\tau^k z} + \tilde{\mu}^N g_+(\tau^N z). \tag{214}$$

Since $\tilde{\mu}$ is bounded and since g_+ is analytic for $|\tau^N z| < 1$ we find that this term behaves like the sum of the singularities. However, by the choice of our η' and ζ contours, all of the singularities have similar size to $1/|1-z|$ and hence $|g_+(z)|$ is bounded by a constant times $1/|1-z|$. Likewise for $g_-(z)$. □

Proof of Lemma 26. By the discussion preceding the statement of this lemma it suffices to consider the expansion without $n_0\log(\zeta/\eta')$ and without the $\log(\epsilon)$ term in m since, as we will see, they exactly cancel out. Therefore, for the sake of this proof we modify the definition of m given in equation (57) to be

$$m = \frac{1}{2}\left[\epsilon^{-1/2}(-a' + \frac{X^2}{2T}) + \frac{1}{2}t + x\right]. \tag{215}$$

where $a' = a + \log 2$ where the $\log 2$ came from the division by 2 in the $\log\epsilon^{-1/2}/2$ term.

The argument now amounts to a Taylor series expansion with control over the remainder term. Let us start by recording the first four derivatives of $\Lambda(\zeta)$:

$$\Lambda(\zeta) = -x\log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m\log\zeta \tag{216}$$

$$\Lambda'(\zeta) = \frac{x}{1-\zeta} + \frac{t}{(1-\zeta)^2} + \frac{m}{\zeta} \tag{217}$$

$$\Lambda''(\zeta) = \frac{x}{(1-\zeta)^2} + \frac{2t}{(1-\zeta)^3} - \frac{m}{\zeta^2} \tag{218}$$

$$\Lambda'''(\zeta) = \frac{2x}{(1-\zeta)^3} + \frac{6t}{(1-\zeta)^4} + \frac{2m}{\zeta^3} \tag{219}$$

$$\Lambda''''(\zeta) = \frac{6x}{(1-\zeta)^4} + \frac{24t}{(1-\zeta)^5} - \frac{6m}{\zeta^4}. \tag{220}$$

We Taylor expand $\Psi(\zeta) = \Lambda(\zeta) - \Lambda(\xi)$ around ξ and then expand in ϵ as ϵ goes to zero and find that

$$\Lambda'(\xi) = \frac{a' + \frac{1}{2}\log\epsilon}{2}\epsilon^{-1/2} + \mathcal{O}(1) \tag{221}$$

$$\Lambda''(\xi) = \mathcal{O}(\epsilon^{-1/2}) \tag{222}$$

$$\Lambda'''(\xi) = \frac{-T}{8}\epsilon^{-3/2} + \mathcal{O}(\epsilon^{-1}) \tag{223}$$

$$\Lambda''''(\xi) = \mathcal{O}(\epsilon^{-3/2}). \tag{224}$$

A Taylor series remainder estimate shows then that

$$\left|\Psi(\zeta) - \left[\Lambda'(\xi)(\zeta-\xi) + \frac{\Lambda''(\xi)}{2!}(\zeta-\xi)^2 + \frac{\Lambda'''(\xi)}{3!}(\zeta-\xi)^3\right]\right| \leq \sup_{t\in B(\xi,|\zeta-\xi|)} \frac{|\Lambda''''(t)|}{4!}|\zeta-\xi|^4, \tag{225}$$

where $B(\xi,|\zeta-\xi|)$ denotes the ball around ξ of radius $|\zeta-\xi|$. Now considering the scaling we have that $\zeta - \xi = c_3^{-1}\epsilon^{1/2}\tilde{\zeta}$ so that when we plug this in along with the estimates on derivatives of Λ at ξ, we find that the equation above becomes

$$\left|\Psi(\zeta) - \left[2^{1/3}a'\tilde{\zeta} - \frac{T}{3}\tilde{\zeta}^3\right]\right| = \mathcal{O}(\epsilon^{1/2}). \tag{226}$$

From this we see that if we included the $\log(\epsilon)$ term in with m it would, as claimed, exactly cancel the $n_0\log(\zeta/\eta')$ term. The above estimate therefore proves the desired first claimed result.

The second result follows readily from the inequality $|e^z - e^w| \leq |z-w|\max\{|e^z|,|e^w|\}$ and the first result, as well as the boundedness of the limiting integrand. □

Proof of Lemma 27. Expanding in ϵ we have that

$$z = \frac{\xi + c_3^{-1}\epsilon^{1/2}\tilde{\zeta}}{\xi + c_3^{-1}\epsilon^{1/2}\tilde{\eta}'} = 1 - \epsilon^{1/2}\tilde{z} + \mathcal{O}(\epsilon) \tag{227}$$

where the error is uniform for our range of $\tilde{\eta}'$ and $\tilde{\zeta}$ and where

$$\tilde{z} = c_3^{-1}(\tilde{\zeta} - \tilde{\eta}'). \tag{228}$$

We now appeal to the functional equation for f, explained in Lemma 33. Therefore we wish to study $\epsilon^{1/2}g_+(z)$ and $\epsilon^{1/2}g_-(z)$ as ϵ goes to 0 and show that they converge uniformly to suitable integrals. First consider the g_+ case. Let us, for the moment, assume that $|\tilde{\mu}| < 1$. We know that $|\tau z| < 1$, thus for any $N \geq 0$, we have

$$\epsilon^{1/2} g_+(z) = \epsilon^{1/2} \sum_{k=1}^{N} \frac{\tilde{\mu}^k}{1-\tau^k z} + \epsilon^{1/2} \tilde{\mu}^N g_+(\tau^N z). \tag{229}$$

Since, by assumption, $|\tilde{\mu}| < 1$, the first sum is the partial sum of a convergent series. Each term may be expanded in ϵ. Noting that

$$1 - \tau^k z = 1 - (1 - 2\epsilon^{1/2} + \mathcal{O}(\epsilon))(1 - \epsilon^{1/2}\tilde{z} + \mathcal{O}(\epsilon)) = (2k + \tilde{z})\epsilon^{1/2} + k\mathcal{O}(\epsilon), \tag{230}$$

we find that

$$\epsilon^{1/2} \frac{\tilde{\mu}^k}{1-\tau^k z} = \frac{\tilde{\mu}^k}{2k+\tilde{z}} + k\mathcal{O}(\epsilon^{1/2}). \tag{231}$$

The last part of the expression for g_+ is bounded in ϵ, thus we end up with the following asymptotics

$$\epsilon^{1/2} g_+(z) = \sum_{k=1}^{N} \frac{\tilde{\mu}^k}{2k+\tilde{z}} + N^2\mathcal{O}(\epsilon^{1/2}) + \tilde{\mu}^N \mathcal{O}(1). \tag{232}$$

It is possible to choose $N(\epsilon)$ which goes to infinity, such that $N^2\mathcal{O}(\epsilon^{1/2}) = o(1)$. Then for any fixed compact set contained in $\mathbb{C} \setminus 2\mathbb{Z}^{<0}$ (where $\mathbb{Z}^{<0} = \{-1, -2, -3, \ldots\}$) we have uniform convergence of this sequence of analytic functions to some function, which is necessarily analytic and equals

$$\sum_{k=1}^{\infty} \frac{\tilde{\mu}^k}{2k+\tilde{z}}. \tag{233}$$

This expansion is valid for $|\tilde{\mu}| < 1$ and for all $\tilde{z} \in \mathbb{C} \setminus 2\mathbb{Z}^{<0}$.

Likewise for $\epsilon^{1/2} g_-(z)$, for $|\tilde{\mu}| > 1$ and for $\tilde{z} \in \mathbb{C} \setminus \mathbb{Z}^{\geq 0}$, we have uniform convergence to the analytic function

$$\sum_{k=-\infty}^{0} \frac{\tilde{\mu}^k}{2k+\tilde{z}}. \tag{234}$$

We now introduce the Hurwitz Lerch transcendental function and relate some basic properties of it which can be found in [27].

$$\Phi(a, s, w) = \sum_{k=0}^{\infty} \frac{a^k}{(w+k)^s} \tag{235}$$

for $w > 0$ real and either $|a| < 1$ and $s \in \mathbb{C}$ or $|a| = 1$ and $\mathrm{Re}(s) > 1$. For $\mathrm{Re}(s) > 0$ it is possible to analytically extend this function using the integral formula

$$\Phi(a, s, w) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-(w-1)t}}{e^t - a} t^{s-1} dt, \tag{236}$$

where additionally $a \in \mathbb{C} \setminus [1, \infty)$ and $\mathrm{Re}(w) > 0$.

Observe that we can express our series in terms of this function as

$$\sum_{k=1}^{\infty} \frac{\tilde{\mu}^k}{2k+\tilde{z}} = \frac{1}{2}\tilde{\mu}\Phi(\tilde{\mu}, 1, 1 + \tilde{z}/2), \tag{237}$$

$$\sum_{k=-\infty}^{0} \frac{\tilde{\mu}^k}{2k-\tilde{z}} = -\frac{1}{2}\Phi(\tilde{\mu}^{-1}, 1, -\tilde{z}/2). \tag{238}$$

These two functions can be analytically continued using the integral formula onto the same region where $\mathrm{Re}(1+\tilde{z}/2) > 0$ and $\mathrm{Re}(-\tilde{z}/2) > 0$ – i.e. where $\mathrm{Re}(\tilde{z}/2) \in (-1, 0)$. Additionally the analytic continuation is valid for all $\tilde{\mu}$ not along $\mathbb{R}^+$.

We wish now to use Vitali's convergence theorem to conclude that $\tilde{\mu} f(\tilde{\mu}, z)$ converges uniformly for general $\tilde{\mu}$ to the sum of these two analytic continuations. In order to do that we need a priori boundedness of $\epsilon^{1/2} g_+$ and $\epsilon^{1/2} g_-$ for compact regions of $\tilde{\mu}$ away from $\mathbb{R}^+$. This, however, can be shown directly as follows. By assumption on $\tilde{\mu}$ we have that $|1 - \tau^k \tilde{\mu}| > c^{-1}$ for some positive constant c. Consider $\epsilon^{1/2} g_+$ first.

$$|\epsilon^{1/2} g_+(z)| \leq \epsilon^{1/2} \tilde{\mu} \sum_{k=0}^{\infty} \frac{|\tau z|^k}{|1 - \tau^k \tilde{\mu}|} \leq c\epsilon^{1/2} \frac{1}{1 - |\tau z|}. \tag{239}$$

We know that $|\tau z|$ is bounded to order $\epsilon^{1/2}$ away from 1 and therefore this show that $|\epsilon^{1/2} g_+(z)|$ has an upperbound uniform in $\tilde{\mu}$. Likewise we can do a similar computation for $\epsilon^{1/2} g_-(z)$ and find the same result, this time using that $|z|$ is bounded to order $\epsilon^{1/2}$ away from 1.

As a result of this apriori boundedness, uniform in $\tilde{\mu}$, we have that for compact sets of $\tilde{\mu}$ away from $\mathbb{R}^+$, uniformly in ϵ, $\epsilon^{1/2} g_+$ and $\epsilon^{1/2} g_-$ are uniformly bounded as ϵ goes to zero. Therefore Vitali's convergence theorem implies that they converge uniformly to their analytic continuation.

Now observe that

$$\frac{1}{2} \tilde{\mu} \Phi(\tilde{\mu}, 1, 1 + \tilde{z}/2) = \frac{1}{2} \int_0^{\infty} \frac{\tilde{\mu} e^{-\tilde{z}t/2}}{e^t - \tilde{\mu}} dt, \tag{240}$$

and

$$-\frac{1}{2} \Phi(\tilde{\mu}^{-1}, 1, -\tilde{z}/2) = -\frac{1}{2} \int_0^{\infty} \frac{e^{-(-\tilde{z}/2 - 1)t}}{e^t - 1/\tilde{\mu}} dt = \frac{1}{2} \int_{-\infty}^{0} \frac{\tilde{\mu} e^{-\tilde{z}t/2}}{e^t - \tilde{\mu}} dt. \tag{241}$$

Therefore, by a simple change of variables in the second integral, we can combine these as a single integral

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\tilde{\mu} e^{-\tilde{z}t/2}}{e^t - \tilde{\mu}} dt = \frac{1}{2} \int_0^{\infty} \frac{\tilde{\mu} s^{-\tilde{z}/2}}{s - \tilde{\mu}} \frac{ds}{s}. \tag{242}$$

The first of the above equations proves the lemma, and for an alternative expression we use the second of the integrals (which followed from the change of variables $e^t = s$) and thus, on the region where $\mathrm{Re}(\tilde{z}/2) \in (-1, 0)$ this integral converges and equals

$$\frac{1}{2} \pi (-\tilde{\mu})^{-\tilde{z}} \csc(\pi \tilde{z}/2). \tag{243}$$

This function is, in fact, analytic for $\tilde{\mu} \in \mathbb{C} \setminus [0, \infty)$ and for all $\tilde{z} \in \mathbb{C} \setminus 2\mathbb{Z}$. Therefore it is the analytic continuation of our asymptotic series. □

3. Weakly asymmetric limit of the corner growth model

Recall the definitions in Section 1.2 of WASEP and its height function (43). For $\epsilon \in (0, 1/4)$, let

$$p = \frac{1}{2} - \frac{1}{2}\epsilon^{1/2}, \qquad q = \frac{1}{2} + \frac{1}{2}\epsilon^{1/2}. \tag{244}$$

For $x \in \mathbb{R}$ and $t \geq 0$ let $Z_\epsilon(t, x)$ denote the rescaled height function;

$$Z_\epsilon(T, X) = \frac{1}{2} \epsilon^{-1/2} \exp\left\{-\lambda_\epsilon h_{\epsilon^{1/2}}(\epsilon^{-2}T, [\epsilon^{-1}X]) + \nu_\epsilon \epsilon^{-2} T\right\} \tag{245}$$

where

$$\begin{aligned}\nu_\epsilon = &\quad p+q-2\sqrt{qp} &= \frac{1}{2}\epsilon+\frac{1}{8}\epsilon^2+\mathcal{O}(\epsilon^3),\\ \lambda_\epsilon = &\quad \tfrac{1}{2}\log(q/p) &= \epsilon^{1/2}+\frac{1}{3}\epsilon^{3/2}+\mathcal{O}(\epsilon^{5/2}),\end{aligned} \tag{246}$$

and the closest integer $[x]$ is given by

$$[x] = \lfloor x+\tfrac{1}{2}\rfloor. \tag{247}$$

First let us describe in simple terms the dynamics in T of $Z_\epsilon(T,X)$ defined in (245). It grows continuously exponentially at rate $\epsilon^{-2}\nu_\epsilon$ and jumps at rates

$$r_-(x) = \epsilon^{-2}q(1-\eta(x))\eta(x+1) = \frac{1}{4}\epsilon^{-2}q(1-\hat\eta(x))(1+\hat\eta(x+1)) \tag{248}$$

to $e^{-2\lambda_\epsilon}Z_\epsilon$ and

$$r_+(x) = \epsilon^{-2}p\eta(x)(1-\eta(x+1)) = \frac{1}{4}\epsilon^{-2}p(1+\hat\eta(x))(1-\hat\eta(x+1)) \tag{249}$$

to $e^{2\lambda_\epsilon}Z_\epsilon$, independently at each site $X\in\epsilon\mathbb{Z}$. We write this as follows,

$$\begin{aligned} dZ_\epsilon(X) \quad &= \quad \left\{\epsilon^{-2}\nu_\epsilon + (e^{-2\lambda_\epsilon}-1)r_-(X) + (e^{2\lambda_\epsilon}-1)r_+(X)\right\}Z_\epsilon(X)dT \\ &\quad + (e^{-2\lambda_\epsilon}-1)Z_\epsilon(X)dM_-(X) + (e^{2\lambda_\epsilon}-1)Z_\epsilon(X)dM_+(X)\end{aligned} \tag{250}$$

where $dM_\pm(T,X) = dP_\pm(T,X) - r_\pm(X)dT$ where $P_-(T,X), P_+(T,X)$, $X\in\epsilon\mathbb{Z}$ are independent Poisson processes running at rates $r_-(T,X), r_+(T,X)$. Let

$$\gamma_\epsilon = 2\sqrt{pq} = 1-\frac{1}{2}\epsilon+\mathcal{O}(\epsilon^2) \tag{251}$$

and Δ_ϵ be the $\epsilon\mathbb{Z}$ Laplacian, $\Delta f(x) = \epsilon^{-2}(f(x+\epsilon)-2f(x)+f(x-\epsilon))$. We also have

$$\tfrac{1}{2}\gamma_\epsilon\Delta_\epsilon Z_\epsilon(X) = \tfrac{1}{2}\epsilon^{-2}\gamma_\epsilon(e^{-\lambda_\epsilon\hat\eta(x+1)}-2+e^{\lambda_\epsilon\hat\eta(x)})Z_\epsilon(X). \tag{252}$$

The parameters have been carefully chosen so that

$$\tfrac{1}{2}\epsilon^{-2}\gamma_\epsilon(e^{-\lambda_\epsilon\hat\eta(X+1)}-2+e^{\lambda_\epsilon\hat\eta(X)}) = \epsilon^{-2}\nu_\epsilon + (e^{-2\lambda_\epsilon}-1)r_-(X) + (e^{2\lambda_\epsilon}-1)r_+(X). \tag{253}$$

Hence [13],[3],

$$dZ_\epsilon = \tfrac{1}{2}\gamma_\epsilon\Delta_\epsilon Z_\epsilon + Z_\epsilon dM_\epsilon \tag{254}$$

where

$$dM_\epsilon(X) = (e^{-2\lambda_\epsilon}-1)dM_-(X) + (e^{2\lambda_\epsilon}-1)dM_+(X) \tag{255}$$

are martingales in T with

$$d\langle M_\epsilon(X), M_\epsilon(Y)\rangle = \epsilon^{-1}\mathbf{1}(X=Y)b_\epsilon(\tau_{-[\epsilon^{-1}X]}\eta)dT \tag{256}$$

where $\tau_x\eta(y) = \eta(y-x)$ and

$$b_\epsilon(\eta) = 1+\hat\eta(1)\hat\eta(0)+\hat b_\epsilon(\eta) \tag{257}$$

where

$$\begin{aligned}\hat b_\epsilon(\eta) \quad &= \quad \epsilon^{-1}\{[p((e^{-2\lambda_\epsilon}-1)^2-4\epsilon)+q((e^{2\lambda_\epsilon}-1)^2-4\epsilon)] \\ &\quad +[q(e^{-2\lambda_\epsilon}-1)^2-p(e^{2\lambda_\epsilon}-1)^2](\hat\eta(1)-\hat\eta(0)) \\ &\quad -[q(e^{-2\lambda_\epsilon}-1)^2+p(e^{2\lambda_\epsilon}-1)^2-\epsilon]\hat\eta(1)\hat\eta(0)\}.\end{aligned} \tag{258}$$

Clearly $b_\epsilon, \hat b_\epsilon \geq 0$. It is easy to check that there is a $C<\infty$ such that

$$\hat b_\epsilon \leq C\epsilon^{1/2} \tag{259}$$

and, for sufficiently small $\epsilon>0$,

$$b_\epsilon \leq 3. \tag{260}$$

Note that (254) is equivalent to the integral equation

$$Z_\epsilon(T,X) = \epsilon \sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T,X-Y)Z_\epsilon(0,Y) + \int_0^T \epsilon \sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T-S,X-Y)Z_\epsilon(S,Y)dM_\epsilon(S,Y) \tag{261}$$

where $p_\epsilon(T,X)$ are the (normalized) transition probabilities for the continuous time random walk with generator $\frac{1}{2}\gamma_\epsilon\Delta_\epsilon$. The normalization is multiplication of the actual transition probabilities by ϵ^{-1} so that

$$p_\epsilon(T,X) \to p(T,X) = \frac{e^{-X^2/2T}}{\sqrt{2\pi T}}. \tag{262}$$

We need some apriori bounds.

Lemma 36. *For $0 < T \le T_0$, and for each $q = 1,2,\ldots$, there is a $C_q = C_q(T_0) < \infty$ such that*

i. $E[Z_\epsilon^2(T,X)] \le C_2 p_\epsilon^2(T,X)$;
ii. $E\left[\left(Z_\epsilon(T,X) - \epsilon\sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T,X-Y)Z_\epsilon(0,Y)\right)^2\right] \le C_2 t p_\epsilon^2(T,X)$;
iii. $E[Z_\epsilon^{2q}(T,X)] \le C_q p_\epsilon^{2q}(T,X)$.

Proof. Within the proof, C will denote a finite number which does not depend on any other parameters except T and q, but may change from line to line. Also, for ease of notation, we identify functions on $\epsilon\mathbb{Z}$ with those on $\mathbb{R}$ by $f(x) = f([x])$.

First, note that

$$Z_\epsilon(0,Y) = \epsilon^{-1/2}\exp\{-\epsilon^{-1}\lambda_\epsilon|Y|\} = \epsilon^{-1/2}\exp\{-\epsilon^{-1/2}|Y| + \mathcal{O}(\epsilon^{1/2})\} \tag{263}$$

is an approximate delta function, from which we check that

$$\epsilon \sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T,X-Y)Z_\epsilon(0,Y) \le Cp_\epsilon(T,X). \tag{264}$$

Let

$$f_\epsilon(T,X) = E[Z_\epsilon^2(T,X)]. \tag{265}$$

From (264), (261) we get

$$f_\epsilon(T,X) \le Cp_\epsilon^2(T,X) + C\int_0^T\int_{-\infty}^{\infty} p_\epsilon^2(T-S,X-Y)f_\epsilon(S,Y)dSdY. \tag{266}$$

Iterating we obtain,

$$f_\epsilon(T,X) \le \sum_{n=0}^{\infty} C^n I_{n,\epsilon}(T,X) \tag{267}$$

where, for $\Delta_n = \Delta_n(T) = \{0 = t_0 \le T_1 < \cdots < T_n < T\}$,$X_0 = 0$,

$$I_{n,\epsilon}(T,X) = \int_{\Delta_n}\int_{\mathbb{R}^n} \prod_{i=1}^n p_\epsilon^2(T_i - T_{i-1}, X_i - X_{i-1}) p_\epsilon^2(T-T_n, X-x_n) \prod_{i=1}^n dX_i dT_i. \tag{268}$$

One readily checks that

$$I_{n,\epsilon}(T,X) \le C^n T^{n/2}(n!)^{-1/2} p_\epsilon^2(T,X). \tag{269}$$

From which we obtain i,

$$f_\epsilon(T,X) \le C\sum_{n=0}^{\infty}(CT)^{n/2}(n!)^{-1/2}p_\epsilon^2(T,X) \le C' p_\epsilon^2(T,X). \tag{270}$$

Now we turn to ii. From (261),

$$E\left[\left(Z_\epsilon(T,X) - \epsilon\sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T,X-Y)Z_\epsilon(0,Y)\right)^2\right] \le C\int_0^T\int_{-\infty}^{\infty} p_\epsilon^2(T-S,X-Y)E[Z_\epsilon^2(S,Y)]dYdS. \tag{271}$$

By i, we have

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} p_\epsilon^2(T-S,X-Y)E[Z_\epsilon^2(S,Y)]dYdS &\le C\int_0^T\int_{-\infty}^{\infty} p_\epsilon^2(T-S,X-Y)p_\epsilon^2(S,Y)dyYdS \\ &= C\sqrt{T}p_\epsilon^2(T,X)\end{aligned} \tag{272}$$

which is ii.

Finally we prove iii. Fix a $q \ge 2$. By standard methods of martingale analysis and (260), we have

$$\begin{aligned}&E\Big[\Big(\int_0^T \epsilon\sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon(T-S,X-Y)Z_\epsilon(S,Y)dM_\epsilon(S,Y)\Big)^{2q}\Big] \\ &\le CE\Big[\Big(\int_0^T \epsilon\sum_{Y\in\epsilon\mathbb{Z}} p_\epsilon^2(T-S,X-Y)Z_\epsilon^2(S,Y)dS\Big)^q\Big].\end{aligned} \tag{273}$$

Let

$$g_\epsilon(T,X) = E[Z_\epsilon^{2q}(T,X)]/p_\epsilon^{2q}(T,X). \tag{274}$$

From the last inequality, and Schwarz's inequality, we have

$$g_\epsilon(T,X) \le C(1+\int_{\Delta_q'(T)}\int_{\mathbb{R}^q}\prod_{i=1}^q p_\epsilon^2(S_i-S_{i-1},X_i-X_{i-1})p_\epsilon^2(S_i,Y_i)g_\epsilon^{1/q}(S_i,Y_i)dY_idS_i). \tag{275}$$

Now use the fact that

$$\prod_{i=1}^q g_\epsilon^{1/q}(S_i,Y_i) \le C\sum_{i=1}^q \frac{\prod_{j\ne i} p_\epsilon^{2/(q-1)}(S_j,Y_j)}{p_\epsilon^2(S_i,Y_i)} g_\epsilon(S_i,Y_i) \tag{276}$$

and iterate the inequality to obtain iii. □

We now turn to the tightness. In fact, although we are in a different regime, the arguments of [3] actually extend to our case. For each $\delta > 0$, let $\mathscr{P}_\epsilon^\delta$ be the distributions of the processes $\{Z_\epsilon(T,X)\}_{\delta\le T}$ on $D([\delta,\infty);D_u(\mathbb{R}))$ where D refers to right continuous paths with left limits. $D_u(\mathbb{R})$ indicates that in space these functions are equipped with the topology of uniform convergence on compact sets. Because the discontinuities of $Z_\epsilon(T,\cdot)$ are restricted to $\epsilon(1/2+\mathbb{Z})$, it is measurable as a $D_u(\mathbb{R})$-valued random function (see Sec. 18 of [4].) Since the jumps of $Z_\epsilon(T,\cdot)$ are uniformly small, local uniform convergence works for us just as well the standard Skhorohod topology. The following summarizes results which are contained [3] but not explicitly stated there in the form we need.

Theorem 37. [3] *There is an explicit $p < \infty$ such that if there exist $C, c < \infty$ for which*

$$\int_{-\infty}^{\infty} Z_\epsilon^p(\delta,X)d\mathscr{P}_\epsilon^\delta \le Ce^{c|X|}, \qquad X\in\epsilon\mathbb{Z}, \tag{277}$$

Then $\{\mathscr{P}^\delta_\epsilon\}_{0\le\epsilon\le 1/4}$ is a tight family. Any limit point $\mathscr{P}^\delta$ is supported $C([\delta,\infty);C(\mathbb{R}))$ and solves the martingale problem for the stochastic heat equation (7) after time δ.

It appears that $p=10$ works in [3], though it almost certainly can be improved to $p=4$. Note that the process level convergence is more than we need for the one-point function. However, it could be useful in the future. Although not explicitly stated there the theorem is proved in [3]. The key point is that all computations in [3] after the initial time are done using the equation (254) for Z_ϵ, which scales linearly in Z_ϵ. So the only input is a bound like (277) on the initial data. In [3], this is made as an assumption, which can easily be checked for initial data close to equilibrium. In the present case, it follows from *iii* of Lemma 36.

The measures $\mathscr{P}^{\delta_1}$ and $\mathscr{P}^{\delta_2}$, $\delta_1<\delta_2$ can be chosen to be consistent on $C[[\delta_2,\infty),C(\mathbb{R})]$ and because of this there is an inverse limit measure $\mathscr{P}$ on $C[(0,\infty),C(\mathbb{R})]$ which is consistent with any $\mathscr{P}^\delta$ on $C[[\delta,\infty),C(\mathbb{R})]$. From the uniqueness of the martingale problem for $t\ge\delta>0$ and the corresponding martingale representation theorem [19] there is a space-time white noise $\dot{\mathscr{W}}$, on a possibly enlarged probability space, $(\bar\Omega,\bar{\mathscr{F}}_T,\bar{\mathscr{P}})$ such that under $\bar{\mathscr{P}}$, for any $\delta>0$,

$$Z(T,X)=\int_{-\infty}^{\infty}p(T-\delta,X-Y)Z(\delta,Y)dY+\int_\delta^T\int_{-\infty}^{\infty}p(T-S,X-Y)Z(S,Y)\mathcal{W}(dY,dS). \quad (278)$$

Finally *ii* of Lemma 36 shows that under $\bar{\mathscr{P}}$,

$$\int_{-\infty}^{\infty}p(T-\delta,X-Y)Z(\delta,Y)dY\to p(T,X) \quad (279)$$

as $\delta\searrow 0$, which completes the proof.

4. Alternative forms of the crossover distribution function

We now demonstrate how the various alternative formulas for $F_T(s)$ given in Theorem 1 are derived from the cosecant kernel formula of Theorem 8.

4.1. **Proof of the crossover Airy kernel formula.**

We prove this by showing that

$$\det(I-K_a^{\mathrm{csc}})_{L^2(\tilde\Gamma_\eta)}=\det(I-K_{\sigma_{T,\tilde\mu}})_{L^2(\kappa_T^{-1}a,\infty)} \quad (280)$$

where $K_{\sigma_{T,\tilde\mu}}$ and $\sigma_{T,\tilde\mu}$ are given in the statement of Theorem 1 and where $\kappa_T=2^{-1/3}T^{1/3}$.

The kernel $K_a^{\mathrm{csc}}(\tilde\eta,\tilde\eta')$ is given by equation (67) as

$$\int_{\tilde\Gamma_\zeta}\exp\left\{-\frac{T}{3}(\tilde\zeta^3-\tilde\eta'^3)+2^{1/3}a(\tilde\zeta-\tilde\eta')\right\}2^{1/3}\left(\int_{-\infty}^{\infty}\frac{\tilde\mu e^{-2^{1/3}t(\tilde\zeta-\tilde\eta')}}{e^t-\tilde\mu}dt\right)\frac{d\tilde\zeta}{\tilde\zeta-\tilde\eta}. \quad (281)$$

For $\mathrm{Re}(z)>0$ we have the following nice identity:

$$\int_a^\infty e^{xz}dx=-\frac{e^{az}}{z}, \quad (282)$$

which, noting that $\mathrm{Re}(\tilde\zeta-\tilde\eta)<0$, we may apply to the above kernel to get

$$-2^{2/3}\int_{\tilde\Gamma_\zeta}\int_{-\infty}^{\infty}\int_a^\infty\exp\left\{-\frac{T}{3}(\tilde\zeta^3-\tilde\eta'^3)-2^{1/3}a\tilde\eta'\right\}\frac{\tilde\mu e^{-2^{1/3}t(\tilde\zeta-\tilde\eta')}}{e^t-\tilde\mu}e^{2^{1/3}(a-x)\tilde\eta}e^{2^{1/3}x\tilde\zeta}dxdtd\tilde\zeta. \quad (283)$$

This kernel can be factored as a product ABC where

$$A : L^2(a,\infty) \to L^2(\tilde\Gamma_\eta), \qquad B : L^2(\tilde\Gamma_\zeta) \to L^2(a,\infty), \qquad C : L^2(\tilde\Gamma_\eta) \to L^2(\tilde\Gamma_\zeta), \tag{284}$$

and the operators are given by their kernels

$$A(\tilde\eta, x) = e^{2^{1/3}(a-x)\tilde\eta}, \qquad B(x,\tilde\zeta) = e^{2^{1/3}x\tilde\zeta}, \tag{285}$$
$$C(\tilde\zeta,\tilde\eta) = -2^{2/3}\int_{-\infty}^{\infty} \exp\left\{-\frac{T}{3}(\tilde\zeta^3 - \tilde\eta^3) - 2^{1/3}a\tilde\eta\right\}\frac{\tilde\mu e^{-2^{1/3}t(\tilde\zeta-\tilde\eta)}}{e^t - \tilde\mu}dt.$$

Since $\det(I - ABC) = \det(I - BCA)$ we consider BCA acting on $L^2(a,\infty)$ with kernel

$$-2^{2/3}\int_{-\infty}^{\infty}\int_{\Gamma_{\tilde\zeta}}\int_{\Gamma_{\tilde\eta}} \exp\left\{-\frac{T}{3}(\tilde\zeta^3 - \tilde\eta'^3) + 2^{1/3}(x-t)\tilde\zeta - 2^{1/3}(y-t)\tilde\eta\right\}\frac{\tilde\mu}{e^t - \tilde\mu}d\tilde\eta d\tilde\zeta dt. \tag{286}$$

Using the formula for the Airy function given by

$$\mathrm{Ai}(r) = \int_{\tilde\Gamma_\zeta} \exp\{-\frac{1}{3}z^3 + rz\}dz \tag{287}$$

and replacing t with $-t$ we find that our kernel equals

$$2^{2/3}T^{-2/3}\int_{-\infty}^{\infty}\frac{\tilde\mu}{\tilde\mu - e^{-t}}\mathrm{Ai}\big(T^{-1/3}2^{1/3}(x+t)\big)\mathrm{Ai}\big(T^{-1/3}2^{1/3}(y+t)\big)dt. \tag{288}$$

We may now change variables in t as well as in x and y to absorb the factor of $T^{-1/3}2^{1/3}$. To rescale x and y we use the fact that $\det(I - K(x,y))_{L^2(ra,\infty)} = \det(I - rK(rx,ry))_{L^2(a,\infty)}$. This completes the proof.

4.2. **Proof of the Gumbel convolution formula.** Before starting we remark that throughout this proof we will dispense with the tilde with respect to $\tilde\mu$ and $\tilde{\mathcal{C}}$. We choose to prove this formula directly from the form of the Fredholm determinant given in the crossover Airy kernel formula of Theorem 1. However, we make note that it is possible, and in some ways simpler (though a little messier) to prove this directly from the csc form of the kernel. Our starting point is the formula for $F_T(s)$ given in equation (20). The integration in μ occurs along a complex contour and even though we haven't been writting it explicitly, the integral is divided by $2\pi i$. We now demonstrate how to squish this contour to the the positive real line (at which point we will start to write the $2\pi i$). The pole in the term $\sigma_{T,\mu}(t)$ for μ along $\mathbb{R}^+$ means that the integral along the positive real axis from above will not exactly cancel the integral from below.

Define a family of contour $\mathcal{C}_{\delta_1,\delta_2}$ parametrized by $\delta_1,\delta_2 > 0$ (small). The contours are defined in terms of three sections

$$\mathcal{C}_{\delta_1,\delta_2} = \mathcal{C}^-_{\delta_1,\delta_2} \cup \mathcal{C}^{circ}_{\delta_1,\delta_2} \cup \mathcal{C}^+_{\delta_1,\delta_2} \tag{289}$$

traversed counterclockwise, where

$$\mathcal{C}^{circ}_{\delta_1,\delta_2} = \{\delta_2 e^{i\theta} : \delta_1 \le \theta \le 2\pi - \delta_1\} \tag{290}$$

and where $\mathcal{C}^\pm_{\delta_1,\delta_2}$ are horizontal lines extending from $\delta_1 e^{\pm i\delta_2}$ to $+\infty$.

We can deform the original μ contour μ to any of these contours without changing the value of the integral (and hence of $F_T(s)$). To justify this we use Cauchy's theorem. However this requires the knowledge that the determinant is an analytic function of μ away from $\mathbb{R}^+$. This may be proved similarly to the proof of Lemma 13 and relies on Lemma 32. As such we do not include this computation here.

Fixing δ_2 for the moment we wish to consider the limit of the integrals over these contours as δ_1 goes to zero. The resulting integral be we written as $I^{circ}_{\delta_2} + I^{line}_{\delta_2}$ where

$$I^{circ}_{\delta_2} = \oint_{|\mu|=\delta_2} \frac{d\mu}{\mu} e^{-\mu} \det(I - K_{T,\mu})_{L^2(\kappa_T^{-1}a,\infty)}, \tag{291}$$

$$I^{line}_{\delta_2} = -\lim_{\delta_1\to 0} \int_{\delta_2}^{\infty} \frac{d\mu}{\mu} e^{-\mu} [\det(I - K_{T,\mu+i\delta_i}) - \det(I - K_{T,\mu-i\delta_i})] \tag{292}$$

Claim 38. *$I^{circ}_{\delta_2}$ exists and $\lim_{\delta_2\to 0} I^{circ}_{\delta_2} = 1$.*

Proof. It is easiest, in fact, to prove this claim by replacing the determinant by the csc determinant: equation (67). From that perspective the μ at 0 and at 2π are on opposite sides of the branch cut for $\log(-\mu)$, but are still defined (hence the $I^{circ}_{\delta_2}$ is clearly defined). As far as computing the limit, one can do the usual Hilbert-Schmidt estimate and show that, uniformly over the circle $|\mu| = \delta_2$, the trace norm goes to zero as δ_2 goes to zero. Thus the determinant goes uniformly to 1 and the claim follows. □

Turning now to $I^{line}_{\delta_2}$, that this limit exists can be seen by going to the equivalent csc kernel (where this limit is trivially just the kernel on different levels of the $\log(-\mu)$ branch cut). Notice now that we can write the operator $K_{T,\mu+i\delta_1} = K^{\rm sym}_{\delta_1} + K^{\rm asym}_{\delta_1}$ and likewise $K_{T,\mu-i\delta_1} = K^{\rm sym}_{\delta_1} - K^{\rm asym}_{\delta_1}$ where $K^{\rm sym}_{\delta_1}$ and $K^{\rm asym}_{\delta_1}$ also act on $L^2(\kappa_T^{-1}a,\infty)$ and are given by their kernels

$$K^{\rm sym}_{\delta_1}(x,y) = \int_{-\infty}^{\infty} \frac{\mu(\mu-b)+\delta_1^2}{(\mu-b)^2+\delta_1^2} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt \tag{293}$$

$$K^{\rm asym}_{\delta_1}(x,y) = \int_{-\infty}^{\infty} \frac{-i\delta_1 b}{(\mu-b)^2+\delta_1^2} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt, \tag{294}$$

where $b = b(t) = e^{-\kappa_T t}$.

From this it follows that

$$K^{\rm sym}(x,y) := \lim_{\delta_1\to 0} K^{\rm sym}_{\delta_1}(x,y) = \mathrm{P.V.} \int \frac{\mu}{\mu - e^{-\kappa_T t}} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt. \tag{295}$$

As far as $K^{\rm asym}_{\delta_1}$, since $\mu - b$ has a unique root at $t_0 = -\kappa_T^{-1}\log\mu$, it follows from the Plemelj formula [10] that

$$\lim_{\delta_1\to 0} K^{\rm asym}_{\delta_1}(x,y) = -\frac{\pi i}{\kappa_T} \mathrm{Ai}(x+t_0)\mathrm{Ai}(y+t_0). \tag{296}$$

With this in mind we define

$$K^{\rm asym}(x,y) = \frac{2\pi i}{\kappa_T} \mathrm{Ai}(x+t_0)\mathrm{Ai}(y+t_0). \tag{297}$$

We see that $K^{\rm asym}$ is a multiple of the projection operator onto the shifted Airy functions.

We may now collect the calculations from above and we find that

$$\begin{aligned} I^{line}_{\delta_2} &= -\frac{1}{2\pi i} \int_{\delta_2}^{\infty} \frac{d\mu}{\mu} e^{-\mu} [\det(I - K^{\rm sym} + \tfrac{1}{2}K^{\rm asym}) - \det(I - K^{\rm sym} - \tfrac{1}{2}K^{\rm asym})] \quad (298) \\ &= -\frac{1}{2\pi i} \int_{\delta_2}^{\infty} \frac{d\mu}{\mu} e^{-\mu} \det(I - K^{\rm sym}) \mathrm{tr}\left((I - K^{\rm sym})^{-1} K^{\rm asym}\right) \end{aligned}$$

where both K^{sym} and K^{asym} act on $L^2(\kappa_T^{-1}a,\infty)$ and where we have used the fact that K^2 is rank one, and if you have A and B, where B is rank one, then

$$\det(I-A+B)=\det(I-A)\det(I+(I-A)^{-1}B)=\det(I-A)\mathrm{tr}\left((I-A)^{-1}B\right). \tag{299}$$

As stated above we've only shown the pointwise convergence of the kernels to K^{sym} and K^2. However, using the decay properties of the Airy function and the exponential decay of σ this can be strengthened to trace-class convergence.

We may now take δ_2 to zero and find that

$$F_T(s)=\lim_{\delta_2\to 0}(I_{\delta_2}^{circ}+I_{\delta_2}^{line})=1-\frac{1}{2\pi i}\int_0^\infty \frac{d\mu}{\mu}e^{-\mu}\det(I-K^1)\mathrm{tr}\left((I-K^1)^{-1}K^{\mathrm{asym}}\right) \tag{300}$$

with K^{sym} and K^{asym} as above acting on $L^2(\kappa_T^{-1}a,\infty)$ and where the integral is improper at zero.

We can simplify our operators so that by changing variables and replacing x by $x+t_0$ and y by $y+t_0$. We can also change variables from μ to e^{-r}. With this in mind we redefine the operators K^{sym} and K^{asym} to act on $L^2(\kappa_T^{-1}(a-r),\infty)$ with kernels

$$\begin{aligned} K^{\mathrm{sym}}(x,y) &= \mathrm{P.V.}\int \sigma(t)\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt \\ K^{\mathrm{asym}}(x,y) &= \mathrm{Ai}(x)\mathrm{Ai}(y), \end{aligned} \tag{301}$$

where $\sigma(t)=\frac{1}{1-e^{-\kappa_T t}}$. In terms of these operators we have

$$F_T(s)=1-\int_{-\infty}^{\infty} e^{-e^{-r}}f(a-r)dr \tag{302}$$

where

$$f(r)=\kappa_T^{-1}\det(I-K^{\mathrm{sym}})_{L^2(\kappa_T^{-1}r,\infty)}\mathrm{tr}\left((I-K^{\mathrm{sym}})^{-1}K^{\mathrm{asym}}\right)_{L^2(\kappa_T^{-1}r,\infty)}. \tag{303}$$

Calling $G(r)=e^{-e^{-r}}$ and observing that $K^{\mathrm{sym}}=K_{\sigma_T}$ and $K^{\mathrm{asym}}=\mathrm{P}_{\mathrm{Ai}}$ this completes the proof of the first part of the Gumbel convolution formula. Turning now to the Hilbert transform formula, we may isolate the singularity of $\sigma_T(t)$ from the above kernel K^{sym} (or K_{σ_T}) as follows. Observe that we may write $\sigma_T(t)$as

$$\sigma_T(t)=\tilde{\sigma}_T(t)+\frac{1}{\kappa_T t} \tag{304}$$

where $\tilde{\sigma}_T(t)$ (given in equation (28)) is a smooth function, non-decreasing on the real line, with $\tilde{\sigma}_T(-\infty)=0$ and $\tilde{\sigma}_T(+\infty)=1$. Moreover, $\tilde{\sigma}_T'$ is an approximate delta function with width $\kappa_T^{-1}=2^{1/3}T^{-1/3}$. The principle value integral of the $\tilde{\sigma}_T(t)$ term can be replaced by a simple integral. The new term gives

$$\mathrm{P.V.}\int \frac{1}{\kappa_T t}\mathrm{Ai}(x+t)\mathrm{Ai}(y+t). \tag{305}$$

This is κ_T^{-1} times the Hilbert transform of the product of Airy functions, which is explicitly computable [34] with the result begin

$$\mathrm{P.V.}\int \frac{1}{\kappa_T t}\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)=\kappa_T^{-1}\pi G_{\frac{x-y}{2}}(\frac{x+y}{2}) \tag{306}$$

where $G_a(x)$ is given in equation (28).

5. Formulas for a class of generalized integrable integral operators

Presently we will consider a certain class of Fredholm determinants and make two computations involving these determinants. The second of these computations closely follows the work of Tracy and Widom and is based on a similar calculation done in [29]. In that case the operator in question is the Airy operator. We deal with the family of operators which arise in considering $F_T(s)$.

Consider the class of Fredholm determinants $\det(I-K)_{L^2(s,\infty)}$ with operator K acting on $L^2(s,\infty)$ with kernel

$$K(x,y)=\int_{-\infty}^{\infty}\sigma(t)\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt, \tag{307}$$

where $\sigma(t)$ is a function which is smooth except at a finite number of points at which it has bounded jumps and which approaches 0 at $-\infty$ and 1 at ∞, exponentially fast. These operators are, in a certain sense, generalizations of the class of integrable integral operators (see [5]).

The kernel can be expressed alternatively as

$$K(x,y)=\int_{-\infty}^{\infty}\sigma'(t)\frac{\varphi(x+t)\psi(y+t)-\psi(x+t)\varphi(y+t)}{x-y}dt, \tag{308}$$

with $\varphi(x)=\mathrm{Ai}(x)$ and $\psi(x)=\mathrm{Ai}'(x)$ and $\mathrm{Ai}(x)$ the Airy function.

This, and the entire generalization we will now develop is analogous to what is known for the Airy operator which is defined by its kernel $K_{\mathrm{Ai}}(x,y)$ on $L^2(-\infty,\infty)$ by

$$K_{\mathrm{Ai}}(x,y)=\int_{-\infty}^{\infty}\chi(t)\mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(x)-\mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}, \tag{309}$$

where presently $\chi(t)=\mathbf{1}_{\{t\geq 0\}}$.

Note that the $\sigma(t)$ in our main result is not exactly of this type. However, one can smooth out the σ, and apply the results of this section to obtain formulas, which then can be shown to converge to the desired formulas as the smoothing is removed. It is straightforward to control the convergence in terms of trace norms, so we will not provide further details here.

5.1. **Symmetrized determinant expression.** It is well known that

$$\det(I-K_{\mathrm{Ai}})_{L^2(s,\infty)}=\det(I-\sqrt{\chi_s}K_{\mathrm{Ai}}\sqrt{\chi_s})_{L^2(-\infty,\infty)} \tag{310}$$

where χ_s is the multiplication operator by $\mathbf{1}_{\{\bullet\geq s\}}$ (i.e., $(\chi_s f)(x)=\mathbf{1}(x\geq s)f(x)$).

The following proposition shows that for our class of determinants the same relation holds, and provides the proof of formula (22) of Theorem 1.

Proposition 39. *For K in the class of operators with kernel as in (307),*

$$\det(I-K)_{L^2(s,\infty)}=\det(I-\hat{K}_s)_{L^2(-\infty,\infty)}, \tag{311}$$

where the kernel for $\hat{K}_s$ is given by

$$\hat{K}_s(x,y)=\sqrt{\sigma(x-s)}K(x,y)\sqrt{\sigma(y-s)}. \tag{312}$$

Proof. Define $L_s : L^2(s,\infty) \to L^2(-\infty,\infty)$ by

$$(L_s f)(x) = \int_s^\infty \mathrm{Ai}(x+y)f(y)dy. \tag{313}$$

Also define $\sigma : L^2(-\infty,\infty) \to L^2(-\infty,\infty)$ by

$$(\sigma f)(x) = \sigma(x)f(x) \tag{314}$$

and $\chi_s : L^2(-\infty,\infty) \to L^2(s,\infty)$ by

$$(\chi_s f)(x) = \mathbf{1}(x \geq s)f(x) \tag{315}$$

Then

$$K = \chi_s L_{-\infty}\sigma L_s. \tag{316}$$

We have

$$\det(I-K)_{L^2(s,\infty)} = \det(I-\tilde{K}_s)_{L^2(-\infty,\infty)} \tag{317}$$

where

$$\tilde{K}_s = \sqrt{\sigma}L_s\chi_s L_{-\infty}\sqrt{\sigma}. \tag{318}$$

The key point is that

$$L_s\chi_s L_{-\infty}(x,y) = K_{\mathrm{Ai}}(x+s,y+s) \tag{319}$$

where K_{Ai} is the Airy kernel. One can also see now that this operator is self-adjoint on the real line. □

5.2. **Painlevé II type integro-differential equation.** We now develop an integro-differential equation expression for $\det(I-K)_{L^2(s,\infty)}$. This provides the proof of Proposition 2.

Recall that $F_{\mathrm{GUE}}(s) = \det(I-K_{\mathrm{Ai}})_{L^2(s,\infty)}$ can be expressed in terms of a non-linear version of the Airy function, known as Painlevé II as follows [29]. Let q be the unique (Hastings-McLeod) solution to Painlevé II:

$$\frac{d^2}{ds^2}q(s) = (s+2q^2(s))q(s) \tag{320}$$

subject to $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$. Then

$$\frac{d^2}{ds^2}\log\det(I-K_{\mathrm{Ai}})_{L^2(s,\infty)} = q^2(s). \tag{321}$$

From this one shows that

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)dx\right). \tag{322}$$

We now show that an analogous expression exists for the class of operators described in (307).

Proposition 40. *For K in the class of operators with kernel as in (307), let $q(t,s)$ be the solution to*

$$\frac{d^2}{ds^2}q_t(s) = \left(s+t+2\int_{-\infty}^\infty \sigma'(r)q_r^2(s)dr\right)q_t(s) \tag{323}$$

subject to $q_t(s) \sim \mathrm{Ai}(t+s)$ as $s \to \infty$. Then we have

$$\begin{aligned} \frac{d^2}{ds^2}\log\det(I-K)_{L^2(s,\infty)} &= \int_{-\infty}^\infty \sigma'(t)q_t^2(s)dt, \\ \det(I-K)_{L^2(s,\infty)} &= \exp\left(-\int_s^\infty (x-s)\int_{-\infty}^\infty \sigma'(t)q_t^2(x)dtdx\right) \end{aligned} \tag{324}$$

Proof. As already mentioned we follow the work of Tracy and Widom [29] very closely, and make the necessary modifications to our present setting. Consider an operator K of the type described in (307).

It will be convenient to think of our operator K as acting, not on (s,∞), but on $(-\infty,\infty)$ and to have kernel

$$K(x,y)\chi_s(y) \tag{325}$$

where χ is the characteristic function of (s,∞). Since the integral operator K is trace-class and depends smoothly on the parameter s, we have the well known formula

$$\frac{d}{ds}\log\det(I-K) = -\mathrm{tr}\left((I-K)^{-1}\frac{\partial K}{\partial s}\right). \tag{326}$$

By calculus

$$\frac{\partial K}{\partial s} \doteq -K(x,s)\delta(y-s). \tag{327}$$

(If L is an operator with kernel $L(x,y)$ we denote this by $L \doteq L(x,y)$.) Substituting this into the above expression gives

$$\frac{d}{ds}\log\det(I-K) = -R(s,s) \tag{328}$$

where $R(x,y)$ is the resolvent kernel of K, i.e. $R=(I-K)^{-1}K \doteq R(x,y)$. The resolvent kernel $R(x,y)$ is smooth in x but discontinuous in y at $y=s$. The quantity $R(s,s)$ is interpreted to mean the limit of $R(s,y)$ as y goes to s from above:

$$\lim_{y\to s^+} R(s,y). \tag{329}$$

5.2.1. *Representation for $R(x,y)$.* If M denotes the multiplication operator, $(Mf)(x) = xf(x)$, then

$$[M,K] \doteq xK(x,y)-K(x,y)y = (x-y)K(x,y) = \int_{-\infty}^{\infty}\sigma'(t)\{\varphi(x+t)\psi(y+t)-\psi(x+t)\varphi(y+t)\}dt. \tag{330}$$

As an operator equation this is

$$[M,K] = \int_{-\infty}^{\infty}\sigma'(t)\{\tau_t\varphi\otimes\tau_t\psi - \tau_t\psi\otimes\tau_t\varphi\}dt. \tag{331}$$

(We define $a\otimes b \doteq a(x)b(y)$ and let $[\cdot,\cdot]$ denote the commutator. The operator τ_t acts as $(\tau_t f)(x) = f(x+t)$) Thus

$$\begin{aligned}\left[M,(I-K)^{-1}\right] &= (I-K)^{-1}[M,K](I-K)^{-1}\\ &= \int\sigma'(t)\{(I-K)^{-1}(\tau_t\varphi\otimes\tau_t\psi-\tau_t\psi\otimes\tau_t\varphi)(I-K)^{-1}\}dt\\ &= \int\sigma'(t)\{Q_t\otimes P_t - P_t\otimes Q_t\}dt \end{aligned}\tag{332}$$

where we have introduced

$$Q_t(x;s) = Q_t(x) = (I-K)^{-1}\tau_t\varphi \quad \text{and} \quad P_t(x;s) = P_t(x) = (I-K)^{-1}\tau_t\psi. \tag{333}$$

(Note an important point here that as K is self-adjoint we can use the transformation $\tau_t\varphi\otimes\tau_t\psi(I-K)^{-1} = \tau_t\varphi\otimes(I-K)^{-1}\tau_t\psi$.)

On the other hand since $(I-K)^{-1} \doteq \rho(x,y) = \delta(x-y) + R(x,y)$,

$$\left[M,(I-K)^{-1}\right] \doteq (x-y)\rho(x,y) = (x-y)R(x,y). \tag{334}$$

Comparing (332) and (334) we see that

$$R(x,y) = \int_{-\infty}^{\infty} \sigma'(t)\{\frac{Q_t(x)P_t(y) - P_t(x)Q_t(y)}{x-y}\}dt, \quad x,y \in (s,\infty). \tag{335}$$

Taking $y \to x$ gives

$$R(x,x) = \int_{-\infty}^{\infty} \sigma'(t)\{Q_t'(x)P_t(x) - P_t'(x)Q_t(x)\}dt \tag{336}$$

where the $'$ denotes differentiation with respect to x.

Introducing

$$q_t(s) = Q_t(s;s) \quad \text{and} \quad p_t(s) = P_t(s;s), \tag{337}$$

we have

$$R(s,s) = \int_{-\infty}^{\infty} \sigma'(t)\{Q_t'(s;s)p_t(s) - P_t'(s;s)q_t(s)\}dt, \quad s < x,y < \infty. \tag{338}$$

5.2.2. *Formulas for $Q_t'(x)$ and $P_t'(x)$.* As we just saw, we need expressions for $Q_t'(x)$ and $P_t'(x)$. If D denotes the differentiation operator, d/dx, then

$$\begin{aligned} Q_t'(x;s) &= D\,(I-K)^{-1}\,\tau_t\varphi \\ &= (I-K)^{-1}\,D\tau_t\varphi + \left[D,(I-K)^{-1}\right]\tau_t\varphi \\ &= (I-K)^{-1}\,\tau_t\psi + \left[D,(I-K)^{-1}\right]\tau_t\varphi \\ &= P_t(x) + \left[D,(I-K)^{-1}\right]\tau_t\varphi. \end{aligned} \tag{339}$$

We need the commutator

$$\left[D,(I-K)^{-1}\right] = (I-K)^{-1}\,[D,K]\,(I-K)^{-1}. \tag{340}$$

Integration by parts shows

$$[D,K] \doteq \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\right) + K(x,s)\delta(y-s). \tag{341}$$

(The δ function comes from differentiating the characteristic function χ.) Using the specific form for φ and ψ ($\varphi' = \psi$, $\psi' = x\varphi$) we compute:

$$\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\right) = \int_{-\infty}^{\infty} \sigma'(t)\tau_t\varphi(x)\tau_t\varphi(y)dt. \tag{342}$$

Thus

$$\left[D,(I-K)^{-1}\right] \doteq -\int_{-\infty}^{\infty} \sigma'(t)Q_t(x)Q_t(y)dt + R(x,s)\rho(s,y). \tag{343}$$

(Recall $(I-K)^{-1} \doteq \rho(x,y)$.) We now use this in (339)

$$\begin{aligned} Q_t'(x;s) &= P_t(x) - \int_{-\infty}^{\infty} \sigma'(\tilde{t})Q_{\tilde{t}}(x)\left(Q_{\tilde{t}},\tau_t\varphi\right)d\tilde{t} + R(x,s)q_t(s) \\ &= P_t(x) - \int_{-\infty}^{\infty} \sigma'(\tilde{t})Q_{\tilde{t}}(x)u_{t,\tilde{t}}(s) + R(x,s)q_t(s) \end{aligned} \tag{344}$$

where the inner product $(Q_{\tilde{t}}, \tau_t\varphi)$ is denoted by $u_{t,\tilde{t}}(s)$ and $u_{t,\tilde{t}}(s) = u_{\tilde{t},t}(s)$. Evaluating at $x = s$ gives

$$Q_t'(s;s) = p_t(s) - \int_{-\infty}^{\infty} \sigma'(\tilde{t}) q_{\tilde{t}}(s) u_{t,\tilde{t}}(s) + R(s,s) q_t(s). \tag{345}$$

We now apply the same procedure to compute P' encountering the one new feature that since $\psi'(x) = x\varphi(x)$ we need to introduce an additional commutator term:

$$\begin{aligned}
P_t'(x;s) &= D\,(I-K)^{-1}\,\tau_t\psi \qquad (346)\\
&= (I-K)^{-1}\,D\tau_t\psi + \left[D,(I-K)^{-1}\right]\tau_t\psi\\
&= (M+t)\,(I-K)^{-1}\,\tau_t\varphi + \left[(I-K)^{-1},M\right]\tau_t\varphi + \left[D,(I-K)^{-1}\right]\tau_t\psi\\
&= (x+t)Q_t(x) + \int_{-\infty}^{\infty}\sigma'(\tilde{t})\,(P_{\tilde{t}}\otimes Q_{\tilde{t}} - Q_{\tilde{t}}\otimes P_{\tilde{t}})\,\tau_t\varphi d\tilde{t} - \int_{-\infty}^{\infty}\sigma(\tilde{t})(Q_{\tilde{t}}\otimes Q_{\tilde{t}})\tau_t\psi d\tilde{t} + R(x,s)p_t(s)\\
&= (x+t)Q_t(x) + \int_{-\infty}^{\infty}\sigma'(\tilde{t})\,\{P_{\tilde{t}}(x)\,(Q_{\tilde{t}},\tau_t\varphi) - Q_{\tilde{t}}(x)\,(P_{\tilde{t}},\tau_t\varphi) - Q_{\tilde{t}}(x)\,(Q_{\tilde{t}},\tau_t\psi)\}\,d\tilde{t} + R(x,s)p_t(s)\\
&= (x+t)Q_t(x) + \int_{-\infty}^{\infty}\sigma'(\tilde{t})\left\{P_{\tilde{t}}(x)u_{t,\tilde{t}}(s) - Q_{\tilde{t}}(x)v_{t,\tilde{t}}(s) - Q_{\tilde{t}}(x)v_{\tilde{t},t}(s)\right\}d\tilde{t} + R(x,s)p_t(s).
\end{aligned}$$

Here $v_{t,\tilde{t}}(s) = (P_{\tilde{t}}, \tau_t\varphi) = (\tau_{\tilde{t}}\psi, Q_t)$. Evaluating at $x = s$ gives

$$P'(s;s) = (s+t)q_t(s) + \int_{-\infty}^{\infty}\sigma'(\tilde{t})\left\{p_{\tilde{t}}(s)u_{t,\tilde{t}}(s) - q_{\tilde{t}}(s)v_{t,\tilde{t}}(s) - q_{\tilde{t}}(s)v_{\tilde{t},t}(s)\right\}d\tilde{t} + R(s,s)p_t(s). \tag{347}$$

Using this and the expression for $Q'(s;s)$ in (338) gives

$$R(s,s) = \int_{-\infty}^{\infty}\sigma'(t)\{p_t^2(s) - sq_t^2(s) - \int_{-\infty}^{\infty}\sigma'(\tilde{t})\{[q_{\tilde{t}}(s)p_t(s) + p_{\tilde{t}}(s)q_t(s)]u_{t,\tilde{t}}(s) - q_{\tilde{t}}(s)q_t(s)[v_{t,\tilde{t}}(s) + v_{\tilde{t},t}(s)]\}\}d\tilde{t}dt. \tag{348}$$

Dropping the s to make it clearer:

$$R(s,s) = \int_{-\infty}^{\infty}\sigma'(t)\{p_t^2 - sq_t^2 - \int_{-\infty}^{\infty}\sigma'(\tilde{t})\{[q_{\tilde{t}}p_t + p_{\tilde{t}}q_t]u_{t,\tilde{t}} - q_{\tilde{t}}q_t[v_{t,\tilde{t}} + v_{\tilde{t},t}]\}\}d\tilde{t}dt. \tag{349}$$

5.2.3. *First order equations for q, p, u and v.* By the chain rule

$$\frac{dq_t}{ds} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial s}\right) Q_t(x;s)\,|_{x=s}. \tag{350}$$

We have already computed the partial of $Q(x;s)$ with respect to x. The partial with respect to s is

$$\begin{aligned}
\frac{\partial}{\partial s}Q_t(x;s) &= (I-K)^{-1}\,\frac{\partial K}{\partial s}\,(I-K)^{-1}\,\tau_t\varphi\\
&= -R(x,s)q_t(s)
\end{aligned}$$

where we used (327). Adding the two partial derivatives and evaluating at $x = s$ gives

$$\frac{dq_t}{ds} = p_t - \int_{-\infty}^{\infty}\sigma'(\tilde{t})q_{\tilde{t}}u_{t,\tilde{t}}d\tilde{t}. \tag{351}$$

A similar calculation gives

$$\frac{dp}{ds} = (s+t)q_t + \int_{-\infty}^{\infty}\sigma'(\tilde{t})\left\{p_{\tilde{t}}u_{t,\tilde{t}} - q_{\tilde{t}}[v_{t,\tilde{t}} + v_{\tilde{t},t}]\right\}d\tilde{t}. \tag{352}$$

We derive first order differential equations for u and v by differentiating the inner products:

$$\begin{aligned} u_{t,\tilde{t}}(s) &= \int_s^\infty \tau_t\varphi(x)Q_{\tilde{t}}(x;s)\,dx, \\ \frac{du_{t,\tilde{t}}}{ds} &= -\tau_t\varphi(s)q_{\tilde{t}}(s) + \int_s^\infty \tau_t\varphi(x)\frac{\partial Q_{\tilde{t}}(x;s)}{\partial s}\,dx \\ &= -\left(\tau_t\varphi(s) + \int_s^\infty R(s,x)\tau_t\varphi(x)\,dx\right) q_{\tilde{t}}(s) \\ &= -\,(I-K)^{-1}\,\tau_t\varphi(s)\,q_{\tilde{t}}(s) \\ &= -q_t q_{\tilde{t}}. \end{aligned}$$

Similarly,

$$\frac{dv_{t,\tilde{t}}}{ds} = -q_t p_{\tilde{t}}. \tag{353}$$

5.2.4. *Integro-differential equation for q_t.* From the first order differential equations for q_t, u_t and $v_{t,\tilde{t}}$ it follows immediately that the derivative in s (these are all functions of s) of

$$\int_{-\infty}^\infty \sigma'(t')u_{t,t'}u_{t',\tilde{t}}dt' - [v_{t,\tilde{t}} + v_{\tilde{t},t}] - q_t q_{\tilde{t}} \tag{354}$$

is zero. Examining the behavior near $s=\infty$ to check that the constant of integration is zero then gives

$$\int_{-\infty}^\infty \sigma'(t')u_{t,t'}u_{t',\tilde{t}}dt' - [v_{t,\tilde{t}} + v_{\tilde{t},t}] = q_t q_{\tilde{t}}, \tag{355}$$

a *first integral.* We now differentiate (351) with respect to s, to get

$$q_t'' = (s+t)q_t + \int_{-\infty}^\infty \sigma'(\tilde{t})\Big\{\int_{-\infty}^\infty \sigma'(t')q_{t'}u_{\tilde{t},t'}dt'u_{t,\tilde{t}} - q_{\tilde{t}}[v_{t,\tilde{t}} + v_{\tilde{t},t}] + q_t q_{\tilde{t}}^2\Big\}d\tilde{t} \tag{356}$$

and then use the first integral to deduce that q satisfies

$$q_t'' = \Big\{s + t + 2\int_{-\infty}^\infty \sigma'(\tilde{t})q_{\tilde{t}}^2 d\tilde{t}\Big\}q_t. \tag{357}$$

Note the boundary condition is

$$q_t(s) \sim \mathrm{Ai}(s+t) \quad \text{as} \quad s\to\infty \tag{358}$$

Since the kernel of $[D,(I-K)^{-1}]$ is $(\partial/\partial x + \partial/\partial y)R(x,y)$, (343) says

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)R(x,y) = -\int_{-\infty}^\infty \sigma'(t)Q_t(x)Q_t(y)dt + R(x,s)\rho(s,y). \tag{359}$$

In computing $\partial Q(x;s)/\partial s$ we showed that

$$\frac{\partial}{\partial s}(I-K)^{-1} \doteq \frac{\partial}{\partial s}R(x,y) = -R(x,s)\rho(s,y). \tag{360}$$

Adding these two expressions,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial s}\right)R(x,y) = -\int_{-\infty}^\infty \sigma'(t)Q_t(x)Q_t(y)dt, \tag{361}$$

and then evaluating at $x=y=s$ gives

$$\frac{d}{ds}R(s,s) = -\int_{-\infty}^\infty \sigma'(t)q_t^2(s)dt. \tag{362}$$

Hence

$$q_t'' = \Big\{s + t - 2R'\Big\} q_t. \tag{363}$$

Integration (and recalling (326)) gives,

$$\frac{d}{ds} \log\det(I-K) = -\int_s^\infty \int_{-\infty}^\infty \sigma'(t) q_t^2(x) dt\, dx; \tag{364}$$

and hence,

$$\log\det(I-K) = -\int_s^\infty \left(\int_y^\infty \int_{-\infty}^\infty \sigma'(t) q_t^2(x) dt\, dx\right) dy = -\int_s^\infty (x-s) \int_{-\infty}^\infty \sigma'(t) q_t^2(x) dt\, dx. \tag{365}$$

so

$$\det(I-K) = \exp\left(-\int_s^\infty (x-s) \int_{-\infty}^\infty \sigma'(t) q_t^2(x) dt\, dx\right) \tag{366}$$

This completes the proof of Proposition 40. □

6. Proofs of Corollaries to Theorem 1

6.1. F_{GUE} asymptotics as $T \nearrow \infty$ (Proof of Corollary 3). We describe how to turn the idea described after Corollary 3 into a rigorous proof. The first step is to cut the $\tilde{\mu}$ contour off outside of a compact region around the origin. Proposition 18 shows that for a fixed T, the tail of the $\tilde{\mu}$ integrand is exponentially decaying in $\tilde{\mu}$. A quick inspection of the proof shows that increasing T only further speeds up the decay. Thus justifies our ability to cut the contour at minimal cost. Of course, the larger the compact region, the smaller the cost (which goes to zero).

We may now assume that $\tilde{\mu}$ is on a compact region. We will show the following critical point: that $\det(I - K_a^{\mathrm{csc}})_{L^2(\Gamma_\eta)}$ converges (uniformly in $\tilde{\mu}$) to the Fredholm determinant with kernel

$$\int_{\Gamma_{\tilde{\zeta}}} \exp\{-\frac{1}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3} s(\tilde{\zeta} - \tilde{\eta}')\} \frac{d\tilde{\zeta}}{(\zeta - \eta')(\zeta - \eta)}. \tag{367}$$

This claim shows that we approach, uniformly, a limit which is independent of $\tilde{\mu}$. Therefore, for large enough T we may make the integral arbitrarily close to the integral of $\frac{e^{-\tilde{\mu}}}{\tilde{\mu}}$ times the above determinant (which is independent of $\tilde{\mu}$), over the cutoff $\tilde{\mu}$ contour. The $\tilde{\mu}$ integral approaches 1 as the contour cutoff moves towards infinity, and the determinant is equal to $F_{\mathrm{GUE}}(2^{1/3}s)$ which proves the corollary. A remark worth making is that the complex contours on which we are dealing are not the same as those of [32], however, owing to the decay of the kernel and the integrand (in the kernel definition), changing the contours to those of [32] has no effect on the determinant.

All that remains, then, is to prove the uniform convergence of the Fredholm determinant claimed above.

The proof of the claim follows in a rather standard manner. We start by taking a change of variables in the equation for K_a^{csc} in which we replace $\tilde{\zeta}$ by $T^{-1/3}\tilde{\zeta}$ and likewise for $\tilde{\eta}$ and $\tilde{\eta}'$. The resulting kernel is then given by

$$T^{-1/3} \int_{\tilde{\Gamma}_\zeta} \exp\{-\frac{1}{3}(\tilde{\zeta}^3 - \tilde{\eta}'^3) + 2^{1/3}(s+a')(\tilde{\zeta} - \tilde{\eta}')\} 2^{1/3} (-\tilde{\mu})^{-2^{1/3} T^{-1/3}(\tilde{\zeta}-\tilde{\eta}')} \pi \csc(\pi 2^{1/3} T^{-1/3}(\tilde{\zeta} - \tilde{\eta}')) \frac{d\tilde{\zeta}}{\tilde{\zeta} - \tilde{\eta}}. \tag{368}$$

Notice that the L^2 space as well as the contour of $\tilde{\zeta}$ integration should have been dilated by a factor of $T^{1/3}$. However, it is possible (using Lemma 31) to show that we may deform these contours back to their original positions without changing the value of the determinant. We have also used the fact that $a = T^{1/3}s - \log\sqrt{2\pi T}$ and hence $T^{-1/3}a = s + a'$ where $a' = -T^{-1/3}\log\sqrt{2\pi T}$.

We may now factor this, just as in Proposition 18, as AB and likewise we may factor our limiting kernel (367) as $K' = A'B'$ where

$$A(\tilde{\zeta},\tilde{\eta}) = \frac{e^{-|\mathrm{Im}(\tilde{\zeta})|}}{\tilde{\zeta}-\tilde{\eta}} \tag{369}$$

$$B(\tilde{\eta},\tilde{\zeta}) = e^{|\mathrm{Im}(\tilde{\zeta})|}\exp\{-\frac{1}{3}(\tilde{\zeta}^3-\tilde{\eta}^3)+2^{1/3}(s+a')(\tilde{\zeta}-\tilde{\eta})\}\frac{\pi 2^{1/3}T^{-1/3}(-\tilde{\mu})^{-2^{1/3}T^{-1/3}(\tilde{\zeta}-\tilde{\eta})}}{\sin(\pi 2^{1/3}T^{-1/3}(\tilde{\zeta}-\tilde{\eta}))}$$

and similarly

$$A'(\tilde{\zeta},\tilde{\eta}) = \frac{e^{-|\mathrm{Im}(\tilde{\zeta})|}}{\tilde{\zeta}-\tilde{\eta}} \tag{370}$$

$$B'(\tilde{\eta},\tilde{\zeta}) = e^{|\mathrm{Im}(\tilde{\zeta})|}\exp\{-\frac{1}{3}(\tilde{\zeta}^3-\tilde{\eta}'^3)+2^{1/3}s)(\tilde{\zeta}-\tilde{\eta}')\}\frac{1}{\tilde{\zeta}-\tilde{\eta}}$$

Notice that $A = A'$. Now we use the estimate

$$|\det(I-K_a^{\mathrm{csc}})-\det(I-K')| \le ||K_a^{\mathrm{csc}}-K'||_1\exp\{1+||K_a^{\mathrm{csc}}||_1+||K'||_1\}. \tag{371}$$

Observe that $||K_a^{\mathrm{csc}}-K'||_1 \le ||AB-AB'||_1 \le ||A||_2||B-B'||_2$. Therefore it suffices to show that $||B-B'||_2$ goes to zero (the boundedness of the trace norms in the exponential also follows from this). This is an explicit calculation and is easily made by taking into account the decay of the exponential terms, and the fact that a' goes to zero. The uniformness of this estimate for compact sets of $\tilde{\mu}$ follows as well. This completes the proof of Corollary 3.

6.2. **Gaussian asymptotics as $T \searrow 0$.**

Proposition 41. *As $T\beta^4 \searrow 0$, $2^{1/2}\pi^{-1/4}\beta^{-1}T^{-1/4}\mathcal{F}_\beta(T,X)$ converges in distribution to a standard Gaussian.*

Proof. We have from (11),

$$\mathcal{F}_\beta(T,X) = \log\Big(1+\beta T^{1/4}G(T,X)+\beta^2T^{1/2}\Omega(\beta,T,X)\Big) \tag{372}$$

where

$$G(T,X) = T^{-1/4}\int_0^T\int_{-\infty}^{\infty}\frac{p(T-S,X-Y)p(S,Y)}{p(T,X)}\mathscr{W}(dY,dS) \tag{373}$$

and

$$\Omega(\beta,T,X) = T^{-1/2}\sum_{n=2}^{\infty}\int_{\Delta_n(T)}\int_{\mathbb{R}^n}(-\beta)^{n-2}p_{t_1,\ldots,t_n}(x_1,\ldots,x_n)\mathscr{W}(dt_1dx_1)\cdots\mathscr{W}(dt_ndx_n). \tag{374}$$

It is elementary to show that for each $T_0 < \infty$ there is a $C = C(T_0) < \infty$ such that, for $T < T_0$

$$E[\Omega^2(\beta,T,X)] \le C. \tag{375}$$

$G(T,X)$ is Gaussian and

$$E[G^2(T,X)] = T^{-1/2}\int_0^T\int_{-\infty}^{\infty}\frac{p^2(T-S,X-Y)p^2(S,Y)}{p^2(T,X)}dY\,dS = \frac{1}{2}\sqrt{\pi}. \tag{376}$$

Hence by Chebyshev's inequality,

$$\begin{aligned} F_T(2^{-1/2}\pi^{1/4}\beta T^{1/4}s) &= P(\beta T^{1/4}G(T,X)+\beta^2T^{1/2}\Omega(\beta,T,X)\le e^{2^{-1/2}\pi^{1/4}\beta T^{1/4}s}-1) \\ &= \int_{-\infty}^{s}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx+\mathcal{O}(\beta T^{1/4}). \end{aligned} \tag{377}$$

□

7. Appendix: Analytic properties of Fredholm Determinants

The following appendix addresses the question of analytic properties of Fredholm Determinants and is based on communications of Percy Deift to IC.

Suppose $T(z)$ is an analytic map from the region $D\in\mathbb{C}$ into the trace-class operators on a (separable) Hilbert space $\mathcal{H}$. Then we have the following result.

Theorem 42. *With $T:D\to\mathcal{B}_1(\mathcal{H})$ as above, the map*

$$z\mapsto\det(1+T(z))=1+\sum_{k=1}^{\infty}\mathrm{tr}(\Gamma^{(k)}(T(z))) \tag{378}$$

is analytic on D and

$$\frac{d}{dz}\det(1+T(z))=\mathrm{tr}\,T'+\mathrm{tr}(T'\otimes T+T\otimes T')+\cdots+\mathrm{tr}(T'\otimes T\otimes\cdots\otimes T+T\otimes T'\otimes\cdots\otimes T+\cdots+T\otimes T\otimes\cdots\otimes T')+\cdots. \tag{379}$$

Remark 43. A word on multivariate algebra: Consider $u_i\in\mathcal{H}$ and define the tensor product $u_1\otimes\cdots\otimes u_n$ by its action on $v_1,\ldots,v_n\in\mathcal{H}$ as

$$u_1\otimes\cdots\otimes u_n(v_1,\ldots,v_n)=\prod_{i=1}^{n}(u_i,v_i). \tag{380}$$

Then $\bigotimes_{i=1}^n\mathcal{H}$ is the span of all such tensor products. There is a vector subspace of this space which is known as the alternating product:

$$\bigwedge^n(\mathcal{H})=\{h\in\bigotimes_{i=1}^{n}\mathcal{H}:\forall\sigma\in S_n,\sigma h=h\}, \tag{381}$$

where $\sigma u_1\otimes\cdots\otimes u_n=u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}$. If $e_1,\ldots,e_n$ is a basis for $\mathcal{H}$ then $e_{i_1}\wedge\cdots\wedge e_{i_k}$ for $1\le i_1<\ldots<i_k\le n$ form a basis of $\bigwedge^n(\mathcal{H})$.

Given an operator $A\in\mathcal{L}(\mathcal{H})$, define

$$\Gamma^n(A)(u_1\otimes\cdots\otimes u_n):=Au_1\otimes\cdots\otimes Au_n. \tag{382}$$

Note that any element in $\bigwedge^n(\mathcal{H})$ can be written as an antisymmetrization of tensor products. Then it follows that $\Gamma^n(A)$ restricts to an operator from $\bigwedge^n(\mathcal{H})$ into $\bigwedge^n(\mathcal{H})$. It is this restriction which we will be using in the subsequent.

Now observe that in the case of finite dimensional $\mathcal{H}$,

$$\begin{aligned} \det(I+A) &= \prod(1+\lambda_i)=1+\sum_i\lambda_i+\sum_{i<j}\lambda_i\lambda_j+\cdots \\ &= 1+\mathrm{tr}\,\Gamma^1(A)+\mathrm{tr}\,\Gamma^2(A)+\cdots. \end{aligned} \tag{383}$$

In the finite dimensional setting we will show the inequality $\mathrm{tr}\,\Gamma^{(n)}(A) \le ||A||_1^n/n!$ and thus establish that this series converges for trace class operators.

Returning to the question at hand, we wish to prove the theorem. In this direction we first prove a very useful Lemma which actually also shows the inequality just previously stated.

Lemma 44. *Suppose* $A_1, \dots, A_k \in \mathcal{B}_1(\mathcal{H})$. *Then*

$$\Gamma(A_1, \dots, A_k) = \sum_{\pi \in S_k} A_{\pi(1)} \otimes \cdots \otimes A_{\pi(k)} \tag{384}$$

maps $\bigwedge^k(\mathcal{H})$ *to* $\bigwedge^k(\mathcal{H})$ *and* $\Gamma(A_1, \dots, A_k) \in \mathcal{B}_1(\bigwedge^k(\mathcal{H}))$ *with norm*

$$||\Gamma(A_1, \dots, A_k)||_1 \le ||A_1||_1 ||A_2||_1 \cdots ||A_k||_1. \tag{385}$$

Proof. Since A_j are trace class, they are also compact. Compact operators have singular value decompositions, which is to say that for each $j \in 1, \dots, k$ there exists a decomposition of A_j as

$$A_j = \sum_{i \ge 1} a_{ji} (\alpha_{ji}, \bullet) \alpha'_{ji}, \tag{386}$$

where $a_{ji} \ge 0$, $\sum_{i=1}^{\infty} a_{ji} < \infty$, and $\{\alpha_{ji}\}$ as well as $\{\alpha'_{ji}\}$ are orthonormal. For $u_1, \dots, u_k \in \mathcal{H}$, we write

$$u_i \wedge \cdots \wedge u_k = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(k)} \in \bigwedge^k(\mathcal{H}). \tag{387}$$

Then

$$\begin{aligned}
&\Gamma(A_1,\ldots,A_k)u_1\wedge u_2\wedge\cdots\wedge u_k\\
&=\frac{1}{\sqrt{k!}}\sum_{\sigma\in S_k}\sum_{\pi\in S_k}\operatorname{sgn}(\sigma)(A_{\pi(1)}\otimes\cdots\otimes A_{\pi(k)})u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(k)}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\frac{1}{\sqrt{k!}}\sum_{\sigma,\pi\in S_k}\operatorname{sgn}(\sigma)\prod_{l=1}^{k}a_{\pi(l),i_l}\bigotimes_{l=1}^{k}((\alpha_{\pi(l),i_l},\bullet)\alpha'_{pi(l),i_l})\bigotimes_{l=1}^{k}u_{\sigma(l)}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\prod_{l=1}^{k}a_{\pi(l),i_l}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{l=1}^{k}(\alpha_{\pi(l),i_l},u_{\sigma(l)})\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_l}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\prod_{l=1}^{k}a_{\pi(l),i_l}\det\left[(\alpha_{\pi(l),i_l},u_m)\right]_{l,m=1}^{k}\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_l}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\operatorname{sgn}(\pi)\prod_{l=1}^{k}a_{l,i_{\pi^{-1}(l)}}\det\left[(\alpha_{l,i_{\pi^{-1}(l)}},u_m)\right]_{l,m=1}^{k}\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_l}\\
&=\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\operatorname{sgn}(\pi)\sum_{i_1,\ldots,i_k\geq 1}\prod_{l=1}^{k}a_{l,i_{\pi^{-1}(l)}}\det\left[(\alpha_{l,i_{\pi^{-1}(l)}},u_m)\right]_{l,m=1}^{n}\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_l}\\
&=\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\operatorname{sgn}(\pi)\sum_{\hat{i}_1,\ldots,\hat{i}_k\geq 1}\prod_{l=1}^{k}a_{l,\hat{i}_l}\det\left[(\alpha_{l,\hat{i}_l},u_m)\right]_{l,m=1}^{n}\bigotimes_{l=1}^{k}\alpha'_{\pi(l),\hat{i}_{\pi(l)}}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\operatorname{sgn}(\pi)\prod_{l=1}^{k}a_{l,i_l}\det\left[(\alpha_{l,i_l},u_m)\right]_{l,m=1}^{k}\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_{\pi(l)}}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\prod_{l=1}^{k}a_{l,i_l}\left((\bigwedge_{l=1}^{k}\alpha_{l,i_l}),(\bigwedge_{l=1}^{k}u_l)\right)\frac{1}{\sqrt{k!}}\sum_{\pi\in S_k}\operatorname{sgn}(\pi)\bigotimes_{l=1}^{k}\alpha'_{\pi(l),i_{\pi(l)}}\\
&=\sum_{i_1,\ldots,i_k\geq 1}\prod_{l=1}^{k}a_{l,i_l}\left((\bigwedge_{l=1}^{k}\alpha_{l,i_l}),(\bigwedge_{l=1}^{k}u_l)\right)\bigwedge\alpha'_{l,i_l}.
\end{aligned}$$

Hence, as linear combinations of $u_1\wedge\cdots\wedge u_k$ are dense in $\bigwedge^k(\mathcal{H})$, we have

$$\Gamma(A_1,\ldots,A_k)=\sum_{i_1,\ldots,i_k\geq 1}a_{1,i_1}\cdots a_{k,i_k}(\alpha_{1,i_1}\wedge\cdots\wedge\alpha_{k,i_k},\bullet)\alpha'_{1,i_1}\wedge\cdots\wedge\alpha'_{k,i_k},\tag{388}$$

which is the generalization of the singular value decomposition to the alternating product of operators.

As $||(u,\bullet)v||_{\mathcal{B}_1}=|(u,v)|\leq||u||\cdot||v||$ for any rank 1 operator in a Hilbert space, we see that

$$||\Gamma(A_1,\ldots,A_k)||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))}\leq\sum_{i_1,\ldots,i_k\geq 1}a_{1,i_1}\cdots a_{k,i_k}=||A_1||_{\mathcal{B}_1}\cdots||A_k||_{\mathcal{B}_1},\tag{389}$$

as

$$||(\alpha_{1,i_1}\wedge\cdots\wedge\alpha_{k,i_k},\bullet)\alpha'_{1,i_1}\wedge\cdots\wedge\alpha'_{k,i_k}||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))}\leq||\alpha_{1,i_1}\wedge\cdots\wedge\alpha_{k,i_k}||\cdot||\alpha'_{1,i_1}\wedge\cdots\wedge\alpha'_{k,i_k}||\leq 1.\tag{390}$$

This proves equation (385). □

Now let $A, B \in \mathcal{B}_1(\mathcal{H})$. For $l, m \geq 0$, $k = l + m$, define

$$\Gamma^{(l,m)}(A,B) = \frac{1}{l!m!}\Gamma(A,\dots,A,B,\dots B), \tag{391}$$

where there are l A's and m B's. Clearly $\Gamma^{(l,m)}(A,B) = \sum c_1 \otimes \cdots \otimes c_k$ where the sum is over all $\binom{m+l}{m}$ ways of designating l of the c_i's as A and the other m as B. As an example, $\Gamma^{(1,2)}(A,B) = A \otimes B \otimes B + B \otimes A \otimes B + B \otimes B \otimes A$.

Corollary 45 (Corollary to Lemma 44)**.**

$$||\Gamma^{(l,m)}(A,B)||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))} \leq \frac{||A||_1^l}{l!}\frac{||B||_1^m}{m!}. \tag{392}$$

We can now proceed with:

Proof of Theorem 42. Fix $z \in D$ and let $T(z+h) = T(z) + \delta = T + \delta$. For $k \geq 1$,

$$\begin{aligned} &T(z+h) \otimes \cdots \otimes T(z+h) \\ &= T + \delta \otimes \cdots \otimes T + \delta \\ &= T \otimes \cdots \otimes T + \Gamma^{(1,k-1)}(\delta, T) + \Gamma^{(2,k-2)}(\delta,T) + \cdots + T^{(l,k-l)}(\delta,T) + \cdots + \delta \otimes \cdots \otimes \delta. \end{aligned} \tag{393}$$

Thus

$$\frac{T(z+h) \otimes \cdots \otimes T(z+h) - T(z) \otimes \cdots \otimes T(z)}{h} = T^{(1,k-1)}(\frac{\delta}{h}, T) + \Delta(h), \tag{394}$$

where by the Corollary,

$$||\Delta(h)||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))} \leq \frac{1}{|h|}\frac{||\delta||_1^2}{2}\frac{||T||_1^{k-2}}{(k-2)!} + \cdots + \frac{1}{|h|}\frac{||\delta||_1^k}{k!}. \tag{395}$$

Observe that $||\delta||_1 = ||T(z+h) - T(z)||_1 = O(h)$. Write

$$T^{(1,k-1)}(\frac{\delta}{h}, T) = \Gamma^{(1,k-1)}(T', T) + \Gamma^{(1,k-1)}(\frac{T(z+h) - T(z)}{h} - T'(z), T), \tag{396}$$

and then observe that by the Corollary

$$\begin{aligned} &||\Gamma^{(1,k-1)}(\frac{T(z+h)-T(z)}{h} - T'(z), T(z))||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))} \\ &\leq ||\frac{T(z+h)-T(z)}{h} - T'(z)||_{\mathcal{B}_1}\frac{1}{(k-1)!}||T(z)||_{\mathcal{B}_1}^{k-1} = O(h). \end{aligned} \tag{397}$$

Combining these observations shows that

$$\frac{T(z+h) \otimes \cdots \otimes T(z+h) - T(z) \otimes \cdots \otimes T(z)}{h} = \Gamma^{(1,k-1)}(T', T) + O(h), \tag{398}$$

and hence the function $z \mapsto T(z) \otimes \cdots \otimes T(z) = \Gamma^{(k)}(T(z))$ is an analytic map from D to $\mathcal{B}_1(\bigwedge^k(\mathcal{H}))$ for all $k \geq 1$ and

$$\frac{d}{dz}T(z) \otimes \cdots \otimes T(z) = \Gamma^{(1,k-1)}(T', T) = T' \otimes T \otimes \cdots \otimes T + \cdots + T \otimes \cdots \otimes T \otimes T'. \tag{399}$$

It then follows that $z \mapsto \operatorname{tr}\Gamma^{(k)}(T(z))$ is analytic for $k \geq 1$ from D to $\mathbb{C}$.

Hence for any $n \geq 1$,

$$1 + \sum_{k=1}^{n} \operatorname{tr}\Gamma^{(k)}(T(z)) \tag{400}$$

is analytic in D and

$$|1+\sum_{k=1}^{n}\operatorname{tr}\Gamma^{(k)}(T(z))| \leq 1+\sum_{k=1}^{n}||\Gamma^{(k)}(T(z))||_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))} \leq 1+\sum_{k=1}^{n}\frac{||T(z)||^k_{\mathcal{B}_1(\bigwedge^k(\mathcal{H}))}}{k!} \leq e^{||T(z)||}, \quad (401)$$

and so so for z in a compact subset of D, the functions $1+\sum_{k=1}^{n}\operatorname{tr}\Gamma^{(k)}(T(z))$ are uniformly bounded in n. It follows by general theory that $z\mapsto \det(I+T(z))=\lim_{n\to\infty}\sum_{k=0}^{n}\operatorname{tr}\Gamma^{(k)}(\mathbb{T}(z))$ is analytic in D and

$$\begin{aligned}\frac{d}{dz}\det(I+T(z)) &= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{d}{dz}\operatorname{tr}\Gamma^{(k)}(T(z))\\ &= \sum_{k=1}^{\infty}\operatorname{tr}(\Gamma^{(1,k-1)}(T'(z),T(z))) \qquad (402)\\ &= \sum_{k=1}^{\infty}\operatorname{tr}(T'(z)\otimes T(z)\otimes\cdots\otimes T(z)+\cdots+T(z)\otimes\cdots\otimes T'(z)).\end{aligned}$$

□

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