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Probability distributions for locations of calling animals, receivers, sound speeds, winds, and data from travel time differences

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Abstract

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Comments

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Probability distributions for locations of calling animals, receivers, sound speeds, winds, and data from travel time differences

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A new nonlinear sequential Monte Carlo technique is used to estimate *posterior* probability distributions for the location of a calling animal, the locations of acoustic receivers, sound speeds, winds, and the differences in sonic travel time between pairs of receivers from measurements of those differences, while adopting realistic *prior* distributions of the variables. Other algorithms in the literature appear to be too inefficient to yield distributions for this large number of variables (up to 41) without recourse to a linear approximation. The new technique overcomes the computational inefficiency of other algorithms because it does not sequentially propagate the joint probability distribution of the variables between adjacent data. Instead, the lower and upper bounds of the distributions are propagated. The technique is applied to commonly encountered problems that were previously intractable such as estimating how accurately sound speed and poorly known initial locations of receivers can be estimated from the differences in sonic travel time from calling animals, while explicitly modeling distributions of *all* the variables in the problem. In both cases, the new technique yields one or two orders of magnitude improvements compared with initial uncertainties. The technique is suitable for accurately estimating receiver locations from animal calls. © *2005 Acoustical Society of America.* [DOI: 10.1121/1.1992708]

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I. INTRODUCTION

Calling animals can be located by measuring the differences in sonic arrival times at pairs of receivers.[1–8] When the speed of sound is spatially homogeneous, the difference in arrival time multiplied by the speed gives the difference in distance of the animal from a pair of receivers. The locus of points in space for which this difference is constant is a hyperboloid. Given sufficient numbers of receivers, one intersects the hyperboloid from each pair to yield location. This so-called hyperbolic location technique has the property that the location of the animal is nonlinearly related to some of the relevant variables in the problem and there are typically many such variables such as the uncertain speed of propagation and the locations of the animal and the receivers. The nonlinearity and, particularly, the large number of variables has apparently made it computationally difficult for *any* approach to yield probability distributions for all the variables without making a linear approximation that is often invalid.[9] Since probability distributions are complete estimates of variables, it is desirable to have a method that can produce them in a computationally feasible manner. It would be desirable to be able to update distributions for the locations of the receivers and the speed of propagation as each new animal call is processed. The receivers could be stationary or mobile, and the speed of propagation could change with time. It would be desirable to use any realistic *prior* probability distributions for locations and speeds. A practical method for accomplishing all these goals is given here without using a linear approximation.

First consider the difficulties associated with nonlinearity. For each pair of receivers at coordinates $\mathbf{r}_i$ and $\mathbf{r}_j$, one wants the possible locations of the animal, $\mathbf{s}$, satisfying

$$\frac{|\mathbf{s}-\mathbf{r}_i|}{c} - \frac{|\mathbf{s}-\mathbf{r}_j|}{c} = \tau_{ij}, \tag{1}$$

where the signal speed is c and the measured lag,

$$\tau_{ij} = t_i - t_j, \tag{2}$$

is the difference in the times for the call to reach receivers i and j, respectively. Each Cartesian coordinate for the source, (s_x, s_y, s_z), and receiver, $(r_i(x), r_i(y), r_i(z))$, is nonlinearly related to the measured lag, τ_{ij}, because

$$|\mathbf{s}-\mathbf{r}_i| = \sqrt{(s_x - r_i(x))^2 + (s_y - r_i(y))^2 + (s_z - r_i(z))^2}.$$

There are three problems stemming from this nonlinearity. First, the probability distribution for the animal's location has no known analytical solution given probability distributions for the lag, receiver locations, and the speed of sound unless some of these distributions are set to an unwavering value. Second, even when one linearizes the relationship between the coordinate of the animal and the other variables, the distribution for its location can be in error by one or two orders of magnitude.[9] Third, a linear estimation scheme can converge to an incorrect solution corresponding to a nonglobal minimum.[10] Thus linear estimation techniques such as least squares, Wiener, and Kalman filters[11] can provide unreliable and inaccurate results. We are thus faced with the problem of how to obtain reliable and robust estimates of probability distributions.

Consider next the difficulties with the large number of pertinent variables. It is assumed that estimated locations of the receivers and the speed of propagation have errors, so their values are to be improved in the light of data. For $\mathcal{R}$ receivers, there are $3\mathcal{R}-6$ associated Cartesian coordinates to estimate. The reduction by 6 merely means we are uninterested in the absolute location and rotation of the coordinate system. Additionally, there are three unknown variables for the animal's location, and one unknown variable for the speed of sound. Summing these there are

$$V = 3\mathcal{R} - 2 \tag{3}$$

variables of interest in hyperbolic location problems. For example, there are 16 variables with six receivers. When the effective speed of the acoustic signal between the animal and each receiver is modified due to winds, refraction of the acoustic path,[12] or propagation through two different media such as the air and water,[13] the hyperbolic location problem can be generalized to obtain locations, but one needs to introduce a different effective speed, c_i, along the acoustic path between the animal and receiver i. This increases the number of variables to be estimated by $\mathcal{R}-1$. The number of relevant variables increases further if the probability distributions of the lags are estimated from the data.

Bayes theorem[14] provides optimal probability distributions but is impractical. Suppose one wants the joint probability distribution, $f(\mathbf{s},\mathbf{r},\mathbf{c}|\tau_{ij})$, of source location, receiver locations, $\mathbf{r}$, and speeds of propagation between the source and each receiver, $\mathbf{c}$, given a lag, τ_{ij}. Bayes theorem supplies the desired result,

$$f(\mathbf{s},\mathbf{r},\mathbf{c}|\tau_{ij}) = \frac{\mathbf{f}(\tau_{ij}|\mathbf{s},\mathbf{r},\mathbf{c})\,\pi(\mathbf{s},\mathbf{r},\mathbf{c})}{\int \mathbf{f}(\tau_{ij}|\mathbf{s},\mathbf{r},\mathbf{c})\,\pi(\mathbf{s},\mathbf{r},\mathbf{c})\mathbf{ds\,dr\,dc}}, \tag{4}$$

in terms of the conditional joint distribution of the data on the source, receivers, and speeds of propagation, $f(\tau_{ij}|\mathbf{s},\mathbf{r},\mathbf{c})$, and the prior joint distribution of the source location, receiver locations, and speeds, $\pi(\mathbf{s},\mathbf{r},\mathbf{c})$. If the distributions on the right side could be evaluated analytically, then evaluating Eq. (4) would be the end of the problem. One could introduce new data and keep finding better estimates of the distribution on the left given updated distributions on the right. Brute force evaluation of the distributions on the right appears to be computationally difficult because there are many variables. For hyperbolic location and six receivers, one needs to estimate the joint distributions of 16 variables for each introduced datum. Suppose each variable is divided into ten bins. Accurate estimation of the joint distribution requires a reliable probability of occurrence in 10^{16} bins. Instead of binning, one can estimate distributions using Gibbs sampling or Markov Chain Monte Carlo approaches,[14,15] but they appear to be computationally impractical for a large number of variables such as 16. Despite the fact that this problem is commonly encountered in the fields of acoustic and electromagnetic tracking, the author is unaware of any publication where the distributions of all the variables (about 16 or more) are estimated without making a linearizing approximation. Instead, the literature appears to treat other problems with elegant approaches where some of the variables are known without error (such as some or all of the receiver coordinates) and/or some linear approximation is adopted.[12,15–20]

The approach taken here has its root in a method for estimating the distribution of an animal's location given realistic *a priori* estimates for the distributions of sound speed, receiver locations, and measurement error.[9] We explain how that approach can be generalized to estimate the distributions of all the variables and how to include dynamical models for the evolution of all variables between the receptions of different animal calls. The approach is more convenient to discuss when given the name "sequential bound estimation" for reasons that are apparent after explaining how it works.

II. SEQUENTIAL BOUND ESTIMATION

Sequential bound estimation could be applied to many problems that are amenable to solution using Bayes theorem. The technique is demonstrated in the context of locating calling animals in situations where one has more receivers than needed to obtain a mathematical solution. The sequential nature is evident because data are sequentially processed from different subsets of the acoustic receivers. The idea of treating these data in subsets and transitioning estimated variables between one datum and the next is analogous to other problems where data are sequentially processed. Section IV explains how sequential bound estimation could be applied in a different location problem.

A. Receiver constellation

We discuss how to treat situations where one has just the right number of receivers to estimate location without mathematical ambiguity. In general, location problems in two spatial dimensions require three or four receivers[21] and three-dimensional problems require four or five receivers.[2] The location of the animal with respect to the receivers dictates the required number.[2,21] A "receiver constellation" is the minimum number of receivers required to obtain an unambiguous location. A systematic way of determining the necessary number of receivers is to use an analytical solution of location based on the lesser number, and use the datum from an additional receiver if needed.

B. Analytical solution for location from each constellation

When the speed of the signal varies between the animal and each receiver, the general solution for the animal's location in three spatial dimensions from a group of four receivers is[9]

$$\vec{s} = \mathbf{R}^{-1}\frac{\vec{b}}{2} - \mathbf{R}^{-1}\vec{f}t_1 - \mathbf{R}^{-1}\vec{g}t_1^2, \tag{5}$$

where

$$\mathbf{R} \equiv \begin{pmatrix} r_2(x) & r_2(y) & r_2(z) \\ r_3(x) & r_3(y) & r_3(z) \\ r_4(x) & r_4(y) & r_4(z) \end{pmatrix}, \quad \vec{b} \equiv \begin{pmatrix} \|\vec{r}_2\|^2 - c_2^2\tau_{21}^2 \\ \|\vec{r}_3\|^2 - c_3^2\tau_{31}^2 \\ \|\vec{r}_4\|^2 - c_4^2\tau_{41}^2 \end{pmatrix},$$
$$\vec{f} \equiv \begin{pmatrix} c_2^2\tau_{21} \\ c_3^2\tau_{31} \\ c_4^2\tau_{41} \end{pmatrix}, \tag{6}$$

and

$$\vec{g} \equiv \frac{1}{2}\begin{pmatrix} c_2^2 - c_1^2 \\ c_3^2 - c_1^2 \\ c_4^2 - c_1^2 \end{pmatrix}, \tag{7}$$

and t_1 is the solution from the quartic equation,

$$a_6 t_1^4 + 2a_5 t_1^3 + (a_3 - a_4 - c_1^2)t_1^2 - a_2 t_1 + \frac{a_1}{4} = 0, \tag{8}$$

where

$$a_1 \equiv (\mathbf{R}^{-1}\vec{b})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{b}), \quad a_2 \equiv (\mathbf{R}^{-1}\vec{b})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{f}),$$

$$a_3 \equiv (\mathbf{R}^{-1}\vec{f})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{f}), \quad a_4 \equiv (\mathbf{R}^{-1}\vec{b})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{g}), \tag{9}$$

$$a_5 \equiv (\mathbf{R}^{-1}\vec{f})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{g}), \quad a_6 \equiv (\mathbf{R}^{-1}\vec{g})^{\mathrm{T}}(\mathbf{R}^{-1}\vec{g}),$$

and $\mathbf{R}^{-1}$ is the inverse of $\mathbf{R}$. Quartic equations have analytical solutions. When the speed is spatially homogeneous, i.e., $c_i = c_1 \forall i$, the cubic and quartic terms in Eq. (8) vanish and the resulting quadratic equation is that found before for hyperbolic location.[3] The analytical solution for location in two spatial dimensions has the same form as above except one assumes the constellation has three receivers so one removes the last rows in the expressions for **R**, **b**, **f**, and **g**.

The signal speed, c_i, is more general than the speed appearing in the scalar wave equation for pressure perturbations. Here, c_i denotes the effective speed of sound between the source and receiver, which is the distance divided by the travel time. Effects of winds, currents, and nonstraight propagation paths due to refraction and diffraction all affect the time for sound to reach a receiver.

Ambiguous solutions for location occur when Eq. (8) yields more than one nonnegative real solution. Such ambiguities occur in practice. For each ambiguity, one generates a model for τ_{51}, which one can do because one knows where receiver 5 is, and chooses the root of t_1 that is closest to that measured.

C. Probability distributions for all variables

We show how one generalizes the idea for obtaining the probability distribution for animal location[9] to obtain probability distributions for all variables including the lags. The lags, τ_{ij}, are treated as random variables because they contain errors, just like all the other variables in the problem such as the locations of the receivers. We also consider situations where there are more receivers than needed to obtain a mathematically unambiguous solution for location.

Since $\mathbf{r}_i$, τ_{ij}, and c_i are random variables, then t_1 and $\mathbf{s}$ are random variables because of Eqs. (5) and (8). One assigns realistic *prior* probability distributions to each of these variables except t_1, whose distribution depends deterministically on the other distributions. For example, in a case of most ignorance, one can assign *prior* uniform distributions such as intervals $[-3000, 3000]$, $[-3000, 3000]$, and $[0, -20]$ m for the (x, y, z) Cartesian coordinates of a snapping shrimp where it is known that one cannot hear its sound at a distance of 3000 m and it must be above a bottom depth of 20 m.

For three-dimensional problems, the number of ways of choosing a four-receiver constellation among $\mathcal{R}$ receivers is

$$N = \frac{\mathcal{R}!}{(\mathcal{R}-4)!4!}. \tag{10}$$

A computer is used to draw a single *configuration* of random variables for the first constellation. Each configuration consists of the set $\{r_i(x), r_i(y), r_i(z), \tau_{ij}, c_i\}$ where i and j are taken from the set of four of the $\mathcal{R}$ receivers for the chosen constellation, and where $i > j$ to avoid using redundant data (e.g., $\tau_{23} = -\tau_{32}$). For each configuration, an animal location is computed from Eq. (5) if that equation yields a unique location. If a configuration yields more than one location, a location is chosen to be that yielding the closest difference in travel time to a randomly chosen fifth receiver and one of the receivers in the constellation.

A *valid configuration* must pass two criteria. First, the animal's location must lie within its *a priori* spatial limits. Second, there must be a real-valued solution for location from Eq. (5) since configurations can yield only complex-valued roots. Configurations not passing these criteria are discarded because they cannot occur in reality.

Valid configurations from a receiver constellation define a cloud of animal locations, receiver locations, estimates for the effective speeds of sound for each path, and the values of each of the lags. Accurate probability distributions for each require a sufficient number of valid configurations.

Some constellations yield better estimates of a variable than others. For example, an animal surrounded by four receivers would be more accurately located than one where four receivers were clumped together along a line at great distance from an animal.

Since each receiver constellation individually yields limits for the upper and lower values of each variable, one can enforce these limits to constrain random selections for that variable when seeking the valid configurations from the next constellation. That is why this method is called sequential bound estimation.

For example, suppose the *prior* distribution for variable k has some shape within the interval $[\check{v}(k)_0, \hat{v}(k)_0]$ with $\check{v}(k)_0 < \hat{v}(k)_0$ and the subscript denotes the bound after using constellation p with 0 denoting *a priori* values. After using data from constellation $p=1$, its *a posteriori* distribution must be contained in the interval, $[\max\{\check{v}(k)_0, \check{v}(k)_1\}, \min\{\hat{v}(k)_0, \hat{v}(k)_1\}]$. In general, the interval following use of constellation p is

$$[\max\{\check{v}(k)_{p-1}, \check{v}(k)_p\}, \min\{\hat{v}(k)_{p-1}, \hat{v}(k)_p\}]. \tag{11}$$

Sequential bound estimation *assigns* a distribution for the variable within these bounds. If one chooses a uniform distribution, then one is not overconstraining its distribution and valid configurations for the next constellation are obtained without prejudice. (In the parlance of information theory, the maximum entropy principle is used to prove that the uniform distribution contains the least information given only a variable's bounds.[22]) If one could implement Bayes theorem, the variable would be contained within the bounds given by Eq. (11), but the distribution would likely not be uniform. It is the author's opinion that it is better to underconstrain the distribution of a variable than to overconstrain it in an incorrect manner, as might occur if one was to assign the variable a narrow Gaussian distribution. Sequential bound estimation could then assign a uniform distribution to each variable after assimilating data from all but the last constellation. After the last constellation, one does not reassign a distribution to the variable, but rather simply computes its probability distribution from its Monte Carlo samples. In fact, one can estimate the joint distribution of the variables from the output of the last constellation. Any desired statistic can be estimated from the final joint distribution such as a percent confidence limit, the mean, and a maximum likelihood value.

A consistent estimator, such as Bayes theorem, tends toward the true distribution as the number of samples goes to infinity. Sequential bound estimation is not a consistent estimator but it can provide distributions that bound the correct distributions. This may not be such a great drawback for two reasons. First, many, if not most, problems of interest are ones where the *prior* distributions of the variables are unknown and need to be guessed. So even if one could implement Bayes theorem, the resulting joint distribution would be in error. When *prior* distributions are in error, methods that work hard to maintain the mathematical rigidity of their assumptions may yield results that are less accurate than sequential bound estimation where the bounds of the variables are enforced with minimum (i.e., uniform) constraints on their distributions at intermediate steps. Second, it is better to have an algorithm that provides distributions of pertinent variables than to have no means to compute them at all.

D. Transitioning variables from one animal call to the next

Like a Kalman filter,[11] sequential bound estimation can incorporate a model to transition variables from one animal call to the next. A few examples are given.

Suppose the speed of sound varies with time. Bounds for the speed between times t_1 and t_2 relative to a reference speed, $c_r(t_1)$, can be modeled as

$$\delta\hat{c}_r(t_2) = \min\left\{ +\left|\delta c_{\max}\right|,\ \hat{c}_r(t_1) - c_r(t_1) + \left|t_2 - t_1\right| \left|\frac{d\delta c_r}{dt}\right| \right\},$$
$$\delta\check{c}_r(t_2) = \max\left\{ -\left|\delta c_{\max}\right|,\ \check{c}_r(t_1) - c_r(t_1) - \left|t_2 - t_1\right| \left|\frac{d\delta c_r}{dt}\right| \right\}, \tag{12}$$

where the path-averaged deviation of speed between the source and receiver r is $\delta c_r(t)$, the maximum deviation is $|\delta c_{\max}|$, and the maximum allowed rate of change of the fluctuation is $|d\delta c_r/dt|$. Equation (12) expresses a method to relax prior constraints on bounds to maximum limits at a specified maximum rate of change. In the absence of further data, the bounds expand to *a priori* values. Bounds for the winds could obey similar relationships.

Similarly, we can transition probability distributions for a mobile animal. Let the prior probability distribution for the animal's location be uniformly distributed in the (x,y,z) Cartesian intervals $[\check{s}_x(t_i^-),\hat{s}_x(t_i^-)]$, $[\check{s}_y(t_i^-),\hat{s}_y(t_i^-)]$, and $[\check{s}_z(t_i^-),\hat{s}_z(t_i^-)]$ for a call received at time t_i. The $-$ superscript denotes the time just prior to assimilation of the data from the call at time t_i. After assimilating the data from time t_i, we have new lower and upper bounds for the animal at time t_i^+ given by $[\check{s}_x(t_i^+),\hat{s}_x(t_i^+)]$, $[\check{s}_y(t_i^+),\hat{s}_y(t_i^+)]$, and $[\check{s}_z(t_i^+),\hat{s}_z(t_i^+)]$ where the $+$ denotes time just after the call. Assign lower and upper bounds, denoted respectively by superscripts $\check{}$ and $\hat{}$, for each component of the Cartesian velocity, (U,V,W), of the animal between calls. Then we know that

$$\begin{aligned}
\check{s}_x(t_{i+1}^-) &= \check{s}_x(t_i^-) + (t_{i+1} - t_i)\check{U},\\
\check{s}_y(t_{i+1}^-) &= \check{s}_y(t_i^-) + (t_{i+1} - t_i)\check{V},\\
\check{s}_z(t_{i+1}^-) &= \check{s}_z(t_i^-) + (t_{i+1} - t_i)\check{W},
\end{aligned} \tag{13}$$

with analogous equations for the transition to the upper bounds. There are numerous ways to transition such bounds.

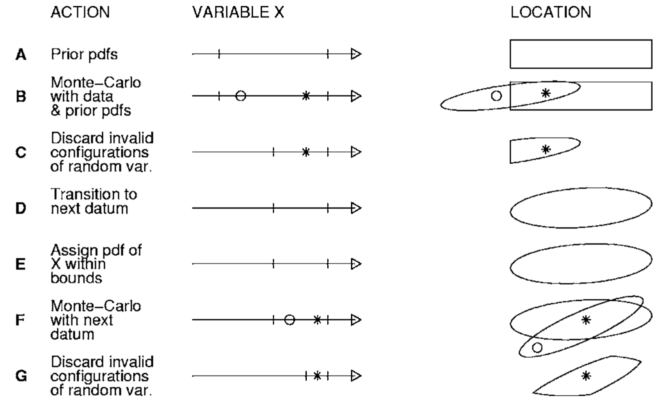

FIG. 1. Flow diagram for sequential bound estimation where location of some object is a function of a variable, x. Probability distribution function is pdf.

E. Flow diagram

Figure 1 summarizes sequential bound estimation for location problems. Step A assigns *prior* distributions and bounds to each variable, e.g., x. Its *prior* bounds are indicated by tic marks on its axis. For hyperbolic locations of calling animals, x could be any Cartesian coordinate of a receiver, an effective speed of propagation, etc. Also, one assigns *prior* bounds (rectangle) for location of the desired object.

In step B, one computes configurations of variables using the first set of data, where the variables are drawn from their *prior* distributions. Each random configuration of variables associates a set of variables with a location. For example, values of x indicated by the circle and asterisks lead to locations given by the circle and asterisks, respectively. All configurations define a cloud of locations in the ellipse.

Step C discards invalid configurations of random variables that are identified by locations outside *prior* bounds (rectangle). Valid configurations have bounds indicated by the new tic marks on the x axis and by the intersection of the

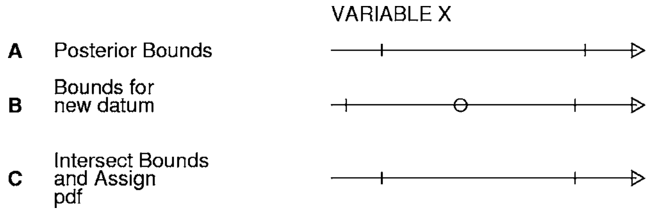

FIG. 2. How sequential bound estimation derives new bounds and probability distribution (pdf) when a variable X is a datum.

ellipse and the rectangle. The asterisk is contained in a set with a valid configuration.

Bounds for the variables are transitioned to the next datum in step D. The bounds for x do not change, but they could in general. For example, if x was a coordinate of a stationary (mobile) receiver, its bounds would not (would) change in the transition.

Step E assigns new distributions to each variable. A uniform distribution is suggested to be the most appropriate to use within the transitioned bounds.

Step F uses the next datum to generate configurations of variables by drawing from the current distributions of variables. Two configurations are indicated by the circle and asterisks. The cloud of locations for the new random configuration is a new ellipse.

Step G discards invalid configurations as in step C, leaving only valid configurations. Not only does one have better bounds for location, but one also has better bounds for each variable. One can estimate distributions of variables from valid configurations.

When variable X is a datum, bounds and probability distributions are determined in a slightly different way. Suppose the posterior bounds for X from a previous step are transitioned to the time just before the next datum, X, is assimilated [Fig. 2(a)]. The next datum value for X [circle, Fig. 2(b)] has some *a priori* probability distribution bounded by the tic marks [Fig. 2(b)]. Sequential bound estimation assigns a probability distribution that is bounded by the intersection of the bounds in panels (a) and (b), i.e., panel (c). Then the distribution in (c) is drawn from to compute configurations of random variables. If all assumptions and data are handled properly, the bounds in (a) and (b) will overlap. When they do not overlap, the algorithm can notify the user that something is wrong. For example, one could have an instrumental error or an incorrect assumption concerning a probability distribution.

III. EXAMPLES

We provide simple examples to show how sequential bound estimation may be used to solve problems of interest that come up when locating calling animals from widely separated receivers. Each example treats more variables than dealt with in the literature. Convergence of distributions is obtained in each example by increasing the number of valid configurations until reaching 4000, at which point no significant change is found.

TABLE I. Cartesian (x,y,z) coordinates for the five receivers shown in Fig. 3 of Ref. 24. Nonzero locations are measured within ±0.05 m.

	Cartesian coordinate (m)		
Receiver	x	y	z
R1	0	0	0
R2	19.76	0	0
R3	17.96	−18.80	0
R4	2.34	−29.92	−0.02
R5	−12.41	−14.35	−0.43

A. Verifying sequential bound estimation with data

We show that probability distributions for location are consistent with data. Previously, an undeveloped form of sequential bound estimation was used to locate a whale.[23] The only variable whose bounds were sequentially modified was the whale's location. Here, all variables are sequentially modified except the distributions of the lags.

On 4 June 1995, five omni-directional microphones recorded sounds from a Red-winged Blackbird in Port Matilda, PA (bird B1 in Fig. 3 of Ref. 24). Isodiachronic location is done using published differences in travel time (Table 2, Ref. 24).

A priori distributions of the variables are uniform. Lags have means as measured with intervals of ±0.000 067 s on either side due to effects of noise and interference between multipath.[24] The speed of sound has a mean of 344 m/s and an interval of ±2 m/s on either side. Cartesian components of the wind have zero means with intervals of ±2, ±2, and ±0.5 m/s on each side for the two horizontal and vertical components, respectively. Cartesian coordinates of the receivers have means given by measured values. The interval on each side of the mean is ±0.05 m except as follows. Receiver 1 is defined to be at the origin of the coordinate system. Receiver 2's location is defined such that the y axis passes through it and its z coordinate is zero. Receiver 3 is defined such that its z coordinate is zero. These definitions define the absolute location and rotation of the coordinate system (Table I). There are five receiver constellations [Eq. (10)].

It is assumed that the location of the Red-winged Blackbird is initially described as a uniform random variable in the Cartesian $x-y-z$ intervals $(-30,30)$ m, $(-30,30)$ m, and $(-5,10)$ m, respectively, where receiver one is about a meter above the ground.

Following sequential bound estimation, Cartesian coordinates of the Red-winged Blackbird have 95% and 100% confidence limits of $(9.7\pm0.1, 7.7\pm1.1, 0.4\pm5.2)$ m and $(9.7\pm0.2, 7.9\pm1.6, 0.9\pm5.9)$ m, respectively. These are consistent with the optical/visual survey for its location which has a 95% confidence limit of $(9.8\pm0.5, 6.8\pm0.5, 2.3\pm1)$ m (Table IV, Ref. 24).

Twenty probability distributions are estimated in this example (nine Cartesian receiver coordinates, three Cartesian source coordinates, three Cartesian wind components, and five variables for the speed of sound between the animal and each receiver).

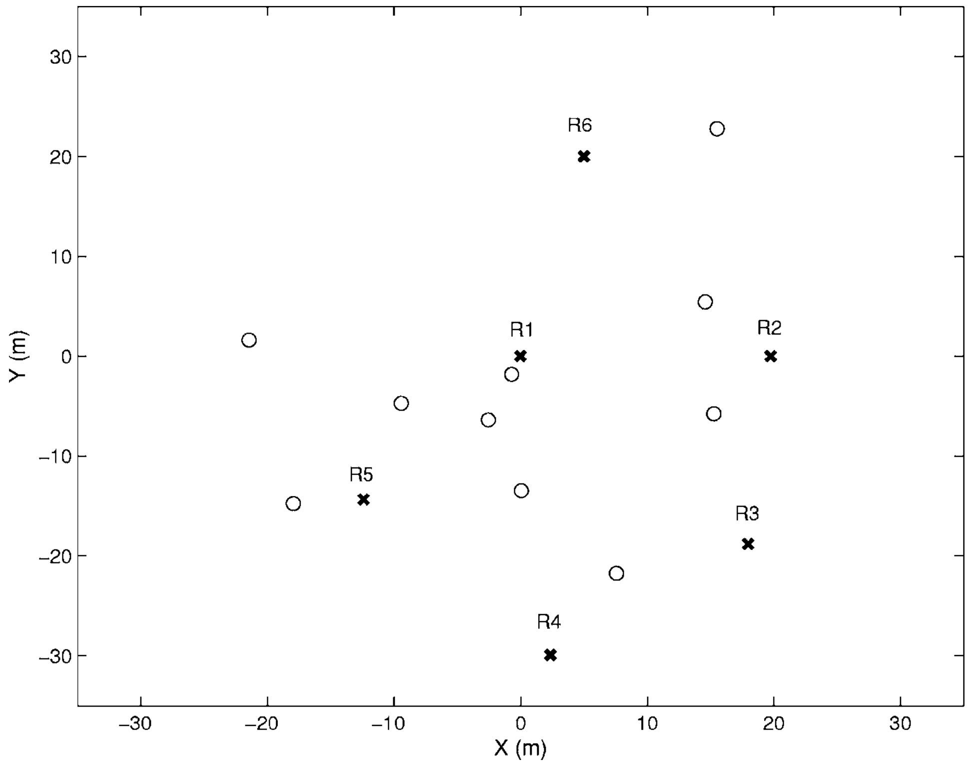

FIG. 3. Horizontal locations of six receivers and ten sources (circles) used to estimate the spatially homogeneous component of sound speed. The receivers have elevations of 2, 2, 2, 1.98, 1.57, and 7.0 m, respectively.

B. Estimating speed of sound: Simulation

We find a probability distribution for the spatially homogeneous component of the sound speed field when the locations of receivers are accurately but imperfectly measured. Six receivers are distributed in air over a region of about 30 m by 30 m (Fig. 3). Their elevations are about 2 m, except for receiver 6 which is at an elevation of 7 m. The initially unknown locations for ten sources are chosen at random (Fig. 3). The sources are assumed to produce sound at 10-s intervals at elevations between between 3 and 6 m.

All random variables have a uniform distribution in the intervals quoted below. Lags are distributed within ±0.000 05 s about their noiseless values.

The initial guesses for the Cartesian coordinates of the receivers are drawn from a uniform distribution within ±0.05 m of their true values. Their initial distributions are taken to have means about these values within an interval of ±0.1 m. An error of zero is assigned to the location of receiver 1 (defined to be the origin), to the y and z coordinates of receiver 2, and the z coordinate of receiver 3 because these define the origin and orientation of the coordinate system.

Mild winds are simulated to be spatially homogeneous with a temporal scale of 5 s. The model assumes their initial x, y, and z components are specified to fluctuate within ±1, ±1, and 0.5 m s^{-1} about a mean of 0. The maximum rate of change of any component is 1 m s^{-2}.

The actual speed of sound is composed of spatially homogeneous and inhomogeneous components. The homogeneous component is about 330 m s^{-1}. It is allowed to vary at a maximum rate of 0.01 m s^{-2}, so in 10 s it can vary by 0.1 m s^{-1}. The limits for the homogeneous component are constrained to be within 330±0.5 m s^{-1}. The inhomogeneous component may vary from section to section within ±0.1 m s^{-1} at a maximum rate of 0.01 m s^{-2}.

In order to test the ability of sequential bound estimation to estimate the speed of sound, the algorithm is given a much wider range of speeds than actually occur. The spatially homogeneous component of sound speed is assumed to have a *prior* distribution that is uniformly distributed within 330±50 m s^{-1}. After sequential bound estimation assimilates simulated data from source 1, the lower and upper bounds for the homogeneous component of sound speed are between 330−6.3 and 330+3.8 m s^{-1}, respectively. Following the use of source 7, these bounds are 330−2.1 and 330+1.3 m s^{-1}, which is equivalent to a variation of ±2.8 °C if due to temperature. The bounds do not change significantly following the use of sources 8–10. The correct value of the speed of sound, about 330 m s^{-1}, falls within the bounds provided by sequential bound estimation. Sequential bound estimation reduces the initial uncertainty of ±50 by a factor of about 29. The 95% confidence limits for the speed of sound along each section are about ±1.3 m s^{-1}, which is equivalent to ±2.1 °C.

Consider adding four additional receivers at Cartesian coordinates (20, 0, 3), (−20 ,22, 4), (25, 21, 2), and (−5, 3, 3) m. Ten new randomly chosen source locations are different than above, but are similarly situated and not chosen to optimize any measure of performance. When the same statistics are assumed as above, it is found that the homogeneous component of the sound speed field has 95% and 100% confidence limits of ±0.6 and ±0.7 m/s, respectively. Sequential bound estimation reduces the initial uncertainty of ±50 m s^{-1} by a factor of about 71. If due to temperature, the limits correspond to ±1.0 and ±1.1 °C, respectively. This is an accurate measurement of average air temperature, and it is about twice as good as the case with six receivers above.

The number of distributions estimated for the six- and ten-receiver simulations are 25 and 41, respectively for each animal. The number of Cartesian receiver coordinates are 12 and 24, respectively, for the six- and ten-receiver simulations. Both simulations use three Cartesian source coordinates, three Cartesian wind components, one homogeneous component of speed, and six and ten variables for each inhomogeneous component of speed corresponding to six and ten receivers, respectively.

C. Estimating wind field: Simulation

Suppose an animal is located at Cartesian coordinate (20,100,7) m and its signals are monitored at five microphones at (0,0,0), (25,0,3), (50,3,5), (30,40,9), and (1,30,4) m, respectively. Each coordinate is initially assumed to be distributed uniformly within ±0.04 m about the true values. This accuracy is typical of that obtained from optical surveys. *A priori* errors are zero for receiver 1, the y and z coordinates of receiver 2, and the z coordinate for receiver 3. These coordinates define the origin and orientation of the coordinate system.

For definiteness, assume the animal's call has a rms bandwidth of 1000 Hz and, following the cross correlation of the signal between each pair of microphones, the peak signal-to-noise ratio is 20 dB. The standard deviation of the peak lag in the cross-correlation function has a standard deviation of 16 μs (Ref. 25), an accuracy that is achieved in practice. Initial errors of the lags are taken to be uniformly distributed within ±32 μs of their noiseless values.

Simulated lags are computed for a speed of sound of 330 m/s and for a horizontal wind blowing at 10 m/s toward the positive y Cartesian axis. The speed of sound has zero variation about a mean of 330 m/s, and the y component of the wind is initially assumed to be uniformly distributed in the interval 0 to +20 m/s. Winds along the x and z axes are set to zero.

After the first constellation, the distributions of the variables (except the lags) are assumed to be uniformly distributed about the most recent values of their sample means from the valid configurations. The limits of the uniform distribution are determined using sequential bound estimation.

With isodiachronic location, 95% confidence limits for the animal are x: 16.6–20.4 m, y: 98.8–101.8 m, and z: 3.3–40.3 m. These are statistically consistent with the correct location at (20,100,7) m. The large variation in z stems from the fact that the animal and receivers are nearly coplanar.

Valid configurations of the y component of the wind have 95% confidence limits of 8.9 and 19 m/s. This is consistent with the true speed of 10 m/s.

The number of distributions estimated in this simulation is 13 (nine Cartesian receiver coordinates, three Cartesian animal coordinates, and one Cartesian wind component).

D. Surveying locations of receivers

1. Simulation

Suppose the calls of 100 animals are used to survey the locations of poorly positioned receivers. For simplicity, suppose the calls occur at 10-s intervals at elevations between 0 and 3 m. These calls could be due to a person walking around and blowing a whistle or due to animals who naturally call within 3 m of the ground. The true locations of the receivers are shown in Fig. 3. Horizontal locations of the calls are chosen to be uniformly distributed within this plotted domain (not shown). *A priori* distributions of all variables are uniformly distributed in the intervals given below.

The x, y, and z components of the wind are distributed within $[-1,+1]$, $[-1,+1]$, and $[-0.5,+0.5]$ m s^{-1}, respectively. Without fluctuations, the mean speed of sound is 330 m s^{-1}. The spatially homogeneous and inhomogeneous fluctuations of sound speed are distributed in $[-0.5,+0.5]$ and $[-0.2,+0.2]$ m s^{-1}, respectively. The initial estimates of the receiver locations are incorrect. Their initial locations are chosen by drawing from a uniform distribution within ±2 m of each of their true Cartesian coordinates. Their error distributions are assumed to be within ±3.94 m of each of their incorrect Cartesian coordinates except as follows. As before, no error is assigned to the location of receiver 1 (defined to be the origin), to the y and z coordinates of receiver 2, and the z coordinate of receiver 3 because these define the origin and orientation of the coordinate system.

The spatially homogeneous and inhomogeneous fluctuations of the sound speed are modeled as in Eq. (12). For homogeneous fluctuations, we take $\delta c_{\max}$ and $d\delta c_r/dt$ equal to 0.5 m s^{-1} and 0.01 m s^{-2}, respectively. The time series for this component has a temporal scale of 500 s. For inhomogeneous fluctuations, we take $\delta c_{\max}$ and $d\delta c_r/dt$ equal to 0.2 m s^{-1} and 0.01 m s^{-2}, respectively. The time series of this component has a temporal scale of 10 s.

The Cartesian components of the wind have temporal variations modeled in the same way as the speed of sound. Maximum values of the x, y, and z components are within the intervals stated above. The maximum rate of change of each component is 1 m s^{-2}. The time series of winds have a temporal scale of 5 s.

Unlike previous examples, the probability distributions of the lags are also updated using sequential bound estimation. The lags are distributed in intervals of ±0.0001 s on either side of their noiseless values.

The 100% confidence limits for the Cartesian x, y, and z coordinates of each receiver are obtained from the lags (e.g., Figs. 4 and 5). Locations improve significantly with the number of calls processed. *A priori* errors of ±3.94 m decrease to values between 0.10 and 3.3 m (Table II). Surveying accuracy is better for receivers 2 and 3. This may occur because *a priori* errors are zero for receiver 2's y and z components and for receiver 3's z component. Receiver 6 is less accurately navigated, perhaps because it is higher, by at least 4 m, than any animals, and the other receivers are within 2 m of elevation of the calls. The animal calls contain some information to locate the vertical coordinates (Table II), but because the geometry is mostly horizontal, vertical coordi-

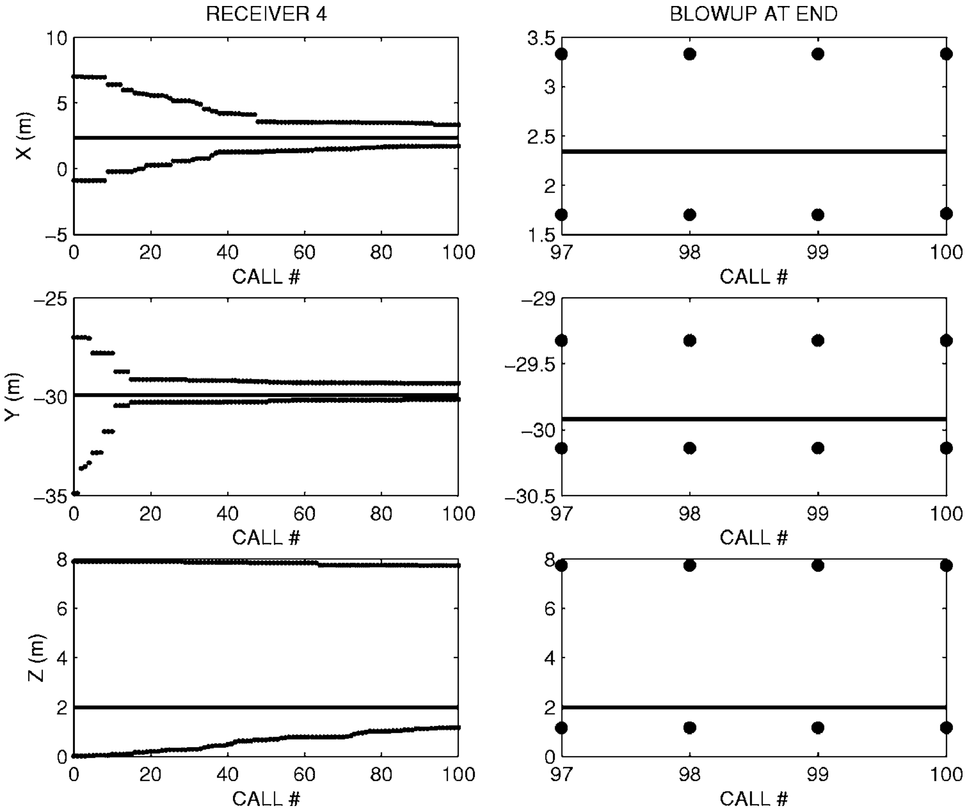

FIG. 4. Example of surveying receiver locations from animal calls at unknown locations near the elevations of the receivers. 100% confidence limits (astericks) for the horizontal (x,y) and vertical z limits for receiver 4 in Fig. 3 are given as a function of the number of calls used for surveying. Each Cartesian coordinate of a receiver is initially known within ±3.94 m. The correct Cartesian coordinates (solid line) are bounded by the 100% limits. Results from the last four calls are shown in the blow-up in the right-column.

nates are not surveyed as accurately. The ratio of *a priori* to final surveying accuracy varies from 39 to 5 for x and y components (Table II).

The number of distributions estimated from each call is 40 (12 Cartesian receiver coordinates, three Cartesian animal coordinates, one homogeneous component of speed, six inhomogeneous components of sound speed, three Cartesian wind components, and 15 lags).

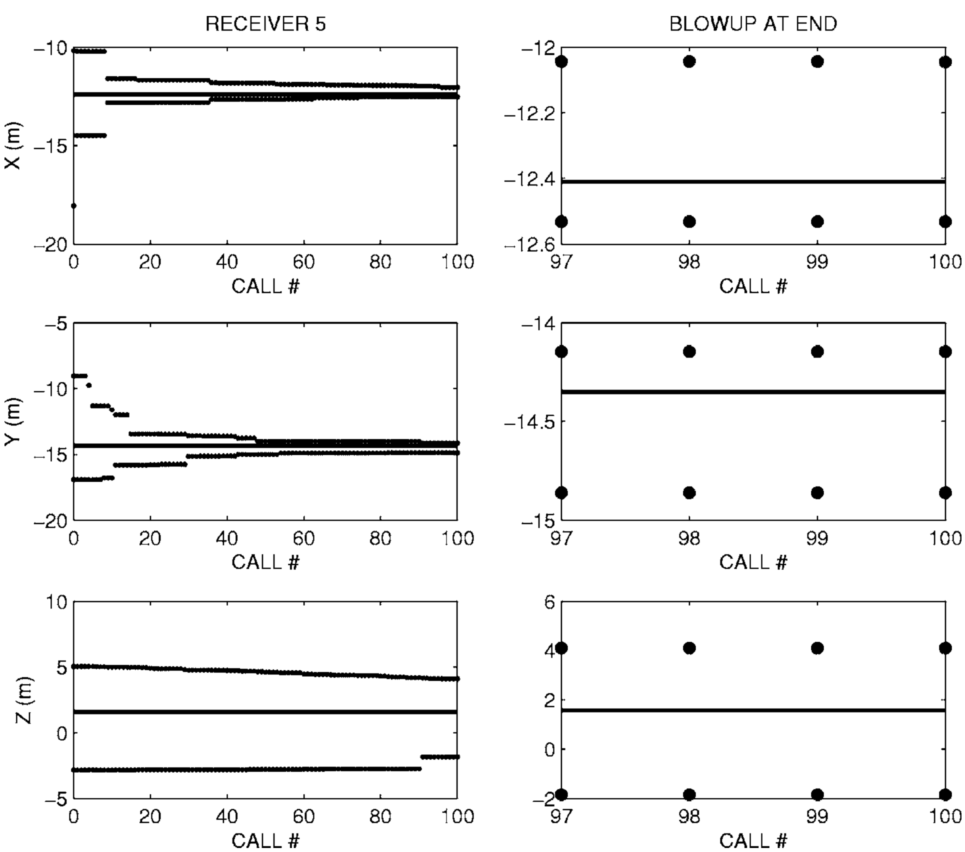

FIG. 5. Same as Fig. 4 except this is receiver 5.

TABLE II. 100% confidence intervals for x, y, and z Cartesian coordinates of receivers 2–6 after using sequential bound estimation on the lags from 100 animal calls at unknown locations (e.g., Figs. 4 and 5). For example, the 100% limits for the x component of receiver 2 are ±0.10 m about a mean value. The mean value is not shown but the mean value plus and minus the indicated limit encompasses the correct coordinate of the receiver in all cases. The*a priori* errors of each Cartesian coordinate are ±3.94 m. By definition, the y coordinate of receiver 2 and the z components of receivers 2 and 3 are 0 (Sec. III D).

Receiver no.	±100% Confidence limits		
	x (m)	y (m)	z (m)
2	0.10	0.00	0.0
3	0.20	0.13	0.0
4	0.81	0.41	3.3
5	0.24	0.36	3.0
6	1.2	0.50	2.9

2. Experiment

Sequential bound estimation is used to provide information about the location of the x coordinate of receiver 2 from Sec. III A (Table I). The information is provided only from a single call of a Red-winged Blackbird recorded on five microphones.

A priori statistics are the same as Sec. III A for isodiachronic location except as follows. The reference location of receiver 2 is changed from (19.76,0,0) to (21,0,0). It's a *a priori* probability distribution is uniform within ±1.5 m of its mean of 21 m. It is important that the algorithm does not know that the x coordinate of receiver 2 is really 19.76±0.05 m.

We find that the 95% confidence limits for the x coordinate of receiver 2 are 19.53 and 21.57 m. This is consistent with the correct answer of 19.76±0.05 m. Its probability distribution is skewed toward the correct answer of 19.76 m (Fig. 6). Sequential bound estimation can estimate joint distributions. For example, the joint distribution of the x coordinates of receiver 2 and the bird are highly correlated (Fig. 7).

IV. ANOTHER KIND OF LOCATION PROBLEM

To illustrate sequential bound estimation in another context, we show a different way to locate a calling whale or other source (Fig. 8). An array is towed along course θ_p at time p when a calling whale is detected at bearing β_p at sound pressure level Z_p from a beamformer. With array center at $\mathbf{r}_p$, the location of the whale is

$$\mathbf{s}_p = \mathbf{r}_p + \rho_p \sin[\theta_p + \beta_p]\hat{\mathbf{i}} + \rho_p \cos[\theta_p + \beta_p]\hat{\mathbf{j}}, \tag{14}$$

where the distance from the array center to the whale is ρ_p, and unit vectors pointing north and east are $\hat{\mathbf{j}}$ and $\hat{\mathbf{i}}$, respectively. At each time p, we take measurements of array location, tow direction, beam angle, and source level. Source level is converted to distance using some algorithm. For $p=1$, we assign upper and lower bounds and *prior* distributions for each measurement and for distance ρ_1 given by $f(\mathbf{r}_p)$, $f(\theta_p)$, $f(\beta_p)$, and $f(\rho_p)$, respectively. For example, the *prior* distribution for distance could be estimated from source level and be uniformly distributed between 10 and 40 000 m. A *prior* region is assigned to the location of the whale, e.g., within a radius of 500 km from the array. Configurations of variables are computed by drawing from their *prior* distributions. Locations for each configuration are obtained from Eq. (14). Invalid configurations are discarded (those with the whale outside its *a priori* bounds), leaving valid configurations. We transition the *posterior* bounds for all variables and spatial bounds for the whale to the time of the next datum, at $p=2$. Since each variable corresponds to a datum, *prior* bounds are assigned to each datum at $p=2$, and these are intersected with those transitioned from the data that were transitioned from $p=1$ (Fig. 2). Then, uniform distributions are assigned to all variables within their new intersected bounds, and configurations are drawn from these distributions. Probability distributions for any variable can be made for each time p. Sequential bound estimation may be as accurate or more accurate than any other Bayes-type estimation scheme since the *prior* distributions of the variables are usually not known very well in this scenario.

V. OTHER CONSIDERATIONS

Bounds taken from the set of valid configurations almost always define a smaller interval than the *prior* bounds for each variable. We would not want to accept a smaller bound if it is likely to have occurred by chance because we draw from a *finite* number of samples. This "bound-creep" can be mitigated using standard statistical techniques. Bound-creep mitigation is used in some of the examples in Sec. III.

Experience with many different kinds of problems is needed to appreciate convergence criteria for distributions. In analogy to many Markov chain Monte Carlo approaches, convergence for sequential bound estimation is checked by increasing the number of valid configurations until no significant change is found. Convergence could be difficult when one seeks distributions for which a pertinent event occurs with small probability. On the other hand, there is no advantage in adopting a Gaussian distribution for a *prior* for

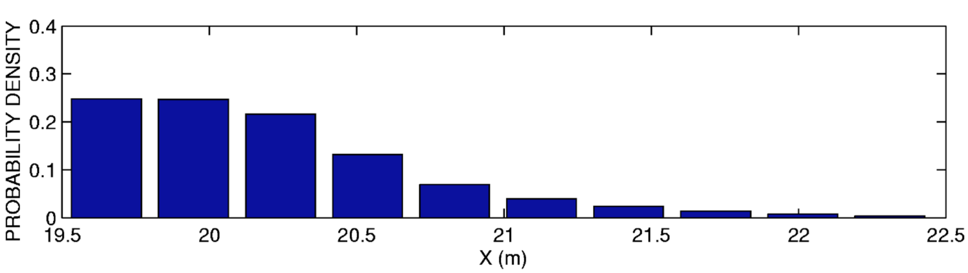

FIG. 6. Probability density function for the Cartesian x coordinate of receiver 2 (Table I) as determined from sequential bound estimation. The correct answer for the x coordinate is 19.76±0.05 m.

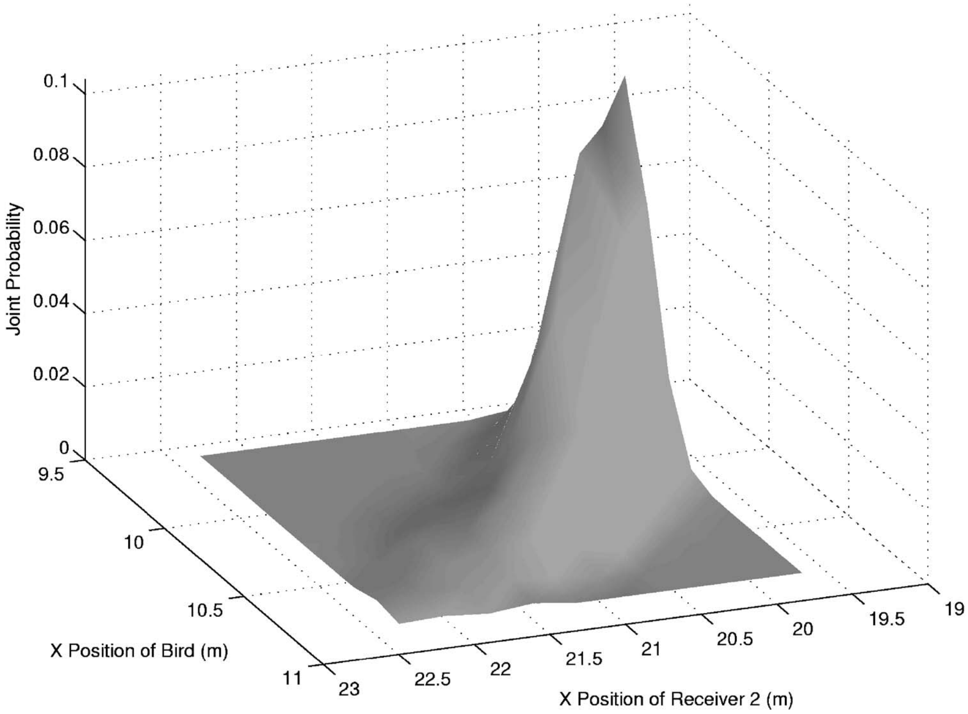

FIG. 7. Joint probability distribution, $f(r_2(x), s(x))$, from valid configurations of random variables following use of data from the fifth receiver constellation. The x coordinates of receiver 2 and the bird location are $r_2(x)$ and $s(x)$, respectively. The marginal distribution for $r_2(x)$ is shown in Fig. 6. The joint distribution shows the probability of joint occurrence for bin intervals of 0.3 and 0.14 m for $r_2(x)$ and $s(x)$, respectively.

the sake of analytical beauty when one distribution is usually as easy to draw from as another, such as a truncated Gaussian. Because data are finite, measurements cannot strictly obey a Gaussian distribution.

We define *efficiency* as the number of valid configurations divided by the number of configurations,

$$E \equiv \mathcal{V}/\mathcal{C}. \tag{15}$$

An efficiency of 1 means that each configuration is a valid configuration. Efficiency increases (decreases) when variables approach statistical independence (dependence). In Sec. III, the efficiency varies roughly between $O(1)$ and $O(10^{-3})$ and depends on the example and receiver constellation in question. Example III D 1 (receiver surveying) has an efficiency of about 10^{-3} and takes about 12 days to run on an Advanced Micro Devices Athlon 1800+ CPU. Example III D 2 (bird location) has an efficiency of about 0.3 and takes 13 min to run. Little effort has been made to optimize run times for the software written in the MATLAB programming language.

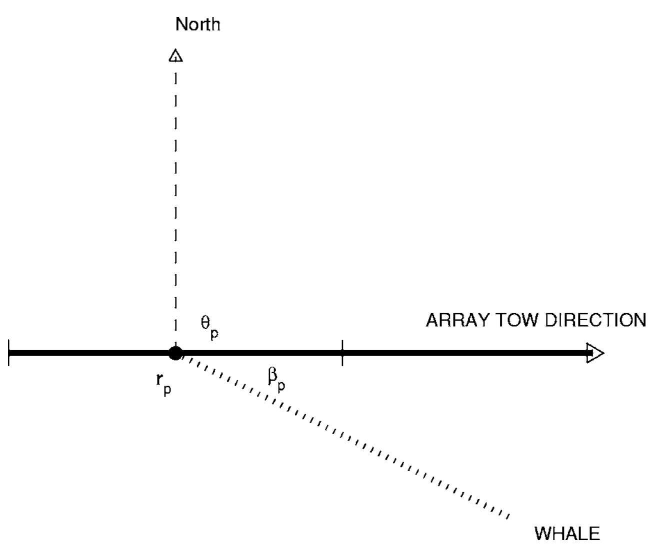

FIG. 8. An array located between the tic marks is towed along course θ_p degrees True at time p. At this time, the center of the array is located at $\mathbf{r}_p$ and the bearing angle to the whale call is β_p.

VI. CONCLUSION

A new sequential Monte Carlo algorithm called "sequential bound estimation" was used to estimate the probability distributions of all pertinent random variables, numbering 13 to 41, in situations where calls from animals were recorded on widely separated receivers. The algorithm was efficient enough to provide distributions for all variables because it did not attempt to propagate joint probability distributions between the use of subsequent data as done using Bayes theorem,[14] Gibbs samplers,[14] and particle filters.[15] These other techniques may be too inefficient to estimate the distributions of so many variables and that is evidently why the literature does not appear to report solutions for the problems dealt with in this paper. Sequential bound estimation was able to estimate the distributions of the receiver locations, speeds of sound, winds, lags, and animal locations by sequentially processing lags between pairs of receivers. The algorithm is robust and leads to distributions that bound the distributions that would be obtained from Bayes theorem if that theorem was practical to implement. Sequential bound estimation need not make any linear approximation between the variables and the data, and it is able to use any realistic *prior* distributions for each of the variables in the problem. Like a Kalman filter, the algorithm is flexible enough to incorporate any model for the transition of all the variables between sequential use of the data.

The covariance intersection method[26] is used for Kalman filters when the correlation between data is unknown, but is boundable.[26] Sequential bound estimation is somewhat analogous to that method in that bounds of distributions are sequentially propagated between data without propagating joint distributions. The latter approach appears to be useful because *prior* distributions are often unknown and it is too inefficient to propagate joint distributions of many variables using other sequential Monte Carlo methods.[15]

According to the new algorithm, the lags of travel time between widely separated receivers contain significant information about the location of a calling animal, the speed of sound in air, and the locations of receivers whose initial locations are poorly known. The algorithm indicates that large *prior* uncertainties for the speed of sound or receiver locations can be improved by factors of one or two orders of magnitude by using the information from the lags. One practical use of the new algorithm may be to use animal calls to accurately survey the locations of receivers that may be too expensive or inconvenient to navigate by other means.

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