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Probability Distributions for the Heights of Maxima and Minima in Fluctuating Nuclear Cross Sections

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Abstract

Probability distributions for the heights of maxima and minima in fluctuating nuclear cross sections have been obtained from synthetic excitation functions. It is shown how these distributions can be used to determine the extent to which experimentally observed large excursions from the average cross sections may deviate from the predictions of simple fluctuation theory. A discussion is given of problems associated with the detection of resonance anomalies in experimental excitation functions.

Introduction

It is well known that, at bombarding energies E such that the mean level width Γ is much greater than the mean level spacing D (that is, $\Gamma/D \gg 1$), nuclear reaction cross sections exhibit fluctuations similar to those described by the simple statistical model of Brink and Stephen (1963) and Ericson (1963). The fluctuations arise from interference between the overlapping states of the compound system. They give rise to distributions of cross sections σ, relative to that of the average cross section $\bar{\sigma}$, which depend on the number N of independently fluctuating cross sections contributing to the reaction. This number depends on the number of magnetic substates involved, which in turn depends on the number of decay channels, the spins of the initial and final states, and the angle of observation. According to the simple statistical model, the cross sections should be correlated over an energy range of $\sim\pi\Gamma$ and an angular range of $\sim(kR)^{-1}$, where k is the wave number of the incident particles and R the interaction radius (Brink *et al.* 1964), but outside of these ranges the cross sections should be uncorrelated. The model also predicts the numbers of maxima per unit energy interval, so that a comparison with experimental data enables one to deduce Γ and, in some cases, the fraction of the direct reaction component of the reaction.

The simple theory contains the assumption that $\bar{\sigma}$ is independent of the bombarding energy. In practice this is almost never the case, owing to effects such as barrier penetration and the opening of new channels, and perhaps to more subtle effects as well. The slow change of $\bar{\sigma}$ over energy regions $\gg\Gamma$ must be taken into account before the raw experimental data can be compared with the model predictions, and this inevitably introduces some uncertainty into the comparison. Nevertheless, when this has been done, one usually obtains quite good accordance with the model predictions. Throughout the paper we assume that such effects have been accounted for.

At present it is of greater interest to find data that disagree rather than agree with the predictions of the fluctuation model. One reason for this is that the observation of large maxima or deep minima, which possess very improbable values of $\sigma/\bar{\sigma}$ and occur in a number of supposedly independent channels, may indicate the presence of a special and unusual state at high excitation. Alternatively, it might indicate that the model itself is inadequate, as was suggested but not followed up by Moldauer (1967). A number of experiments, mainly with heavy-ion projectiles, have recently been carried out to search for such anomalies.

Although a few fairly definite anomalies have been observed, it is generally very difficult to establish the existence of such behaviour since, in order to obtain good values of $\bar{\sigma}$, it is necessary to cover a large energy range. Also, a number of the many (usually hundreds) of open reaction channels must be investigated and these must be randomly chosen in the statistical sense. However, a special nuclear state may only populate a limited number of channels. Hence, while an obvious violation of the model might appear if these channels (which are *a priori* unknown) were studied alone, if many other channels were also included in the analysis, there might appear to be little if any violation. Another difficulty is that, in a sufficiently large range of data, a number of rather improbable peaks must be expected.

We are concerned here with the more limited problem of determining the expected numbers of peaks (and dips) per unit energy range having maxima (or minima) with heights greater (or less) than given values of $\sigma/\bar{\sigma}$. These values are required if one is to make a quantitative assessment of a possible deviation from model predictions. As far as we are aware this problem has not been solved previously. An analytical solution would be very difficult because the cross sections are correlated over an energy region $\sim\pi\Gamma$ (P. A. P. Moran, personal communication). We have therefore adopted the approach of generating synthetic excitation functions using a digital computer.

Method and Results

Synthetic excitation functions were generated by means of the Univac 1108 computer of the Australian National University. The first method, used for the case $N = 1$, was that of Brink and Stephen (1964), in which we have

$$\sigma = |S|^2, \tag{1}$$

where

$$S = \sum_{\lambda} A_\lambda/(E-E_\lambda+\tfrac{1}{2}\mathrm{i}\Gamma). \tag{2}$$

The amplitudes A_λ were all taken to be real and of equal magnitude, with signs chosen at random. The compound levels were assumed to be equally spaced with an energy separation D of one energy unit, while the coherence energy Γ was taken to be 10 energy units (that is, $\Gamma/D = 10$). The summation range was $1 \leqslant \lambda \leqslant 201$, or 20Γ in extent, which according to Van der Woude (1966) and Dallimore and Hall (1966) is a reasonable compromise between reliability of results and computer time.

The initial value of σ was generated from 201 random signs for the A_λ as input data. The signs were obtained by determining $(-1)^x$ from a set of random numbers x given by Abramowitz and Stegun (1964). Each subsequent value of σ was obtained by decreasing the indices of the A_λ by 1 and assigning amplitude A_{201} a new random sign, which was chosen to be positive or negative according as the least significant digit in the antecedent value of σ was even or odd. In this manner, synthetic excitation functions which extended over 2×10^4 coherence widths (2×10^5 values of σ) were generated. A small sample of the data is shown in Fig. 1.

Since it was impracticable to save 2×10^5 numbers in the computer, only the three most recently calculated values of σ were stored and updated every time a new value of σ was calculated. After each calculation, the appropriate channel of the spectrum of cross sections was incremented. Also the three stored values of σ were investigated to see if the previously calculated one had been a maximum or minimum. If this was the case, the register corresponding to the range of heights within which the maximum occurred was incremented.

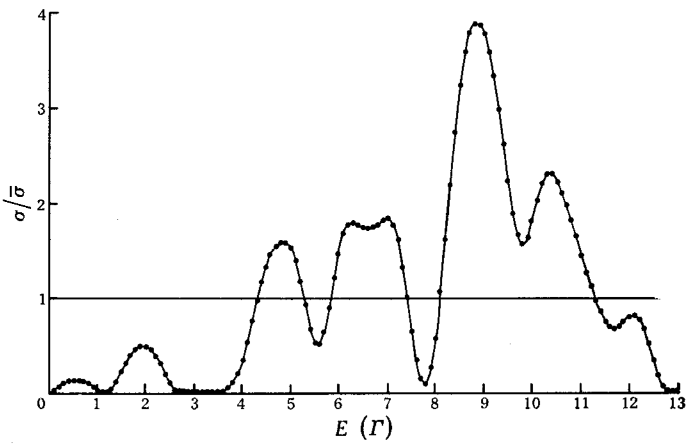


Fig. 1. Example of a synthetic excitation function with $N = 1$, in which $\bar{\sigma}$ is the average for the entire set of data ($2\times10^4\,\Gamma$ in extent).

The distribution P of the derived cross sections for $\Gamma/D = 10$ is shown in Fig. 2*a*, where it is compared with the theoretical distribution function (Ericson 1963)

$$P(\sigma/\bar{\sigma}) = \exp(\sigma/\bar{\sigma}), \tag{3}$$

which applies when $\Gamma/D = \infty$. It is clear that the present results deviate from the theoretical distribution (3) by increasing amounts as $\sigma/\bar{\sigma}$ increases. To investigate this further we generated synthetic excitation functions for $\Gamma/D = 2{\cdot}5$, 5 and 20. The resulting distributions are included in Fig. 2*a*, from which it can be seen that P only approaches the theoretical result ($\Gamma/D = \infty$) for very large Γ/D. Since it was impracticable to use a value of $\Gamma/D \gtrsim 10$, an empirical correction was applied to the points in the $\Gamma/D = 10$ excitation function in order to compare the results with the theory. After application of this correction, the distribution agreed with the form (3) to an accuracy of 5% for $\sigma/\bar{\sigma} \leqslant 7$.

It should be pointed out that a value for Γ/D of 10 is perhaps more typical of realistic cases than one of ∞. For equal nonzero level spacings, it is obvious that the exponential distribution (3) cannot be achieved and that P must vanish for some finite value of $\sigma/\bar{\sigma}$. Even if more realistic distributions were used for the A_λ and for the level spacings (e.g. the Wigner distribution), it would seem likely that the exponential distribution would overestimate the probability density for large values of $\sigma/\bar{\sigma}$. We did not investigate the effect of different distributions for the A_λ and D but Van der Woude (1966) and Dallimore and Hall (1966) obtained almost identical results when they used the Wigner distribution for D instead of constant D. In view of these considerations, the adopted procedure of applying an empirical correction probably overestimates the number of maxima to be expected above any level greater than $\sigma/\bar{\sigma} \approx 3$.

Analysis of the data on maxima and minima showed that the number n of maxima per energy interval Γ was $0 \cdot 77$, which does not agree well with the theoretical estimates of $0 \cdot 587$ (Stephen 1963) and $0 \cdot 55$ (Dallimore and Hall 1966, quoted as a personal communication from D. M. Brink). However, as is shown in Fig. 2*b*, it does agree with the result of Van der Woude (1966) for $\Gamma/D = 10$, which was obtained by a similar method to ours. To examine this situation further, we computed values of n for values of $\Gamma = 2 \cdot 5$, 5 and 20 energy units ($D/\Gamma = 0 \cdot 4$, $0 \cdot 2$ and $0 \cdot 05$) keeping the summation range $1 \leqslant \lambda \leqslant 201$. The results of these calculations are also shown (full circles) in Fig. 2*b*. The decrease in the number of maxima observed for $D/\Gamma > 0 \cdot 3$

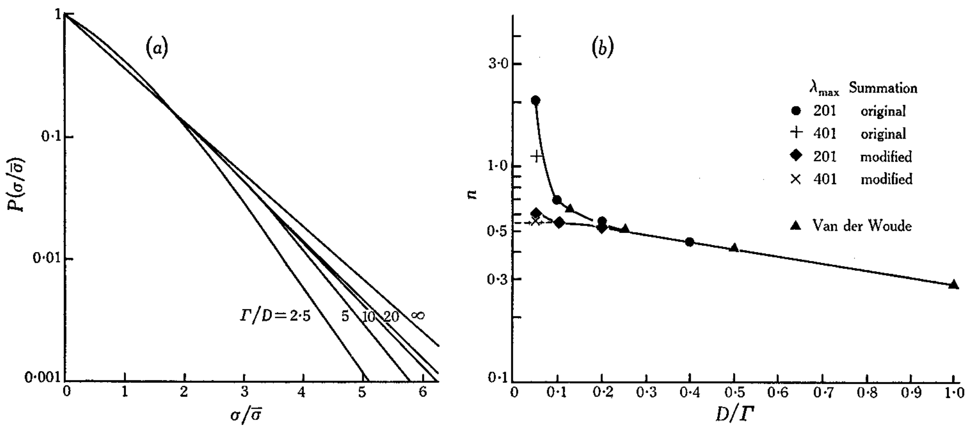
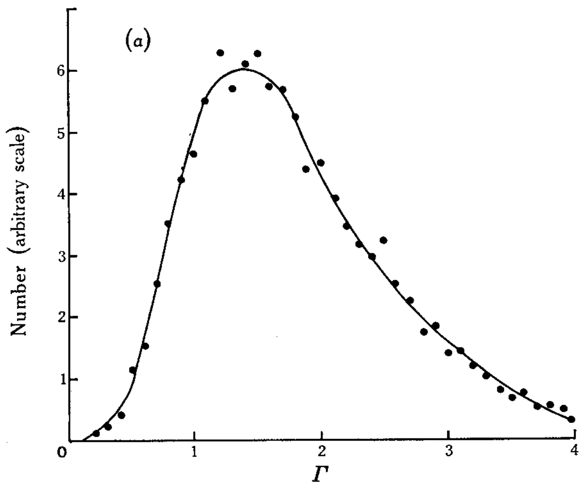


Fig. 2. Results obtained from the synthetic excitation functions:
(*a*) distribution of cross section for the indicated values of Γ/D and
(*b*) average number n of maxima observed per energy interval Γ as a function of D/Γ. The sharp rise in n predicted both by Van der Woude (1966) and the unmodified present results is an artefact of the method of generating the excitation functions.

is probably largely due to the fact that, if maxima are more closely spaced than $\sim 2D$, they cannot be observed by the method described here. However, this explanation is unlikely to account for the rapid increase observed below $D/\Gamma = 0 \cdot 1$, since it would imply that many maxima were separated by less than $0 \cdot 1\,\Gamma$. This result is not expected from the form of equation (2) nor is it obvious from plots of the excitation functions (e.g. Fig. 1). Furthermore, if the summation range is increased to $1 \leqslant \lambda \leqslant 401$, the number of maxima is found to decrease by a factor of ~ 2 for $D/\Gamma = 0 \cdot 05$, as is shown by the plus sign in Fig. 2*b*. The sharp increase in the expected number of maxima at low values of D/Γ is thus an artefact of the method used to generate the synthetic excitation functions. The effect arises because of the small discontinuous changes that are made to the cross section each time a new amplitude is introduced to and removed from the summation (2). Such discontinuities can produce spurious maxima and minima when $\partial\sigma/\partial E$ is small, i.e. near to genuine maxima and minima. The number of spurious maxima and minima that can be generated in this way clearly increases as the step length D/Γ decreases.

To reduce the effect of discontinuities at the ends of summation (2), the terms with $\lambda = 1, 2, 3, \ldots, 10$ were multiplied by $0\cdot1, 0\cdot2, 0\cdot3, \ldots, 1\cdot0$ and those with $\lambda = 191, 192, 193, \ldots, 201$ by $1\cdot0, 0\cdot9, 0\cdot8, \ldots, 0\cdot1$ respectively. Results obtained from this modified summation are shown by full diamonds (lower curve) in Fig. 2*b*. This curve has a much shallower rise for $D/\Gamma < 0\cdot1$. Increasing the range of the modified summation to $1 \leqslant \lambda \leqslant 401$ has a negligible effect on all calculated points except that for $D/\Gamma = 0\cdot05$ (cross in Fig. 2*b*). We therefore consider that the $\Gamma/D = 10$ data



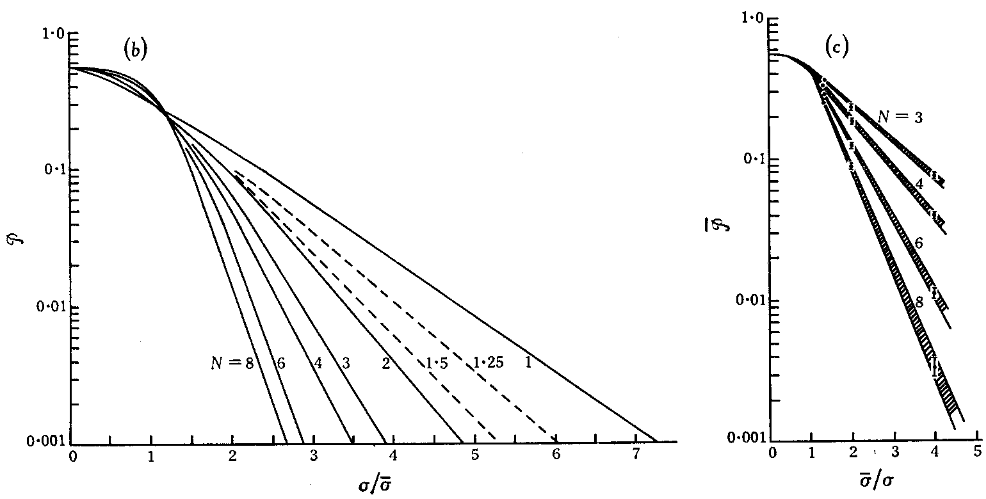


Fig. 3. Properties of maxima in fluctuating cross sections calculated by the modified summation method:
(*a*) distribution of the separations between maxima as a function of Γ for $N = 3$ (the distribution is not significantly sensitive to changes in N),
(*b*) probability $\mathscr{P}$ of the presence of a maximum greater in height than $\sigma/\bar{\sigma}$ in an energy interval Γ as a function of $\sigma/\bar{\sigma}$ and
(*c*) probability $\bar{\mathscr{P}}$ of the presence of a minimum smaller in depth than $\sigma/\bar{\sigma}$ in an energy interval Γ as a function of $\bar{\sigma}/\sigma$.

generated in this manner contain an insignificant number of spurious maxima and minima. The distribution of the separations between maxima are shown in Fig. 3*a*. From this figure it is evident that the number of maxima to be expected with separations less than $0\cdot2\,\Gamma$ is negligible, which indicates that the $\Gamma/D = 10$ data should give a good estimate of the number of maxima per energy interval Γ. This estimate was found to be $0\cdot56$, which agrees well with the prediction of $0\cdot55$ (Dallimore and Hall 1966). The modified method for generating the synthetic excitation functions was used in deriving the results presented below.

Excitation functions for $N = 1{\cdot}25$, $1{\cdot}5$, 2, 3, 4, 6 and 8 were obtained by adding together the appropriate number of corrected independent cross sections for $N = 1$. The distribution functions for $\sigma/\bar{\sigma}$ were found to be in good agreement with the theory, which predicts that they should behave like χ^2 functions for $2N$ degrees of freedom. The distribution functions for maxima separation were found to be independent of N within the limits of accuracy of our method. They have the shape shown in Fig. 3*a*. The distributions for the heights of maxima and minima which we obtained are shown in Figs 3*b* and 3*c*.

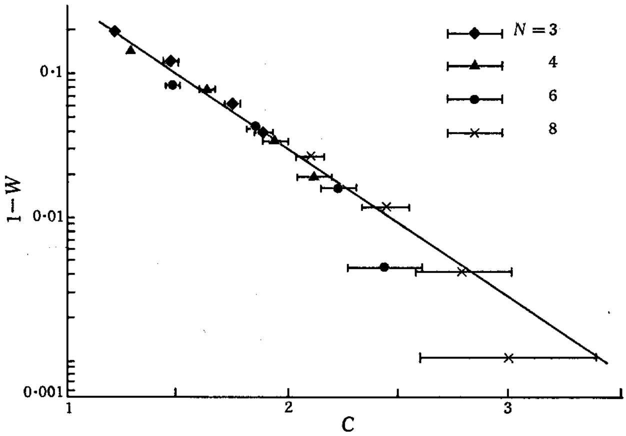


Fig. 4. Dependence of the factor C on $1-W$.

It was found empirically that, for $W < 0{\cdot}1$, where W is the probability integral of the χ^2 distribution for $2N$ degrees of freedom taken over the range $2N(\sigma/\bar{\sigma})$ to ∞, the expected number of maxima per coherence width Γ with height $\geqslant \sigma/\bar{\sigma}$ is given by

$$\mathscr{P} = (1{\cdot}2 \pm 0{\cdot}2)W.$$

The error is not statistical and includes the extreme results required to describe all of the data shown in Fig. 3*b*. Consequently it might be thought that the number of minima observed below a given level per interval Γ is given by an expression of the form

$$\bar{\mathscr{P}} = C(1-W),$$

where C is constant. It was found, however, that C is not constant but varies as shown in Fig. 4. Within the present limits of accuracy, C appears to be independent of N and is given approximately by the relation

$$C = 0{\cdot}52 - 0{\cdot}98 \log_{10}(1-W).$$

Hence empirical values for the number of minima below a given value of $\sigma/\bar{\sigma}$ can be obtained easily. Our results have poor statistical accuracy for the lower values of $(1-W)$, so that it is uncertain whether the above relationship for C holds accurately over the whole range of the present data. There is some slight indication that C may become independent of $(1-W)$ as $(1-W)$ becomes smaller. It would certainly be unwise to use the expression to extrapolate to very much smaller values of $(1-W)$ than we have considered.

Discussion

To show the use of the results of the previous section, we consider two examples. Halbert *et al.* (1967) measured excitation functions for the $^{12}C(^{16}O,\alpha)^{24}Mg$ reaction leading to the first seven states of ^{24}Mg. They noted that, when the data are summed over all seven α-groups, strong peaks are seen at 31·7 MeV bombarding energy at a number of observation angles. At 0° and 20° (lab.) they obtained values for $\sigma/\bar{\sigma}$ of $2\cdot72\pm0\cdot19$ and $2\cdot70\pm0\cdot08$ with $N = 6$ and 22 respectively. From the present results it follows that the expected number of peaks with heights greater than those observed in the measured data-range of 84Γ is 0·017 and $1\cdot0\times10^{-5}$ for 0° and 20° respectively. Data at these two angles would be expected to be independent according to the model, and hence the probability of two such peaks being coincident within $\pm\frac{1}{2}\Gamma$ is given by $0\cdot017\times1\cdot0\times10^{-5}\times84^{-1} = 2\cdot0\times10^{-9}$.

A different type of anomaly was observed by Stokstad *et al.* (1972) in the same reaction. They observed a deep minimum for $E = 46\cdot0$ MeV in the summed excitation function for 12 α-groups taken at 0°. Taking $N = 12$, it follows that the expected number of minima with depths $\sigma/\bar{\sigma} \leqslant 0\cdot18$ (the observed value) is $\sim5\times10^{-4}$ for their range of data ($\sim25\Gamma$).

In both of the above cases it is clear from the present results that the observed anomalies are extremely unlikely to be Ericson fluctuations. Hence they may be a consequence of special nuclear properties such as quasi-molecular states, as has been proposed. We have thus demonstrated how the probability functions presented here can be used to investigate the possibility that an anomaly exists in a set of data. It seems likely, however, that a quantitative investigation of anomalies should also involve consideration of their width as well as their height or depth. This might be of considerable help in assessing cases which are not so clear cut as those above.

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