

PROBABILITY DISTRIBUTIONS RELATED TO RANDOM MAPPINGS

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1. Introduction and Summary. A Random Mapping Space (X, \mathfrak{J}, P) is a triplet, where X is a finite set of elements x of cardinality n , \mathfrak{J} is a set of transformations T of X into X , and P is a probability measure over \mathfrak{J} .

In this paper, four choices of \mathfrak{J} are considered

(I) \mathfrak{J} is the set of all transformations of X into X .

(II) \mathfrak{J} is the set of all transformations of X into X such that for each $x \in X$ $Tx \neq x$.

(III) \mathfrak{J} is the set of one-to-one mappings of X onto X .

(IV) \mathfrak{J} is the set of one-to-one mappings of X onto X , such that for each $x \in X$, $Tx \neq x$.

In each case P is taken as the uniform probability distribution over \mathfrak{J} .

If $x \in X$ and $T \in \mathfrak{J}$, we will define $T^k x$ as the k th iteration of T on x , where k is an integer, i.e. $T^k x = T(T^{k-1}x)$, and $T^0 x = x$ for all x . The reader should note that, in general, $T^k x$, $k < 0$, may not exist or may not be uniquely determined.

If for some $k \geq 0$, $T^k x = y$, then y is said to be a k th image of x in T . The set of successors of x in T , $S_T(x)$ is the set of all images of x in T , i.e.,

$$S_T(x) = \{x, Tx, T^2x, \dots, T^{n-1}x\},$$

which need not be all distinct elements.

If for some $k \leq 0$, $T^k x = y$, y is said to be a k th inverse of x in T . The set of all k th inverses of x in T is $T^{(k)}(x)$ and

$$P_T(x) = \bigcup_{k=-n}^0 T^{(k)}(x)$$

is the set of predecessors of x in T .

If there exists an $m > 0$, such that $T^m x = x$, then x is a cyclical element of T and the set of elements $x, Tx, T^2x, \dots, T^{m-1}x$ is the cycle containing x , $C_T(x)$. If m is the smallest positive integer for which $T^m x = x$, then $C_T(x)$ has cardinality m .

We note further an interesting equivalence relation induced by T . If there exists a pair of integers k_1, k_2 such that

$$T^{k_1}x = T^{k_2}y,$$

then $x \sim y$ under T .

It is readily seen that this is in fact an equivalence, and hence decomposes X

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into equivalence classes, which we shall call the components of X in T ; and designate by $K_T(x)$ the component containing x .

We define $s_T(x)$ to be the number of elements in $S_T(x)$, $p_T(x)$ to be the number of elements in $P_T(x)$, and $l_T(x)$ to be the number of elements in the cycle contained in $K_T(x)$ (i.e. $l(x) =$ the number of elements in $C_T(x)$ if x is cyclical). We designate by q_T the number of elements of X cyclical in T , and by r_T the number of components of X in T .

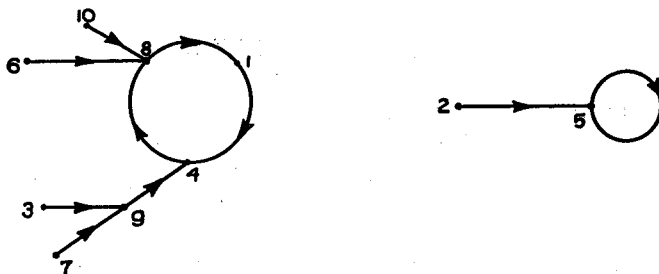
Rubin and Sitgreaves [9] in a Stanford Technical Report have obtained the distributions of s, p, l, q , and have given a generating function for the distribution of r in case I. Folkert [3], in an unpublished doctoral dissertation has obtained the distribution of r in cases I and II. The distribution of r in case III is classical and may be found in Feller [2], Gontcharoff [4], and Riordan [8]. In the present paper, a number of these earlier results are rederived and extended. Specifically, for cases I and II, we compute the probability distributions of s, p, l, q and r . In cases III and IV the distributions of l and r are given. In addition some asymptotic distributions and low order moments are obtained.

For the convenience of the reader, an index of notations having a fixed meaning is provided in the appendix to the paper.

2. Representation of T as a directed graph. It will be convenient to represent elements of \mathfrak{S} as directed graphs. For example, if $n = 10, X = \{1, 2, 3, 4, \dots, 10\}$, and

$$\begin{aligned} T(1) &= 4, & T(2) &= 5, & T(3) &= 9, & T(4) &= 8, & T(5) &= 5, \\ T(6) &= 8, & T(7) &= 9, & T(8) &= 1, & T(9) &= 4, & T(10) &= 8, \end{aligned}$$

Then T has the representation below:



3. Probability Distribution for Case I. In case I, $P(T) = 1/n^n$ for all $T \in \mathfrak{S}$. We now turn to the computation of the probability distributions of s and l , the number of elements in $S_T(x)$ and the number of elements in the cycle contained in $K_T(x)$ respectively.

Then, for any choice of x , we have:

$$P(s = k, l = j) = P\{T^r x \neq x, T x, \dots, T^{r-1} x (0 < r \leq k - 1); T^k x = T^{k-j} x\}$$

Hence

$$(3.1) \quad P(s = k, l = j) = \frac{(n-1)!}{(n-k)!n^k}, \quad 1 \leq j \leq k \leq n,$$

and summing over j , we have

$$(3.2) \quad P(s = k) = \frac{(n-1)!k}{(n-k)!n^k},$$

$$(3.3) \quad P(l = j) = \sum_{k=j}^n \frac{(n-1)!}{(n-k)!n^k}.$$

From consideration of symmetry, we note trivially that

$$(3.4) \quad E(l) = E[(s+1)/2].$$

We now obtain the asymptotic probability densities of s and l . In (3.1) let $k = \sqrt{nx}$, $j = \sqrt{ny}$, and replace factorials by Stirling's approximation. Then we have

$$(3.5) \quad P(s = \sqrt{nx}, l = \sqrt{ny}) \sim \frac{n^{n-\sqrt{nx}-\frac{1}{2}} e^{-\sqrt{nx}}}{(n-\sqrt{nx})^{n-\sqrt{nx}+\frac{1}{2}}} \\ = \frac{n^{n-\sqrt{nx}-\frac{1}{2}} e^{-\sqrt{nx}}}{n^{n-\sqrt{nx}+\frac{1}{2}} \left(1 - \frac{x}{\sqrt{n}}\right)^{n-\sqrt{nx}+\frac{1}{2}}}$$

Write

$$\left(1 - \frac{x}{\sqrt{n}}\right)^{n-\sqrt{nx}+\frac{1}{2}} = \exp\left[\left(n - \sqrt{nx} + \frac{1}{2}\right) \log\left(1 - \frac{x}{\sqrt{n}}\right)\right],$$

and expand $\log\left(1 - \frac{x}{\sqrt{n}}\right)$ in a power series, obtaining

$$P(s = \sqrt{nx}, l = \sqrt{ny}) \sim n^{-1} e^{-\frac{1}{2}x^2}.$$

Thus, the asymptotic density of $(s/\sqrt{n}, l/\sqrt{n})$ is

$$(3.6) \quad f(x, y) = e^{-\frac{1}{2}x^2}, \quad 0 < y \leq x < \infty.$$

The marginal distributions $f_1(x)$, $f_2(y)$ give the asymptotic densities of s/\sqrt{n} , l/\sqrt{n} respectively and are easily obtained by integration.

$$(3.7) \quad f_1(x) = x e^{-\frac{1}{2}x^2}, \quad 0 < x,$$

$$(3.8) \quad f_2(y) = \sqrt{2\pi} (1 - \Phi(y)), \quad 0 < y,$$

where $\Phi(y) = \int_{-\infty}^y (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx$.

In numerical computations, the cumulative distribution function $F_2(y)$ is probably more useful than the density function $f_2(y)$ and is therefore given below:

$$(3.9) \quad F_2(y) = P(Y \leq y) = 1 - e^{-y^2/2} + y\sqrt{2\pi}(1 - \Phi(y)).$$

We note further that

$$(3.10) \quad EY^r = \frac{1}{r+1} EX^r = \frac{2^{r/2}}{r+1} \Gamma\left(\frac{r+2}{2}\right).$$

Hence

$$(3.11) \quad E(l) \sim \frac{1}{4}(2\pi n)^{\frac{1}{2}}, \quad \sigma_l^2 \sim n[(2/3) - (2\pi/16)].$$

Formulas (3.1), (3.2), (3.3), and (3.7) have been obtained by Rubin and Sitgreaves [9].

Rubin and Sitgreaves have also shown that

$$(3.12) \quad P\{q = j\} = \frac{(n-1)!j}{(n-j)!n^j}, \quad j = 1, 2, \dots, n.$$

We now prove this using a partition argument due to Katz [7].

Consider the directed graph representation of T and partition X as follows. Let $M_0(T)$ be those elements of X cyclical under T . Define $M_1(T)$ to be those elements of X whose images are cyclical under T , but are not themselves cyclical. Let $M_2(T)$ be those elements of X whose images under T are in $M_1(T)$. Continuing in this manner until X is exhausted, the $n - j$ non-cyclical elements of X are partitioned into $m(T)$ sets each non-empty for $j \neq n$. Designate the cardinality of $M_j(T)$ by $n_j(T)$, $j = 1, 2, \dots, m(T)$.

The number of decompositions of X for n_1, n_2, \dots, n_m fixed is

$$(3.13) \quad \frac{n!}{j!n_1!n_2!\dots n_m!} j!j^{n_1}n_1^{n_2}\dots n_{m-1}^{n_m},$$

where $\sum_{i=1}^m n_i = n - j$. Hence

$$(3.14) \quad P(q = j) = n^{-n} \sum \frac{n!}{n_1!n_2!\dots n_m!} j^{n_1}n_1^{n_2}\dots n_{m-1}^{n_m}, \quad j \neq n,$$

where the sum is taken over all non-empty m -part partitions of $n - j$.

Katz [7] has shown that

$$(3.15) \quad \sum \frac{n!}{j!n_1!n_2!\dots n_m!} j^{n_1}n_1^{n_2}\dots n_{m-1}^{n_m} = \frac{n!n^{n-j-1}}{(j-1)!(n-j)!},$$

from which we obtain (3.12) for $j \neq n$. We have $j = n$, if and only if T is one-to-one and onto; hence

$$(3.16) \quad P(q = n) = n!/n^n,$$

which coincides with (3.12) for $j = n$.

It is curious to note that this is exactly the same as the distribution of s given in (3.2), and hence has the same asymptotic distribution and asymptotic moments.

The distribution of p has been obtained by Rubin and Sitgreaves [9],

$$(3.17) \quad P\{p = j\} = \frac{(n-1)!j^{j-2}(n-j)^{n-j}}{(n-j)!(j-1)!n^{n-1}}.$$

We establish this as follows:

Let X_{j-1} be $j - 1$ specified elements of X say x_1, x_2, \dots, x_{j-1} . Let x be a distinguished element of X not in X_{j-1} . Then define \mathfrak{J}_1 as those transformations T in \mathfrak{J} such that $T(X - (X_{j-1} \cup x)) = X - (X_{j-1} \cup x)$. Define \mathfrak{J}_2 as those transformations in \mathfrak{J} such that $T(X_{j-1}) = X_{j-1} \cup x$, and $T^k x_i = x$ for some $k > 0$ and $i = 1, 2, \dots, j - 1$. We further define $\mathfrak{J}^* = \mathfrak{J}_1 \cap \mathfrak{J}_2$. Then

$$(3.18) \quad \binom{n-1}{j-1} P(T \in \mathfrak{J}^*) = P(p = j),$$

and

$$(3.19) \quad P(T \in \mathfrak{J}^*) = P(T \in \mathfrak{J}_1)P(T \in \mathfrak{J}_2).$$

We readily see that

$$(3.20) \quad P(T \in \mathfrak{J}_1) = [(n - j)/n]^{n-j}.$$

Hence we have only to compute $P(T \in \mathfrak{J}_2)$. For any $T \in \mathfrak{J}_2$, we can, by restricting attention to X_{j-1} define an associated transformation T_{j-1}^* which has $T_{j-1}^* x_i = T x_i, i = 1, 2, \dots, j - 1$, and $T_{j-1}^* x = x$. Let N_{j-1} be the number of distinct transformations which can be constructed in this manner from $T \in \mathfrak{J}_2$. Since, in \mathfrak{J}_2, x has n equally likely images under T , we have

$$(3.21) \quad P(T \in \mathfrak{J}_2) = N_{j-1}/n^{j-1}$$

and N_{j-1} is readily obtained by Katz's Lemma and the partition argument used in (3.13). Hence

$$(3.22) \quad N_{j-1} = \frac{1}{j} \sum \frac{j!}{1! n_1! \dots n_m!} 1^{n_1} n_1^{n_2} \dots n_{m-1}^{n_m} = j^{j-2}, \quad j > 1,$$

and, trivially, $N_0 = 1$.

In (3.22), the sum is over all non-empty m -part partitions of $j - 1$, and the factor $(1/j)$ is obtained by distinguishing the element x . Hence

$$P(p = j) = \binom{n-1}{j-1} \frac{j^{j-2}}{n^{j-1}} \left(\frac{n-j}{n}\right)^{n-j}$$

and (3.17) is established.

We now note an interesting relationship,

$$(3.23) \quad E(S) = E(p).^1$$

This is established at once by symmetry. For any $T \in \mathfrak{J}$ such that y is a successor of x , there is a corresponding $T \in \mathfrak{J}$ with x a predecessor of y ; the correspondence is accomplished by interchanging x and y in the directed graphs.

We may also note an interesting physical property of directed graphs of this type, which holds for every $T \in \mathfrak{J}$. For any $T \in \mathfrak{J}$, let r_j be the number of ele-

¹ This was pointed out by D. Blackwell in a private conversation with the author.

ments x for which $T^{-1}x$ has j elements. Then,

$$(3.24) \quad \sum_{j=0}^n jr_j = n.$$

Also,

$$(3.25) \quad \sum_{j=0}^n r_j = n.$$

Thus,

$$(3.26) \quad r_0 = \sum_{j=1}^{n-1} jr_{j+1}.$$

From this it follows at once that

$$(3.27) \quad E(p^{(1)}) = 1,$$

where $p^{(1)}$ is the number of elements in $T^{-1}(x)$.

The distribution of $p^{(1)}$ is readily seen to be

$$(3.28) \quad P\{p^{(1)} = j\} = \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j}.$$

We proceed now to the question of the probability distribution of r , the number of components of X in T . Folkert [3] has obtained the distribution and has shown

$$(3.29) \quad P\{r = j\} = \frac{1}{n^n} \sum_{\mu=j}^n \frac{S_{\mu}^j}{\mu!} \sum_{k_1, k_2, \dots, k_{\mu}} \frac{n!}{k_1! k_2! \dots k_{\mu}!} k_1^{k_1} k_2^{k_2} \dots k_{\mu}^{k_{\mu}},$$

where S_{μ}^r are Stirling's Numbers of the First Kind, and the sum over k_1, k_2, \dots, k_{μ} is over all choices of k_1, k_2, \dots, k_{μ} with $k_i > 0$ ($i = 1, 2, \dots, \mu$) and $\sum_{i=1}^{\mu} k_i = n$.

In this paper we obtain a probability generating function for the number of components, which has a good deal of intrinsic interest because of its relation to Faà de Bruno's formula (Jordan [6]) and the exponential polynomials of Bell [1].

Let k_i denote the number of components with exactly i elements. Then every $T \in \mathfrak{S}$ determines an n -tuple (k_1, k_2, \dots, k_n) . Hence, for every specification of (k_1, k_2, \dots, k_n) we have a set of transformations $\mathfrak{S}_{k_1, k_2, \dots, k_n}$ in \mathfrak{S} .

Then

$$(3.30) \quad P(T \in \mathfrak{S}_{k_1, k_2, \dots, k_n}) = \frac{n! I_1^{k_1} I_2^{k_2} \dots I_n^{k_n}}{1!^{k_1} 2!^{k_2} \dots n!^{k_n} k_1! k_2! \dots k_n! n^n},$$

where I_j/j^j ($j = 1, 2, \dots, n$) is the probability that a transformation T_j on j elements X_j is indecomposable, i.e. $K_{T_j}(x) = X_j$ for all $x \in X_j$, where $0 \leq k_i \leq n$

and $\sum_{i=1}^n ik_i = n$. We have

$$\frac{I_j}{j^j} = \sum_{i=1}^j P(q = i, K_{T_j}(x) = X_j) = \sum_{i=1}^j \frac{(j-1)! i(i-1)!}{(j-i)! j^i i!}.$$

Hence

$$(3.31) \quad I_j = \sum_{i=0}^{j-1} \frac{(j-1)! j^i}{i!}.$$

This result has been obtained earlier by both Katz [7] and Rubin and Sitgreaves [9]. Then, the generating function of k_1, k_2, \dots, k_n is given by

$$(3.32) \quad G(x_1, x_2, \dots, x_n) = \sum_{k_1, k_2, \dots, k_n} \frac{n!(I_1 x_1)^{k_1} (I_2 x_2)^{k_2} \dots (I_n x_n)^{k_n}}{1!^{k_1} 2!^{k_2} \dots n!^{k_n} k_1! k_2! \dots k_n! n^n},$$

since the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} = P(T \in \mathfrak{J}_{k_1, k_2, \dots, k_n})$ for $\sum_{i=1}^n ik_i = n$.
 Since $r = \sum_{i=1}^n k_i$,

$$G(x_1, x_2, \dots, x_n) = \sum_{k_1, k_2, \dots, k_n} \frac{n! r! (I_1 x_1)^{k_1} (I_2 x_2)^{k_2} \dots (I_n x_n)^{k_n}}{n^n r! k_1! k_2! \dots k_n! 1!^{k_1} 2!^{k_2} \dots n!^{k_n}},$$

and

$$(3.33) \quad G(x_1, x_2, \dots, x_n) = \sum_r \frac{n!}{r! n^n} \left(\frac{I_1 x_1}{1!} + \frac{I_2 x_2}{2!} + \dots + \frac{I_n x_n}{n!} \right)^r.$$

We can extend the definition to $G(x_1, x_2, \dots)$ with no loss of generality, since this will in no way affect the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}$. Hence

$$(3.34) \quad G(x_1, x_2, \dots) = \frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i x_i}{i!}.$$

If in (3.34), we replace x_i by x^i , the coefficient of x^n in $G(x, x^2, \dots)$ is 1 for all n . Thus, we have

$$(3.35) \quad \frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i x^i}{i!} = \frac{n!}{n^n} \sum_{i=0}^{\infty} \frac{i^i x^i}{i!}$$

and

$$(3.36) \quad \sum_{i=1}^{\infty} I_i x^i / i! = \log \sum_{i=0}^{\infty} (i^i / i!) x^i.$$

Replacing x_i by $t^i x_i$ in (3.34) we obtain:

$$(3.37) \quad G(tx_1, t^2 x_2, \dots) = \frac{n!}{n^n} \exp \sum_{i=1}^{\infty} \frac{I_i t^i x_i}{i!}.$$

In (3.37) the coefficient of t^n gives the probability of any possible decomposition of X into components with the exponents of x_i indexing the decomposition.

Finally we observe that, replacing x_i by tx^i , we get

$$(3.38) \quad G(tx, tx^2, \dots) = \frac{n!}{n^n} \exp t \sum_{i=1}^{\infty} \frac{I_i x^i}{i!},$$

or equivalently

$$(3.39) \quad G(tx, tx^2, \dots) = \frac{n!}{n^n} \left[\sum_{i=0}^{\infty} \frac{i^i x^i}{i!} \right]^t,$$

and the coefficient of $t^k x^n$ in $G(tx, tx^2, \dots) = P\{r = k\}$.

We now employ the generating function given above to obtain Folkert's formula (3.29). From (3.38), we have

$$(3.40) \quad \text{coefficient of } t^k = \frac{n!}{n^n k!} \left(\sum_{i=1}^{\infty} \frac{I_i x^i}{i!} \right)^k,$$

and from (3.36) we have

$$(3.41) \quad \text{coefficient of } t^k = \frac{n!}{n^n k!} \left[\log \left(1 + \sum_{i=1}^{\infty} \frac{i^i}{i!} x^i \right) \right]^k,$$

Since

$$\log(1 + u)^k = \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_{\mu}^k u^{\mu};$$

see, for example, Jordan [6], p. 146. Employing this in (3.41), we get

$$(3.42) \quad \text{coefficient of } t^k = \frac{n!}{n^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_{\mu}^k \left[\sum_{i=1}^{\infty} \frac{i^i}{i!} x^i \right]^{\mu},$$

and, expansion by the multinomial theorem gives

$$(3.43) \quad \begin{aligned} \text{coefficient of } t^k &= \frac{n!}{n^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_{\mu}^k \\ &\cdot \sum_{\substack{k_1, k_2, \dots, k_n \geq 0 \\ \sum_{i=1}^n k_i = \mu}} \frac{\mu!}{k_1! k_2! \dots k_n!} \left(\frac{1}{1!} x \right)^{k_1} \left(\frac{2^2}{2!} x^2 \right)^{k_2} \dots \left(\frac{n^n}{n!} x^n \right)^{k_n}. \end{aligned}$$

To find the coefficient of x^n in (3.43) it suffices to restrict the second sum to non-negative n -tuples (k_1, k_2, \dots, k_n) with $\sum_{i=1}^n k_i = \mu$, $\sum_{i=1}^n i k_i = n$; hence

$$(3.44) \quad P(r = k) = \frac{n!}{n^n k!} \sum_{\mu=k}^n S_{\mu}^k \sum_{k_1, k_2, \dots, k_n} \frac{1}{k_1! k_2! \dots k_n!} \left(\frac{1}{1!} \right)^{k_1} \left(\frac{2^2}{2!} \right)^{k_2} \dots \left(\frac{n^n}{n!} \right)^{k_n},$$

which coincides with (3.29), except that partitions of n are enumerated without regard to order in (3.44), and thus we have obtained an alternate form of Folkert's formula. Rubin and Sitgreaves [9] noted that $n^{-1}E(s) = E(s^{-1})$. We remark, further, that it is even more curious that

$$(3.45) \quad n^{-1}E(s) = P(r = 1) = P(l = 1) = n^{-1}E(p) = n^{-1}E(q).$$

4. Probability Distribution for Case II. In case II, $P(T) = (n - 1)^{-n}$ for all $T \in \mathfrak{S}$. As in case I, we first consider the probability distribution of s and l .

Computing exactly as in Section 3, we obtain

$$(4.1) \quad P(s = k, l = j) = \frac{(n - 2)!}{(n - 1)^{k-1}(n - k)!}, \quad 2 \leq j \leq k \leq n,$$

and

$$(4.2) \quad P(s = k) = \frac{(n - 2)!(k - 1)}{(n - 1)^{k-1}(n - k)!},$$

$$(4.3) \quad P(l = j) = \sum_{k=j}^n \frac{(n - 2)!}{(n - 1)^{k-1}(n - k)!}, \quad 2 \leq j \leq n.$$

Comparing these results with (3.1), (3.2) and (3.3), we have

$$(4.4) \quad \begin{aligned} P(s = k | I, n) &= P(s = k + 1 | II, n + 1) \\ P(s = k, l = j | I, n) &= P(s = k + 1, l = j + 1 | II, n + 1), \end{aligned}$$

and

$$P(l = j | I, n) = P(l = j + 1 | II, n + 1).$$

Hence

$$(4.5) \quad E(l | II, n + 1) = E(l | I, n) + 1$$

and

$$(4.6) \quad E(s | II, n + 1) = E(s | I, n) + 1.$$

From (3.4) we have

$$(4.7) \quad \frac{1}{2}E(s | II, n) + 1 = E(l | II, n).$$

Then, by analogy with (3.6), we note that the asymptotic density of $(s/\sqrt{n-1}, l/\sqrt{n-1})$ is

$$(4.8) \quad f(x, y) = e^{-\frac{1}{2}x^2}, \quad 0 \leq y \leq x < \infty,$$

giving the same marginal density functions as (3.7) and (3.8).

Now consider the probability distribution of the number of elements of X cyclical under T . We show that

$$(4.9) \quad P(q = j) = n^{n-j} D_j \binom{n-1}{j-1} / (n-1)^n,$$

where D_j is the j th derangement number, i.e., D_j is the nearest integer to $j!/e$, $j \neq 0$, and $D_0 = 1$.

The proof is identical with the proof of (3.12) except that the $j!$ in the numerator of (3.13) is replaced by D_j . Hence an application of Katz's lemma

gives

$$(4.10) \quad P(q = j) = \frac{1}{(n - 1)^n} \sum \frac{n!}{j!n_1!n_2! \cdots n_m!} D_j j^{n_1} n_1^{n_2} \cdots n_{m-1}^{n_m}, \quad j \neq n,$$

the sum being taken over all non-empty m -part partitions of $n - j$. Hence

$$(4.11) \quad P(q = j) = \frac{1}{(n - 1)^n} D_j \frac{n!n^{n-j-1}}{(j - 1)!(n - j)!}, \quad j \neq n,$$

and thus $P(q = j)$ is given by (4.9). The case $j = n$, is given trivially by (4.9).

The asymptotic distribution is obtained by replacing D_j by $j!/e$, and replacing factorials by Stirling's approximation. Then, letting $j = \sqrt{ny}$, we get

$$(4.12) \quad f(y) = y^{-1/2}, \quad 0 < y < \infty,$$

for the asymptotic density of qn^{-1} . The agreement of (4.12) with (3.7) can hardly be surprising in view of the agreement of (3.2) and (3.12).

We now obtain the distribution of p ,

$$(4.13) \quad \begin{aligned} P(p = q) &= \frac{(n - 1)!(n - j - 1)^{n-j} j^{j-2}}{(j - 1)!(n - j)!(n - 1)^{n-1}}, \quad j = 1, 2, \dots, n - 2, \\ P(p = n - 1) &= 0, \\ P(p = n) &= 1 - \sum_{j=1}^{n-2} P(p = j) = \frac{n!}{(n - 1)^n} \sum_{j=2}^n \frac{n^{n-j-2}}{(n - j)!}. \end{aligned}$$

This is established as follows. Define X_{j-1} , \mathfrak{J}_1 , \mathfrak{J}_2 , and \mathfrak{J}^* , as in case I. Let x be a distinguished element of X . Then, as before,

$$(4.14) \quad P(p = j) = \binom{n - 1}{j - 1} P(T \varepsilon \mathfrak{J}^*),$$

and

$$(4.15) \quad P(T \varepsilon \mathfrak{J}^*) = P(T \varepsilon \mathfrak{J}_1)P(T \varepsilon \mathfrak{J}_2).$$

Then

$$(4.16) \quad P(T \varepsilon \mathfrak{J}_1) = [(n - j - 1)/(n - 1)]^{n-j}.$$

Exactly as in (3.22), we can employ Katz's Lemma to obtain

$$(4.17) \quad P(T \varepsilon \mathfrak{J}_2) = j^{j-2}/(n - 1)^{j-1}.$$

Combining these we have

$$P(p = j) = \binom{n - 1}{j - 1} \left(\frac{n - j - 1}{n - 1} \right)^{n-j} \frac{j^{j-2}}{(n - 1)^{j-1}}.$$

The condition $Tx \neq x$ for all $x \in X$, precludes the possibility of $p = n - 1$. There remains the case $p = n$.

$$\begin{aligned}
 P(p = n | x) &= \sum_{j=2}^n P(q = j, K_T(x) = X, C_T(x) \neq 0) \\
 (4.18) \qquad &= \sum_{j=2}^n \frac{n^{n-j-1} D_j n!}{(n-1)^n (j-1)! (n-j)!} \cdot \frac{(j-1)!}{D_j} \cdot \frac{j}{n}
 \end{aligned}$$

Inasmuch as (3.23) depends only on invariance under the symmetric group operating on X and (3.26) is a property of the directed graphs in general, both of these apply in II.

The distribution of $p^{(1)}$ is obtained trivially,

$$(4.19) \qquad P(p^{(1)} = j) = \binom{n-1}{j} \left(\frac{1}{n-1}\right)^j \left(\frac{n-2}{n-1}\right)^{n-j}$$

The distribution of r , the number of components, has been computed by Folkert [3], and shown to be

$$\begin{aligned}
 (4.20) \qquad P(r = k) &= \frac{1}{(n-1)^n} \sum_{\mu=k}^{\lfloor n/2 \rfloor} \frac{S_\mu^k}{\mu!} \sum_{k_1, k_2, \dots, k_\mu} \\
 &\cdot \frac{n!}{k_1! k_2! \dots k_\mu!} (k_1 - 1)^{k_1} (k_2 - 1)^{k_2} \dots (k_\mu - 1)^{k_\mu},
 \end{aligned}$$

where the sum over k_1, k_2, \dots, k_μ is over all μ -tuples with $k_i > 1$ and $\sum_1^\mu k_i = n$.

We will now develop a probability generating function for the number of components, and obtain an alternate derivation of (4.20). The argument parallels the same discussion in Case I and hence will only be sketched briefly.

As in case I,

$$(4.21) \qquad P(T \in \mathfrak{S}_{k_1, k_2, \dots, k_n}) = \frac{n! I_2^{k_2} I_3^{k_3} \dots I_n^{k_n}}{2!^{k_2} 3!^{k_3} \dots n!^{k_n} k_2! k_3! \dots k_n! (n-1)^n},$$

where $I_j/(j-1)^j$ is the probability that a transformation T_j on j elements X_j is indecomposable, i.e., $K_{T_j}(x) = X_j$ for all $x \in X_j$, $0 \leq k_i \leq n$ and $\sum_{i=2}^n i k_i = n$.

$$\begin{aligned}
 (4.22) \qquad \frac{I_j}{(j-1)^j} &= \sum_{i=2}^j P(q = i, K_{T_j}(x) = X_j) = \sum_{i=2}^j \frac{j^{j-i-1} j!}{(j-1)^i (j-1)!} \\
 &= \sum_{i=0}^{j-2} \frac{(j-1)! j^i}{(j-1)^{i+1}}.
 \end{aligned}$$

(4.22) has previously been established by Katz [7] using a somewhat different argument. Then

$$(4.23) \qquad G(x_2, x_3, \dots, x_n) = \sum_{k_2, k_3, \dots, k_n} \frac{n! (I_2 x_2)^{k_2} (I_3 x_3)^{k_3} \dots (I_n x_n)^{k_n}}{2!^{k_2} 3!^{k_3} \dots n!^{k_n} k_2! k_3! \dots k_n! (n-1)^n}$$

is the generating function of k_2, k_3, \dots, k_n in the same manner as (3.32).

Since $r = \sum_{i=1}^n k_i$, we obtain, after extending the definition to $G(x_2, x_3, \dots)$,

$$(4.24) \quad G(x_2, x_3, \dots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x_i}{i!}$$

If, in (4.24), we replace x_i by x^i , we obtain

$$(4.25) \quad \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \frac{n!}{(n-1)^n} \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}.$$

Thus

$$(4.26) \quad \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} = \log \sum_{j=0}^{\infty} \frac{(j-1)^j x^j}{j!}.$$

Replacing x_i by $t^i x_i$ in (4.24), we obtain:

$$(4.27) \quad G(t^2 x_2, t^3 x_3, \dots) = \frac{n!}{(n-1)^n} \exp \sum_{i=2}^{\infty} \frac{I_i t^i x_i}{i!}.$$

Then the coefficient of t^n in (4.27) gives the probability of every possible decomposition of X into components, in the same manner as (3.37). If we replace x_i by tx^i , we get

$$(4.28) \quad G(tx^2, tx^3, \dots) = \frac{n!}{(n-1)^n} \exp \left[t \sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right]$$

or

$$(4.29) \quad G(tx^2, tx^3, \dots) = \frac{n!}{(n-1)^n} \left[\sum_{i=0}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^t,$$

giving

$$\text{coefficient of } t^k x^n \text{ in } G(tx^2, tx^3, \dots) = P\{r = k\}.$$

We now employ the generating function to obtain an alternate form of Folkert's formula (4.20). From (4.28) we have

$$(4.30) \quad \text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \left[\sum_{i=2}^{\infty} \frac{I_i x^i}{i!} \right]^k,$$

and from (4.26) we have

$$(4.31) \quad \text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \left[\log \left(1 + \sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!} \right) \right]^k.$$

Hence

$$\text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_{\mu}^k \left[\sum_{i=1}^{\infty} \frac{(i-1)^i x^i}{i!} \right]^{\mu},$$

and, as in (3.43), we get

$$(4.32) \quad \text{coefficient of } t^k = \frac{n!}{(n-1)^n k!} \sum_{\mu=k}^{\infty} \frac{k!}{\mu!} S_{\mu}^k \sum_{\substack{k_1, k_2, \dots, k_n \geq 0 \\ \sum_{i=1}^n k_i = \mu}} \frac{\mu!}{k_1! k_2! \dots k_n!} \left[\frac{(1-1)x^1}{1!} \right]^{k_1} \left[\frac{(2-1)^2 x^2}{2!} \right]^{k_2} \dots \left[\frac{(n-1)^n x^n}{n!} \right]^{k_n}$$

To find the coefficient of x^n in (4.32), it is sufficient to restrict the second sum to non-negative n -tuples (k_1, k_2, \dots, k_n) with $\sum_{i=1}^n k_i = \mu$ and $\sum_{i=1}^n i k_i = n$. Thus, we have

$$(4.33) \quad P(r = k) = \frac{n!}{(n-1)^n} \sum_{\mu=k}^{\lfloor n/2 \rfloor} S_{\mu}^k \sum_{k_2, k_3, \dots, k_n} \frac{1}{k_2! k_3! \dots k_n!} \left(\frac{1^2}{2!} \right)^{k_2} \left(\frac{2^3}{3!} \right)^{k_3} \dots \left(\frac{(n-1)^n}{n!} \right)^{k_n},$$

with the second sum restricted to $k_2, k_3, \dots, k_n \geq 0, \sum_{i=2}^n k_i = \mu$, and $\sum_{i=2}^n i k_i = n$ by the deletion of zero terms, and thus we have an alternate form of Folkert's formula.

5. Probability Distributions for Cases III and IV. In case III, we have $P(T) = n!^{-1}$, and in case IV, we have $P(T) = D_n^{-1}$. In both cases, every $x \in X$ is cyclic, since \mathfrak{J} is a collection of mappings which are one-to-one and onto in each case. As a consequence

$$(5.1) \quad S_T(x) = P_T(x) = C_T(x) = K_T(x).$$

Therefore, many of the probability distributions considered for cases I and II coincide in cases III and IV. Hence, we consider only the distributions of l and r . Then, in case III, we have

$$(5.2) \quad P(l = j) = \left[\binom{n}{j} (j-1)! j(n-j)! \right] / nn! = 1/n.$$

Gontcharoff [4] has shown that the probability that the number of components r of T is k is given by

$$(5.3) \quad P(r = k) = \text{coefficient of } t^k \text{ in } \frac{t(t+1) \dots (t+n-1)}{n!},$$

and therefore is given by the well-known result,

$$(5.4) \quad P(r = k) = |S_n^k|/n!,$$

which may be found in Riordan [8]. Gontcharoff [4] has also shown that the distribution of $(r - Er)/\sigma_r$ is asymptotically normally distributed with mean 0 and variance 1. Feller [2] and Greenwood [5] have also computed Er and σ_r^2 . We show an alternative computation using (5.2).

Let m_j be the number of components of T with exactly j elements. Then

$$(5.5) \quad Er = E \sum_{j=1}^n m_j = \sum_{j=1}^n Em_j.$$

From (5.2) we note that $Em_j = 1/j$, and hence

$$(5.6) \quad Er = \sum_{j=1}^n \frac{1}{j} \sim \log n + \gamma,$$

where γ is Euler's constant.

The variance has been shown to be

$$(5.7) \quad \sigma_r^2 = \sum_{j=1}^n \frac{1}{j} - \left(\sum_{j=1}^n \frac{1}{j^2} \right) \sim \log n + \gamma - \frac{\pi^2}{6}.$$

In case IV, we have

$$(5.8) \quad P(l = j) = \left[\binom{n}{j} (j-1)! j D_{n-j} \right] / [n D_n] \\ = \frac{(n-1)! D_{n-j}}{(n-j)! D_n}.$$

For large n ; and $j \geq 2$ and sufficiently small compared to n ,

$$(5.9) \quad P(l = j) \sim 1/n, \quad 2 \leq j.$$

Furthermore

$$P(l = n) \sim e/n \\ P(l = n - 1) = 0 \\ P(l = n - 2) \sim e/2n \\ P(l = n - 3) \sim e/3n.$$

To get the probability distribution of the number of components, we employ the same type of generating function used earlier. First we note that

$$(5.10) \quad P(K_r(x) = X) = (n-1)!/D_n$$

since all $(n-1)!$ n -cycles belong to \mathfrak{J} . From this, we obtain:

$$(5.11) \quad P(T \in \mathfrak{J}_{k_1, k_2, \dots, k_n}) = \frac{n! 1!^{k_2} 2!^{k_3} \dots (n-1)!^{k_n}}{D_n 2!^{k_2} 3!^{k_3} \dots n!^{k_n} k_2! k_3! \dots k_n!} \\ = \frac{n!}{D_n 2^{k_2} 3^{k_3} \dots n^{k_n} k_2! \dots k_n!}$$

where $0 \leq k_i \leq n$ and $\sum_{i=2}^n ik_i = n$. Then

$$\begin{aligned}
 (5.12) \quad G(x_2, x_3, \dots, x_n) &= \sum_{k_2, k_3, \dots, k_n} \frac{n!}{D_n} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \dots \left(\frac{x_n}{n}\right)^{k_n} \frac{1}{k_2! k_3! \dots k_n!} \\
 &= \sum_r \frac{n!}{D_n} \left(\frac{x_2}{2} + \frac{x_3}{3} + \dots + \frac{x_n}{n}\right)^r / r!,
 \end{aligned}$$

where $r = \sum_{i=2}^n k_i$. Proceeding as before, we have

$$(5.13) \quad G(x_2, x_3, \dots) = \frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i}.$$

Replacing x_i by x^i , we have

$$(5.14) \quad \frac{n!}{D_n} \exp \sum_{i=2}^{\infty} \frac{x_i}{i} = \frac{n!}{D_n} \sum_{j=0}^{\infty} \frac{D_j}{j!} x^j.$$

If we replace x_i by tx^i in (5.13), we obtain

$$(5.15) \quad G(tx^2, tx^3, \dots) = \frac{n!}{D_n} \exp t \sum_{i=2}^{\infty} \frac{x^i}{i} = \frac{n!}{D_n} \left[\sum_{j=0}^{\infty} \frac{D_j}{j!} x^j \right]^t.$$

Since

$$\sum_{i=2}^{\infty} \frac{x^i}{i} = -\log(1-x) - x,$$

we have

$$\begin{aligned}
 (5.16) \quad G(tx^2, tx^3, \dots) &= \frac{n!}{D_n} \exp^{-t(\log(1-x)+x)} \\
 &= \frac{n!}{D_n} \frac{e^{-tx}}{(1-x)^t}.
 \end{aligned}$$

Then

$$(5.17) \quad \text{coefficient of } t^k x^n \text{ in } G(tx^2, tx^3, \dots) = P(r = k).$$

From (5.15) we have

$$(5.18) \quad \text{coefficient of } t^k = \frac{n!}{D_n k!} \left(\sum_{i=2}^{\infty} \frac{x^i}{i} \right)^k,$$

and, expanding by the multinomial theorem, we get

$$\begin{aligned}
 (5.19) \quad &\text{coefficient of } t^k \\
 &= n! / D_n k! \sum_{\substack{k_2, k_3, \dots, k_n \geq 0 \\ \sum_{i=2}^n k_i = k \\ 1}} \frac{k!}{k_2! k_3! \dots k_n!} \left(\frac{x^2}{2}\right)^{k_2} \left(\frac{x^3}{3}\right)^{k_3} \dots \left(\frac{x^n}{n}\right)^{k_n},
 \end{aligned}$$

and thus

$$(5.20) \quad P(r = k) = \frac{n!}{D_n} \sum_{k_2, k_3, \dots, k_n} [k_2! k_3! \dots k_n! 2^{k_2} 3^{k_3} \dots n^{k_n}]^{-1},$$

the sum over all non negative $n - 1$ tuples k_2, k_3, \dots, k_n with $\sum_{i=2}^n k_i = k$ and $\sum_{i=2}^n ik_i = n$. Another form of the same result is obtained from (5.16). Here we have

$$(5.21) \quad \text{coefficient of } x^n = n!/D_n \sum_{j=0}^n \frac{(-1)^{n-j} t^{n-j} (-1)^{2j} t(t+1) \dots (t+j-1)}{(n-j)! j!}.$$

Hence, we find that the coefficient of $t^k x^n$ is

$$(5.22) \quad P(r = k) = n!/D_n \sum_{j=0}^n \frac{(-1)^{n+j} |S_j^{k-n+j}|}{(n-j)! j!},$$

or

$$(5.23) \quad P(r = k) = n!/D_n \sum_{j=0}^n \frac{(-1)^j |S_{n-j}^{k-j}|}{(n-j)! j!}.$$

Since

$$(5.24) \quad Er = \text{coefficient of } x^n \text{ in } (d/dt)G(tx^2, tx^3, \dots) \Big|_{t=1}$$

and

$$(5.25) \quad Er(r-1) = \text{coefficient of } x^n \text{ in } (d^2/dt^2)G(tx^2, tx^3, \dots) \Big|_{t=1},$$

then

$$(5.26) \quad Er = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(-\log(1-x) - x),$$

$$(5.27) \quad Er(r-1) = \text{coefficient of } x^n \text{ in } (n!/D_n)(e^{-x}/(1-x))(\log(1-x) + x)^2.$$

Expanding (5.26) and (5.27) in a power series we obtain

$$(5.28) \quad Er = \frac{n!}{D_n} \sum_{s=2}^n \frac{D_{n-s}}{s(n-s)!},$$

$$(5.29) \quad \begin{aligned} Er(r-1) &= \frac{n!}{D_n} \sum_{s=4}^n \frac{D_{n-s}}{(n-s)!} \sum_{\substack{j, k \geq 2 \\ j+k=s}} \frac{1}{jk} \\ &= \frac{n!}{D_n} \sum_{s=4}^n \frac{D_{n-s}}{(n-s)!} \sum_{j=2}^{s-2} \frac{1}{j(s-j)}. \end{aligned}$$

Since $D_j = (j!/e) + O(1)$,

$$Er \sim e \sum_{s=2}^n \frac{(n-s)!}{s(n-s)!} + O(1) \sim \sum_{s=2}^n \frac{1}{s} + O(1).$$

Hence

$$(5.30) \quad Er \sim \log n + O(1).$$

Similarly

$$(5.31) \quad \begin{aligned} Er(r-1) &= \frac{n!}{D_n} \sum_{s=4}^n \frac{D_{n-s}}{(n-s)!} \frac{1}{s} \sum_{j=2}^{s-2} \left[\frac{1}{j} + \frac{1}{s-j} \right] \\ &\sim e \sum_{s=4}^n \left[\frac{(n-s)! + O(1)}{e(n-s)!} \right] \frac{2}{s} \left[\log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right) \right] \end{aligned}$$

where γ is Euler's constant. Hence

$$Er(r-1) \sim \sum_{s=4}^n \frac{2}{s} \left[\log s - 1 - \frac{1}{s-1} - \frac{1}{s} + \gamma + O\left(\frac{1}{s}\right) \right] + O(1),$$

and

$$(5.32) \quad Er(r-1) \sim \log^2 n + 2(\gamma - 1) \log n + O(1).$$

Thus

$$(5.33) \quad \sigma_r^2 \sim (2\gamma - 1) \log n + O(1).$$

6. Miscellaneous Remarks. The problem of random mappings is of interest in various studies of human behaviors. We produce one such example. If we ask each of n individuals in a group to name his best friend from among the members of the group, the individual asked is the element x , and his choice Tx . In this case we have $Tx \neq x$, and the hypothesis of "randomness" leads to case II.

APPENDIX

Index of Notations Having a Fixed Meaning

(X, \mathfrak{J}, P) -random mapping space

X -a finite set of n elements

\mathfrak{J} -a set of transformations T of X into X

P -a probability measure over \mathfrak{J}

Case I-is set of all transformations of X into X , $P(T) = n^{-n}$ for each $T \in \mathfrak{J}$

Case II-is set of all transformations of X into X with $Tx \neq x$ for each $x \in X$,

$$P(T) = (n-1)^{-n} \text{ for each } T \in \mathfrak{J}$$

Case III-is set of all one-to-one mappings of X onto X , $P(T) = n!^{-1}$ for each $T \in \mathfrak{J}$

Case IV-is set of all one-to-one mappings of X onto X with $Tx \neq x$ for each $x \in X$, $P(T) = D_n^{-1}$, D_n is the n th derangement number for each $T \in \mathfrak{J}$

$S_T(x)$ -the set of all images of x in T .

$P_T(x)$ -the set of predecessors of x in T .

$C_T(x)$ -the cycle containing x .

$K_T(x)$ -the component containing x .

$s_T(x)$, s —the number of elements in $S_T(x)$.

$p_T(x)$, p —the number of elements in $P_T(x)$.

$l_T(x)$, l —the number of elements in cycle contained in $K_T(x)$.

$q_T(x)$, q —the number of cyclical elements of X in T .

r_T , r —the number of components of T .

S_n^k —Stirling's Numbers of the First Kind.

$\mathfrak{J}_{k_1, k_2, \dots, k_n}$ the subset of \mathfrak{J} with k_i components with exactly i elements,
 $i = 1, 2, \dots, n$.

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