# Probability of Reflection by a Random Laser

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A theory is presented (and supported by numerical simulations) for phase-coherent reflection of light by a disordered medium which either absorbs or amplifies radiation. The distribution of reflection eigenvalues is shown to be the Laguerre ensemble of random-matrix theory. The statistical fluctuations of the albedo (the ratio of reflected and incident power) are computed for arbitrary ratio of sample thickness, mean free path, and absorption or amplification length. On approaching the laser threshold all moments of the distribution of the albedo diverge. Its modal value remains finite, however, and acquires an anomalous dependence on the illuminated surface area.

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Recent experiments on turbid laser dyes [1–4] have drawn attention to the remarkable properties of disordered media which are optically active. The basic issue is to understand the interplay of phase-coherent multiple scattering and amplification (or absorption) of radiation. A quantity which measures this interplay is the albedo $a$, which is the power reflected by the medium divided by the incident power. A thick disordered slab which is optically passive has $a = 1$. Absorption leads to $a < 1$ and amplification to $a > 1$. As the amplification increases the laser threshold is reached, at which the average albedo becomes infinitely large [5]. Such a generator was referred to by its inventor Letokhov as a "laser with incoherent feedback" [6], because the feedback of radiation is provided by random scattering and not by mirrors—as in a conventional laser.

The current renewed interest in random lasers owes much to the appreciation that randomness is not the same as incoherence. Early theoretical work on this problem was based on the equation of radiative transfer [7], which ignores phase coherence. Zyuzin [8] and Feng and Zhang [9] considered interference effects on the average albedo $\overline{a}$, averaged over different configurations of the scattering centra. Their prediction of a sharpening of the backscattering peak in the angular distribution of the average reflected intensity has now been observed [3]. The other basic interference effect is the appearance of large, sample-specific fluctuations of the albedo around its average. These diverge faster than the average on approaching the laser threshold [10], so that $\overline{a}$ is no longer characteristic for the albedo of a given sample. In the present Letter we show that while all moments of the distribution function $P(a)$ of the albedo diverge at the laser threshold, its modal value $a_{\max}$ remains finite. The modal value is the value of $a$ at which $P(a)$ is maximal, and hence it is the most probable value measured in a single experiment. The diagrammatic perturbation theory of Refs. [8–10] can give only the first few moments of $a$, and hence cannot determine $a_{\max}$. Here we develop a nonperturbative random-matrix theory for the entire distribution of the reflection matrix, from which $P(a)$ can be computed directly.

We contrast the two cases of absorption and amplification. In the case of absorption, $P(a)$ is a Gaussian with a width $\delta a$ smaller than the average $\overline{a}$ by a factor $\sqrt{N}$, where $N \simeq S/\lambda^2 \gg 1$ is the number of modes associated with an illuminated area $S$ and wavelength $\lambda$. In the case of amplification, both $\delta a$ and $\overline{a}$ increase strongly on approaching the laser threshold—in a manner which we compute precisely. Below threshold, the mean and modal value of $a$ coincide. Above threshold, the mean is infinite while the modal value is found to be

$$a_{\max} = 1 + 0.8\gamma N . \tag{1}$$

Here $\gamma$ denotes the amplification per mean free path, assumed to be in the range $N^{-2} \ll \gamma \ll 1$. The existence of a finite $a_{\max}$ is due to the finiteness of the number of modes $N$ in a surface area $S$ (ignored in radiative transfer theory). Since $a_{\max}$ scales with $N$ and hence with $S$, and the incident power scales with $S$, it follows that the reflected power scales *quadratically* rather than *linearly* with the illuminated area. We suggest the name "superreflection" for this phenomenon. To measure the albedo in the unstable regime above the laser threshold we propose a time-resolved experiment, consisting of illumination by a short intense pulse to pump the medium beyond threshold, rapidly followed by a weak pulse to measure the reflected intensity before spontaneous emission has caused substantial relaxation.

Our work on this problem was motivated by a recent paper by Pradhan and Kumar [11] on the case $N = 1$ of a single-mode waveguide. We discovered the anomalous scaling with area in an attempt to incorporate the effects of mode coupling into their approach.

We consider the reflection of a monochromatic plane wave (frequency $\omega$, wavelength $\lambda$) by a slab (thickness $L$, area $S$) consisting of a disordered medium (mean free path $l$) which either amplifies or absorbs the radiation. We denote by $\sigma$ the amplification per unit length, a negative

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value of $\sigma$ indicating absorption. The parameter $\gamma = \sigma l$ is the amplification (or absorption) per mean free path. We treat the case of a scalar wave amplitude, and leave polarization effects for future study. A discrete number $N$ of scattering channels is defined by imbedding the slab in an optically passive waveguide without disorder (see Fig. 1, inset). The number $N$ is the number of modes which can propagate in the waveguide at frequency $\omega$. The $N \times N$ reflection matrix $r$ contains the amplitudes $r_{mn}$ of waves reflected into mode $m$ from an incident mode $n$. (The basis states of $r$ are normalized such that each carries unit power.) The reflection eigenvalues $R_n$ ($n = 1, 2, \ldots, N$) are the eigenvalues of the matrix product $rr^\dagger$. The matrix $r$ is determined by the $R_n$'s and by a unitary matrix $U$,

$$r_{mn} = \sum_k U_{mk} U_{nk} \sqrt{R_k}\,. \tag{2}$$

Note that $r_{mn} = r_{nm}$ because of time-reversal symmetry. From $r$ one can compute the albedo $a$ of the slab, which is the ratio of the reflected and incidental power:

$$a = \sum_m |r_{mn}|^2 = \sum_k U_{nk} U_{nk}^* R_k\,. \tag{3}$$

For a statistical description we consider an ensemble of slabs with different configurations of scatterers. As in earlier work on optically passive media [12], we make the isotropy assumption that $U$ is uniformly distributed in the unitary group. This assumption breaks down if the transverse dimension $W$ of the slab is much greater than its thickness $L$, but is expected to be reasonable if $W \lesssim L$. As a consequence of isotropy, $a$ becomes statistically independent of the index $n$ of the incident mode. We further assume that the wavelength $\lambda$ is much smaller than both the mean free path $l$ and the amplification length $1/\sigma$. The evolution of the reflection eigenvalues with increasing $L$ can then be described by a Brownian motion process. To describe this evolution it is convenient to use the parametrization

$$R_n = 1 + \mu_n^{-1}, \qquad \mu_n \in (-\infty, -1) \cup (0, \infty)\,. \tag{4}$$

The $L$ dependence of the distribution $P(\mu_1, \mu_2, \ldots, \mu_N)$ of the $\mu$'s is governed by the Fokker-Planck equation

$$l \frac{\partial P}{\partial L} = \frac{2}{N+1} \sum_{i=1}^{N} \frac{\partial}{\partial \mu_i} \mu_i (1 + \mu_i) \times \left[ \frac{\partial P}{\partial \mu_i} + P \sum_{j \neq i} \frac{1}{\mu_j - \mu_i} + \gamma (N+1) P \right], \tag{5}$$

with initial condition $\lim_{L \to 0} P = N \prod_i \delta(\mu_i + 1)$. In the single-channel case ($N = 1$), the term $\sum_{j \neq i}$ is absent and Eq. (5) reduces to the differential equation studied by Pradhan and Kumar [11,13]. The multichannel case is essentially different due to the coupling of the eigenvalues by the term $\sum_{j \neq i} (\mu_j - \mu_i)^{-1}$. This term induces a *repulsion* of closely separated eigenvalues. Equation (5) with $\gamma = 0$ is known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [14,15], and has been studied extensively in the context of electronic conduction [16]. We have generalized it to $\gamma \neq 0$, by adapting the approach of Ref. [15] to a nonunitary scattering matrix.

The average $\bar{a} \equiv \langle a \rangle$ and the variance $\mathrm{Var}\, a \equiv \langle (a - \bar{a})^2 \rangle$ of the albedo (3) can be computed by first averaging $U$ over the unitary group and then evaluating moments of the $R_k$'s by means of Eq. (5) [17]. In the limit $N \to \infty$ we obtain the differential equations

$$l \frac{d}{dL} \bar{a} = (\bar{a} - 1)^2 + 2\gamma \bar{a}, \tag{6a}$$

$$l \frac{d}{dL} \mathrm{Var}\, a = 4(\bar{a} - 1 + \gamma) \mathrm{Var}\, a + 2N^{-1} \bar{a} (\bar{a} - 1)^2. \tag{6b}$$

Corrections are smaller by a factor $|\gamma N^2|^{-1/2}$, which we assume to be $\ll 1$. Equation (6a) for the average albedo is an old result of radiative transfer theory [18]. Equation (6b) for the variance is new. It describes the sample-specific fluctuations of the albedo due to interference of multiply scattered waves. Integration of Eq. (6) yields

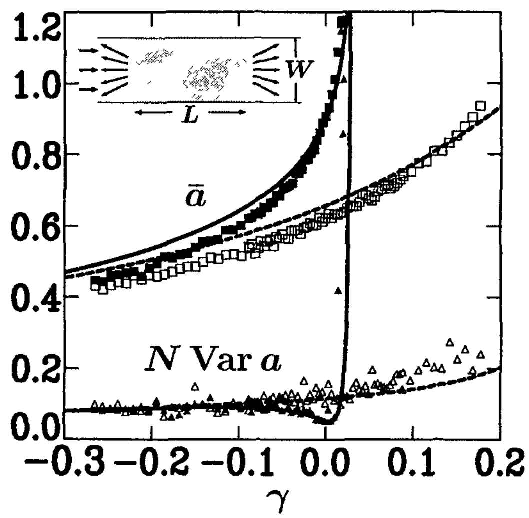


FIG. 1. Comparison between theory and simulation of the average albedo $\bar{a}$ (upper curves, squares) and Var $a$ (lower curves, triangles) for $L/l = 1.92$ (dashed curves, open markers) and $L/l = 9.58$ (solid curves, filled markers). Negative $\gamma$ corresponds to absorption, positive $\gamma$ to amplification. The curves are the theoretical result (7). The data points are a numerical simulation of a two-dimensional lattice ($L = 50d$ and $250d$, $W = 51d$, $N = 21$), averaged over 100 realizations of the disorder. The inset shows schematically the system under consideration.

$$\bar{a} = 1 - \gamma + (2\gamma - \gamma^2)^{1/2} \tan t, \tag{7a}$$

$$\text{Var } a = (8N \cos^4 t)^{-1}\{4\gamma(1 - 2\gamma)L/l + 2\gamma(1 + \gamma) - 4\gamma^2 \cos 2t + 2\gamma(1 - \gamma)\cos 4t + (2 - \gamma)^{-1}(2\gamma - \gamma^2)^{1/2}[4\gamma(1 - \gamma)\sin 2t - (1 - 4\gamma + 2\gamma^2)\sin 4t]\}. \tag{7b}$$

We have abbreviated $t = (2\gamma - \gamma^2)^{1/2}L/l - \arcsin(1 - \gamma)$.

Plots of Eq. (7) as a function of $\gamma$ are shown in Fig. 1, for two values of $L/l$. (The data points are numerical simulations, discussed later.) In the case of absorption ($\gamma < 0$), the large-$L$ limit

$$\bar{a}_\infty = 1 - \gamma - (\gamma^2 - 2\gamma)^{1/2}, \tag{8a}$$

$$\text{Var } a_\infty = \frac{1}{2N} \frac{\bar{a}_\infty(1 - \bar{a}_\infty)^2}{1 - \gamma - \bar{a}_\infty}, \tag{8b}$$

can be obtained directly from Eq. (6) by equating the right-hand side to zero. The limit (8) is reached when $L/l \gg (\gamma^2 - 2\gamma)^{-1/2}$. In the case of amplification ($\gamma > 0$), Eq. (7) holds for $L$ smaller than the critical length

$$L_c = l(2\gamma - \gamma^2)^{-1/2} \arccos(\gamma - 1) \tag{9}$$

at which $\bar{a}$ and Var $a$ diverge. This is the laser threshold [5,18]. For $\gamma < 0$ the large-$L$ limit of the probability distribution $P(a)$ of the albedo is well described by a Gaussian, with mean and variance given by Eq. (8). (The tails are non-Gaussian, but carry negligible weight.) The modal value $a_{\max}$ of the albedo equals $\bar{a}$. For $\gamma > 0$ the large-$L$ limit of $P(a)$ cannot be reconstructed from its moments, but needs to be determined directly. We will see that while $\bar{a}$ diverges, $a_{\max}$ remains finite.

The large-$L$ limit $P_\infty(\mu_1, \mu_2, \ldots, \mu_N)$ of the distribution of the $\mu$'s is obtained by equating to zero the expression between square brackets in Eq. (5). The result is

$$P_\infty = C \prod_i \exp[-\gamma(N + 1)\mu_i] \prod_{i<j} |\mu_j - \mu_i|, \tag{10}$$

with $C$ a normalization constant. Equation (10) holds for both positive and negative $\gamma$, but the support of $P_\infty$ depends on the sign of $\gamma$: All $\mu$'s have to be $>0$ for $\gamma > 0$ (amplification) and $<-1$ for $\gamma < 0$ (absorption). In what follows we take $\gamma > 0$. The function (10) is known in random-matrix theory as the distribution of the *Laguerre ensemble* [19]. The density $\rho(\mu) = \langle \sum_i \delta(\mu - \mu_i) \rangle$ of the $\mu$'s is a series of Laguerre polynomials, hence the name. For $\gamma N^2 \gg 1$ one has asymptotically

$$\rho(\mu) = (N/\pi)(2\gamma/\mu - \gamma^2)^{1/2}, \quad 0 < \mu < 2/\gamma. \tag{11}$$

The square-root singularity at $\mu = 0$ is cut off in the exact density [20], such that $\rho = \gamma N^2$ if $\mu \lesssim 1/\gamma N^2$. The cumulative density is plotted in Fig. 2, together with the numerical simulations (discussed below).

We seek the probability distribution of the albedo

$$P(a) = \left\langle \delta\left(a - 1 - \sum_k U_{nk} U_{nk}^* \mu_k^{-1}\right)\right\rangle. \tag{12}$$

The average $\langle \cdots \rangle$ consists of the average of $U$ over the unitary group followed by the average of the $\mu_k$'s over the Laguerre ensemble. The averages can be done analytically for $N^{-2} \ll \gamma \ll 1$ (in the continuum approximation [21], i.e., by ignoring the discreteness of the eigenvalues), and numerically for any $N$, $\gamma$ (by Monte Carlo integration, i.e., by randomly sampling the Laguerre ensemble).

The analytical result is an inverse Laplace transform,

$$P(a) = \frac{1}{2\gamma N} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \exp[\tfrac{1}{2} s(a - 1)/\gamma N - 2f(s)] \times [1 + \tfrac{1}{4} f(s)]^2, \tag{13a}$$

where $f$ is an implicit function of the Laplace variable $s$:

$$(s - \tfrac{1}{2} f + \tfrac{1}{2}\sqrt{4f + f^2})^{-1/2} + 2(f - \sqrt{4f + f^2})^{-1} + 1 = 0. \tag{13b}$$

The continuum approximation (13) is plotted in the inset of Fig. 3 (dashed curve). It is close to the exact numerical

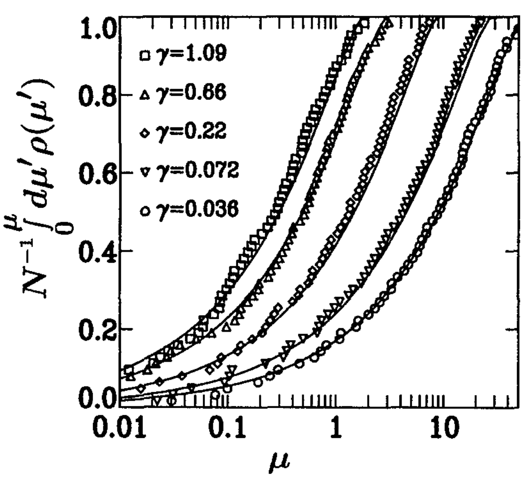


FIG. 2. Comparison between theory and simulation of the cumulative density of the variables $\mu_n$ (related to the reflection eigenvalues by $R_n = 1 + \mu_n^{-1}$). Curves are computed from the density (11) of the Laguerre ensemble; data points are from the simulation ($L = 500d = 19.2l$, $W = 151d$, $N = 63$), for a single realization of the disorder.

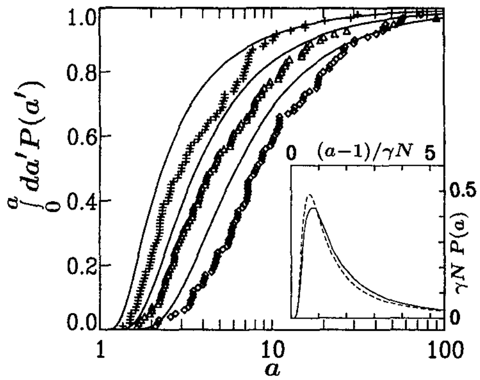


FIG. 3. Comparison between theory and simulation of the cumulative probability distribution of the albedo ($L = 500d = 19.2l$, $\gamma = 0.07$). Solid curves are obtained by numerically averaging over the Laguerre ensemble; data points are the results of the simulation, averaged over 100 realizations of the disorder. The three sets of data are for $W = 25d$, $N = 10$ (plusses), $W = 51d$, $N = 21$ (triangles), and $W = 101d$, $N = 42$ (diamonds). The inset compares the continuum approximation (13) for $P(a)$ (dashed) with the exact large-$N$ limit of the Laguerre ensemble (solid).

large-$N$ result (solid curve). The modal value $a_{\rm max}$ of the albedo is given by Eq. (1). The distribution $P(a)$ drops off $\propto \exp[-2\gamma N/(a - 1)]$ for smaller $a$ and $\propto a^{-5/3}$ for larger $a$, so that all moments diverge.

To test these predictions of random-matrix theory on a model system, we have carried out numerical simulations of the analogous electronic Anderson model with a complex scattering potential, using the recursive Green's function technique [22]. The disordered medium is modeled by a two-dimensional square lattice (lattice constant $d$, length $L$, width $W$). The relative dielectric constant $\varepsilon = \varepsilon_1 + i\varepsilon_2$ (relative to the value outside the disordered region) has a real part $\varepsilon_1$ which fluctuates from site to site between $1 \pm \delta\varepsilon$, and a nonfluctuating imaginary part $\varepsilon_2$. The multiple scattering of a scalar wave $\psi$ (wave number $k = 2\pi/\lambda$) is described by discretizing the Helmholtz equation $(\nabla^2 + k^2\varepsilon)\Psi = 0$. The mean free path $l$ which enters in Eq. (5) is obtained from the average albedo $\overline{a} = (1 + l/L)^{-1}$ without amplification ($\varepsilon_2 = 0$). We choose $k^2 = 1.5d^{-2}$, $\delta\varepsilon = 1$, leading to $l = 26.1d$. The parameter $\sigma$ (and hence $\gamma = \sigma l$) is obtained from the analytical solution of the discretized Helmholtz equation in the absence of disorder ($\delta\varepsilon = 0$). The complex longitudinal wave number $k_n$ of transverse mode $n$ then satisfies the dispersion relation

$$\cos(k_n d) + \cos(n\pi d/W) = 2 - \tfrac{1}{2}(kd)^2(1 + i\varepsilon_2), \tag{14}$$

and leads to $\sigma = -2N^{-1}\mathrm{Im}\sum_n k_n$. The albedo (3) is computed for normal incidence. Data points in Figs. 1–3 are the numerical results. The agreement with the curves from random-matrix theory is quite remarkable, given that there are *no adjustable parameters.*

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[1] N. M. Lawandy, R. M. Balachandran, A. S. L. Gomes, and E. Sauvain, Nature (London) **368**, 436 (1994).
[2] W. L. Sha, C.-H. Liu, and R. R. Alfano, Opt. Lett. **19**, 1922 (1994).
[3] D. S. Wiersma, M. P. van Albada, and A. Lagendijk, Phys. Rev. Lett. **75**, 1739 (1995).
[4] For a critical discussion of random lasers, see the article by A. Z. Genack and J. M. Drake [Nature (London) **368**, 400 (1994)] and the correspondence between the authors of Refs. [1] and [3] [Nature (London) **373**, 203 (1995)].
[5] V. S. Letokhov, Pis'ma Zh. Eksp. Teor. Fiz. **5**, 262 (1967) [JETP Lett. **5**, 212 (1967)].
[6] V. S. Letokhov, Zh. Eksp. Teor. Fiz. **53**, 1442 (1967) [Sov. Phys. JETP **26**, 835 (1968)]; N. N. Lavrinovich and V. S. Letokhov, *ibid.* **67**, 1609 (1974) [*ibid.* **40**, 800 (1975)].
[7] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
[8] A. Yu. Zyuzin, Europhys. Lett. **26**, 517 (1994); Pis'ma Zh. Eksp. Teor. Fiz. **61**, 961 (1995) [JETP Lett. **61**, 990 (1995)].
[9] S. Feng and Z.-Q. Zhang (to be published).
[10] A. Yu. Zyuzin, Phys. Rev. E **51**, 5274 (1995).
[11] P. Pradhan and N. Kumar, Phys. Rev. B **50**, 9644 (1994).
[12] P. A. Mello, E. Akkermans, and B. Shapiro, Phys. Rev. Lett. **61**, 459 (1988).
[13] The $N = 1$ limit of Eq. (5) was actually obtained as early as 1959 in the course of a radio-engineering problem: M. E. Gertsenshtein and V. B. Vasil'ev, Teor. Veroyatn. Primen. **4**, 424 (1959); **5**, 3(E) (1960) [Theor. Probab. Appl. **4**, 391 (1959); **5**, 340(E) (1960)].
[14] O. N. Dorokhov, Pis'ma Zh. Eksp. Teor. Fiz. **36**, 259 (1982) [JETP Lett. **36**, 318 (1982)].
[15] P. A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988).
[16] C. W. J. Beenakker, Mod. Phys. Lett. B **8**, 469 (1994).
[17] This "method of moments" was developed for the DMPK equation by P. A. Mello and A. D. Stone, Phys. Rev. B **44**, 3559 (1991).
[18] A. C. Selden, Opt. Commun. **10**, 1 (1974).
[19] B. V. Bronk, J. Math. Phys. (N.Y.) **6**, 228 (1965).
[20] T. Nagao and K. Slevin, J. Math. Phys. (N.Y.) **34**, 2317 (1993).
[21] F. J. Dyson, J. Math. Phys. (N.Y.) **3**, 157 (1962).
[22] H. U. Baranger, D. P. DiVincenzo, R. A. Jalabert, and A. D. Stone, Phys. Rev. B **44**, 10 637 (1991).