Problems and results on judicious partitions

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Abstract

We present a few results and a larger number of questions concerning partitions of graphs or hypergraphs, where the objective is to maximize or minimize several quantities simultaneously. We consider a variety of extremal problems; many of these also have algorithmic counterparts.

1 Introduction

Many classical partitioning problems in combinatorics ask for a single quantity to be maximized or minimized over a set of partitions of a combinatorial object. For instance, Max Cut asks for the largest bipartite subgraph of a graph G or, equivalently, the minimum of $e(V_1) + e(V_2)$, over partitions $V(G) = V_1 \cup V_2$; Max Bisection asks for the maximum of $e(V_1, V_2)$ over partitions with $|V_1| \leq |V_2| \leq |V_1| + 1$. Note that there are two problems here: the extremal problem of determining the largest or smallest subgraph we can guarantee given certain graph parameters, such as the number of edges; and the algorithmic problem of finding efficient procedures to produce such a subgraph.

In this paper we shall be concerned with partitioning problems where we seek to maximize or minimize several quantities simultaneously. We shall refer to such problems as *judicious partitioning problems*. For instance, given

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a graph G, we can ask for the minimum of $\max\{e(V_1), \ldots, e(V_k)\}$ over partitions $V(G) = \bigcup_{i=1}^k V_i$. (Note that this contrasts to Max k-Cut, where we ask for the minimum of $\sum_{i=1}^k e(V_i)$; we can therefore think of this judicious partitioning problem as minimizing the l_∞ norm rather than the l_1 norm of $(e(V_i))_{i=1}^k$.) This problem has been extensively investigated by Porter ([35], [36], [37]), Porter and Bing Yang [38], Shahrokhi and Székely [40], Porter and Székely [39], and in [11] and [17]; optimal bounds in terms of $e(G)$ are given in [13] (see also section 2 below). Another example of a judicious partitioning problem is that of partitioning a set: given a set S of size n, find a partition $S = \bigcup_{i=1}^k S_i$ with $\max_{1\le i\le k} |S_i|$ as small as possible. Of course this is trivial, but the weighted version of the problem is the NP-hard Bin Packing problem (for extremal results see [46]). Both of these problems ask for partitions in which each vertex class has "small weight". One of the precursors of the judicious partitioning problem is the conjecture of Bollobás and Thomason, which asserts that every r-uniform hypergraph with m edges has a vertex partition into r classes such that every class meets at least $rm/(2r-1)$ edges (see [10], [14]). We return to this below.

In previous papers ([11], [12], [14], [13], [17]) we have examined a variety of problems concerning judicious partitions. Usually the arguments involve a mixture of probabilistic and extremal methods. For instance, we often generate a partition by first partitioning a small number of "difficult" vertices deterministically and then extending that to a full partition in a probabilistic manner. A frequent feature of judicious partitioning problems is that stronger results hold for large graphs or hypergraphs than for small ones. For instance, every graph with m edges has a bipartition in which each vertex class contains at most $m/3$ edges, and K_3 shows that this is best possible. However, K_3 is the unique extremal graph: in fact, every graph with m edges has a bipartition in which each vertex class contains $(1 + o(1))m/4$ edges. It is therefore sometimes sensible to ask both for a simple extremal bound that should hold for all graphs and for the asymptotic extremal behaviour for large graphs.

The aim of this paper is to present a number of problems and conjectures concerning judicious partitions. In section 2 we are concerned with judicious partitioning problems for graphs; in section 3 we turn to hypergraph problems.

Unless otherwise indicated, throughout the paper G will denote a graph with n vertices, m edges and maximum degree Δ.

2 Partitioning graphs

We begin by considering partitions of a graph such that no vertex class contains too many edges. In a random partition of a graph G into k sets $V_1, \ldots, V_k$, we expect $e(G)/k^2$ edges in each set. This gives an immediate bound for Max k-Cut: $\mathbb{E} \sum_{i=1}^{k} e(V_i) = e(G)/k$, so there is a partition with $\sum_{i=1}^{k} e(V_i) \leq e(G)/k$. However, the quantities $e(V_1), \ldots, e(V_k)$ are not independent, and so this tells us nothing about $\max\{e(V_1), \ldots, e(V_k)\}$. Indeed, it is not at all obvious that, given a graph with m edges, there is a partition into k sets, each of which contains at most $(1+o(1))m/k^2$ edges (the sceptical reader may like to try the exercise of proving that, for large enough m, every graph with m edges has a bipartition in which each class contains at most $5m/16$ edges).

It was proved in [13] that every graph with m edges has a bipartition in which each vertex class contains at most

$$\frac{m}{4} + \sqrt{\frac{m}{32} + \frac{1}{256}} - \frac{1}{16} \tag{1}$$

edges; more generally, there is a vertex partition into k vertex classes such that every class contains at most

$$\frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right) \tag{2}$$

edges. These bounds are sharp, as can be seen by considering complete graphs on $kn + 1$ vertices.

In fact, for bipartite graphs we can demand slightly more than (1). Edwards ([20], [21]) proved that every graph G with m edges has a bipartition $V(G) = V_1 \cup V_2$ with

$$e(V_1, V_2) \geq \left\lceil \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8} \right\rceil; \tag{3}$$

this bound is sharp for complete graphs (though for extensions see [2], [4], [15]). It was shown in [13] that every graph G has a bipartition such that both vertex classes satisfy (1) and the bipartition also satisfies (3). It would be interesting to know whether there is an equivalent result for the k-partite case. The analogue of (3) for k-cuts is the assertion (proved in [15]) that every graph with m edges has a k-cut of size at least

$$\left(1-\frac{1}{k}\right)m+\frac{k-1}{2k}\sqrt{2m+\frac{1}{4}}+O(k^2). \tag{4}$$

We are led to the following problem.

Problem 1. *Does every graph with m edges have a partition into k sets such that every vertex class satisfies* (2) *and the k-cut they define has size at least* (4)*?*

As a possibly easier question, can we achieve a k-cut of at least average size: in other words, can we demand that (2) is satisfied for each vertex class and $\sum_{i=1}^{k} e(V_i) \leq m/k$? As noted in [13], when k is a power of 2 this follows from the existence of a partition satisfying (1) and (3).

The bound (3) of Edwards is sharp for complete graphs. However, Erdős conjectured that (3) can be arbitrarily far from the correct value. Alon [2] showed that (3) can be improved by $cm^{1/4}$ when m is about halfway between $\binom{n}{2}$ and $\binom{n+1}{2}$, while considering unions of complete graphs shows that it is never out by more than $O(m^{1/4})$; in fact, the optimal bound in (3) is known to within a constant for every m (see [4], [15]). It seems likely that the optimal bound in (1) exhibits a similar behaviour. Perhaps it would be possible to determine the optimal value for all sufficiently large m. For $m \geq 1$, let $g(m)$ be the smallest integer such that every graph with m edges has a bipartition in which each vertex class contains at most $g(m)$ edges. A first step in determining the behaviour of $g(m)$ more precisely than (1) would be the following.

Conjecture 2. $m/4+\sqrt{m/32}-g(m)$ *is unbounded as* $m \to \infty$.

A good starting point here might be graphs with $m = \binom{n}{2}+\binom{k}{2}$ edges, where $0 \leq \binom{k}{2} < n$ and n is large. Such graphs are known to contain a bipartite subgraph with at least

$$\min\left\{\left\lfloor\frac{(n+1)^2}{4}\right\rfloor, \left\lfloor\frac{n^2}{4}\right\rfloor+\left\lfloor\frac{k^2}{4}\right\rfloor\right\}$$

edges, which would be best possible, as can be seen by considering $K_n \cup K_k$, and K_{n+1} with $\binom{n+1}{2}-\binom{n}{2}-\binom{k}{2}$ edges deleted. Can anything similar be said for judicious partitions?

In general, it is not possible to improve (1) beyond $(1+o(1))m/4$, as can be seen by considering complete graphs. However, for some classes of graphs it is possible to do better. Alon, Bollobás, Krivelevich and Sudakov [3] have shown that if G has a large cut then G also has a judicious partition that is better than (1). In particular, suppose that G is a graph with m edges and largest cut of size $m/2+\delta$. If $\delta \le m/30$ then there is a bipartition $V(G) = V_1 \cup V_2$ with

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m},$$

while if $\delta \ge m/30$ then there is a bipartition with

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - \frac{m}{100}.$$

They used this result to demonstrate the existence of good judicious partitions in graphs without short cycles. Large cuts in graphs without short cycles had been investigated previously by various authors, including Erdős [23], Erdős and Lovász (see [24], [22]), Shearer [41], Poljak and Tuza [33] and Alon [2]. Alon, Bollobás, Krivelevich and Sudakov [3] showed that for every integer $r \ge 4$ there is $c_r > 0$ such that every graph G with $m > 0$ edges and girth at least r has a bipartition $V(G) = V_1 \cup V_2$ with

$$\max\{e(V_1), e(V_2)\} \le \frac{m}{4} - c_r m^{r/(r+1)}.$$

We can improve on (1) when our graphs have bounded degree. For example, it is shown in [17] that if G has m edges and maximum degree d, where d is odd, then there is a bipartition $V(G) = V_1 \cup V_2$ such that, for $i = 1, 2$,

$$e(V_i) \le \frac{d-1}{4d} m + \frac{d-1}{4} \tag{5}$$

and, in addition,

$$e(V_1, V_2) \ge \frac{d+1}{2d} m; \tag{6}$$

the extremal graphs are $(2s+1)K_d \cup tK_{d+1}$, for $s, t \ge 0$. (Note that this immediately implies a result for graphs with even maximal degree d, with extremal graphs $(2s+1)K_{d+1}$.)

Stronger inequaltities than (6) hold for regular graphs. Staton [44] and Locke [31] showed that every cubic graph G containing no K_4 has a cut of size at least $7e(G)/9$, with equality holding for infinitely many connected graphs. Hopkins and Staton [29] showed that if G is cubic and triangle-free then G has a cut of size at least $4e(G)/5$. This was extended by Bondy and Locke [18], who showed that every triangle-free graph G with maximal degree at most 3 has a cut of size at least $4e(G)/5$: the only triangle-free cubic graphs for which equality holds are the Petersen graph and the dodecahedron.

For regular graphs, it is also possible to improve on (5). Locke [31] showed that every cubic graph G that does not contain K_4 has a partition $V(G) = V_1 \cup V_2$ into two sets of equal size such that $e(V_1, V_2) \geq 11e(G)/15$. Since such a partition has $e(V_1) = e(V_2)$, it follows that $\max\{e(V_1), e(V_2)\} \leq 2e(G)/15$. By considering copies of K_4 separately, it follows that any cubic graph G has a cut in which each vertex class contains at most $e(G)/6$ edges. More generally, it was shown in [17] that, for d odd, every d-regular graph G has a bipartition in which each vertex class contains at most $(d-1)e(G)/4d$ edges; if d is even then there is a bipartition in which which each class contains at most $de(G)/4(d+1)$ edges is $|G|$ is even or $de(G)/4(d+1)+d/4$ edges if $|G|$ is odd.

If G is cubic and has a maximum cut $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2|$, then $V_1 \cup V_2$ is clearly an optimal judicious partition. However, any maximum cut of the Petersen graph P (which has 15 edges and maximum cut size $4e(P)/5 = 12$) has one vertex class of size 4 and one class of size 6. Thus the smaller vertex class is an independent set and the larger class contains three edges; so any maximum cut of the Petersen graph is some way from being a good judicious partition. It would be interesting to know what other graphs have a maximum cut that is a bad judicious partition. Note that if G is cubic and $V(G) = V_1 \cup V_2$ is a maximum cut then $G[V_2]$ has maximum degree at most 1. If V_1 is an independent set then $e(G) = e(V_1, V_2) + e(V_2)$ and $e(V_1, V_2) \geq 2|V_2| \geq 4e(V_2)$, and so $e(V_1, V_2) \geq 4e(G)/5$. Therefore if G has a maximum cut with one vertex class independent then it must have maximum cut size at least $4e(G)/5$.

Problem 3. *What cubic graphs G have a maximum cut size $4e(G)/5$ and have a maximum cut such that one vertex class is an independent set?*

Note that if $V_1 \cup V_2$ is a maximum cut such that $e(V_1) = 0$ and $e(V_1, V_2) = 4e(G)/5$ then $G[V_2]$ consists of a matching. Replacing each $v \in V_2$ with an edge between its neighbours in V_1 gives a cubic graph C on V_1. We can recover G from C by subdividing each edge once, and adding a suitable matching between the subdividing vertices. For instance, starting with K_4, if we subdivide each edge once and add a matching between the subdividing vertices on pairs of nonincident edges then we obtain the Petersen graph!

We now turn to judicious partitions in which we consider more than one vertex class at a time. It was shown in [11] that every graph G with m edges has a partition $V(G) = \bigcup_{i=1}^{k} V_i$ with

$$\max\{e(V_1), \dots, e(V_k)\} \le e(G)/\binom{k+1}{2}. \tag{7}$$

The unique extremal graph is K_{k+1}, but we know from (2) that we can do much better if m is large, when we can demand

$$\max\{e(V_1), \dots, e(V_k)\} \le (1 + o(1))m/k^2.$$

However, perhaps we can add additional terms on the left hand side of (7). For instance, with $k = 2$, consider a partition $V(G) = V_1 \cup V_2$ with $e(V_1, V_2)$ maximal. Every vertex $v \in V_1$ has $|\Gamma(v) \cap V_2| \ge |\Gamma(v) \cap V_1|$, or else moving v from V_1 to V_2 increases the size of the cut. Summing over V_1 we obtain $2e(V_1) \le e(V_1, V_2)$, and so

$$3e(V_1) + e(V_2) \le e(V_1) + e(V_1, V_2) + e(V_2) = m.$$

By a similar argument

$$e(V_1) + 3e(V_2) \le m.$$

Note that this gives an improvement on (7) (and that, in addition, we have $e(V_1) + e(V_2) \le m/2$, and so $e(V_1, V_2) \ge m/2$). It would be interesting to prove results of this type for $k > 2$.

Problem 4. *For $k \ge 3$, what is the largest $c(k)$ such that, for every graph G with m edges there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ with*

$$\binom{k+1}{2} e(V_i) + c(k) \sum_{j \ne i} e(V_j) \le m$$

for $1 \le i \le k$?

By considering large complete graphs, it is clear that $c(k) \le k/2$. Can this bound be achieved?

Another possible extension to (2) and (3) comes from considering unions of more than one vertex class. For instance, given $k > 2$, what is the smallest $f(m)$ such that every graph G with m edges has a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that $e(V_i \cup V_j) \le f(m)$ whenever $i \ne j$? Considering $K_{1,n}$ and K_{k+2} leads to the following conjecture.

Conjecture 5. *For $k \geq 2$, every graph G with m edges and n vertices has a partition into k sets $V_1, \ldots V_k$, such that*

$$\max_{1 \leq i < j \leq k} e(V_i \cup V_j) \leq 6m / \binom{k+2}{2} + O(n). \tag{8}$$

Perhaps there is a constant c_k such that if $\delta(G) \geq c_k n$ then (8) holds without the $O(n)$ term.

Similarly, considering K_{kn+2} suggests that a possible analogue for (2) when we take unions of two classes is

$$\frac{4}{k^2}m + \frac{3(k-2)}{k^2}\sqrt{2m + \frac{1}{4}} + \frac{(k-2)(2k-5)}{10} + O(n). \tag{9}$$

Note that we would expect (8), like (2), to be a good bound only for small graphs; for larger graphs (9) would be a much better bound. Of course, the same questions can be asked when we consider the union of $t \geq 3$ classes. Once again, natural conjectures are suggested by considering complete graphs and complete bipartite graphs. More ambitiously, it would be interesting to have simultaneous bounds on $\max_{1 \leq i \leq k} e(V_i)$ and $\max_{1 \leq i < j \leq k} e(V_i \cup V_j)$. Is it possible that (2) and (8) or (9) could be satisfied simultaneously? What about bounds on unions of all different sizes?

So far we have been concerned only with upper bounds. We now consider partitions $V(G) = \bigcup_{i=1}^{k} V_i$ in which $e(V_i)$ is bounded above and below. For instance, when can we demand a partition with

$$e(V_i) = (1 + o(1))\frac{m}{k^2} \tag{10}$$

for $1 \leq i \leq k$? An easy bound comes from considering the vertex exposure martingale. Colouring each vertex in turn, we see that $\mathbb{E}e(V_i) = m/k^2$, while a vertex v affects $e(V_i)$ by at most $d(v)$. The Azuma-Hoeffding inequality (see [5], [28], [32], [8], [9]) implies

$$\mathbb{P}(|e(V_i) - \frac{m}{k^2}| > t) \leq 2\exp(-t^2/2\sum_{j=1}^{n} d_j^2). \tag{11}$$

Now $\sum d_j^2 \leq 2\Delta m$, so as long as $t > (2\Delta m \log 2k)^{1/2}$ we see that there is a positive probability that every vertex class satisifies $|e(V_i) - m/k^2| \leq t$. So if $\Delta = o(m)$ then there is a partition satisfying (10).

How far can the restriction $\Delta(G) = o(\sqrt{n})$ be weakened? The star $K_{1,n}$ has n edges while every bipartition has a vertex class with no edges. Similarly, for $\epsilon > 0$ and $t > (1+\epsilon)3n/5$, consider a star on t vertices together with $(n-t)/2$ independent edges: every bipartition has a vertex class with at most $(n-t)/2$ edges, which is at most $(1-17\epsilon/10)m/4 + O(1)$. Clearly, $\delta(G) = \omega(n)$ is enough for (10) to hold, but how close do we get when δ is fixed? Perhaps (10) is true provided δ is sufficently large. For instance, does (10) hold when $\delta = 2$? Note that the graphs that cannot be dealt with by the martingale argument above have $O(n)$ edges and at least on vertex of large degree.

A slightly different class of judicious partitioning problems arises when we restrict the set of partitions we consider. We say that a partition $V(G) = \bigcup_{i=1}^k V_i$ is *balanced* if $|V_1| \leq \cdots \leq |V_k| \leq |V_1|+1$. The classical Min k-Section and Max k-Section problems ask for the minimum and maximum size respectively of k-cuts generated by balanced partitions into k sets. The analogue of this for judicious partitions asks for the minimum of $\max\{e(V_1), \ldots, e(V_k)\}$ over balanced partitions $V(G) = \bigcup_{i=1}^k V_i$. Considering $K_{1,n}$, we see that we cannot expect to do better than $m/k + O(1)$, while considering random partitions shows that we can achieve a bound of this type. However, increasing the minimal degree should improve the constant. For instance, the following conjecture may not be too difficult.

Conjecture 6. *Every graph with m edges and minimal degree at least 2 has a balanced bipartition with at most $m/3$ edges in each vertex class.*

More generally, under what conditions can we guarantee a balanced bipartition in which each class contains at most $(1+o(1))m/4$ edges? Perhaps $\delta(G) \to \infty$ as $n \to \infty$ is enough, or else $\Delta(G) = o(n)$.

Problem 7. *What is the smallest constant $c(k)$ such that every graph G with minimal degree at least k has a balanced bipartition with at most $c(k)e(G)$ edges in each class?*

It is worth noting that extremal problems for balanced partitions have been relatively little investigated. For instance, although there has been significant work on algorithms for Max Bisection and Min Bisection, there are no results analogous to that of Edwards for Max Cut. This is an interesting problem in itself: for a graph G with n vertices and m edges, what are the largest and smallest cuts that we can guarantee with balanced bipartitions?

(Of course, Min Bisection would be trivial if we allowed any bipartition, since we could then choose to have all vertices in the same class.) Note that Max Bisection for G corresponds to Min Bisection for $\overline{G}$. We remark that, Max Bisection is very different from Max Cut, since a cut satisfying the inequality (3) of Edwards can be found by a compression argument: we repeatedly identify pairs of nonadjacent vertices until we obtain a complete multigraph; then a random balanced bipartition of this multigraph gives a bipartite graph of expected size at least (3). This can be derandomized, and easily translates back into a bipartition of the original graph. However, this approach does not work for balanced bipartitions, since a balanced partition of the compressed graph need not correspond to a balanced partition of the original graph.

One starting point both for the extremal Max k-Section and Min k-Section problems and the judicious partitioning analogues may be to consider balanced partitions which satisfy local conditions. (This has been a useful approach for the Max Cut problem; see for instance [30], [45].) For instance, every graph G has a partition $V(G) = V_1 \cup V_2$ in which $|\Gamma(v) \cap V_1| \leq |\Gamma(v) \cap V_2|$ for all $v \in V_1$ and $|\Gamma(v) \cap V_2| \leq |\Gamma(v) \cap V_1|$ for all $v \in V_2$ (see for instance [11], [19], and for infinite graphs see [1], [7], [42]). We cannot demand this for balanced partitions, as shown by considering $K_{2l+1,m}$, where $m \geq 2l+3$. However, perhaps we can demand the following.

Conjecture 8. *Every graph G has a balanced bipartition $V(G) = V_1 \cup V_2$ with*

$$|\Gamma(v) \cap V_1| \leq |\Gamma(v) \cap V_2| + 1 \tag{12}$$

for $v \in V_1$ and

$$|\Gamma(v) \cap V_2| \leq |\Gamma(v) \cap V_1| + 1 \tag{13}$$

for $v \in V_2$.

Even proving the bounds to within $O(1)$ would imply the existence of a balanced bipartition with at most $m/3 + O(n)$ edges in each vertex class, by summing (12) over V_1 and (13) over V_2. A similar question applies for partitions into k parts.

A well-known result of Hajnal and Szemerédi [26] asserts that if $\Delta(G) \leq r$ and $|G| = s(r+1)$ then G has an $(r+1)$-colouring with all vertex classes of equal size. Can we control the distribution of edges as well? For instance, is there a partition into $r+1$ classes such that there is essentially the same number of edges between every pair of vertex classes? Perhaps we can also demand that all vertex classes be the same size.

Finally in this section we mention the following problem.

Problem 9. *Let G be a graph with n vertices and $p\binom{n}{2}$ edges. What is the smallest $c(p,n)$ such that there is an ordering $v_1,\dots,v_n$ of the vertices with*

$$|e(\{v_1,\dots,v_t\}) - p\binom{t}{2}| < c(p,n)$$

for $1 \le t \le n$?

We might think of this as a form of *graded discrepancy* (see the surveys by Sós [43] and Beck and ós [6], and also [16]). Graphs consisting of a complete graph plus some isolated vertices are probably close to extremal here.

3 Partitioning hypergraphs

While there are reasonable bounds for many judicious partitioning problems for graphs, the analogous problems for r-uniform hypergraphs seem to be much more difficult. There are some asymptotic results for 3-uniform hypergraphs ([12], [14]), but for $r \ge 4$ virtually nothing is known.

A random bipartition of an r-uniform hypergraph into k sets has an expected m/k^r edges contained in each vertex class. As we have seen, for $r = 2$ it is known that every graph with m edges has a partition into k sets in which each class contains at most $m/k^2 + O(\sqrt{m})$ edges. For $r = 3$ it was proved in [12] that every 3-uniform hypergraph has a partition into k sets, each of which contains at most $m/k^3 + o(m)$ edges; it would be very interesting to determine the correct size of the $o(m)$ term. For $r \ge 4$, it is conjectured in [12] that for fixed k and r, every r-uniform hypergraph with m edges can be partitioned into k classes, each of which contains at most

$$\frac{m}{k^r} + o(m) \tag{14}$$

edges. The best upper bound that we have in general is the following.

Theorem 10. *Let G be an r-uniform hypergraph with m edges and n vertices, and let $k \ge 2$ be an integer. Then there is a vertex-partition $V(G) = \bigcup_{i=1}^k V_i$ such that, for $i = 1,\dots,k$,*

$$e(V_i) \le \frac{a(r)}{k^r} m + b(r) m^{\frac{2r}{2r+1}},$$

where

$$a(r) = \frac{(r+1)^{\frac{2r+2}{r-1}}}{2^{\frac{2r}{r-1}}(r+1)^{\frac{r+1}{r-1}}r^{\frac{r}{r-1}} - r^{\frac{2r}{r-1}}2^{\frac{2r+2}{r-1}}}$$

and

$$b(r) = 2(2r^2 \log k)^{\frac{r}{2r+1}}/r!^{\frac{1}{2r+1}}.$$

Proof. Let $c = (r!^2 2r^2 \log k)^{1/(2r+1)} m^{-(2r-1)/(2r+1)}$, and let V_1 be the set of $\lfloor cm \rfloor$ vertices of highest degree (note that $cm \leq n$ if $n \geq 2r^2 \log k$, since $m \leq \binom{n}{r}$; if $n < 2r^2 \log k$ then $b(r)m^{2r/(2r+1)} \geq m$, so any partition will do). Note that

$$e(V_1) \leq \binom{|V_1|}{r} \leq \frac{(cm)^r}{r!} = \frac{b(r)}{2} m^{2r/(2r+1)}.$$

It suffices to find a partition $V_1 = \bigcup_{i=1}^k W_i$ and probabilities $p_1, \ldots, p_k$ such that, extending the partition of V_1 to a partition $V(G) = \bigcup_{i=1}^k Y_i$, where each vertex from $V(G)\backslash V_1$ independently belongs to Y_i with probability p_i for each i, and writing X_i for the number of edges in Y_i that are not contained in V_1, we have

$$\mathbb{E}(X_i) \leq \frac{a(r)}{k^r} m. \tag{15}$$

The result then follows by applying the Azuma-Hoeffding inequality as in (11): writing $(d_i)_{i=1}^n$ for the degree sequence (in increasing order), we see that

$$\mathbb{P}(X_i < \mathbb{E}X_i + t) \leq \exp(-t^2/2 \sum_{i=cm+1}^{n} d_i^2).$$

Now

$$\sum_{i=\lfloor cm \rfloor+1}^{n} d_i^2 \leq d_{\lfloor cm \rfloor+1} \sum_{i=cm+1}^{n} d_i \leq \frac{rm}{cm} rm = mr^2/c,$$

so if $t^2 > (mr^2/c)\log k$ then $X_i < \mathbb{E}X_i + t$ for all i with positive probability. Thus we may take $t = r\sqrt{(m/c)\log k} = b(r)m^{2r/(2r+1)}/2$ and so $e(Y_i) \leq X_i + e(V_i) \leq a(r)m/k^r + b(r)m^{2r/(2r+1)}$.

We now show that there are $p_1, \ldots, p_k$ such that (15) is satisfied. For $S \subset V_1$, define

$$w(S) = |\{e \in E(G) : e \cap V_1 = S\}|$$

and, for $0 \leq i \leq r$, define

$$w_i(S) = \sum_{T \in S^{(i)}} w(T).$$

(Here $S^{(i)}$ is the set of all subsets of S of size i.) Thus $w_i(V_1)$ is the number of edges meeting V_1 in i vertices. Now let $W_1, \ldots, W_k$ be a random partition of V_1, where each vertex of V_1 is assigned independently to each W_i with equal probability. Then, for $0 \le j < r$,

$$\mathbb{E} \sum_{i=1}^{k} w_j(W_i) = k^{1-j} w_j(V_1).$$

(Note that this holds for $j = 0$, since $w_0(S) = w_0(V_1) = e(V \setminus V_1)$ for every $S \subset V_1$.) Then

$$\mathbb{E} \sum_{j=0}^{r-1} k^{j-1} \sum_{i=1}^{k} w_j(W_i) = \sum_{j=0}^{r-1} w_j(V_1) \le m.$$

Thus we can pick a partition such that

$$\sum_{i=1}^{k} \sum_{j=0}^{r-1} k^{j-1} w_j(W_i) \le m.$$

This will be our partition of V_1. We now pick our probabilities $p_1, \ldots, p_k$.

Let

$$m_i = \sum_{j=0}^{r-1} k^{j-1} w_j(W_i). \tag{16}$$

If $m_i = 0$ for some i, then take $p_i = 1$ and $p_j = 0$ for $j \neq i$. Otherwise, define q_i by

$$\sum_{j=0}^{r-1} q_i^{r-j} w_j(W_i) = \frac{a(r)}{k^r} m. \tag{17}$$

Note that q_i is the maximum value of p_i consistent with (15) (since the expected value of X_i is $\sum_{j=0}^{r-1} p_i^{r-j} w_j(W_i)$). Now if $\sum_{i=1}^{k} q_i \ge 1$ then we are done, since we may take $p_j = q_j / \sum_{i=1}^{k} q_i \le q_j$ for $j = 1, \ldots, k$. We show that this is the case. Indeed, by (16),

$$\begin{aligned}
\sum_{j=0}^{r-1} q_i^{r-j} w_j(W_i) &= \frac{1}{k^{r-1}} \sum_{j=0}^{r-1} (kq_i)^{r-j} k^{j-1} w_j(W_i) \\
&\le \frac{1}{k^{r-1}} \max \left\{ \sum_{j=0}^{r-1} (kq_i)^{r-j} t_j : t_i \ge 0 \ \forall i, \sum_{j=0}^{r-1} t_j = m_i \right\} \\
&= \frac{m_i}{k^{r-1}} \max \left\{ kq_i, (kq_i)^r \right\}.
\end{aligned}$$

Thus, by (17),

$$\max\{kq_i, (kq_i)^r\} \geq \frac{a(r)}{k}\frac{m}{m_i},$$

and so, writing $r_i = m_i/m$, we have

$$q_i \geq \min\left\{\frac{a(r)}{k^2 r_i}, \frac{1}{k}\left(\frac{a(r)}{kr_i}\right)^{1/r}\right\}.$$

Therefore

$$\sum_{i=1}^{k} q_i \geq \sum_{i=1}^{k} \min\left\{\frac{a(r)}{k^2 r_i}, \frac{1}{k}\left(\frac{a(r)}{kr_i}\right)^{1/r}\right\},$$

where $\sum_{i=1}^{k} r_i \leq 1$. Now setting $s_i = kr_i$, we get

$$\sum_{i=1}^{k} q_i \geq \frac{1}{k}\sum_{i=1}^{k} \min\left\{\frac{a(r)}{s_i}, \left(\frac{a(r)}{s_i}\right)^{1/r}\right\}, \tag{18}$$

where $(1/k)\sum_{i=1}^{k} s_i \leq 1$. Now the right hand side is just a convex combination of points on the curves $y = a/x$ and $y = (a/x)^{1/r}$. The tangent to $y = a/x$ at x_0 is

$$y = \frac{2a}{x_0} - \frac{ax}{x_0^2}$$

and the tangent to $y = (a/x)^{1/r}$ at x_1 is

$$y = \frac{r+1}{r}\left(\frac{a}{x_1}\right)^{\frac{1}{r}} - \frac{1}{rx_1}\left(\frac{a}{x_1}\right)^{\frac{1}{r}} x.$$

These coincide when

$$\frac{2a}{x_0} = \frac{r+1}{r}\left(\frac{a}{x_1}\right)^{\frac{1}{r}}$$

and

$$\frac{a}{x_0^2} = \frac{1}{rx_1}\left(\frac{a}{x_1}\right)^{\frac{1}{r}},$$

which gives

$$x_0 = \frac{ar^{\frac{r}{r-1}}2^{\frac{r+1}{r-1}}}{(r+1)^{\frac{r+1}{r-1}}}.$$

Thus setting $f(x) = (2a/x_0) - x(a/x_0^2)$, we see that from (18) we get

$$\begin{aligned}\sum_{i=1}^k q_i &\geq \frac{1}{k}\sum_{i=1}^k f(s_i)\\ &= f(\frac{1}{k}\sum_{i=1}^k s_i)\\ &\geq f(1).\end{aligned}$$

Thus we need only show that

$$\frac{2a}{x_0} - \frac{a}{x_0^2} \geq 1,$$

that is

$$\frac{(r+1)^{\frac{r+1}{r-1}}}{r^{\frac{r}{r-1}}2^{\frac{2}{r-1}}} - \frac{(r+1)^{\frac{2r+2}{r-1}}}{ar^{\frac{2r}{r-1}}2^{\frac{2r+2}{r-1}}} \geq 1.$$

Rearranging, we require

$$a \geq \frac{(r+1)^{\frac{2r+2}{r-1}}}{(r+1)^{\frac{r+1}{r-1}}r^{\frac{r}{r-1}}2^{\frac{2r}{r-1}} - r^{\frac{2r}{r-1}}2^{\frac{2r+2}{r-1}}},$$

which holds by definition. □

Note that the value for $a(r)$ is quite good for small values of r: $a(2) = 1.0355\cdots$, $a(3) = 1.098\cdots$ and $a(4) = 1.167\cdots$. Asymptotically, as $r \to \infty$, we obtain

$$\frac{a(r)m}{k^r} = (1+o(1))\frac{r}{4\log r}\frac{m}{k^r}.$$

Even proving a bound of form $(1+o(1))cm/k^r$ would be a start!

A question concerning a different variety of judicious partition was raised by Bollobás and Thomason, who conjectured that every r-uniform hypergraph with m edges has a vertex partition into r classes such that each class meets *at least* $rm/(2r-1)$ edges. For $r = 3$, Bollobás, Reed and Thomason [10] proved that every 3-uniform hypergraph has a tripartition, each class of which meets at least $(1 - 1/e)m/3 \approx 0.21m$ edges. It was shown in [14] that there is a tripartition in which each class meets at least $(5m-1)/9$ edges; for $r > 3$, it was shown that every r-uniform hypergraph with m edges has a vertex partition into r classes each of which meets at least $0.27m$ edges. Note that for $r = 2$, the conjectured bound follows immediately from (7). It is conjectured in [14] that every graph with m edges has a partition into k sets, each of which meets at least $2m/(2k-1)$ edges. It is also conjectured that, for fixed $r, k \geq 2$, every r-uniform hypergraph with m edges has a partition into k sets, each of which meets at least

$$(1+o(1))\left(1-(1-\frac{1}{k})^r\right)m$$

edges.

A possible approach to proving results about r-uniform hypergraphs is through proving 'mixed' partitioning results, which are also interesting in their own right. The idea is that after removing a vertex, or set of vertices, the remaining graph can be considered as a hypergraph with edges of size at most r. Thus if we remove v from $H = (V, E)$, the new hypergraph is $(V \setminus v, \{e \setminus v : e \in E\})$. It was proved in [14] that a hypergraph with m_i edges of size i for $i = 1, \ldots, k$ has a bipartition in which each vertex class meets at least

$$\frac{m_1 - 1}{3} + \frac{2m_2}{3} + \frac{3m_3}{4} + \cdots + \frac{km_k}{k+1}$$

edges. Perhaps this can be improved to

$$\frac{m_1}{2} + \frac{3m_2}{4} + \cdots + (1 - \frac{1}{2^k})m_k + o(m), \tag{19}$$

where $m = \sum_{i=1}^k m_i$. Indeed, it may be possible that we can take the $o(m)$ term to be $O(\sqrt{m_2} + 1)$.

The first interesting case of (19) is when $k = 2$. It seems likely that every hypergraph with all edges of size at most 2 (i.e., a graph together with some chosen vertices) has a bipartition in which each set meets at least

$$\frac{m_1 - 1}{2} + \frac{2m_2}{3}$$

edges. The hypergraph consisting of all edges and vertices of K_3 shows that this would be sharp. It is also of interest to consider the weighted version, which would ask for a bipartition in which each class meets edges of total weight at least

$$\frac{w_1 - \Delta_1}{2} + \frac{2w_2}{3},$$

where w_i is the total weight of edges of size i, and Δ_1 is the maximum weight of an edge of size 1.

In the unweighted case, the following related question asks for an extension of (7).

Conjecture 11. *For fixed k, every hypergraph with $m = m_1 + m_2$ edges, of which m_1 have size 1 and m_2 have size 2, has a partition into k classes, each of which contains at most*

$$m_1/k + m_2/\binom{k+1}{2} + O(1)$$

edges.

Perhaps as $m \to \infty$ we can demand $m_1/k + m_2/k^2 + O(\sqrt{m})$.

4 Conclusion

We have presented a variety of problems concerning judicious partitions of graphs and hypergraphs. We have concentrated here on judicious partitions from the perspective of extremal combinatorics. However, there is also a general problem of finding efficient algorithms for obtaining judicious partitions. We note that a polynomial time algorithm can be read out of [11] that, given a graph G, finds a bipartition $V(G) = V_1 \cup V_2$ which satisfies (1) and (3).

It would also be of interest to find good approximation algorithms for the problems above. For instance, Goemans and Williamson [25] have found an algorithm for Max Cut that approximates the optimum within a factor 1.1383. On the other hand, Håstad [27] has shown that it is NP-hard to approximate within a factor smaller than 17/16. It follows that the problem of minimizing $\max\{e(V_1), e(V_2)\}$ over bipartitions of a graph G is also NP-hard to approximate, since finding a good judicious partition of $2G$ is equivalent to finding a good cut of G. It would be very interesting to find a judicious partitioning analogue for the results of Goemans and Williamson.

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