PROBLEMS AT RESONANCE FOR FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS VIA LYAPUNOV-LIKE FUNCTIONS

by

J. J. Nieto*
Department of Mathematics
The University of Texas at Arlingotn
Arlington, Texas 76019

Technical Report #181

April, 1982

*Research partially supported by U. S. Army Research Grant #DAAG29-80-C-0060.

1. INTRODUCTION

Recently [7,8,10] an attempt is made successfully to combine the two basic techniques namely, Lyapunov-Schmidt method and the method of upper and lower solutions, to investigate the existence of periodic solutions of differential equations. When we wish to extend such results for systems of differential equations there are two possibilities to follow. One is to generalize the method of upper and lower solutions using a suitable cone, and the other is to utilize the concept of Lyapunov-like functions and the theory of differential inequalities. In the paper, we shall discuss the existence of problems at resonance for first and second order differential systems by the second approach developing necessary theory of differential inequalities for problems at resonance.

2. AN ABSTRACT EXISTENCE RESULT

Let E be a real Hilbert Space. Consider the nonlinear operator equation

$$Lu = Nu \tag{2.1}$$

where $L: D(L) \subset E \longrightarrow E$ is a linear operator and $N: D(N) \subset E \longrightarrow E$ a nonlinear operator with $D(L) \cap D(N) \neq \emptyset$. Let $E_0 = N(L)$ be such that $\dim E_0 < \infty$ and $E = E_0 \oplus E_1$. Furthermore, we assume that E_1 is also the range of L. Suppose that $P: E \longrightarrow E_0$ is the idempotent projection operator and $H: E_1 \longrightarrow E_1$ the compact inverse of L on E_1. Then it is well known [3] that the problem (2.1) is equivalent to the coupled system of operator equations

$$u_1 = H(I-P)N(u_0+u_1) \tag{2.2}$$

$$0 = PN(u_0+u_1) \tag{2.3}$$

Concerning the problem (2.1), we have the following result [4].

Theorem 2.1: Suppose that

(i) $\|Nu\| \leq J_0$ for all $u \in D(N)$

(ii) There exists r_0, $R_0 > 0$ such that

$$\langle N(u_0+u_1), u_0 \rangle \geq 0 \quad (\text{or} \leq 0) \quad \text{whenever} \quad \|u_0\| = R_0$$

and $\|u_1\| = r_0$, where $u_0 \in E_0$ and $u_1 \in E_1$.

Then the problem (2.1) admits at least one solution.

3. COMPARISON PRINCIPLES

Given $g \in C\Big[[0,2\pi] \times \mathbb{R}, \mathbb{R}\Big]$ and $v \in C\Big[[0,2\pi], \mathbb{R}\Big]$, we say that the function $\tilde{g}(t,u)$ is a modified function relative to v, if

$$\tilde{g}(t,u) = g(t,p(t,u)) + \frac{p(t,u)-u}{1+u^2}. \tag{3.1}$$

If $g \in C\Big[[0,2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\Big]$ we define

$$\tilde{g}(t,u,u') = g(t,p(t,u),u') + \frac{u-p(t,u)}{1+u^2}. \tag{3.2}$$

This function will be called the modification of $g(t,u,u')$ relative to v. Here, in (3.1) and (3.2), $p(t,u) = \max\{v(t),u\}$.

Let us consider the periodic boundary value problem (PBVP for short).

$$u' = g(t,u) \qquad u(0) = u(2\pi). \tag{3.3}$$

We have the following comparison results:

Theorem 3.1. Let $g \in C\left[[0,2\pi] \times \mathbb{R}, \mathbb{R}\right]$ and assume that

(i) $D^{-}m(t) = \limsup_{h \to 0^{-}} \frac{m(t+h)-m(t)}{h} \leq g(t,m(t))$ for all $t \in (0,2\pi]$, where $m \in C\left[[0,2\pi], \mathbb{R}\right]$ and $m(0) \leq m(2\pi)$

(ii) There exists $\beta \in C\left[[0,2\pi],\mathbb{R}\right]$ such that

$$D^{-}\beta(t) \geq g(t,\beta(t)) \text{ for all } t \in (0,2\pi] \text{ and } \beta(0) \geq \beta(2\pi)$$

(iii) g is strictly decreasing in u for each $t \in [0,2\pi]$.

Then $m(t) \leq \beta(t)$ on $[0,2\pi]$.

Proof. Suppose that $m(t) \leq \beta(t)$ on $[0,2\pi]$ is not true. Then there exists a $t_0 \in [0,2\pi]$ and an $\varepsilon > 0$ such that

(3.4) $$m(t_0) = \beta(t_0) + \varepsilon \text{ and } m(t) \leq \beta(t) + \varepsilon,\ t \in [0,2\pi].$$

If $t_0 \in (0,2\pi]$, we have $D^{-}m(t_0) \geq D^{-}\beta(t_0)$. Hence,

$$g(t_0,m(t_0)) \geq D^{-}m(t_0) \geq D^{-}\beta(t_0) \geq g(t_0,\beta(t_0))$$

and because of (iii) we see that $\beta(t_0) \geq m(t_0)$ which is a contradiction to (3.4).

If $t_0 = 0$, we have for small $h > 0$, $m(2\pi-h) - m(2\pi) \leq m(2\pi-h) - m(0) \leq \beta(2\pi-h) + \varepsilon - \beta(0) - \varepsilon \leq \beta(2\pi-h) - \beta(2\pi)$, which implies $D^{-}m(2\pi) \geq D^{-}\beta(2\pi)$. It then follows that $\beta(2\pi) \geq m(2\pi)$, and we obtain $\beta(0) \geq \beta(2\pi) \geq m(2\pi) \geq m(0)$ which again is a contradiction. Thus, we have established that $m(t) \leq \beta(t)$ on $[0,2\pi]$.

Theorem 3.2. Let $g \in C\left[[0,2\pi] \times \mathbb{R}, \mathbb{R}\right]$ and suppose that (i) of Theorem 3.1 holds. Also assume:

(a) $m(0) = m(2\pi)$

(b) The PBVP (3.3) has a maximal solution $r(t)$

(c) For every lower solution of (3.3) v, the modified PBVP

$$u' = \tilde{g}(t,u) \qquad u(0) = u(2\pi) \tag{3.5}$$

where $\tilde{g}$ is defined by (3.1), has a solution $u(t)$. Then $m(t) \le r(t)$ on $[0,2\pi]$.

Proof. Let $\tilde{g}$ the modified function relative to $m(t)$ and let $u(t)$ be a solution of the modified PBVP (3.5) guaranteed by (c). We shall first show that $m(t) \le u(t)$ on $[0,2\pi]$. If this is false, then there exists a smaller $\varepsilon > 0$ such that

$$m(t) \le u(t) + \varepsilon, \quad t \in [0,2\pi]$$

and at least one $t_0 \in [0,2\pi]$ satisfying

$$m(t_0) = u(t_0) + \varepsilon.$$

If $t_0 \in (0,2\pi]$, we have $m(t_0) > u(t_0)$ and $D^- m(t_0) \ge u'(t_0)$. On the other hand, we have in view of the definition of $\tilde{g}$ and $p(t,u)$

$$0 \le D^- m(t_0) - u'(t_0) \le g(t_0,m(t_0)) - g(t_0,p(t_0,u(t_0))) - \frac{p(t_0,u(t_0))-u(t_0)}{1+u^2(t_0)}$$

$$= \frac{u(t_0)-m(t_0)}{1+u^2(t_0)} < 0 \quad \text{which is a contradiction.}$$

If $t_0 = 0$, we obtain $D^- m(2\pi) \ge u'(2\pi)$, and from (a), we get $p(2\pi,u(2\pi)) = \max(m(2\pi),u(2\pi)) = \max(m(0), u(0)) = m(0)$, and then

$$0 \le D^- m(2\pi) - u'(2\pi) \le g(2\pi, m(2\pi)) - g(2\pi, p(2\pi, u(2\pi))) - \frac{p(2\pi, u(2\pi)) - u(2\pi)}{1+u^2(2\pi)} =$$

$$= \frac{u(0)-m(0)}{1+u^2(0)} < 0 \quad \text{which again is a contradiction.}$$

Thus we have established that $m(t) \le u(t)$ on $[0,2\pi]$ and so $u(t)$ is actually a solution of the PBVP (3.3). It, therefore, follows from the definition of maximal solutions that $m(t) \le r(t)$ on $[0,2\pi]$, and this proves the theorem.

In the sequel of this paragraph we shall consider the boundary value problem (BVP for short).

$$u'' = g(t,u,u') \qquad u'(0) = u'(2\pi) = 0 \tag{3.6}$$

where $g \in C\left[[0,2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right]$.

We then have the following comparison result for the BVP (3.6) which is analogous to the Theorem 3.2.

<u>Theorem 3.3.</u> Assume that

(a) $m \in C^2\left[[0,2\pi], \mathbb{R}\right]$ and $m''(t) \ge g(t,m(t),m'(t))$ for all $t \in [0,2\pi]$

(b) $m'(0) \ge 0 \ge m'(2\pi)$

(c) The BVP (3.6) has a maximal solution $r(t)$

(d) For every lower solution v, the modified BVP

$$u'' = \tilde{g}(t,u,u') \qquad u'(0) = u'(2\pi) = 0 \tag{3.7}$$

where $\tilde{g}(t,u,u')$ is defined by (3.2), has a solution $u(t)$.

Then $m(t) \le r(t)$ on $[0,2\pi]$.

<u>Proof.</u> Let us consider the modified BVP (3.7) relative to $m(t)$ and let $u(t)$

be a solution which exists by (d). If $m(t) \leq u(t)$ is not true, there exists a $t_0 \in [0,2\pi]$ and $\varepsilon > 0$ such that $m(t_0) = u(t_0) + \varepsilon$ and $m(t) \leq u(t) + \varepsilon$ for all $t \in [0,2\pi]$.

If $t_0 \in (0,2\pi)$, we get $m'(t_0) = u'(t_0)$ and $p(t_0,u(t_0)) = m(t_0)$ and we have

$$m''(t_0) - u''(t_0) = \frac{m(t_0)-u(t_0)}{1+u^2(t_0)} > 0 \text{ which is a contradiction.}$$

If $t_0 = 0$, we have $m'(0) \leq u'(0)$ and from (b) it follows that $m'(0) = u'(0) = 0$. Hence we obtain as before $m''(0) - u''(0) > 0$ which again is a contradiction. A similar contradiction holds at $t_0 = 2\pi$. Thus, we have established that $m(t) \leq u(t)$ on $[0,2\pi]$ and we then have $m(t) \leq r(t)$ on $[0,2\pi]$, and this proves the theorem.

Using similar reasonings we shall prove the following comparison result.

Theorem 3.4. Let $g \in C\left[[0,2\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right]$ and suppose that (a) and (b) of Theorem 3.3 hold and there exists $\beta \in C\left[[0,2\pi],\mathbb{R}\right]$ such that

$$\beta''(t) \leq g(t,\beta(t),\beta'(t)) \quad \text{for all} \quad t \in [0,2\pi], \quad \text{and} \quad \beta'(0) \leq 0 \leq \beta'(2\pi).$$

If, in addition, we have that $g(t,u,u')$ is increasing in u for each $(t,u') \in [0,2\pi] \times \mathbb{R}$, $m(t) \leq \beta(t)$ on $[0,2\pi]$.

Proof. Suppose that the assertion $m(t) \leq \beta(t)$ on $[0,2\pi]$ is false. Then there exists a $t_0 \in [0,2\pi]$ and an $\varepsilon > 0$ satisfying (3.4). If $t_0 \in (0,2\pi)$, we have $m'(t_0) = \beta'(t_0)$ and

$$0 \geq m''(t_0) - \beta''(t_0) \geq g(t_0,m(t_0),m'(t_0)) - g(t_0,\beta(t_0), \beta'(t_0))$$

and from the fact that g is increasing in u, we get $\beta(t_0) \geq m(t_0)$ which is a contradiction to (3.4).

If $t_0 = 0$, $m'(0) \le \beta'(0)$ and we must have $m'(0) = \beta'(0) = 0$ and, as before, we obtain $\beta(0) \ge m(0)$ which again is a contradiction. A similar argument holds at $t_0 = 2\pi$. Thus, we have proved that $m(t) \le \beta(t)$ on $[0,2\pi]$.

4. EXISTENCE FOR THE MODIFIED PROBLEMS

Consider the periodic boundary value problem

$$u' = f(t,u), \; u(0) = u(2\pi) \tag{4.1}$$

where $f \in C\left[[0,2\pi] \times \mathbb{R}^n, \mathbb{R}^n\right]$.

Let $M > 0$ and δ be such that $\delta \in C[\mathbb{R}^+,\mathbb{R}^+]$, and $\delta(s) = 1$, $0 < \delta(s) < 1$ and $\delta(s) = 0$ accordingly as $0 \le s \le M$, $M < s < M + 1$, and $s \ge M + 1$. Then define

$$F(t,u) = \delta(\|u\|) \cdot f(t,u) \quad \text{for all } (t,u) \in [0,2\pi] \times \mathbb{R}^n \tag{4.2}$$

We shall say that $F(t,u)$ is a modified function of $f(t,u)$ relative to δ, M. Clearly the function $F(t,u)$ is continuous and bounded on $[0,2\pi] \times \mathbb{R}^n$.

Let us now consider the modified PBVP

$$u' = F(t,u) \qquad u(0) = u(2\pi) \tag{4.3}$$

We let $E = (L_2[0,2\pi])^n$, $Lu = u'$, $D(L) = \{u \in E\colon\ u$ is absolutely continuous, $u' \in E$ and $u(0) = u(2\pi)\}$ and N be the nonlinear operator generated by F. Then the PBVP (4.3) is equivalent to the operator equation (2.1). We note that $E_0 = N(L)$ consists of constant functions and hence E_1, where $E = E_0 \oplus E_1$, is the class of all functions whose average is zero. The projection operator P can be defined by $Pu = \frac{1}{2\pi}\int_0^{2\pi} u(s)ds$. It is easily seen that the partial inverse operator H also exists and it is compact.

Because the function F is bounded, the hypothesis (i) of Theorem 2.1 is satisfied.

On the other hand, there exists a $\gamma > 0$ such that $\gamma(\|x_0\|+\|x_1\|) \leq \|x_0 + x_1\|$ whenever $x_0 \in E_0$, $x_1 \in E_1$. If we choose $R_0 = (M+1) \cdot \gamma^{-1}$, we get for all $x_0 \in E_1$ with $\|x_0\| = R_0$ and $x_1 \in E_1$

$$\langle N(x_0+x_1),x_0\rangle = \sum_{i=1}^{n} x_0^i \int_0^{2\pi} F_i(t,x_0+x_1(t))dt =$$

$$= \sum_{i=1}^{n} x_0^i \int_0^{2\pi} \delta(\|x_0+x_1(t)\|) \cdot f_i(t,x_0+x_1(t))dt.$$

But, $\|x_0 + x_1(t)\| \geq \gamma(\|x_0\|+\|x_1(t)\|) \geq M + 1$, and hence we have $\delta(\|x_0\|+\|x_1(t)\|) = 0$, which implies $\langle N(x_0+x_1),x_0\rangle = 0$ whenever $x_0 \in E_0$, $\|x_0\| = R_0$ and $x_1 \in E_1$. Thus, the hypothesis (ii) of the Theorem 2.1 is also satisfied.

We have just proved the following result.

Theorem 4.1. The modified boundary value problem (4.3) has at least one solution.

Now, let us consider the boundary value problem

$$(4.4) \qquad u'' = f(t,u,u') \qquad u'(0) = u'(2\pi) = 0$$

where $f \in C\left[[0,2\pi] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right]$.

Let $\delta \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$ such that $\delta(t,s) \leq 1$ for all $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$ and

$$\delta(t,s) = \begin{cases} 1 & \text{if} \quad t \leq M_1 \quad \text{and} \quad s \leq M_2 \\ 0 & \text{if} \quad t \geq M_1 + 1 \quad \text{or} \quad s \geq M_2 + 1 \end{cases}$$

where M_1 and M_2 are positive numbers.

We define the modified function F by

$$F(t,u,u') = \delta(\|u\|,\|u'\|) \cdot f(t,u,u') \tag{4.5}$$

and consider the modified problem

$$u'' = F(t,u,u') \qquad u'(0) = u'(2\pi) = 0 \tag{4.6}$$

Clearly F is continuous and bounded on $[0,2\pi] \times \mathbb{R}^n \times \mathbb{R}^n$. We let $E = (L_2[0,2\pi])^n$, $Lu = u''$, $D(L) = \{u \in E:\ u,u'$ are absolutely continuous, $u'' \in E$ and $u'(0) = u'(2\pi) = 0\}$ and N be the Nemytskii operator generated by F. Using similar arguments as before one gets

$$\langle N(u_0+u_1),u_0\rangle = 0 \quad \text{whenever} \quad u_0 \in E_0,\ \|u_0\| = R_0 = (M_1+1) \cdot \gamma^{-1}$$

and $u_1 \in E_1$.

Thus, the assumptions required in Theorem 2.1 are satisfied and we have established the following existence result.

Theorem 4.2. The modified boundary value problem (4.6) has at least one solution.

5. EXISTENCE

Now, we shall prove the existence of solution of the PBVP (4.1) and of the BVP (4.4).

Relative to the PBVP (4.1), let us list the following assumptions for convenience.

(5.1) There exists $V \in C\left[[0,2\pi] \times \mathbb{R}^n, \mathbb{R}^+\right]$ and $g \in C\left[[0,2\pi] \times \mathbb{R}^+, \mathbb{R}\right]$ such that

$$D_-V_f(t,u) = \liminf_{h\to 0^-} \frac{V(t+h,u+hf(t,u))-V(t,u)}{h} \le g(t,V(t,u))$$

and V is locally Lipschitzian in u.

(5.2) The boundary conditions of (4.1) imply $V(0,u(0)) \le V(2\pi,u(2\pi))$

(5.3) $D_-V_{\lambda f}(t,u) \le g(t,V(t,u))$ for all λ such that $0 \le \lambda \le 1$

(5.4) $V(t,u) \to \infty$ as $\|u\| \to \infty$ uniformly in $t \in [0,2\pi]$

(5.5) The PBVP (3.3) has a maximal solution $r(t)$, and for every lower solution v, the modified PBVP (3.5), where $\tilde{g}$ is defined by (3.1), has a solution $u(t)$.

(5.6) The function g is strictly decreasing in u for each $t \in [0,2\pi]$, and there exists $\beta \in C\left[[0,2\pi],\mathbb{R}\right]$ such that $D^-\beta(t) \ge g(t,\beta(t))$ for all $t \in (0,2\pi]$, and $\beta(0) \ge \beta(2\pi)$. Also, the boundary conditions of (4.1) imply $V(0,u(0)) = V(2\pi,u(2\pi))$.

As an application of Theorem 4.1 and the comparison Theorems 3.1 and 3.2, we shall prove the following result.

Theorem 5.1. Consider the PBVP (4.1) and let (5.1)-(5.4) hold. Then any one of the conditions (5.5), (5.6) implies that there exists a solution u of (4.1).

Proof. Let us consider the modified PBVP (4.3). By Theorem 4.1, it has a solution $u(t)$. Setting $m(t) = V(t,u(t))$, for sufficiently small $h < 0$, we have

$$m(t+h) - m(t) = V(t+h,u(t)+hf(t,u(t)) + \varepsilon(h)) - V(t,u(t))$$

where $\varepsilon(h)/h \to 0$ as $h \to 0$. Since $V(t,u)$ is locally Lipschitzian in u, we get, using (5.1) and (5.3), the inequality

$$D_-m(t) \le g(t,m(t)).$$

From (5.5), applying Theorem 3.1, we have

$$m(t) \le r(t) \quad \text{on} \quad [0,2\pi]$$

Similarly, using Theorem 3.2, from (5.6) we get

$$m(t) \le \beta(t) \quad \text{on} \quad [0,2\pi].$$

In any case, we have a $\kappa > 0$ such that $0 \le m(t) \le \kappa$ for all $t \in [0,2\pi]$. Then, by (5.4), there exists a $M > 0$ which satisfies $u(t) \le M$ on $[0,2\pi]$. This M is the constant that is used in the definition of δ for the modified function (4.2). So $u(t)$ is actually a solution of the PBVP (4.1). The proof is thus complete.

In order to prove an existence result for the BVP (4.4) we need the following result [11].

<u>**Theorem 5.2.**</u> Let $h(s)$ be positive nondecreasing continuous function such that $\lim_{s\to\infty} \frac{s^2}{h(s)} = \infty$ and let M_1 be a positive number. Then there exists a positive constant M_2 depending only on M_1 and h such that if $u \in C^2\left[[0,2\pi],\mathbb{R}^n\right]$ is such that $\|u\| \le M_1$ and $\|u''\| \le h(\|u'\|)$, then $\|u'(t)\| \le M_2$ on $[0,2\pi]$.

Let us now list some assumptions:

(5.7) $\qquad V \in C^2\left[[0,2\pi] \times \mathbb{R}^n, \mathbb{R}^+\right]$ and $g \in C\left[[0,2\pi] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}\right]$

(5.8) $\qquad$ For $0 \le \lambda \le 1$, $V''_{\lambda f} \ge g(t,V(t,u),V'(t,u))$

where $V''_{\lambda f}(t,u) = U(t,u,u') + \lambda V_u(t,u)f(t,u,u')$ and $U(t,u,u') = V_{tt}(t,u) + 2V_{tu}(t,u) \cdot u' + V_{uu}(t,u) \cdot u' \cdot u'$. Here we have used the facts that if $V \in C^2\left[[0,2\pi] \times \mathbb{R}^n, \mathbb{R}^+\right]$ then $V'(t,u) = V_t(t,u) + V_u(t,u) \cdot u'$ and $V''(t,u) = U(t,u,u') + V_u(t,u) \cdot u''$, where $V_{uu}(t,u)$ is the $n \times n$ matrix $\left(\frac{\partial^2 V(t,u)}{\partial u_i \partial u_j}\right)$ $i,j = 1,2,\ldots,n$

(5.9) $\qquad V(t,u) \to \infty$ as $\|u\| \to \infty$ uniformly in $t \in [0,2\pi]$

(5.10) The boundary conditions of (4.4) imply

$$\frac{\partial V}{\partial t}(0,u(0)) \geq 0 \quad \text{and} \quad \frac{\partial V}{\partial t}(2\pi,u(2\pi)) \leq 0$$

(5.11) There exists a positive nondecreasing continuous function h such that $\lim_{s\to\infty} \frac{s^2}{h(s)} = \infty$ and

$$\|f(t,u,u')\| \leq h(\|u'\|) \quad \text{for all} \quad (t,u) \in [0,2\pi] \times \mathbb{R}^n.$$

(5.12) The BVP (3.6) has a maximal solution $r(t)$, and for every lower solution v, the modified BVP (3.7) has a solution $u(t)$.

(5.13) The function g is increasing in u for each $(t,u') \in [0,2\pi] \times \mathbb{R}^n$, and there exists $\beta \in C\left[[0,2\pi],\mathbb{R}\right]$ such that $\beta''(t) \geq g(t,\beta(t), \beta'(t))$ on $[0,2\pi]$ and $\beta'(0) \leq 0 \leq \beta'(2\pi)$.

We are now in a position to prove the following result:

Theorem 5.3. Assume that (5.7)-(5.11) hold. Then any one of the conditions (5.12), (5.13) implies that there exists a solution for the BVP (4.4).

Proof. Consider the modified problem (4.6) of (4.4) and let $u(t)$ be a solution guaranteed by Theorem 4.2. Defining $m(t) = V(t,u(t))$ and using the assumptions (5.7) and (5.8) we get

$$m''(t) \geq g(t,m(t),m'(t))$$

and by (5.10) we have

$$m'(0) \geq 0 \geq m'(2\pi)$$

Hence, by comparison Theorems 3.3 and 3.4, it follows that

$$(5.14)\quad \begin{aligned} m(t) &\le r(t) \quad \text{on} \quad [0,2\pi] \\ m(t) &\le \beta(t) \quad \text{on} \quad [0,2\pi] \end{aligned}$$

when the assumptions (5.12), (5.13) are satisfied respectively. Either of the inequalities of (5.14) implies $m(t) \le k$ for some $k > 0$. Now the assumption (5.9) implies that there is an $M_1 > 0$ such that $\|u(t)\| \le M_1$ on $[0,2\pi]$. Choosing this M_1 in Theorem 5.2, we obtain $\|u'(t)\| \le M_2$ on $[0,2\pi]$. These are the constants M_1 and M_2 that are used in the definition of δ. In view of the definition (4.5) of F, it follows that $u(t)$ is actually a solution of the BVP (4.4) and the proof is complete.

Remark 5.1. There are other possibilities of proving Theorem 5.3. For example, (5.8) would be improved to

$$V''_{\lambda f}(t,u) \ge g(t,v(t,u),\ V'(t,u)) + \sigma\|f(t,u,u')\|,\ \sigma > 0$$

with suitable conditions on g, in which case the assumptions of Theorem 5.2 become superfluous. We do not intend to discuss these possibilities and refer the reader to [1]. See also [2,6].

Remark 5.2. Consider the PBVP

$$(5.15)\qquad u'' = f(t,u,u') \quad u(0) = u(2\pi) \quad \text{and} \quad u'(0) = u'(2\pi)$$

where $f \in C\left[[0,2\pi] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n\right]$.

We then have the following result concerning the existence of solution of the PBVP (5.15). We only give minimal details in the proof.

Theorem 5.4. Suppose that

(i) The conditions (5.7)-(5.9), (5.11) and (5.12) hold.

(ii) $V(0,x) \ge V(2\pi,x)$, $V_t(0,x) \ge V_t(2\pi,x)$ and $V_u(0,x) = V_u(2\pi,x)$ for all x.

Then the PBVP (5.15) has a solution.

Proof. Consider the modified PBVP

$$u'' = F(t,u,u') \quad u(0) = u(2\pi) \quad \text{and} \quad u'(0) = u'(2\pi)$$

where F is defined by (4.5).

We let $E = (L_2[0,2\pi])^n$, $Lu = u''$, $D(L) = \{u \in E\colon\ u,u'$ are absolutely continuous, $u'' \in E$, $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)\}$, and N be the nonlinear operator generated by F. Then, as in Theorem 4.2, using the comparison Theorem 3.3 with (ii) instead of (b), we have the existence of one solution $u(t)$, which is actually a solution of (5.15) because (5.9), (5.11), (5.12) and (ii) hold. Thus, the proof is complete.

It is possible to prove the existence of solution of the PBVP (5.15) with small modifications in the comparison Theorem 3.4 and using a similar assumption to (ii) of Theorem (5.4).

ACKNOWLEDGMENT

The author is grateful to Professor V. Lakshmikantham for many stimulating discussions and helpful suggestions during this work.

REFERENCES

1. Bernfeld, S., Ladde, G., and Lakshmikantham, V., "Existence of solutions of two point boundary value problems for nonlinear systems." J. Diff. Eqns., 18(1975), 103-110.
2. Bernfeld, S. and Lakshmikantham, V., "An Introduction to Nonlinear Boundary Value Problems", Academic Press, New York, 1974.
3. Cesari, L., "Functional analysis, nonlinear differential equations, and the alternative method, Proc. of Conf. on Nonlinear Functional Analysis and Differential Equations (Eds. Cesari, Kannan, and Schuur), Marcel Dekker, Inc., New York (1976), 1-197.
4. Cesari, L. and Kannan, R., "An abstract theorem at resonance," Proc. Amer. Math. Soc., 63 (1977), 221-225.
5. Chandra, J., Lakshmikantham, V., and Leela, S., "Comparison principle and theory of Nonlinear Boundary Value Problems," The University of Texas at Arlington, Technical Report #137 (1980).
6. Hartman, P., "On two point boundary value problems for nonlinear second order systems." SIAM J. Math. Analysis, 5(1974), 172-177.
7. Kannan, R. and Lakshmikantham, V., "Periodic Solutions of Nonlinear Boundary Value Problems", Nonlinear Analysis, TM&A, Vol. 6, No. 1, pp. 1-10, (1982).
8. Kannan, R. and Lakshmikantham, V., "Periodic solution of nonlinear boundary value problems," Proc. Inter. Conf. on "Recent methods in nonlinear analysis and applications", Naples, 1980, Liquori Editore, Naples (1981), 99-127.
9. Lakshmikantham, V. and Leela, S., "Differential and integral inequalities, Vol. I., Academic Press, New York, 1969.

10. Lakshmikantham, V. and Leela, S., "Existence and monotone method for periodic solutions of first order differential equations," to appear in Jour. Math. Anal. Appl.

11. Schmitt, K. and Thompson, R., "Boundary value problems for infinite systems of second order differential equations," J. Diff. Eqns., 8 (1975), 277-295.