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A PROCEDURE FOR ESTIMATING STABILITY AND CONTROL PARAMETERS FROM FLIGHT TEST DATA BY USING MAXIMUM LIKELIHOOD METHODS EMPLOYING A REAL-TIME DIGITAL SYSTEM

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16. Abstract

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A PROCEDURE FOR ESTIMATING STABILITY AND CONTROL PARAMETERS FROM FLIGHT TEST DATA BY USING MAXIMUM LIKELIHOOD METHODS EMPLOYING A REAL-TIME DIGITAL SYSTEM

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SUMMARY

A maximum likelihood parameter estimation procedure and program were developed for the extraction of the stability and control derivatives of aircraft from flight test data. Nonlinear six-degree-of-freedom equations describe the aircraft dynamics and are used to derive the sensitivity equations for the method of quasilinearization. The maximum likelihood function with quasilinearization was used to derive the parameter change equations, the covariance matrices for the parameters and measurement noise, and the performance index function.

The maximum likelihood estimator was mechanized into an iterative estimation procedure utilizing a real-time digital computer and graphic display system. This program was developed for 8 measured state variables and 40 parameters. Test cases were conducted with pseudo or simulated data for validation of the estimation procedure and program. This program has been applied to a V/STOL tilt-wing aircraft, a military fighter airplane, and a light single-engine airplane.

The appendixes describe in detail the particular nonlinear equations of motion, derivation of the sensitivity equations, addition of accelerations into the algorithm, operational features of the real-time digital system, and test cases.

INTRODUCTION

The problem of estimating stability and control parameters of an aircraft from flight test data dates from the early days of flight. The results of early investigations were frequently limited, however, due primarily to insufficient estimation technology and restricted computational resources. Since 1960 there has been significant progress in

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correcting these deficiencies; therefore, it seems appropriate to reevaluate, in a more general setting, the problem of estimating aerodynamic coefficients from flight data.

Parameter estimation is the process of determining the parameters in a mathematical model after having been supplied measured values for the variables of state and the input to the dynamic system. The accuracy of the resulting estimate is degraded by a combination of measurement, modeling, and numerical errors. Obtaining this estimate is a problem in inverse computation and the matter of existence and uniqueness of solution must be resolved at least to some relative degree. The fact that in the defining of the discipline it is necessary to refer to modeling errors and the existence and uniqueness of solution suggests that the expression "parameter estimation" is not adequate to describe the task, the task being a study undertaken by an analyst using the parameter estimation program. A meaningful consideration must include studies of the model definition in relation to the engineering application, the accuracy and precision to which the parameters are computed, and some indication of the uniqueness of solution. It is suggested that the expression "system identification" better represents the task being considered and more accurately implies the interrelated disciplines the study requires. A survey of the general problem of identifying a dynamic system from input-output measurements is given in reference 1.

The objectives of this paper are twofold:

(1) To present the development of an estimation procedure, based on maximum likelihood statistics, suitable for extracting stability and control parameters of an aircraft from flight test data for realistic aircraft models

(2) To develop a computer program and provide operational features of the estimation procedure when implemented on the Langley real-time simulation system

The general approach adopted in this paper was based on a maximum likelihood output error method. The case where process noise is present is not considered. The assumed mathematical model for the aircraft was based on a standard six-degree-of-freedom rigid body description with linearized aerodynamic forces and moments. The interactive role of the analyst is discussed as well as various program options which are available. Also included in the paper is a demonstration of the performance of the estimation procedure and the computer program using pseudo flight data. The program developed has been successfully applied to the analysis of flight data for generically different aircraft (refs. 2 and 3).

SYMBOLS

A — sensitivity coefficient matrix ($n \times p'$)

$a_{X,cg}, a_{Y,cg}, a_{Z,cg}$ — longitudinal, lateral, and vertical accelerations at center of gravity

$a_{X,I}, a_{Y,I}, a_{Z,I}$ — longitudinal, lateral, and vertical accelerations at instrument location

B — sensitivity coefficient matrix for N data points ($nN \times p'$)

b — wing span

C_l, C_m, C_n — rolling-, pitching-, and yawing-moment coefficients

$C_{l,o}, C_{n,o}$ — rolling-moment and yawing-moment coefficients at $\beta = \delta_a = \delta_r = 0$

$C_{m,o}$ — pitching-moment coefficient at $\alpha_a = \delta_e = 0$

C_X, C_Y, C_Z — longitudinal-, lateral-, and normal-force coefficients

$C_{X,o}, C_{Z,o}$ — longitudinal-force and normal-force coefficients at $\alpha_a = \delta_e = 0$

$C_{Y,o}$ — lateral-force coefficient at $\beta = \delta_a = \delta_r = 0$

$\bar{c}$ — mean aerodynamic chord

E — mathematical expectation

$\vec{F}$ — vector function defined in equations of motion ($n \times 1$)

F_j — components of $\vec{F}$, where $j = 1, 2, \ldots, n$

G — sensitivity equation matrix ($n \times n$)

g — gravity

$\vec{H}$ — measurement noise vector for N data points ($nN \times 1$)

I_X,I_Y,I_Z	moment of inertia about X-, Y-, and Z-axis, respectively
I_{XZ}	product of inertia
i_w	wing tilt angle
J_N	performance index function
L	maximum likelihood function
M_X,M_Y,M_Z	rolling, pitching, and yawing moments
m	mass
m'	dimension of control deflection vector
N	number of data points
n	dimension of system
p'	number of parameters
p,q,r	roll, pitch, and yaw angular velocities
R_1	measurement noise covariance matrix $(n \times n)$
S	wing area
T	flight test time period
T_X,T_Y,T_Z	thrust along X-, Y-, and Z-axis, respectively
t	time
t_i	data point time, where $i = 1, 2, \ldots, N$
u,v,w	longitudinal, lateral, and vertical velocity components

V velocity

V_{SS} slipstream velocity

W_1 weighting matrix ($nN \times nN$)

$\vec{x}$ state vector ($n \times 1$)

x_k components of $\vec{x}$, where $k = 1, 2, \ldots, n$

$\vec{\alpha}$ parameter vector ($p' \times 1$)

α_a, β angles of attack and sideslip

α_i components of $\vec{\alpha}$, where $i = 1, 2, \ldots, p'$

$\overrightarrow{\Delta\alpha}$ parameter change vector ($p' \times 1$)

$\Delta\alpha_i$ components of $\overrightarrow{\Delta\alpha}$, where $i = 1, 2, \ldots, p'$

$\vec{\delta}$ control deflection vector ($m' \times 1$)

$\delta_a, \delta_e, \delta_r$ aileron, elevator, and rudder control deflections

δ_{ij} Kronecker delta

$\vec{\eta}$ measurement noise vector ($n \times 1$)

η_k components of $\vec{\eta}$

ρ atmospheric density

$\rho_{\alpha_i \alpha_j}$ correlation coefficient of α_i and α_j

σ_{α_i} standard deviation of α_i

ϕ, θ, ψ roll, pitch, and yaw angles

$$C_{X_q} = \frac{\partial C_X}{\partial \frac{q\bar{c}}{2V}}$$

$$C_{X_{\alpha_a}} = \frac{\partial C_X}{\partial \alpha_a}$$

$$C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$$

$$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$$

$$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{l_{\dot{\beta}}} = \frac{\partial C_l}{\partial \frac{\dot{\beta}b}{2V}}$$

$$C_{l_{\delta_a}} = \frac{\partial C_l}{\partial \delta_a}$$

$$C_{l_{\delta_r}} = \frac{\partial C_l}{\partial \delta_r}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

$$C_{Y_{\dot{\beta}}} = \frac{\partial C_Y}{\partial \frac{\dot{\beta}b}{2V}}$$

$$C_{Y_{\delta_r}} = \frac{\partial C_Y}{\partial \delta_r}$$

$$C_{m_q} = \frac{\partial C_m}{\partial \frac{q\bar{c}}{2V}}$$

$$C_{m_{\alpha_a}} = \frac{\partial C_m}{\partial \alpha_a}$$

$$C_{m_{\dot{\alpha}_a}} = \frac{\partial C_m}{\partial \frac{\dot{\alpha}\bar{c}}{2V}}$$

$$C_{m_{\delta_e}} = \frac{\partial C_m}{\partial \delta_e}$$

$$C_{Z_q} = \frac{\partial C_Z}{\partial \frac{q\bar{c}}{2V}}$$

$$C_{Z_{\alpha_a}} = \frac{\partial C_Z}{\partial \alpha_a}$$

$$C_{Z_{\delta_e}} = \frac{\partial C_Z}{\partial \delta_e}$$

$$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$$

$$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{n_{\dot{\beta}}} = \frac{\partial C_n}{\partial \frac{\dot{\beta}b}{2V}}$$

$$C_{n_{\delta_a}} = \frac{\partial C_n}{\partial \delta_a}$$

$$C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r}$$

Matrix exponents:

T indicates transpose matrix operation

-1 indicates inverse matrix operation

The superscript M denotes measured value.

The superscript o indicates nominal evaluation.

Dot over a symbol denotes time derivative.

Arrow over symbol indicates vector.

BACKGROUND DISCUSSION

Early methods for identifying aircraft stability and control parameters are typically characterized as equation error methods. They were basically least-squares estimators which minimize the equation error and are known to give biased estimates in the presence of measurement noise. Details of these various methods can be found in references 4 to 7. Since the unknown parameters enter the equations of motion in a linear fashion, equation error methods are characterized computationally as single step processes and, therefore, are simple to deal with.

More recent parameter identification methods are generally classified as output error techniques. These methods minimized the output error (measurement noise) between the measurements and the true states of the dynamic system and are often used to modify the initial estimates obtained by equation error methods. Unlike equation error methods, the identification problem using output error methods is nonlinear and this requires an iterative solution. Standard output error methods include the Newton-Raphson iteration method (ref. 8), modified Newton-Raphson method (ref. 9), quasilinearization method (refs. 10 to 14), and various forms of gradient-dependent methods (ref. 15). The quasilinearization and modified Newton-Raphson methods can be shown to be identical. The Kalman filter, a sequential estimation method, can be shown for certain restrictive cases to be equivalent to the two techniques just mentioned and, therefore, can be considered an output error method. Output error methods are known to produce unbiased parameter estimates under realistic conditions on the measurement noise. However, if a significant amount of process noise exists, that is, gusts and modeling errors, then the validity of estimates obtained by using output error methods is subject to serious question.

PROBLEM STATEMENT AND ASSUMPTIONS

Assume that the equations of motion of an aircraft are in the form

$$\dot{\vec{x}} = \vec{F}\left(\vec{x},\vec{\alpha},\vec{\delta}(t)\right) + \vec{\mu}(t)$$

with a measurement model defined as

$$\vec{x}^M(t) = \vec{h}\left(\vec{x}(\vec{\alpha},t)\right) + \vec{\eta}(t)$$

$$\vec{\delta}^M(t) = \vec{\delta}(t) + \vec{\gamma}(t)$$

The variables are defined as follows:

$\vec{x}$	aircraft state vector
$\vec{\delta}(t)$	aircraft control vector
$\vec{\alpha}$	unknown parameter vector
$\vec{x}^M$	state measurement vector
$\vec{\delta}^M(t)$	control measurement vector
$\vec{\mu}(t)$	process noise vector
$\vec{\eta}(t)$	measurement noise vector
$\vec{\gamma}(t)$	measurement noise on input
$\vec{h}\left(\vec{x}(\vec{\alpha},t)\right)$	measurement output vector

The essential problem is to estimate the parameter vector $\vec{\alpha}$ when given the equations of motion of the dynamic system and the measurements $\vec{x}^M(t)$ and $\vec{\delta}^M(t)$.

The solution of the problem as posed in the preceding paragraph is probably not practical at the present time. Various approaches to the general problem have been attempted (for example, refs. 16 to 18). The results of these studies indicate that for dynamic systems as complicated as aircraft, a solution of the estimation problem requires substantial computational effort. In order to proceed with a solution of the estimation problem which is computationally feasible and for which theoretical techniques have been developed, the following assumptions are made:

(1) Rigid body aircraft (six degrees of freedom)

(2) Only measurement noise $\left(\vec{\mu}(t) = \vec{\gamma}(t) = 0\right)$

(3) The measurement noise $\vec{\eta}(t_i)$ is a sequence of independent Gaussian random variables with

$$E\left[\vec{\eta}(t_i)\right] = 0$$

$$E\left[\vec{\eta}(t_i)\vec{\eta}^T(t_j)\right] = R_1\delta_{ij}$$

where the matrix R_1 is unknown.

The foregoing assumptions imply the following conditions:

(1) The aircraft maneuvers do not exceed a dynamic range consistent with linearization of the aerodynamic forces and moments.

(2) Wind gusts, modeling errors, and inaccuracies of measuring physical movement of a control surface are considered sufficiently small to warrant the use of an output error method.

(3) Alinement and location of rate gyros and angle of attack and sideslip instrumentation are of sufficient quality to preclude their inclusion in the measurement model. Anomalies introduced by accelerometer location are included in the measurement model.

PARAMETER ESTIMATION PROCEDURE

The parameter estimation procedure, using the method of maximum likelihood with quasilinearization, is diagramed in figure 1. The basic components are the (1) equations of motion, (2) sensitivity equations, (3) maximum likelihood estimation equations, (4) performance index, and (5) flight test data. These components are described next and the procedure in figure 1 is explained in a subsequent section.

Equations of Motion

The equations of motion are for a six-degree-of-freedom rigid body aircraft and are stated in detail in appendix A. The equations of motion are written in a general vector form for simplicity in formulating the parameter estimation algorithm.

The nonlinear equations of motion in vector form are

$$\dot{\vec{x}} = \vec{F}\left(\vec{x},\vec{\alpha},\vec{\delta}\right) = \left[F_1, F_2, \ldots, F_n\right]^T \tag{1}$$

where n is the dimension of the system ($n \leqq 12$).

The state vector is defined as

$$\vec{x} = \vec{x}(\vec{\alpha},t) = \left[x_1, x_2, \ldots, x_n\right]^T \qquad (2)$$

where $\vec{x}(\vec{\alpha},t)$ is the solution of the equations of motion and $\vec{x}(\vec{\alpha}^o,t) = \vec{x}^o(t)$ is the nominal solution.

The parameter vector, the components of which are the system parameters, is

$$\vec{\alpha} = \left[\alpha_1, \alpha_2, \ldots, \alpha_{p'}\right]^T \qquad (3)$$

where p' is the total number of system parameters and $\vec{\alpha}^o$ is the nominal parameter vector. These parameters are the aerodynamic coefficients (stability and control derivatives) and the initial conditions of the state. These parameters must be initialized for the first iteration of the algorithm. The coefficients are initialized by a prior estimate (wind-tunnel data or analysis) and the state is initialized from the flight test data.

The input to the system is

$$\vec{\delta} = \vec{\delta}(t) = \left[\delta_1, \delta_2, \ldots, \delta_{m'}\right]^T \qquad (4)$$

where $\vec{\delta}$ is the control deflection vector with dimension m'. The control vector $\vec{\delta}$ is set equal to the measured control input to the aircraft

$$\vec{\delta}(t_i) = \vec{\delta}^M(t_i) \qquad (i = 1, 2, \ldots, N) \qquad (5)$$

with linear interpolation between points.

Sensitivity Equations

The sensitivity equations form a basis for the method of quasilinearization and are derived by formally differentiating the equations of motion with respect to the parameters. The sensitivity equations are integrated in parallel with the equations of motion to yield the sensitivity coefficients, which reflect the sensitivity of the nominal solution with respect to each parameter. The method of quasilinearization uses these coefficients to linearize the change in the solution of the nonlinear equations of motion due to a change in the system parameters. Reference 19 describes the use of these parameter sensitivity coefficients in dynamic systems.

Sensitivity coefficients.- Used in the method of quasilinearization is a linear approximation expressing the change in the state vector as a linear function of the changes in the parameters. The expansion of the nominal solution about the nominal parameter vector, neglecting second and higher order terms, is

$$\vec{x}\left(\vec{\alpha}^{o}+\overrightarrow{\Delta\alpha}, t\right) = \vec{x}\left(\vec{\alpha}^{o}, t\right) + \sum_{i=1}^{p'}\left(\frac{\partial\vec{x}(t)}{\partial\alpha_i}\right)^{o} \Delta\alpha_i \tag{6}$$

where each partial derivative (sensitivity coefficient vector) is evaluated along the nominal solution.

The equation in matrix form is

$$\vec{x}\left(\vec{\alpha}^{o}+\overrightarrow{\Delta\alpha}, t\right) - \vec{x}\left(\vec{\alpha}^{o}, t\right) = A(t)\,\overrightarrow{\Delta\alpha} \tag{7}$$

where

$$A(t) = \left[\frac{\partial\vec{x}}{\partial\alpha_1}, \frac{\partial\vec{x}}{\partial\alpha_2}, \ldots, \frac{\partial\vec{x}}{\partial\alpha_{p'}}\right]^{o}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial\alpha_1} & \frac{\partial x_1}{\partial\alpha_2} & \cdots & \frac{\partial x_1}{\partial\alpha_{p'}} \\ \frac{\partial x_2}{\partial\alpha_1} & \frac{\partial x_2}{\partial\alpha_2} & \cdots & \frac{\partial x_2}{\partial\alpha_{p'}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_n}{\partial\alpha_1} & \frac{\partial x_n}{\partial\alpha_2} & \cdots & \frac{\partial x_n}{\partial\alpha_{p'}} \end{bmatrix}^{o}$$

and

$$\overrightarrow{\Delta\alpha} = \left[\Delta\alpha_1, \Delta\alpha_2, \ldots, \Delta\alpha_{p'}\right]^{T}$$

This matrix equation expresses the change in the nominal solution as a linear function of the parameter changes and the sensitivity coefficients. This equation is used in the expansion of the maximum likelihood function about the nominal parameter vector.

Derivation.- The sensitivity equations are derived from the equations of motion by taking the partial derivative of each equation with respect to each parameter. The sensitivity equations corresponding to the equations of motion in appendix A are derived in detail in appendix B.

By assuming that the derivatives are continuous in $\vec{\alpha}$ and t (ref. 20),

$$\frac{d}{dt}\left(\frac{\partial\vec{x}}{\partial\alpha_i}\right) = \frac{\partial}{\partial\alpha_i}\left(\frac{d\vec{x}}{dt}\right) \tag{8}$$

that is, the order of differentiation can be interchanged. This result is used to derive the sensitivity equation

$$\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) = \sum_{k=1}^{n} \frac{\partial \vec{F}}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \alpha_i} \qquad (i = 1, 2, \ldots, p') \qquad (9)$$

This equation in matrix form is

$$\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) = G(t)\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \alpha_i} \qquad (i = 1, 2, \ldots, p') \qquad (10)$$

where

$$G(t) = \left[\frac{\partial \vec{F}}{\partial x_1}, \frac{\partial \vec{F}}{\partial x_2}, \ldots, \frac{\partial \vec{F}}{\partial x_n}\right]$$

$$= \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial \vec{F}}{\partial \alpha_i} = \left[\frac{\partial F_1}{\partial \alpha_i}, \frac{\partial F_2}{\partial \alpha_i}, \ldots, \frac{\partial F_n}{\partial \alpha_i}\right]^T$$

This system of equations represents p' sets of n simultaneous first-order linear differential equations with time varying coefficients. The solutions of this system are the sensitivity coefficients $\partial x_k/\partial \alpha_i$ where $i = 1, 2, \ldots, p'$ and $k = 1, 2, \ldots, n$. These coefficients are the elements of the matrix $A(t)$ used in the linear approximation of the change in the nominal solution.

The initial conditions of the sensitivity coefficients corresponding to the aerodynamic parameters are

$$\left.\frac{\partial x_k}{\partial \alpha_i}\right|_{t=0} = 0$$

and the initial conditions of the sensitivity coefficients corresponding to the initial conditions of the state (parameters) are

$$\left.\frac{\partial x_k}{\partial \alpha_i}\right|_{t=0} = \left.\frac{\partial x_k}{\partial x_i(0)}\right|_{t=0} = \delta_{ik}$$

where δ_{ik} is the Kronecker delta.

Maximum Likelihood Estimation

The maximum likelihood method is used to estimate the stability and control parameters of the aircraft from flight test data. This method has the asymptotic properties of unbiased and minimum variance estimates (ref. 18). Maximum likelihood estimation requires initial parameter values to start the algorithm, but it assumes that the covariance matrix of the measurement noise is unknown.

Maximization of the likelihood function yields the parameter change equations and the covariance matrix for the parameters. These equations yield the changes in the nominal parameters to improve the fit between the measured and calculated variables of state. The covariance matrix gives the variances (or standard deviations) of the parameters, or in an inverse sense the sensitivity of the parameters in the equations of motion. This matrix also indicates the dependency or correlation among the parameters.

Maximization of the likelihood function also yields the covariance matrix for the measurement noise based on the current nominal solution. This matrix gives the variances (or standard deviations) of the difference of the measured state and the nominal solution. The inverse of this matrix is used in the parameter change equations as a weighting matrix. The performance index function is derived by substituting the covariance matrix of the measurement noise into the likelihood function (ref. 21). The performance index function derived is the determinant of the covariance matrix of the measurement noise which is defined as the criterion for the maximum likelihood method.

Measurement noise.- Let the measurement noise be

$$\begin{aligned}\vec{\hat{\eta}}(t_i) &= \vec{x}^M(t_i) - \vec{x}(\vec{\alpha},t_i) \\ &= \left[\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_n\right]^T \qquad (i = 1, 2, \ldots, N) \qquad (11)\end{aligned}$$

where N is the number of data points, $\vec{x}^M(t_i)$ is the measured data, $\vec{x}(\vec{\alpha},t_i)$ is the calculated solution for the true $\vec{\alpha}$ values, and $\vec{\hat{\eta}}(t_i)$ are independent Gaussian variables with zero mean. The noise is assumed to have the statistical properties of

$$E\left[\vec{\hat{\eta}}(t_i)\right] = 0 \qquad (12)$$

$$E\left[\vec{\hat{\eta}}(t_i)\vec{\hat{\eta}}^T(t_j)\right] = R_1\delta_{ij} \qquad (13)$$

where E denotes the mathematical expectation and R_1 is the unknown covariance matrix of the measurement noise.

Maximum likelihood function.- The maximum likelihood estimation of $\overrightarrow{\Delta\alpha}$ and R_1 is obtained by maximizing the likelihood function $L\left(\vec{\alpha}^o+\overrightarrow{\Delta\alpha}, R_1\right)$ with respect to $\overrightarrow{\Delta\alpha}$ and R_1, respectively. Although the term $\overrightarrow{\Delta\alpha}$ is not explicit in the function $L\left(\vec{\alpha},R_1\right)$, its use is evident in the maximization procedure.

The likelihood function for the state is (ref. 18)

$$L\left(\vec{\alpha},R_1\right) = -\frac{1}{2}\sum_{i=1}^{N}\left\|\vec{x}^M(t_i) - \vec{x}\left(\vec{\alpha},t_i\right)\right\|^2_{R_1^{-1}} - \frac{N}{2}\ln\left|R_1\right| \tag{14}$$

where $\left|R_1\right|$ denotes the determinant, R_1^{-1} denotes the inverse of the symmetric covariance matrix R_1, and $\|\vec{x}\|^2_{R_1^{-1}} = \vec{x}^T R_1^{-1}\vec{x}$.

Parameter change estimation.- The maximum likelihood function for the nominal solution is

$$L\left(\vec{\alpha}^o,R_1\right) = -\frac{1}{2}\sum_{i=1}^{N}\vec{\eta}^T(t_i)R_1^{-1}\vec{\eta}(t_i) - \frac{N}{2}\ln\left|R_1\right| \tag{15}$$

where

$$\vec{\eta}(t_i) = \vec{x}^M(t_i) - \vec{x}^o(t_i)$$

and R_1 is estimated from the nominal solution. The likelihood function is expressed for a change in the nominal solution by using equation (7) as

$$\begin{aligned}\vec{x}^M(t_i) - \vec{x}\left(\vec{\alpha}^o+\overrightarrow{\Delta\alpha}, t_i\right) &= \vec{x}^M(t_i) - \vec{x}^o(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\\ &= \vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\end{aligned} \tag{16}$$

Hence,

$$L\left(\vec{\alpha}^o+\overrightarrow{\Delta\alpha},R_1\right) = -\frac{1}{2}\sum_{i=1}^{N}\left\{\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]^T R_1^{-1}\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]\right\} - \frac{N}{2}\ln\left|R_1\right| \tag{17}$$

To maximize this equation, the partial derivatives of $L\left(\vec{\alpha}^o+\overrightarrow{\Delta\alpha},R_1\right)$ with respect to each parameter change $\Delta\alpha_i$ are set to zero; that is,

$$\begin{aligned}\frac{\partial L}{\partial\overrightarrow{\Delta\alpha}}\left(\vec{\alpha}^o+\overrightarrow{\Delta\alpha},R_1\right) &= -\frac{1}{2}\sum_{i=1}^{N}\left[-2A^T(t_i)R_1^{-1}\vec{\eta}(t_i) + 2A^T(t_i)R_1^{-1}A(t_i)\overrightarrow{\Delta\alpha}\right]\\ &= 0\end{aligned} \tag{18}$$

This operation implies that

$$\sum_{i=1}^{N}\left[A^T(t_i)R_1^{-1}A(t_i)\right]\overrightarrow{\Delta\alpha} = \sum_{i=1}^{N}\left[A^T(t_i)R_1^{-1}\vec{\eta}(t_i)\right] \tag{19}$$

In matrix form the equations are

$$\left[B^TW_1B\right]\overrightarrow{\Delta\alpha} = B^TW_1\vec{H} \tag{20}$$

where

$$B = \begin{bmatrix} A(t_1) \\ A(t_2) \\ \vdots \\ A(t_N) \end{bmatrix}_{(nN,p')} \qquad \vec{H} = \begin{bmatrix} \vec{\eta}(t_1) \\ \vec{\eta}(t_2) \\ \vdots \\ \vec{\eta}(t_N) \end{bmatrix}_{(nN,1)} \qquad W_1 = \begin{bmatrix} R_1 & & & 0 \\ & R_1 & & \\ & & \ddots & \\ 0 & & & R_1 \end{bmatrix}^{-1}_{(nN,nN)}$$

For $p' > n$, there exists more unknowns than equations; the system is overdetermined by calculating the nominal solution and sensitivity coefficients at N data points. The solution of $\overrightarrow{\Delta\alpha}$ is given by

$$\overrightarrow{\Delta\alpha} = \left[B^TW_1B\right]^{-1}B^TW_1\vec{H} \tag{21}$$

where the new nominal parameter vector is incremented by $\overrightarrow{\Delta\alpha}$. A necessary condition for this to be the estimate to maximize the likelihood function is that the matrix B^TW_1B must be positive definite.

Maximum likelihood methods (ref. 18) give asymptotically unbiased estimates and the inverse of $\left[B^TW_1B\right]$ is the error covariance matrix for the following estimated parameters:

$$E\left[\overrightarrow{\Delta\alpha}\right] = 0 \tag{22}$$

for

$$E\left[\vec{H}\right] = 0$$

and

$$E\left[\overrightarrow{\Delta\alpha}\;\overrightarrow{\Delta\alpha}^T\right] = \left[B^TW_1B\right]^{-1} \tag{23}$$

for

$$E\left[\vec{H}\vec{H}^T\right] = W_1^{-1}$$

The error covariance matrix for the estimated parameters is (ref. 22)

$$\left[B^T W_1 B\right]^{-1} = \begin{bmatrix} \sigma_{\alpha_1}^2 & \rho_{\alpha_1\alpha_2}\sigma_{\alpha_1}\sigma_{\alpha_2} & \cdots & \rho_{\alpha_1\alpha_{p'}}\sigma_{\alpha_1}\sigma_{\alpha_{p'}} \\ \rho_{\alpha_2\alpha_1}\sigma_{\alpha_2}\sigma_{\alpha_1} & \sigma_{\alpha_2}^2 & \cdots & \rho_{\alpha_2\alpha_{p'}}\sigma_{\alpha_2}\sigma_{\alpha_{p'}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \rho_{\alpha_{p'}\alpha_1}\sigma_{\alpha_{p'}}\sigma_{\alpha_1} & \rho_{\alpha_{p'}\alpha_2}\sigma_{\alpha_{p'}}\sigma_{\alpha_2} & \cdots & \sigma_{\alpha_{p'}}^2 \end{bmatrix} \tag{24}$$

where $\sigma_{\alpha_i}^2$ is the variance of the estimate, σ_{α_i} is the standard deviation, and $\rho_{\alpha_i\alpha_j}$ is the correlation coefficient for α_i and α_j.

Estimation of measurement noise statistics.- The likelihood function of equation (17) for the change in the nominal solution is

$$L\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, R_1\right) = -\frac{1}{2}\sum_{i=1}^{N}\left\{\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]^T R_1^{-1}\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]\right\} - \frac{N}{2}\ln\left|R_1\right| \tag{25}$$

where R_1 denotes the unknown covariance matrix. The estimated value of the covariance matrix R_1 is denoted by $R_1'(N)$.

In a similar maximization procedure as for $\overrightarrow{\Delta\alpha}$, the likelihood function is maximized with respect to R_1. The derivative with respect to a matrix is defined as follows:

$$\frac{\partial L}{\partial R_1} = \left[\frac{\partial L}{\partial r_{ij}}\right] \qquad \left(R_1 = \left[r_{ij}\right]\right)$$

The derivative of $L\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, R_1\right)$ with respect to R_1 is set to zero. But (ref. 21)

$$\frac{\partial L}{\partial R_1}\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, R_1\right) = 0 \tag{26}$$

is equivalent to

$$\frac{\partial L}{\partial R_1^{-1}}\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, R_1\right) = 0 \tag{27}$$

and, hence,

$$\frac{N}{2}\left\{R_1'(N) - \frac{1}{N}\sum_{i=1}^{N}\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]^T\right\} = 0 \tag{28}$$

or

$$R_1'(N) = \frac{1}{N}\sum_{i=1}^{N}\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]^T \tag{29}$$

where $R_1'(N)$ is the predicted estimate of the covariance matrix for N data points due to the change in the nominal solution.

The equation for calculating the estimate of the covariance matrix used in this algorithm is

$$R_1^o(N) \triangleq \text{Estimate of } R_1$$

$$= \frac{1}{N}\sum_{i=1}^{N} \vec{\eta}(t_i)\vec{\eta}^T(t_i) \tag{30}$$

The schemes for calculating the estimate of R_1 are discussed later.

The matrix $R_1^o(N)$ is written as

$$R_1^o(N) = \begin{bmatrix} \sigma_{\eta_1}^2 & \sigma_{\eta_1\eta_2} & \cdots & \sigma_{\eta_1\eta_n} \\ \sigma_{\eta_2\eta_1} & \sigma_{\eta_2}^2 & \cdots & \sigma_{\eta_2\eta_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \sigma_{\eta_n\eta_1} & \sigma_{\eta_n\eta_2} & \cdots & \sigma_{\eta_n}^2 \end{bmatrix} \tag{31}$$

where $\sigma_{\eta_k}^2$ is the variance of η_k, and $\sigma_{\eta_k\eta_j}$ is the covariance of η_k and η_j.

Performance index.- The performance index or index function evaluation gives a measure of performance for the iterative estimation procedure. Selection of the index function for the maximum likelihood estimator is an important condition as to whether the estimation procedure converges to the true parameter values. (See ref. 21.)

The index function is derived from the likelihood function (ref. 21). Substituting the covariance matrix estimate $R_1^o(N)$ (eq. (30)) into the likelihood function $L\left(\vec{\alpha}^o, R_1\right)$ (eq. (15)) gives

$$L\left(\vec{\alpha}^o, R_1^o(N)\right) = -\frac{1}{2}(nN) - \frac{N}{2} \ln \left|R_1^o(N)\right| \tag{32}$$

Thus maximization of the likelihood function is equivalent to the minimization of

$$\begin{aligned} J_N\left(\vec{\alpha}^o\right) &= \left|R_1^o(N)\right| \\ &= \det\left\{\frac{1}{N}\sum_{i=1}^{N} \vec{\eta}(t_i)\vec{\eta}^T(t_i)\right\} \end{aligned} \tag{33}$$

which is defined as the index function for the maximum likelihood estimation procedure. The minimization of $R_1'(N)$ with respect to $\overrightarrow{\Delta\alpha}$ yields equivalent parameter change equations as in equation (19).

Flight Test Data

The flight test data are composed of the onboard instrument measurements of the aircraft behavior and are assumed to be the output of the aircraft mathematical model superimposed with instrument noise. These data contain many individual aircraft maneuvers stored on one magnetic tape, with each maneuver easily accessible to the central memory of the computer. These data are used for comparison with the mathematical model output and for initialization of and control input to the equations of motion. The measurements, $\vec{x}^M(t_i)$ and $\vec{\delta}^M(t_i)$ for $i = 1, 2, \ldots, N$, are known for all states and control deflections corresponding to the equations of motion.

Steps in Procedure

The steps in the maximum likelihood estimation procedure, corresponding to figure 1, are as follows:

(1) Initialize the system parameters, where $\vec{\alpha}^o$ denotes the nominal or current values of the parameters.

(2) Integrate the equations of motion and the sensitivity equations to obtain the nominal solution and the sensitivity coefficient matrix, respectively.

(3) Form the comparisons of the flight test data and nominal solution for each data point time t_i, where $i = 1, 2, \ldots, N$ and $t_1 = 0$ and $t_N = T$.

(4) Form the maximum likelihood estimation equations from the comparisons in step (3) and the sensitivity coefficient matrix in step (2) (dash lines indicate accumulation of information over the flight test time period T).

(5) Calculate the performance index $J_N(\vec{\alpha}^o)$.

(6) Calculate the parameter changes $\overrightarrow{\Delta\alpha}$ and the statistical information matrix $R_1^o(N)$.

(7) Update the nominal parameter values in step (1) to start the next iteration procedure and repeat steps until convergence.

Each iteration of the procedure extends over the flight test time period T and results in the update of the parameters. Evaluations within the period are at specified data point times t_i, where the intervals $t_{i+1} - t_i$ are integer multiples of the integration step size. The integration step size is made compatible with the flight data intervals and the problem dynamics.

Acceleration measurements and equations were added later to improve the estimation procedure and the addition is presented in appendix C.

COMPUTATIONAL CONSIDERATIONS

The flight tests do not always necessitate the use of all the state variables and parameters in the estimation algorithm (n state variables and p' parameters) for specific cases. These cases involve only a specific portion of the program, as with an excitation of only the longitudinal motion of the aircraft. The total number of differential equations in the algorithm for evaluation and integration is $n + np'$; in appendixes A and B, $n = 8$ and $p' = 40$.

The parameter estimation algorithm was programed with a variable dimensioning capability with respect to the number of state variables and parameters necessary for any specific case. The analyst through the operational control features (appendix D) could select any subsets of the state variables and parameters to be active in the parameter estimation algorithm. Computer program parameters were activated for each state variable and each parameter desired. This operation generated two sequences of numbers specifying the state variables and parameters which were active in the algorithm and neglected the remaining ones.

This variable dimensioning of the algorithm allowed flexibility in the parameter identification study in that the analyst could alter the program easily for each specific aircraft maneuver or computer run. In addition, the number of integrations was reduced.

In the calculation of $\overrightarrow{\Delta\alpha}$ the matrix equations (eq. (20)) were

$$\left[B^T W_1 B\right] \overrightarrow{\Delta\alpha} = B^T W_1 \vec{H}$$

If the matrix B and the vector $\vec{H}$ were calculated for N data points, this would result in the storage of an $nN \times p'$ matrix and an nN vector. The storage would increase as the number of data points N increased. To eliminate the storage being dependent on N, equation (19), which is restated, was used:

$$\sum_{i=1}^{N} \left[A^T(t_i) R_1^{-1} A(t_i)\right] \overrightarrow{\Delta\alpha} = \sum_{i=1}^{N} \left[A^T(t_i) R_1^{-1} \vec{\eta}(t_i)\right]$$

where

$$B^T W_1 B = \sum_{i=1}^{N} A^T(t_i) R_1^{-1} A(t_i) \tag{34}$$

and

$$B^T W_1 \vec{H} = \sum_{i=1}^{N} A^T(t_i) R_1^{-1} \vec{\eta}(t_i) \tag{35}$$

In these equations the matrix products are formed as a function of time and the dimensions of the matrix products were not a function of the number of data points N. In fact, the dimensions depended only on p' and n, the number of parameters and state variables.

In the calculation of the measurement noise covariance matrix, three computational schemes can be used. The matrix can be updated (1) on the same iteration as $\overrightarrow{\Delta\alpha}$, (2) one iteration behind $\overrightarrow{\Delta\alpha}$, and (3) with a two step procedure.

Scheme (1) uses equation (29):

$$R_1'(N) = \frac{1}{N} \sum_{i=1}^{N} \left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]\left[\vec{\eta}(t_i) - A(t_i)\overrightarrow{\Delta\alpha}\right]^T$$

In this equation the matrix $A(t_i)$ and the vector $\vec{\eta}(t_i)$ must be stored for each increment of time until $\overrightarrow{\Delta\alpha}$ is calculated.

Scheme (2) is easy to incorporate into the program by using equation (30):

$$R_1^o(N) \triangleq \text{Estimate of } R_1$$

$$= \frac{1}{N} \sum_{i=1}^{N} \vec{\eta}(t_i) \vec{\eta}^T(t_i)$$

In this equation the covariance matrix is calculated from the nominal solution and not with the predicted change in the nominal solution. The matrix is one iteration behind in the algorithm but the effect is negligible when the change in the solution is small, that is, when convergence is achieved.

Scheme (3) uses a two step process for each parameter update. The first step is to calculate $\overrightarrow{\Delta\alpha}$ in the usual manner. The second step is to update the parameters and integrate only the equations of motion. Thus, the covariance matrix is

$$R_1(N) = \frac{1}{N} \sum_{i=1}^{N} \left[\vec{x}^M(t_i) - \vec{x}\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, t_i\right)\right]\left[\vec{x}^M(t_i) - \vec{x}\left(\vec{\alpha}^o + \overrightarrow{\Delta\alpha}, t_i\right)\right]^T \tag{36}$$

Scheme (3) is similar to scheme (1) in that scheme (1) uses the predicted change, whereas scheme (3) uses the calculated change in the nominal solution. Scheme (3) approaches scheme (2) when $\overrightarrow{\Delta\alpha}$ approaches zero.

Scheme (2) was used in the parameter estimation program with the option of using scheme (3). In test cases using schemes (2) and (3), the indication was that the difference is not significant.

TEST PROCEDURE

The testing procedure used pseudo flight data in checking the maximum likelihood estimation algorithm. The data were generated by integrating the equations of motion and then adding measurement noise, all states assumed being measured. The measurement noise was sequences of pseudorandom numbers (random within the capability of a digital computer) having the normal (Gaussian) distribution with zero mean and known standard deviation. These data were assumed to be the flight test data or measured data for specified parameter values. The parameter values were then offset to become the nominal parameter values for the parameter estimation algorithm.

Test cases using the pseudo flight data were conducted for the longitudinal motion of the aircraft and are presented in appendix E. The maximum likelihood algorithm computed the standard deviation of the measurement noise and the parameter values and their standard deviations; no statistical information was assumed concerning the noise. Results obtained from the test cases indicated that the calculated parameter values and standard deviations of the noise were converging to the true values.

CONCLUDING REMARKS

A maximum likelihood parameter estimation procedure and program have been developed and validated for the extraction of the stability and control derivatives of aircraft from flight test data. A nonlinear six-degree-of-freedom aircraft mathematical model was used in the derivation of the sensitivity equations. Instrument measurement noise was accounted for by the maximum likelihood estimator. The program was developed for 8 measured state variables and 40 parameters, from which subsets could be selected for program operation. Real-time digital simulation and graphic display provided the analyst with interactive control and display capabilities during the study. The program has been applied to a V/STOL tilt-wing aircraft, a military fighter airplane, and a light single-engine airplane.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., April 6, 1972.

APPENDIX A

EQUATIONS OF MOTION

The equations of motion are stated for reference and in particular for the derivation of the sensitivity equations in appendix B. The following equations of motion are for a V/STOL tilt-wing aircraft:

$$\dot{u} = -qw + rv - g \sin \theta + \frac{1}{2} \frac{\rho}{m} V^2 S\left(C_{X,o} + C_{X_{\alpha_a}} \alpha_a + C_{X_q} \frac{q\bar{c}}{2V}\right) + \frac{T_X}{m} \tag{A1}$$

$$\dot{v} = -ru + pw + g \cos \theta \sin \phi + \frac{1}{2} \frac{\rho}{m} V^2 S\left(C_{Y,o} + C_{Y_\beta}\beta + C_{Y_{\dot{\beta}}} \frac{\dot{\beta} b}{2V} + C_{Y_p} \frac{pb}{2V} + C_{Y_r} \frac{rb}{2V} + C_{Y_{\delta_r}} \delta_r\right) + \frac{T_Y}{m} \tag{A2}$$

$$\dot{w} = -pv + qu + g \cos \theta \cos \phi + \frac{1}{2} \frac{\rho}{m} V^2 S\left(C_{Z,o} + C_{Z_{\alpha_a}} \alpha_a + C_{Z_q} \frac{q\bar{c}}{2V} + C_{Z_{\delta_e}} \delta_e\right) + \frac{T_Z}{m} \tag{A3}$$

$$\dot{p} = qr\left(\frac{I_Y - I_Z}{I_X}\right) + \frac{I_{XZ}}{I_X}(pq + \dot{r}) + \frac{1}{2} \frac{\rho}{I_X} V^2 Sb\left(C_{l,o} + C_{l_\beta}\beta + C_{l_{\dot{\beta}}} \frac{\dot{\beta} b}{2V} + C_{l_p} \frac{pb}{2V} + C_{l_r} \frac{rb}{2V} + C_{l_{\delta_r}} \delta_r\right) + \frac{1}{2} \frac{\rho}{I_X}(V_{SS} + V)^2 Sb\left(C_{l_{\delta_a}} \cos i_w - C_{n_{\delta_a}} \sin i_w\right)\delta_a + \frac{M_X}{I_X} \tag{A4}$$

$$\dot{q} = pr\left(\frac{I_Z - I_X}{I_Y}\right) + \frac{I_{XZ}}{I_Y}(r^2 - p^2) + \frac{1}{2} \frac{\rho}{I_Y} V^2 S\bar{c}\left(C_{m,o} + C_{m_{\alpha_a}} \alpha_a + C_{m_{\dot{\alpha}_a}} \frac{\dot{\alpha}_a \bar{c}}{2V} + C_{m_q} \frac{q\bar{c}}{2V} + C_{m_{\delta_e}} \delta_e\right) + \frac{M_Y}{I_Y} \tag{A5}$$

$$\dot{r} = pq\left(\frac{I_X - I_Y}{I_Z}\right) + \frac{I_{XZ}}{I_Z}(\dot{p} - qr) + \frac{1}{2} \frac{\rho}{I_Z} V^2 Sb\left(C_{n,o} + C_{n_\beta}\beta + C_{n_{\dot{\beta}}} \frac{\dot{\beta} b}{2V} + C_{n_p} \frac{pb}{2V} + C_{n_r} \frac{rb}{2V} + C_{n_{\delta_r}} \delta_r\right) - \frac{1}{2} \frac{\rho}{I_Z}(V_{SS} + V)^2 Sb\left(C_{l_{\delta_a}} \sin i_w + C_{n_{\delta_a}} \cos i_w\right)\delta_a + \frac{M_Z}{I_Z} \tag{A6}$$

$$\dot{\theta} = q \cos \phi - r \sin \phi \tag{A7}$$

$$\dot{\phi} = p + (q \sin \phi + r \cos \phi)\tan \theta \tag{A8}$$

and

$$\left.\begin{aligned}
\dot{\psi} &= \frac{q \sin \phi + r \cos \phi}{\cos \theta} \\
V &= \sqrt{u^2 + v^2 + w^2} \\
\alpha_a &= \tan^{-1} \frac{w}{u} \\
\dot{\alpha}_a &\approx \frac{\dot{w}}{u} \\
\beta &= \sin^{-1} \frac{v}{V} \\
\dot{\beta} &\approx \frac{\dot{v}}{V}
\end{aligned}\right\} \tag{A9}$$

Trim conditions of the aircraft are added by substituting, respectively, for α_a, β, δ_a, δ_e, and δ_r in equations (A1) to (A6) the following terms:

$$\alpha_a - \alpha_{a,t}$$
$$\beta - \beta_t$$
$$\delta_a - \delta_{a,t}$$
$$\delta_e - \delta_{e,t}$$
$$\delta_r - \delta_{r,t}$$

where the subscript t denotes the trim conditions (values) of the aircraft.

The equations of motion are altered in two ways: (1) solve for $\dot{v}$ (eq. (A2)) explicitly, and (2) decouple the $\dot{p}$ (eq. (A4)) and $\dot{r}$ (eq. (A6)) equations. The equations of motion are then written in simplified notation for use in the derivation of the sensitivity equations.

The equations of motion are

$$\begin{aligned}
\dot{\vec{x}} &= \vec{F}\left(\vec{x}, \vec{\alpha}, \vec{\delta}, V, \alpha_a, \dot{\alpha}_a, \beta, \dot{\beta}\right) \\
&= \left[F_1, F_2, \ldots, F_8\right]^T
\end{aligned} \tag{A10}$$

The state vector is

$$\vec{x} = \left[x_1, x_2, \ldots, x_8\right]^T$$
$$= \left[u, v, w, p, q, r, \theta, \phi\right]^T \tag{A11}$$

The parameter vector is

$$\vec{\alpha} = \left[\alpha_1, \alpha_2, \ldots, \alpha_{40}\right]^T$$
$$= \left[u(0), C_{X,o}, \ldots, C_{n_{\delta_a}}\right]^T \tag{A12}$$

The control deflection vector is

$$\vec{\delta} = \left[\delta_a, \delta_e, \delta_r\right]^T \tag{A13}$$

Equations (A1) through (A8) are written as

$$\dot{u} = F_1\left(\vec{x}, \vec{\alpha}, \vec{\delta}, V, \alpha_a\right)$$
$$= -qw + rv - g \sin \theta + a_1 V^2 \left(C_{X1} + C_{X2}\right) + \frac{T_X}{m}$$
$$= -qw + rv - g \sin \theta + a_1 V^2 \left(C_{X,o} + C_{X_{\alpha_a}} \alpha_a\right) + a_2 V \left(C_{X_q} q\right) + \frac{T_X}{m} \tag{A14}$$

$$\dot{v} = F_2\left(\vec{x}, \vec{\alpha}, \vec{\delta}, V, \beta\right)$$
$$= \frac{-ru + pw + g \cos \theta \sin \phi + a_1 V^2 \left(C_{Y1} + C_{Y2}\right) + \frac{T_Y}{m}}{1 - a_3 C_{Y_{\dot{\beta}}}}$$
$$= \frac{-ru + pw + g \cos \theta \sin \phi + a_1 V^2 \left(C_{Y,o} + C_{Y_\beta} \beta + C_{Y_{\delta_r}} \delta_r\right) + a_3 V \left(C_{Y_p} p + C_{Y_r} r\right) + \frac{T_Y}{m}}{1 - a_3 C_{Y_{\dot{\beta}}}} \tag{A15}$$

$$\dot{w} = F_3\left(\vec{x}, \vec{\alpha}, \vec{\delta}, V, \alpha_a\right)$$
$$= -pv + qu + g \cos \theta \cos \phi + a_1 V^2 \left(C_{Z1} + C_{Z2}\right) + \frac{T_Z}{m}$$
$$= -pv + qu + g \cos \theta \cos \phi + a_1 V^2 \left(C_{Z,o} + C_{Z_{\alpha_a}} \alpha_a + C_{Z_{\delta_e}} \delta_e\right) + a_2 V \left(C_{Z_q} q\right) + \frac{T_Z}{m} \tag{A16}$$

$$\dot{p} = F_4\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right)$$

$$= a_6F_4'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right) + b_1F_6'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right)$$

$$= a_6\left[b_2qr + I_{XZ}pq + a_4V^2\left(C_{l1} + C_{l2}\right) + a_4V_S^{\,2}\left(C_{l3}\right) + M_X\right]$$

$$+ b_1\left[b_3pq - I_{XZ}qr + a_4V^2\left(C_{n1} + C_{n2}\right) - a_4V_S^{\,2}\left(C_{n3}\right) + M_Z\right]$$

$$= a_6\left[b_2qr + I_{XZ}pq + a_4V^2\left(C_{l,o} + C_{l_\beta}\beta + C_{l_{\delta_r}}\delta_r\right) + a_5V\left(C_{l_{\dot{\beta}}}\dot{\beta} + C_{l_p}p + C_{l_r}r\right)\right.$$

$$\left. + a_4V_S^{\,2}\left(C'_{l_{\delta_a}}\delta_a\right) + M_X\right] + b_1\left[b_3pq - I_{XZ}qr + a_4V^2\left(C_{n,o} + C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r\right)\right.$$

$$\left. + a_5V\left(C_{n_{\dot{\beta}}}\dot{\beta} + C_{n_p}p + C_{n_r}r\right) - a_4V_S^{\,2}\left(C'_{n_{\delta_a}}\delta_a\right) + M_Z\right] \qquad \text{(A17)}$$

$$\dot{q} = F_5\left(\vec{x},\vec{\alpha},\vec{\delta},V,\alpha_a,\dot{\alpha}_a\right)$$

$$= a_7pr + b_4\left(r^2 - p^2\right) + a_8V^2\left(C_{m1} + C_{m2}\right) + \frac{M_Y}{I_Y}$$

$$= a_7pr + b_4\left(r^2 - p^2\right) + a_8V^2\left(C_{m,o} + C_{m_{\alpha_a}}\alpha_a + C_{m_{\delta_e}}\delta_e\right)$$

$$+ a_9V\left(C_{m_{\dot{\alpha}_a}}\dot{\alpha}_a + C_{m_q}q\right) + \frac{M_Y}{I_Y} \qquad \text{(A18)}$$

$$\dot{r} = F_6\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right)$$

$$= b_1F_4'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right) + b_5F_6'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right) \qquad \text{(A19)}$$

$$\dot{\theta} = F_7(\vec{x})$$

$$= q \cos \phi - r \sin \phi \qquad \text{(A20)}$$

$$\dot{\phi} = F_8(\vec{x})$$

$$= p + (q \sin \phi + r \cos \phi)\tan \theta \qquad \text{(A21)}$$

where

$$a_1 = \frac{1}{2}\frac{\rho}{m}S \qquad a_2 = a_1\left(\frac{\bar{c}}{2}\right) \qquad a_3 = a_1\left(\frac{b}{2}\right)$$

$$a_4 = \frac{1}{2}\rho Sb \qquad a_5 = a_4\left(\frac{b}{2}\right) \qquad a_6 = \frac{I_Z}{I_X I_Z - I_{XZ}^2}$$

$$a_7 = \frac{I_Z - I_X}{I_Y} \qquad a_8 = \frac{1}{2}\frac{\rho}{I_Y}S\bar{c} \qquad a_9 = a_8\left(\frac{\bar{c}}{2}\right)$$

$$b_1 = \frac{I_{XZ}}{I_X I_Z - I_{XZ}^2} \qquad b_2 = I_Y - I_Z \qquad b_3 = I_X - I_Y$$

$$b_4 = \frac{I_{XZ}}{I_Y} \qquad b_5 = \frac{I_X}{I_X I_Z - I_{XZ}^2}$$

and

$$V_S = V_{SS} + V$$

$$C_{X1} = C_{X,o} + C_{X_{\alpha_a}}\alpha_a \qquad C_{X2} = C_{X_q}\frac{q\bar{c}}{2V}$$

$$C_{Y1} = C_{Y,o} + C_{Y_\beta}\beta + C_{Y_{\delta_r}}\delta_r \qquad C_{Y2} = \frac{b}{2V}\left(C_{Y_p}p + C_{Y_r}r\right)$$

$$C_{Z1} = C_{Z,o} + C_{Z_{\alpha_a}}\alpha_a + C_{Z_{\delta_e}}\delta_e \qquad C_{Z2} = C_{Z_q}\frac{q\bar{c}}{2V}$$

$$C_{l1} = C_{l,o} + C_{l_\beta}\beta + C_{l_{\delta_r}}\delta_r \qquad C_{l2} = \frac{b}{2V}\left(C_{l_{\dot{\beta}}}\dot{\beta} + C_{l_p}p + C_{l_r}r\right)$$

$$C'_{l_{\delta_a}} = C_{l_{\delta_a}}\cos i_w - C_{n_{\delta_a}}\sin i_w \qquad C_{l3} = C'_{l_{\delta_a}}\delta_a$$

$$C_{m1} = C_{m,o} + C_{m_{\alpha_a}}\alpha_a + C_{m_{\delta_e}}\delta_e \qquad C_{m2} = \frac{\bar{c}}{2V}\left(C_{m_{\dot{\alpha}_a}}\dot{\alpha}_a + C_{m_q}q\right)$$

$$C_{n1} = C_{n,o} + C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r \qquad C_{n2} = \frac{b}{2V}\left(C_{n_{\dot{\beta}}}\dot{\beta} + C_{n_p}p + C_{n_r}r\right)$$

$$C'_{n_{\delta_a}} = C_{l_{\delta_a}}\sin i_w + C_{n_{\delta_a}}\cos i_w \qquad C_{n3} = C'_{n_{\delta_a}}\delta_a$$

APPENDIX B

DETAILS OF DERIVATION OF SENSITIVITY EQUATIONS

The sensitivity equations for the method of quasilinearization are derived in detail for the equations of motion in appendix A.

The sensitivity equations are

$$\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) = \left(\frac{\partial \dot{\vec{x}}}{\partial \alpha_i}\right)$$

$$= \left\{\sum_{k=1}^{8} \frac{\partial \vec{F}}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial V}\sum_{k=1}^{3}\frac{\partial V}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \alpha_a}\sum_{k=1}^{3}\frac{\partial \alpha_a}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right)\right.$$

$$+ \frac{\partial \vec{F}}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial x_1}\left(\frac{\partial x_1}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \beta}\left[\frac{\partial \beta}{\partial x_2}\left(\frac{\partial x_2}{\partial \alpha_i}\right) + \frac{\partial \beta}{\partial V}\sum_{k=1}^{3}\frac{\partial V}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right)\right]$$

$$\left. + \frac{\partial \vec{F}}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial V}\sum_{k=1}^{3}\frac{\partial V}{\partial x_k}\left(\frac{\partial x_k}{\partial \alpha_i}\right)\right\} + \left[\frac{\partial \vec{F}}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3}\left(\frac{\partial \dot{x}_3}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\left(\frac{\partial \dot{x}_2}{\partial \alpha_i}\right)\right] + \frac{\partial \vec{F}}{\partial \alpha_i}$$

$$= G'(t)\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) + \left[\frac{\partial \vec{F}}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3}\left(\frac{\partial \dot{x}_3}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\left(\frac{\partial \dot{x}_2}{\partial \alpha_i}\right)\right] + \frac{\partial \vec{F}}{\partial \alpha_i} \qquad \text{(B1)}$$

where

$$G'(t) = \left[g'_{jk}(t)\right] \qquad (j, k = 1, 2, \ldots, 8)$$

The functions F_2 and F_3 do not contain $\dot{\alpha}_a$ or $\dot{\beta}$. Thus,

$$\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) = \left[G'(t)\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3}\sum_{k=1}^{8} g'_{3k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial \vec{F}}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\sum_{k=1}^{8} g'_{2k}\left(\frac{\partial x_k}{\partial \alpha_i}\right)\right]$$

$$+ \left(\frac{\partial \vec{F}}{\partial \alpha_i} + \frac{\partial \vec{F}}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3}\frac{\partial F_3}{\partial \alpha_i} + \frac{\partial \vec{F}}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\frac{\partial F_2}{\partial \alpha_i}\right)$$

$$= G(t)\left(\frac{\partial \vec{x}}{\partial \alpha_i}\right) + \vec{\bar{F}}(\alpha_i) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B2)}$$

where

$$\frac{\partial \vec{x}}{\partial \alpha_i} = \left[\frac{\partial u}{\partial \alpha_i}, \frac{\partial v}{\partial \alpha_i}, \frac{\partial w}{\partial \alpha_i}, \frac{\partial p}{\partial \alpha_i}, \frac{\partial q}{\partial \alpha_i}, \frac{\partial r}{\partial \alpha_i}, \frac{\partial \theta}{\partial \alpha_i}, \frac{\partial \phi}{\partial \alpha_i}\right]^T$$

$$G(t) = \left[g_{jk}(t)\right] \qquad (j, k = 1, 2, \ldots, 8)$$

$$\vec{\tilde{F}}(\alpha_i) = \left[\tilde{F}_1(\alpha_i), \tilde{F}_2(\alpha_i), \ldots, \tilde{F}_8(\alpha_i)\right]^T$$

The following equations are used in the derivation of the sensitivity equations:

(1-8) (1-6) (1,3,5) (5) (2,4,6) (2,4,6)

$$g_{jk} = \left(\frac{\partial F_j}{\partial x_k} + \frac{\partial F_j}{\partial V}\frac{\partial V}{\partial x_k} + \frac{\partial F_j}{\partial \alpha_a}\frac{\partial \alpha_a}{\partial x_k} + \frac{\partial F_j}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial x_k} + \frac{\partial F_j}{\partial \beta}\frac{\partial \beta}{\partial x_k} + \frac{\partial F_j}{\partial \beta}\frac{\partial \beta}{\partial V}\frac{\partial V}{\partial x_k}\right.$$

(4,6) (5) (4,6)

$$\left. + \frac{\partial F_j}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial V}\frac{\partial V}{\partial x_k}\right) + \left(\frac{\partial F_j}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3} g_{3k} + \frac{\partial F_j}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} g_{2k}\right)$$

$$= g'_{jk} + \frac{\partial F_j}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3} g_{3k} + \frac{\partial F_j}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} g_{2k} \qquad (j, k = 1, 2, \ldots, 8) \qquad \text{(B3)}$$

and

(1-6) (5) (4,6)

$$\tilde{F}_j(\alpha_i) = \frac{\partial F_j}{\partial \alpha_i} + \frac{\partial F_j}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{x}_3}\frac{\partial F_3}{\partial \alpha_i} + \frac{\partial F_j}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\frac{\partial F_2}{\partial \alpha_i} \qquad \begin{pmatrix} j = 1, 2, \ldots, 8; \\ i = 1, 2, \ldots, p' \end{pmatrix} \qquad \text{(B4)}$$

The numbers in the parentheses above each term indicate the derivations in which they are used; this is in reference to the equations of motion used in the derivation of sensitivity equations.

(1) Sensitivity equations derived from $\dot{u}$ equation (eq. (A14)):

$$\frac{d}{dt}\left(\frac{\partial u}{\partial \alpha_i}\right) = \sum_{k=1}^{8} g_{1k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial F_1}{\partial \alpha_i} \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B5)}$$

where

$$g_{11} = a_1 u\left(2C_{X1} + C_{X2}\right) - a_1 V^2 C_{X_{\alpha_a}} \frac{w}{u^2 + w^2}$$

$$g_{12} = r + a_1 v\left(2C_{X1} + C_{X2}\right)$$

$$g_{13} = -q + a_1 w\left(2C_{X1} + C_{X2}\right) + a_1 V^2 C_{X_{\alpha_a}} \frac{u}{u^2 + w^2}$$

$$g_{14} = 0$$

$$g_{15} = -w + a_2 V C_{X_q}$$

$$g_{16} = v$$

$$g_{17} = -g \cos \theta$$

$$g_{18} = 0$$

and

$$\frac{\partial F_1}{\partial C_{X,o}} = a_1 V^2 \qquad \frac{\partial F_1}{\partial C_{X_{\alpha_a}}} = a_1 V^2 \alpha_a \qquad \frac{\partial F_1}{\partial C_{X_q}} = a_2 V q$$

(2) Sensitivity equations derived from $\dot{v}$ equation (eq. (A15)):

$$\frac{d}{dt}\left(\frac{\partial v}{\partial \alpha_i}\right) = \sum_{k=1}^{8} g_{2k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial F_2}{\partial \alpha_i} \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B6)}$$

where

$$g_{21} = \frac{-r + a_1 u\left[2C_{Y1} + C_{Y2} - C_{Y_\beta} v\left(u^2 + w^2\right)^{-1/2}\right]}{1 - a_3 C_{Y_{\dot{\beta}}}}$$

$$g_{22} = \frac{a_1 v\left[2C_{Y1} + C_{Y2} - C_{Y_\beta} v\left(u^2 + w^2\right)^{-1/2}\right] + a_1 V^2 C_{Y_\beta}\left(u^2 + w^2\right)^{-1/2}}{1 - a_3 C_{Y_{\dot{\beta}}}}$$

$$g_{23} = \frac{p + a_1 w\left[2C_{Y1} + C_{Y2} - C_{Y_\beta} v\left(u^2 + w^2\right)^{-1/2}\right]}{1 - a_3 C_{Y_{\dot{\beta}}}}$$

$$g_{24} = \frac{w + a_3VC_{Y_p}}{1 - a_3C_{Y_{\dot{\beta}}}}$$

$$g_{25} = 0$$

$$g_{26} = \frac{-u + a_3VC_{Y_r}}{1 - a_3C_{Y_{\dot{\beta}}}}$$

$$g_{27} = \frac{-g \sin\theta \sin\phi}{1 - a_3C_{Y_{\dot{\beta}}}}$$

$$g_{28} = \frac{g \cos\theta \cos\phi}{1 - a_3C_{Y_{\dot{\beta}}}}$$

and

$$\frac{\partial F_2}{\partial C_{Y,o}} = \frac{a_1V^2}{1 - a_3C_{Y_{\dot{\beta}}}} \qquad \frac{\partial F_2}{\partial C_{Y_\beta}} = \frac{a_1V^2\beta}{1 - a_3C_{Y_{\dot{\beta}}}} \qquad \frac{\partial F_2}{\partial C_{Y_{\dot{\beta}}}} = \frac{a_3\dot{v}}{1 - a_3C_{Y_{\dot{\beta}}}}$$

$$\frac{\partial F_2}{\partial C_{Y_p}} = \frac{a_3Vp}{1 - a_3C_{Y_{\dot{\beta}}}} \qquad \frac{\partial F_2}{\partial C_{Y_r}} = \frac{a_3Vr}{1 - a_3C_{Y_{\dot{\beta}}}} \qquad \frac{\partial F_2}{\partial C_{Y_{\delta_r}}} = \frac{a_1V^2\delta_r}{1 - a_3C_{Y_{\dot{\beta}}}}$$

(3) Sensitivity equations derived from $\dot{w}$ equation (eq. (A16)):

$$\frac{d}{dt}\left(\frac{\partial w}{\partial \alpha_i}\right) = \sum_{k=1}^{8} g_{3k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \frac{\partial F_3}{\partial \alpha_i} \qquad (i = 1, 2, \ldots, 40) \tag{B7}$$

where

$$g_{31} = q + a_1u\left(2C_{Z1} + C_{Z2}\right) - a_1V^2C_{Z_{\alpha_a}}\frac{w}{u^2 + w^2}$$

$$g_{32} = -p + a_1v\left(2C_{Z1} + C_{Z2}\right)$$

$$g_{33} = a_1w\left(2C_{Z1} + C_{Z2}\right) + a_1V^2C_{Z_{\alpha_a}}\frac{u}{u^2 + w^2}$$

$$g_{34} = -v$$

$$g_{35} = u + a_2VC_{Z_q}$$

$$g_{36} = 0$$

$$g_{37} = -g \sin \theta \cos \phi$$

$$g_{38} = -g \cos \theta \sin \phi$$

and

$$\frac{\partial F_3}{\partial C_{Z,o}} = a_1 V^2 \qquad \frac{\partial F_3}{\partial C_{Z_{\alpha_a}}} = a_1 V^2 \alpha_a \qquad \frac{\partial F_3}{\partial C_{Z_q}} = a_2 V_q \qquad \frac{\partial F_3}{\partial C_{Z_{\delta_e}}} = a_1 V^2 \delta_e$$

(4) Sensitivity equations derived from $\dot{p}$ equation (eq. (A17)):

$$\dot{p} = F_4\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right)$$

$$= a_6 F_4'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right) + b_1 F_6'\left(\vec{x},\vec{\alpha},\vec{\delta},V,\beta,\dot{\beta}\right)$$

Equation (B3) for g_{4k} becomes

$$g_{4k} = a_6\left(\frac{\partial F_4'}{\partial x_k} + \frac{\partial F_4'}{\partial V}\frac{\partial V}{\partial x_k} + \frac{\partial F_4'}{\partial \beta}\frac{\partial \beta}{\partial x_k} + \frac{\partial F_4'}{\partial \beta}\frac{\partial \beta}{\partial V}\frac{\partial V}{\partial x_k} + \frac{\partial F_4'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial V}\frac{\partial V}{\partial x_k}\right)$$

$$+ b_1\left(\frac{\partial F_6'}{\partial x_k} + \frac{\partial F_6'}{\partial V}\frac{\partial V}{\partial x_k} + \frac{\partial F_6'}{\partial \beta}\frac{\partial \beta}{\partial x_k} + \frac{\partial F_6'}{\partial \beta}\frac{\partial \beta}{\partial V}\frac{\partial V}{\partial x_k} + \frac{\partial F_6'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial V}\frac{\partial V}{\partial x_k}\right)$$

$$+ a_6\left(\frac{\partial F_4'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} g_{2k}\right) + b_1\left(\frac{\partial F_6'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} g_{2k}\right)$$

$$= a_6 g_{4k}'' + b_1 g_{6k}'' + \left(a_6 \frac{\partial F_4'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} + b_1 \frac{\partial F_6'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\right) g_{2k}$$

$$= g_{4k}' + \left(a_6 \frac{\partial F_4'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2} + b_1 \frac{\partial F_6'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\right) g_{2k} \tag{B8}$$

and equation (B4) for $\tilde{F}_4(\alpha_i)$ becomes

$$\tilde{F}_4(\alpha_i) = a_6\left(\frac{\partial F_4'}{\partial \alpha_i} + \frac{\partial F_4'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\frac{\partial F_2}{\partial \alpha_i}\right) + b_1\left(\frac{\partial F_6'}{\partial \alpha_i} + \frac{\partial F_6'}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{x}_2}\frac{\partial F_2}{\partial \alpha_i}\right)$$

$$= a_6 \tilde{F}_4'(\alpha_i) + b_1 \tilde{F}_6'(\alpha_i) \tag{B9}$$

Then,

$$\frac{d}{dt}\left(\frac{\partial p}{\partial \alpha_i}\right) = a_6\left[\sum_{k=1}^{8}\left(g''_{4k} + \frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \left(\frac{\partial F'_4}{\partial \alpha_i} + \frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}}\frac{\partial F_2}{\partial \alpha_i}\right)\right]$$

$$+ b_1\left[\sum_{k=1}^{8}\left(g''_{6k} + \frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \left(\frac{\partial F'_6}{\partial \alpha_i} + \frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}}\frac{\partial F_2}{\partial \alpha_i}\right)\right]$$

$$= \sum_{k=1}^{8}\left(g'_{4k} + a_6\frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k} + b_1\frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right)$$

$$+ a_6\tilde{F}'_4(\alpha_i) + b_1\tilde{F}'_6(\alpha_i)$$

$$= \sum_{k=1}^{8} g_{4k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \tilde{F}_4(\alpha_i) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B10)}$$

where

$$g''_{41} = a_4 u\left[2C_{l1} + C_{l2} + 2C_{l3}\frac{V_S}{V} - C_{l\beta} v\left(u^2 + w^2\right)^{-1/2} - C_{l\dot{\beta}}\frac{\dot{\beta}b}{2V}\right]$$

$$g''_{42} = a_4 v\left[2C_{l1} + C_{l2} + 2C_{l3}\frac{V_S}{V} - C_{l\beta} v\left(u^2 + w^2\right)^{-1/2} - C_{l\dot{\beta}}\frac{\dot{\beta}b}{2V}\right]$$

$$+ a_4 V^2 C_{l_\beta}\left(u^2 + w^2\right)^{-1/2}$$

$$g''_{43} = a_4 w\left[2C_{l1} + C_{l2} + 2C_{l3}\frac{V_S}{V} - C_{l\beta} v\left(u^2 + w^2\right)^{-1/2} - C_{l_{\dot{\beta}}}\frac{\dot{\beta}b}{2V}\right]$$

$$g''_{44} = I_{XZ} q + a_5 V C_{l_p}$$

$$g''_{45} = b_2 r + I_{XZ} p$$

$$g''_{46} = b_2 q + a_5 V C_{l_r}$$

$$g''_{47} = g''_{48} = 0$$

and

$$g''_{61} = a_4 u\left[2C_{n1} + C_{n2} - 2C_{n3}\frac{V_S}{V} - C_{n_\beta} v(u^2 + w^2)^{-1/2} - C_{n_{\dot\beta}}\frac{\dot\beta b}{2V}\right]$$

$$g''_{62} = a_4 v\left[2C_{n1} + C_{n2} - 2C_{n3}\frac{V_S}{V} - C_{n_\beta} v(u^2 + w^2)^{-1/2} - C_{n_{\dot\beta}}\frac{\dot\beta b}{2V}\right]$$
$$+ a_4 V^2 C_{n_\beta}(u^2 + w^2)^{-1/2}$$

$$g''_{63} = a_4 w\left[2C_{n1} + C_{n2} - 2C_{n3}\frac{V_S}{V} - C_{n_\beta} v(u^2 + w^2)^{-1/2} - C_{n_{\dot\beta}}\frac{\dot\beta b}{2V}\right]$$

$$g''_{64} = b_3 q + a_5 V C_{n_p}$$

$$g''_{65} = b_3 p - I_{XZ} r$$

$$g''_{66} = -I_{XZ} q + a_5 V C_{n_r}$$

$$g''_{67} = g''_{68} = 0$$

and

$$\frac{\partial F'_4}{\partial C_{l,o}} = \frac{\partial F'_6}{\partial C_{n,o}} = a_4 V^2$$

$$\frac{\partial F'_4}{\partial C_{l_\beta}} = \frac{\partial F'_6}{\partial C_{n_\beta}} = a_4 V^2 \beta$$

$$\frac{\partial F'_4}{\partial C_{l_{\dot\beta}}} = \frac{\partial F'_6}{\partial C_{n_{\dot\beta}}} = a_5 V \dot\beta$$

$$\frac{\partial F'_4}{\partial C_{l_p}} = \frac{\partial F'_6}{\partial C_{n_p}} = a_5 V p$$

$$\frac{\partial F'_4}{\partial C_{l_r}} = \frac{\partial F'_6}{\partial C_{n_r}} = a_5 V r$$

$$\frac{\partial F'_4}{\partial C_{l_{\delta_r}}} = \frac{\partial F'_6}{\partial C_{n_{\delta_r}}} = a_4 V^2 \delta_r$$

$$\frac{\partial F'_4}{\partial C_{l_{\delta_a}}} = -\frac{\partial F'_6}{\partial C_{n_{\delta_a}}} = a_4 {V_S}^2 \delta_a \cos i_w$$

$$\frac{\partial F'_4}{\partial C_{n_{\delta_a}}} = \frac{\partial F'_6}{\partial C_{l_{\delta_a}}} = -a_4 {V_S}^2 \delta_a \sin i_w$$

and

$$\frac{\partial F'_4}{\partial \dot\beta}\frac{\partial \dot\beta}{\partial \dot v} = a_5 C_{l_{\dot\beta}}$$

$$\frac{\partial F'_6}{\partial \dot\beta}\frac{\partial \dot\beta}{\partial \dot v} = a_5 C_{n_{\dot\beta}}$$

(5) Sensitivity equations derived from $\dot{q}$ equation (eq. (A18)):

$$\frac{d}{dt}\left(\frac{\partial q}{\partial \alpha_i}\right) = \sum_{k=1}^{8}\left(g'_{5k} + \frac{\partial F_5}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{w}} g_{3k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \left(\frac{\partial F_5}{\partial \alpha_i} + \frac{\partial F_5}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{w}}\frac{\partial F_3}{\partial \alpha_i}\right)$$

$$= \sum_{k=1}^{8} g_{5k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \tilde{F}_5(\alpha_i) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B11)}$$

where

$$g'_{51} = a_8 u\left(2C_{m1} + C_{m2}\right) - a_8 V^2 C_{m\alpha_a}\frac{w}{u^2 + w^2} - a_9 V C_{m\dot{\alpha}_a}\frac{\dot{w}}{u^2}$$

$$g'_{52} = a_8 v\left(2C_{m1} + C_{m2}\right)$$

$$g'_{53} = a_8 w\left(2C_{m1} + C_{m2}\right) + a_8 V^2 C_{m\alpha_a}\frac{u}{u^2 + w^2}$$

$$g'_{54} = a_7 r - 2b_4 p$$

$$g'_{55} = a_9 V C_{mq}$$

$$g'_{56} = a_7 p + 2b_4 r$$

$$g'_{57} = g'_{58} = 0$$

and

$$\frac{\partial F_5}{\partial C_{m,o}} = a_8 V^2 \qquad \frac{\partial F_5}{\partial C_{m\alpha_a}} = a_8 V^2 \alpha_a \qquad \frac{\partial F_5}{\partial C_{m\dot{\alpha}_a}} = a_9 V \dot{\alpha}_a$$

$$\frac{\partial F_5}{\partial C_{mq}} = a_9 V q \qquad \frac{\partial F_5}{\partial C_{m\delta_e}} = a_8 V^2 \delta_e$$

and

$$\frac{\partial F_5}{\partial \dot{\alpha}_a}\frac{\partial \dot{\alpha}_a}{\partial \dot{w}} = \frac{a_9 V C_{m\dot{\alpha}_a}}{u}$$

(6) Sensitivity equations derived from $\dot{r}$ equation (eq. (A19)):

$$\frac{d}{dt}\left(\frac{\partial r}{\partial \alpha_i}\right) = b_1\left[\sum_{k=1}^{8}\left(g''_{4k} + \frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \left(\frac{\partial F'_4}{\partial \alpha_i} + \frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}}\frac{\partial F_2}{\partial \alpha_i}\right)\right]$$

$$+ b_5\left[\sum_{k=1}^{8}\left(g''_{6k} + \frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \left(\frac{\partial F'_6}{\partial \alpha_i} + \frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}}\frac{\partial F_2}{\partial \alpha_i}\right)\right]$$

$$= \sum_{k=1}^{8}\left(g'_{6k} + b_1\frac{\partial F'_4}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k} + b_5\frac{\partial F'_6}{\partial \dot{\beta}}\frac{\partial \dot{\beta}}{\partial \dot{v}} g_{2k}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right)$$

$$+ b_1\tilde{F}'_4(\alpha_i) + b_5\tilde{F}'_6(\alpha_i)$$

$$= \sum_{k=1}^{8} g_{6k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) + \tilde{F}_6(\alpha_i) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B12)}$$

where all the terms have been defined in the derivation of the sensitivity equations for $\dot{p}$ equation.

(7) Sensitivity equations derived from $\dot{\theta}$ equation (eq. (A20)):

$$\frac{d}{dt}\left(\frac{\partial \theta}{\partial \alpha_i}\right) = \sum_{k=1}^{8} g_{7k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B13)}$$

where

$$g_{71} = g_{72} = g_{73} = g_{74} = 0$$

$$g_{75} = \cos\phi$$

$$g_{76} = -\sin\phi$$

$$g_{77} = 0$$

$$g_{78} = -\dot{\psi}\cos\theta$$

(8) Sensitivity equations derived from $\dot{\phi}$ equation (eq. (A21)):

$$\frac{d}{dt}\left(\frac{\partial \phi}{\partial \alpha_i}\right) = \sum_{k=1}^{8} g_{8k}\left(\frac{\partial x_k}{\partial \alpha_i}\right) \qquad (i = 1, 2, \ldots, 40) \qquad \text{(B14)}$$

where

$$g_{81} = g_{82} = g_{83} = 0$$

$$g_{84} = 1$$

$$g_{85} = \sin \phi \tan \theta$$

$$g_{86} = \cos \phi \tan \theta$$

$$g_{87} = \frac{\dot{\psi}}{\cos \theta}$$

$$g_{88} = \dot{\theta} \tan \theta$$

APPENDIX C

ADDITION OF ACCELERATIONS INTO ALGORITHM

The acceleration measurements and equations were included in the parameter estimation algorithm to improve the extraction process. They are used with or can replace the linear velocities u, v, and w. The acceleration equations were transformed to the instrument location from the center of gravity.

The equations for the center of gravity are

$$\begin{bmatrix} a_{X,cg} \\ a_{Y,cg} \\ a_{Z,cg} \end{bmatrix} = \frac{1}{g} \begin{bmatrix} \dot{u} + qw - rv + g \sin\theta \\ \dot{v} + ru - pw - g\cos\theta\sin\phi \\ \dot{w} + pv - qu - g\cos\theta\cos\phi \end{bmatrix} \tag{C1}$$

The equations are transformed to the instrument location (ref. 23), that is,

$$\begin{bmatrix} a_{X,I} \\ a_{Y,I} \\ a_{Z,I} \end{bmatrix} = \begin{bmatrix} a_{X,cg} \\ a_{Y,cg} \\ a_{Z,cg} \end{bmatrix} + \frac{1}{g} \begin{bmatrix} -(q^2 + r^2) & (pq - \dot{r}) & (pr + \dot{q}) \\ (pq + \dot{r}) & -(p^2 + r^2) & (qr - \dot{p}) \\ (pr - \dot{q}) & (qr + \dot{p}) & -(p^2 + q^2) \end{bmatrix} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \tag{C2}$$

where x_a, y_a, z_a are the center-of-gravity offsets of the accelerometer measurements.

The acceleration sensitivity equations were derived in terms of the sensitivity equations and coefficients stated in appendix B and need only to be evaluated and not integrated.

(1) Sensitivity equations derived from $a_{X,I}$ equation:

$$\frac{\partial a_{X,I}}{\partial \alpha_i} = \frac{1}{g}\left[-r\left(\frac{\partial v}{\partial \alpha_i}\right) + q\left(\frac{\partial w}{\partial \alpha_i}\right) + \left(y_a q + z_a r\right)\left(\frac{\partial p}{\partial \alpha_i}\right) + \left(-2x_a q + y_a p + w\right)\left(\frac{\partial q}{\partial \alpha_i}\right)\right.$$
$$+ \left(-2x_a r + z_a p - v\right)\left(\frac{\partial r}{\partial \alpha_i}\right) + g\cos\theta\left(\frac{\partial \theta}{\partial \alpha_i}\right)$$
$$\left. + \left(\frac{\partial \dot{u}}{\partial \alpha_i}\right) + z_a\left(\frac{\partial \dot{q}}{\partial \alpha_i}\right) - y_a\left(\frac{\partial \dot{r}}{\partial \alpha_i}\right)\right] \qquad (i = 1, 2, \ldots, 40) \tag{C3}$$

(2) Sensitivity equations derived from $a_{Y,I}$ equation:

$$\frac{\partial a_{Y,I}}{\partial \alpha_i} = \frac{1}{g}\left[r\left(\frac{\partial u}{\partial \alpha_i}\right) - p\left(\frac{\partial w}{\partial \alpha_i}\right) + \left(-w + x_a q - 2y_a p\right)\left(\frac{\partial p}{\partial \alpha_i}\right) + \left(x_a p + z_a r\right)\left(\frac{\partial q}{\partial \alpha_i}\right)\right.$$

$$+ \left(u - 2y_a r + z_a q\right)\left(\frac{\partial r}{\partial \alpha_i}\right) + g \sin\theta \sin\phi\left(\frac{\partial \theta}{\partial \alpha_i}\right) - g \cos\theta \cos\phi\left(\frac{\partial \phi}{\partial \alpha_i}\right)$$

$$\left. + \left(\frac{\partial \dot{v}}{\partial \alpha_i}\right) - z_a\left(\frac{\partial \dot{p}}{\partial \alpha_i}\right) + x_a\left(\frac{\partial \dot{r}}{\partial \alpha_i}\right)\right] \qquad (i = 1, 2, \ldots, 40) \qquad \text{(C4)}$$

(3) Sensitivity equations derived from $a_{Z,I}$ equation:

$$\frac{\partial a_{Z,I}}{\partial \alpha_i} = \frac{1}{g}\left[-q\left(\frac{\partial u}{\partial \alpha_i}\right) + p\left(\frac{\partial v}{\partial \alpha_i}\right) + \left(v + x_a r - 2z_a p\right)\left(\frac{\partial p}{\partial \alpha_i}\right) + \left(-u + y_a r - 2z_a q\right)\left(\frac{\partial q}{\partial \alpha_i}\right)\right.$$

$$+ \left(x_a p + y_a q\right)\left(\frac{\partial r}{\partial \alpha_i}\right) + g \sin\theta \cos\phi\left(\frac{\partial \theta}{\partial \alpha_i}\right) + g \cos\theta \sin\phi\left(\frac{\partial \phi}{\partial \alpha_i}\right)$$

$$\left. + \left(\frac{\partial \dot{w}}{\partial \alpha_i}\right) + y_a\left(\frac{\partial \dot{p}}{\partial \alpha_i}\right) - x_a\left(\frac{\partial \dot{q}}{\partial \alpha_i}\right)\right] \qquad (i = 1, 2, \ldots, 40) \qquad \text{(C5)}$$

The maximum likelihood function was modified to include the accelerations in the algorithm. The likelihood function is

$$L\left(\vec{\alpha}^{o}, R_2\right) = -\frac{1}{2}\sum_{i=1}^{N} \vec{\eta}'^{T}(t_i) R_2^{-1} \vec{\eta}'(t_i) - \frac{N}{2} \ln \left|R_2\right| \qquad \text{(C6)}$$

where

$$\vec{\eta}'(t_i) = \begin{bmatrix} \vec{\eta}(t_i) \\ \vec{\nu}(t_i) \end{bmatrix}$$

$$\vec{\nu}(t_i) = \begin{bmatrix} a^{M}_{X,I}(t_i) - a^{o}_{X,I}(t_i) \\ a^{M}_{Y,I}(t_i) - a^{o}_{Y,I}(t_i) \\ a^{M}_{Z,I}(t_i) - a^{o}_{Z,I}(t_i) \end{bmatrix}$$

and R_2 is measurement noise covariance matrix with accelerations included; that is,

$$R_2^o(N) \triangleq \text{Estimate of } R_2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \vec{\eta}'(t_i)\vec{\eta}'^{T}(t_i)$$

The maximization procedure is similar to the previous developments and similar estimation equations can be derived.

APPENDIX D

OPERATIONAL FEATURES

The computer program was written in FORTRAN IV language (75 000 octal locations) and run on the Control Data series 6000 digital computer complex, a major application being real-time simulation (RTS) (ref. 24). Incorporated in the RTS system are the cathode ray tube (CRT) graphic display units.

The computer program was mechanized into an iterative estimation procedure with manual interactive control through the utilization of the RTS system. The operational diagram of the RTS system is shown in figure 2, the main components being the computer complex, control console, and CRT. The remaining equipment is for output of information and monitoring the program. Figure 3(a) shows a photograph of the program control station and figure 3(b) shows a closeup of the control console.

The maximum likelihood estimation program resides in the central memory of the computer. The analyst investigating the stability and control derivatives of the aircraft has direct control of the computer program through the control console. The control console has mode control switches for program operation, a data entry keyboard for inputing program parameters and logic controls, logical switches for program options, and indicator lights for program status. The digital decimal display was used to monitor continuously any selected parameter or variable in the program, particularly the performance index function.

The CRT displayed the flight test maneuver at the start of each iteration. The response of the equations of motion was plotted simultaneously as it was computed in the digital program and was plotted with the flight test maneuver for direct comparison. This display permitted quick analysis of each flight test case on an iteration to iteration basis. Figure 4 shows three CRT displays; they are a portion of the dynamic check. (Note that symbols on CRT display in figure 4 are not the standard symbols defined in the Symbols section.) Permanent pictures of the CRT displays were obtained directly from the hard copy unit in the facility or from postprocessing of the plotting routine in the computer program. The plotting routine generated figure 4 by plotting the CRT display and adding the additional labeling on the right.

The information output consisted mainly of calculated data preselected by the analyst and routed to the high-speed printer. The information could be printed for any iteration by activating a logical control switch. The printer is located in the proximity of the program control station and easily accessible to the computer operator. The output consisted of the following information:

(1) run and iteration numbers

(2) covariance matrix of the measurement noise $\left(R_1^o(N)\right)$, its determinant, and its inverse

(3) variables of state and parameters active in algorithm

(4) nominal parameter values α_i^o (i = 1, 2, . . ., p')

(5) calculated changes in the nominal values $\Delta\alpha_i$ (i = 1, 2, . . ., p')

(6) covariance matrix for the parameters in a modified form more readable to the analyst

$$\begin{bmatrix} \sigma_{\alpha_1} & \rho_{\alpha_1\alpha_2} & \cdots & \rho_{\alpha_1\alpha_{p'}} \\ \rho_{\alpha_2\alpha_1} & \sigma_{\alpha_2} & \cdots & \rho_{\alpha_2\alpha_{p'}} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \rho_{\alpha_{p'}\alpha_1} & \rho_{\alpha_{p'}\alpha_2} & \cdots & \sigma_{\alpha_{p'}} \end{bmatrix}$$

in that the standard derivations and correlation coefficients are expressed explicitly.

The integration scheme that was used for the parameter estimation procedure was second-order Adams-Bashforth, a 1-pass integration scheme. The real-time system provides the option of four integration schemes: (1) second-order Runge-Kutta (2 pass), (2) fourth-order Runge-Kutta (4 pass), (3) second-order Adams-Moulton (2 pass), and (4) fourth-order Adams-Moulton (2 pass). The Adams-Bashforth scheme was obtained by program logic limiting scheme (3) to a 1-pass operation; this thus reduced the computation time for the integration of the equations of motion and sensitivity equations. The dynamic check was run by using the Adams-Bashforth scheme and scheme (2); the indication was that the Adams-Bashforth scheme was adequate for the parameter estimation procedure.

APPENDIX E

TEST CASES

The test cases were for the longitudinal motion of the aircraft and included the effects of measurement noise on the pseudo data.

The equations of motion are

$$\dot{u} = -qw - g \sin \theta + \frac{1}{2}\frac{\rho}{m} V^2 S\left(C_{X,o}\right) \tag{E1}$$

$$\dot{w} = qu + g \cos \theta + \frac{1}{2}\frac{\rho}{m} V^2 S\left(C_{Z,o} + C_{Z_{\alpha_a}}\alpha_a + C_{Z_{\delta_e}}\delta_e\right) \tag{E2}$$

$$\dot{q} = \frac{1}{2}\frac{\rho}{I_Y} V^2 S\bar{c}\left(C_{m,o} + C_{m_{\alpha_a}}\alpha_a + C_{m_q}\frac{q\bar{c}}{2V} + C_{m_{\delta_e}}\delta_e\right) \tag{E3}$$

$$\dot{\theta} = q \tag{E4}$$

where

$$\delta_e = \begin{cases} 0.1 \sin 2.5t & (0 \leqq t \leqq \pi/1.25) \\ 0 & (t \geqq \pi/1.25) \end{cases}$$

$$V = \sqrt{u^2 + w^2}$$

$$\alpha_a = \tan^{-1} \frac{w}{u}$$

The test cases were for different noise levels of 1, 2, 5, and 10 percent on the variables u, w, q, and θ. Table I shows the known and calculated standard deviations of the noise for each percent level. The calculated standard deviations agreed closely with the known input, with an error of less than 1 percent. Table II shows the true and calculated parameter values and their standard deviations at each noise level. The calculated parameter values indicate convergence to within one standard deviation of the true values based on a fixed number of iterations. Figure 5 shows the CRT display of the converged solution and the pseudo flight data. (Note that symbols on CRT display in figure 5 are not the standard symbols defined in the Symbols section.)

REFERENCES

1. Åström, K. J.; and Eykhoff, P.: System Identification, A Survey. Identification and Process Parameter Estimation, Pt. I, 0.1, Int. Fed. Automat. Contr., June 1970.

2. Steinmetz, George G.; Parrish, Russell V.; and Bowles, Roland L.: Longitudinal Stability and Control Derivatives of a Jet Fighter Airplane Extracted From Flight Test Data by Utilizing Maximum Likelihood Estimation. NASA TN D-6532, 1972.

3. Suit, William T.: Aerodynamic Parameters of the Navion Airplane Extracted From Flight Data. NASA TN D-6643, 1972.

4. Gerlach, O. H.: Determination of Performance and Stability Parameters From Nonsteady Flight Test Maneuvers. [Preprint] 700236, Soc. Automot. Eng., Mar. 1970.

5. Greenberg, Harry: A Survey of Methods for Determining Stability Parameters of an Airplane From Dynamic Flight Measurements. NACA TN 2340, 1951.

6. Milliken, William F., Jr.: Progress in Dynamic Stability and Control Research. J. Aeronaut. Sci., vol. 14, no. 9, Sept. 1947, pp. 493-519.

7. Shinbrot, Marvin: On the Analysis of Linear and Nonlinear Dynamical Systems From Transient-Response Data. NACA TN 3288, 1954.

8. Ortega, J. M.; and Rheinboldt, W. C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, Inc., 1970.

9. Taylor, Lawrence W., Jr.; Iliff, Kenneth W.; and Powers, Bruce G.: A Comparison of Newton-Raphson and Other Methods for Determining Stability Derivatives From Flight Data. AIAA Paper No. 69-315, Mar. 1969.

10. Kumar, K. S. Prasanna; and Sridhar, R.: On the Identification of Control Systems by the Quasi-Linearization Method. IEEE Trans. Automat. Contr., vol. AC-9, no. 2, Apr. 1964, pp. 151-154.

11. Larson, Duane B.: Identification of Parameters by the Method of Quasilinearization. CAL Rep. No. 164, Cornell Aeronautical Lab., Inc., May 14, 1968.

12. Denery, Dallas G.: An Identification Algorithm Which Is Insensitive to Initial Parameter Estimates. AIAA Paper No. 70-34, Jan. 1970.

13. Young, Peter C.: Process Parameter Estimation and Self Adaptive Control. Theory of Self-Adaptive Control Systems, P. H. Hammond, ed., Plenum Press, 1966, pp. 118-140.

14. Dolbin, Benjamin H., Jr.: A Differential Correction Method for the Identification of Airplane Parameters From Flight Test Data. Proceedings of the National Electronics Conference, Vol. XXV, 1969, pp. 90-94.

15. Bard, Yonathan: Comparison of Gradient Methods for the Solution of Nonlinear Parameter Estimation Problems. SIAM J. Numerical Anal., vol. 7, no. 1. Mar. 1970, pp. 157-186.

16. Kerr, R. B.: Bayesian Identification of System Parameters. J. Eng. Math., vol. 4, no. 3, July 1970, pp. 273-281.

17. Tyler, James S.; Powell, J. David; and Mehra, Raman K.: The Use of Smoothing and Other Advanced Techniques for VTOL Aircraft Parameter Identification. Contract No. N00019-69-C-0534, Systems Control Inc., June 30, 1970.

18. Mehra, Raman K.: Maximum Likelihood Identification of Aircraft Parameters. Eleventh Joint Automatic Control Conference of the American Automatic Control Council, Amer. Soc. Mech. Eng., c.1970, pp. 442-444.

19. Meissinger, Hans F.: The Use of Parameter Influence Coefficients in Computer Analysis of Dynamic Systems. Simulation, vol. 3, no. 2, Aug. 1964, pp. 53-63.

20. Bekey, George A.: Optimization of Multiparameter Systems by Hybrid Computer Techniques. Pt. I. Simulation, vol. 2, no. 2, Feb. 1964, pp. 19-32.

21. Kashyap, R. L.: Maximum Likelihood Identification of Stochastic Linear Systems. IEEE Trans. Automat. Contr., vol. AC-15, no. 1, Feb. 1970, pp. 25-34.

22. Compton, Harold R.: A Study of the Accuracy of Estimating the Orbital Elements of a Lunar Satellite by Using Range and Range-Rate Measurements. NASA TN D-3140, 1966.

23. Wagner, William E.; and Serold, Arno C.: Formulation on Statistical Trajectory Estimation Programs. NASA CR-1482, 1970.

24. White, Ellis: Eastern Simulation Council Meeting. Simulation, vol. 12, no. 2, Feb. 1969, pp. 53-56.

TABLE I.- MEASUREMENT NOISE STATISTICS USING MAXIMUM LIKELIHOOD ESTIMATION

Percent noise level	Standard deviation of –							
	u		w		q		θ	
	Known	Calculated	Known	Calculated	Known	Calculated	Known	Calculated
1	0.05	0.049511	0.3	0.29877	0.002	0.0019988	0.002	0.0019973
2	.10	.099036	.6	.59736	.004	.0039965	.004	.0039946
5	.25	.24763	1.5	1.4930	.010	.0099891	.010	.0099855
10	.50	.49529	3.0	2.9854	.020	.019977	.020	.019971

TABLE II.- PARAMETER VALUES AND STANDARD DEVIATIONS USING MAXIMUM LIKELIHOOD ESTIMATION

Parameter	True value	1-Percent noise level		2-Percent noise level		5-Percent noise level		10-Percent noise level	
		Calculated value	Standard deviation	Calculated value	Standard deviation	Calculated value	Standard deviation	Calculated value	Standard deviation
$C_{X,o}$	0.112	0.11181	0.0001748	0.11162	0.0003494	0.11105	0.0008720	0.11010	0.001739
$C_{Z,o}$	-1.29	-1.2899	.0007873	-1.2898	.001574	-1.2895	.003925	-1.2892	.007820
$C_{Z_{\alpha_a}}$	-4.59	-4.5763	.02246	-4.5616	.04488	-4.5159	.1120	-4.4353	.2231
$C_{Z_{\delta_e}}$	-4.93	-4.9332	.04083	-4.9331	.08167	-4.9313	.2043	-4.9286	.4090
$C_{m,o}$	.0199	.019855	.00008130	.019830	.0001624	.019747	.0004048	.019600	.0008054
$C_{m_{\alpha_a}}$	-.836	-.83457	.001664	-.83316	.003326	-.82889	.008300	-.82162	.01655
C_{m_q}	-32.0	-32.102	.1222	-32.176	.2446	-32.401	.6130	-32.798	1.231
$C_{m_{\delta_e}}$	-3.1	-3.1018	.005563	-3.1001	.01113	-3.0948	.02785	-3.0863	.05584

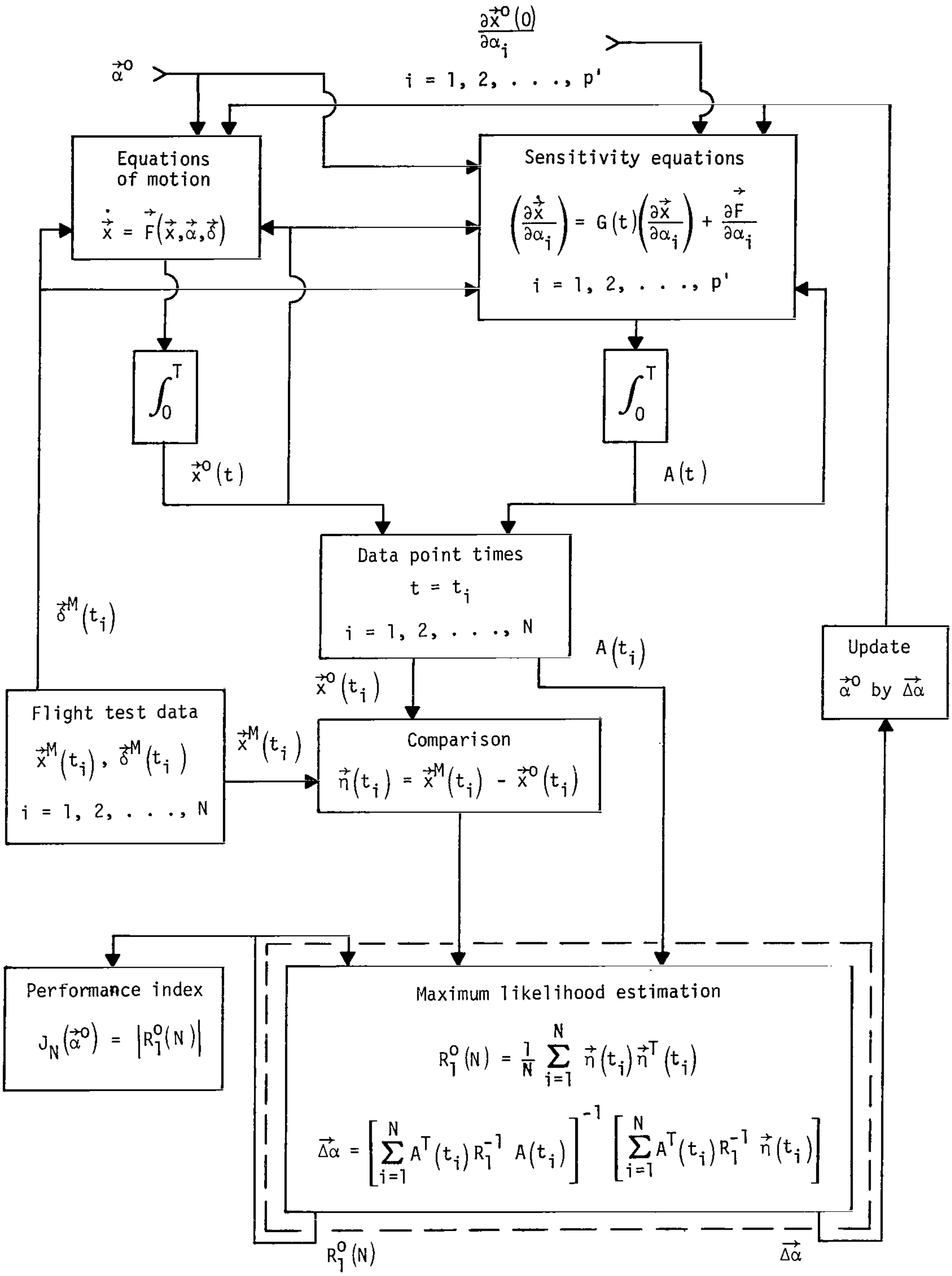


Figure 1.- Maximum likelihood parameter estimation procedure.

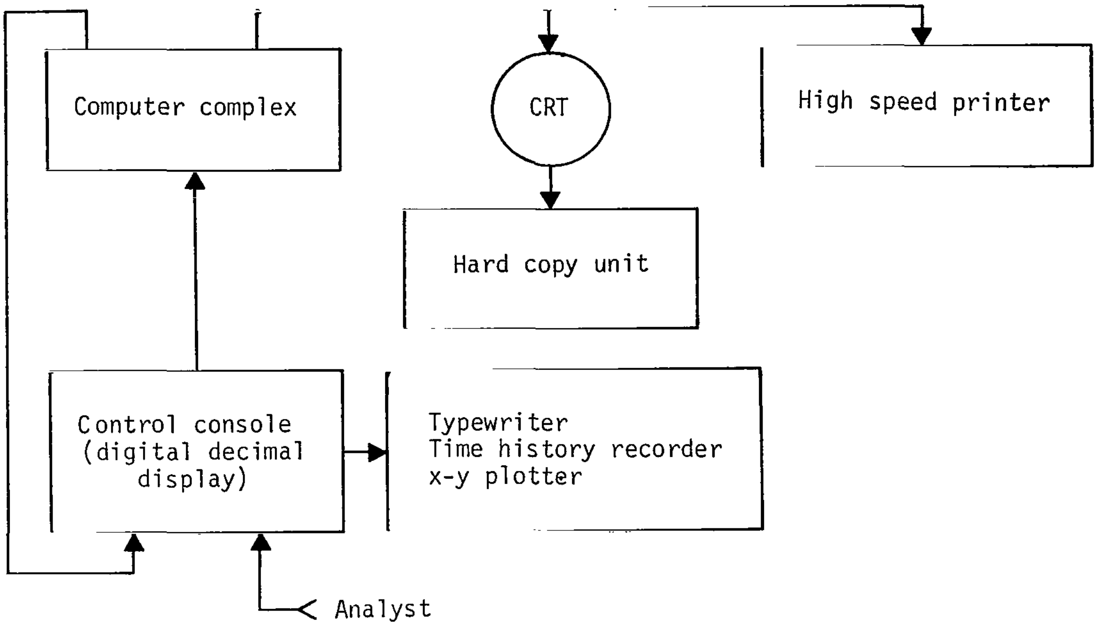
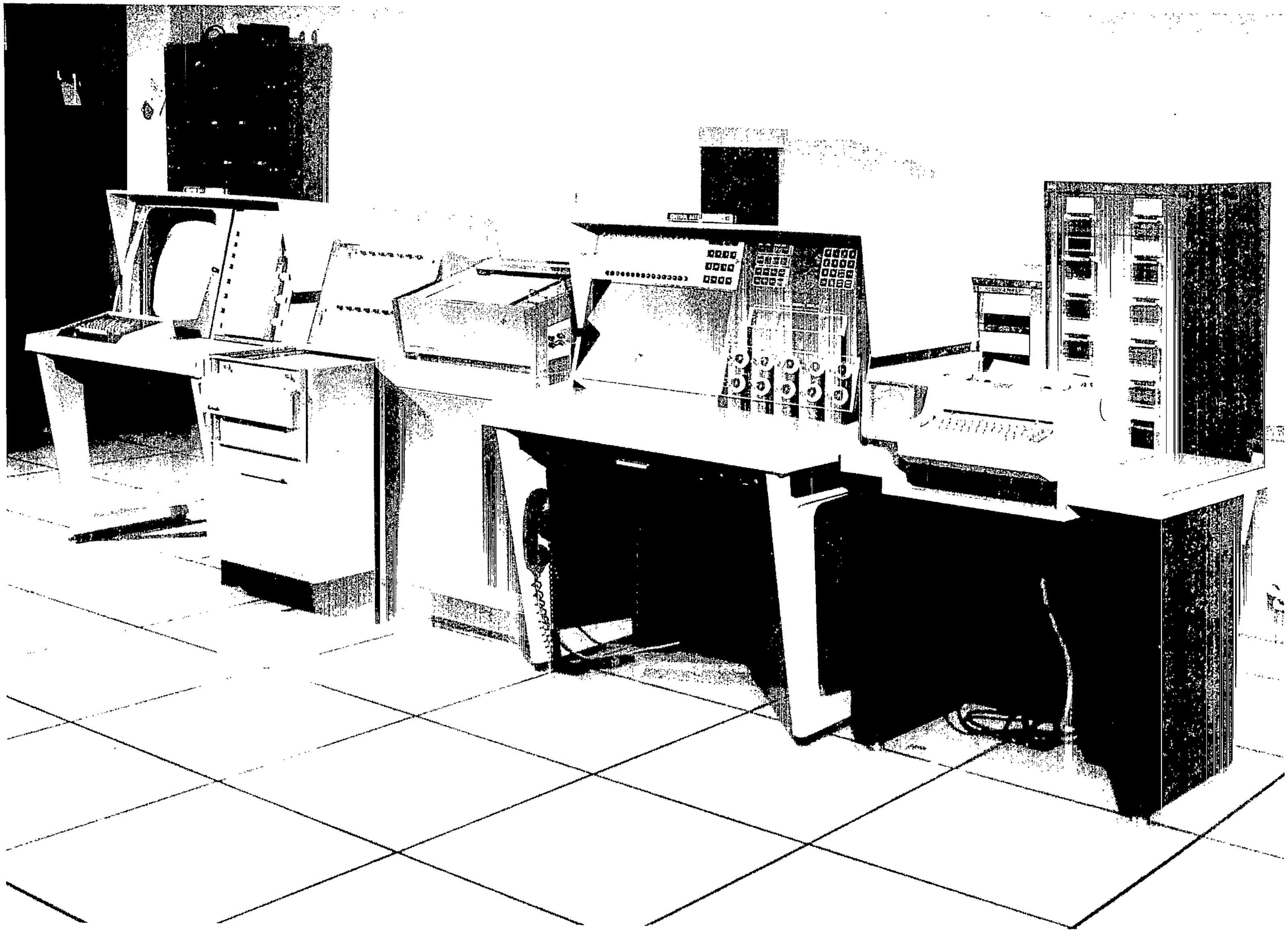


Figure 2.- Operational diagram of Langley real-time simulation system for parameter estimation.



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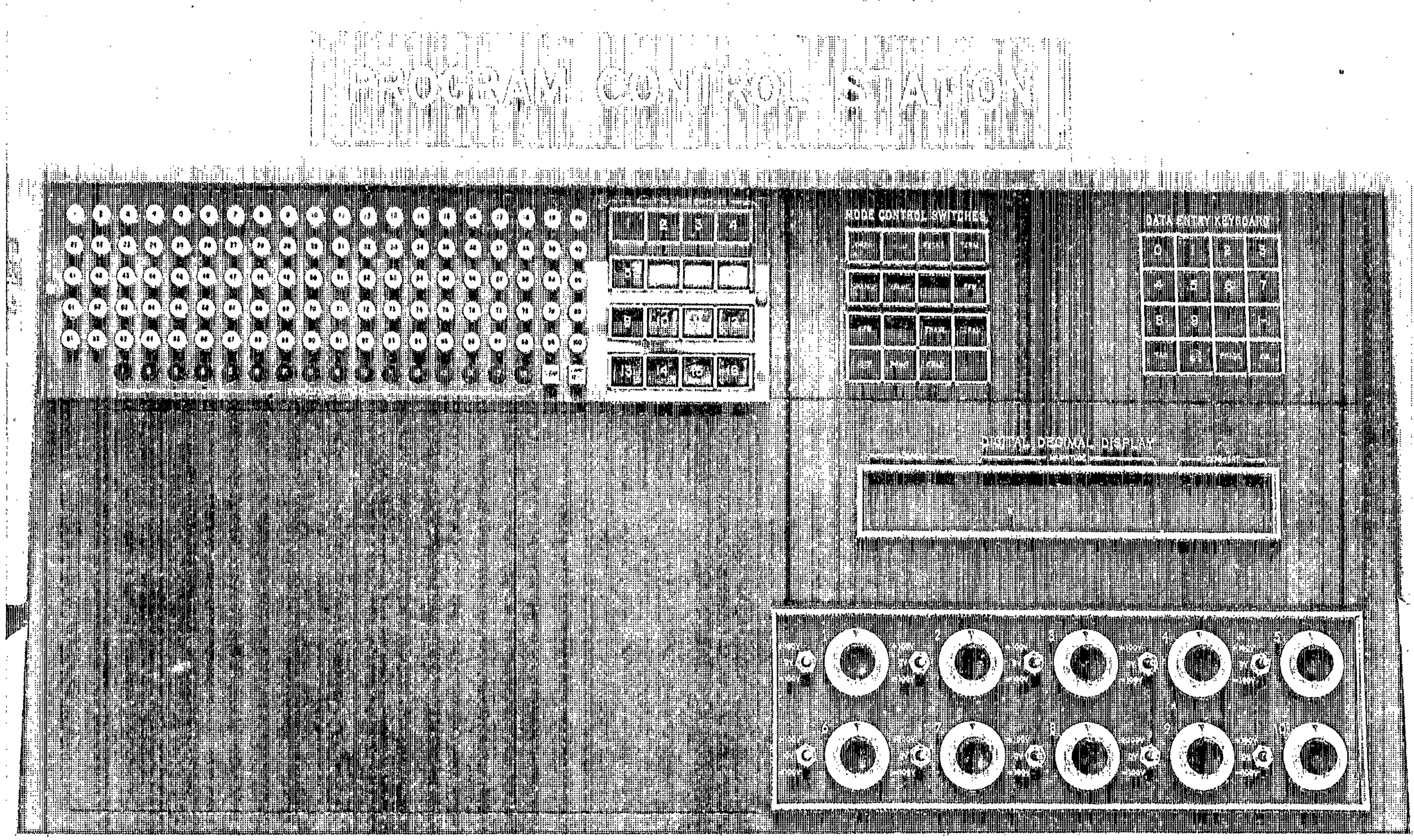
(a) Typical program control station.

Figure 3.- Operational control features.



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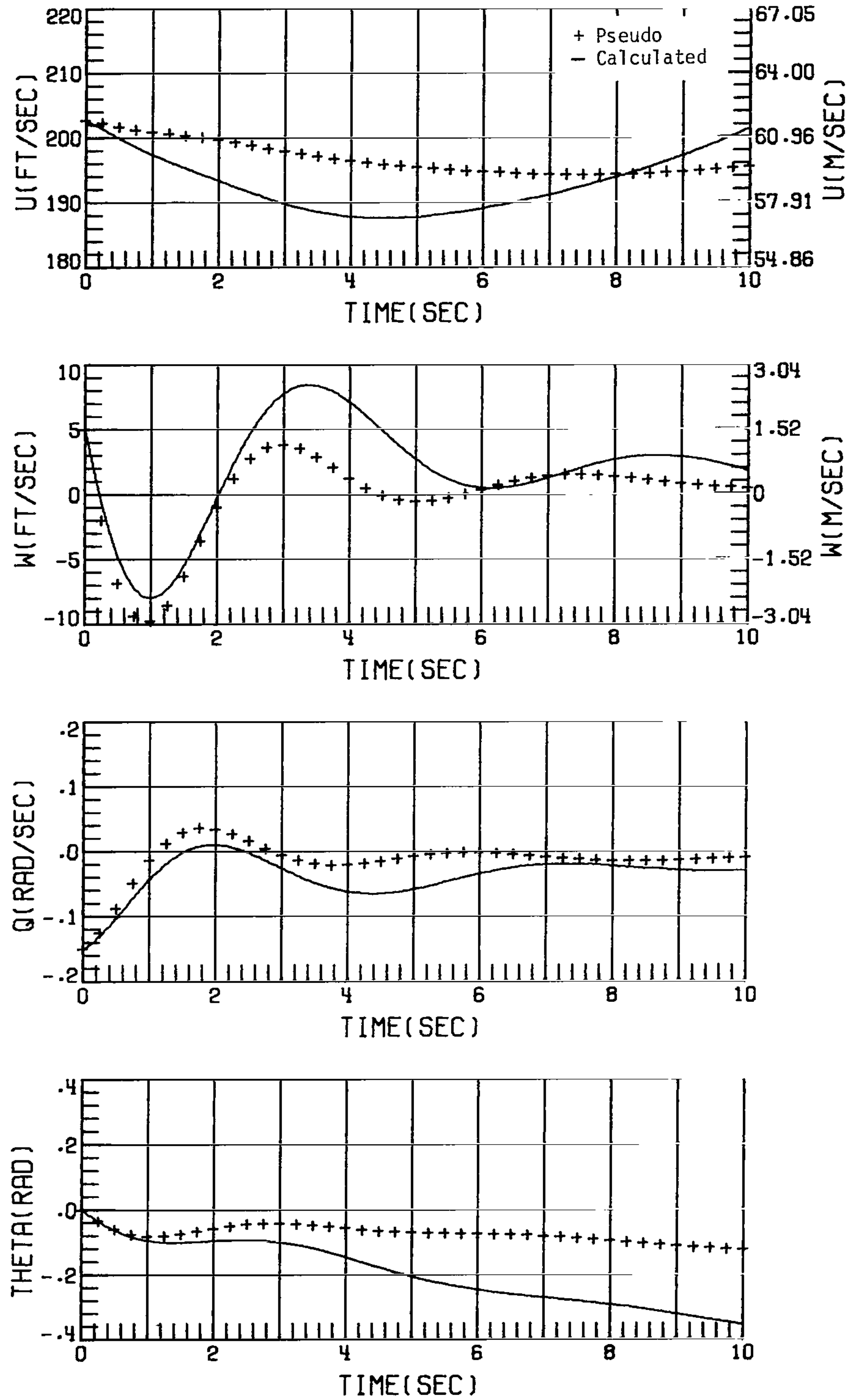
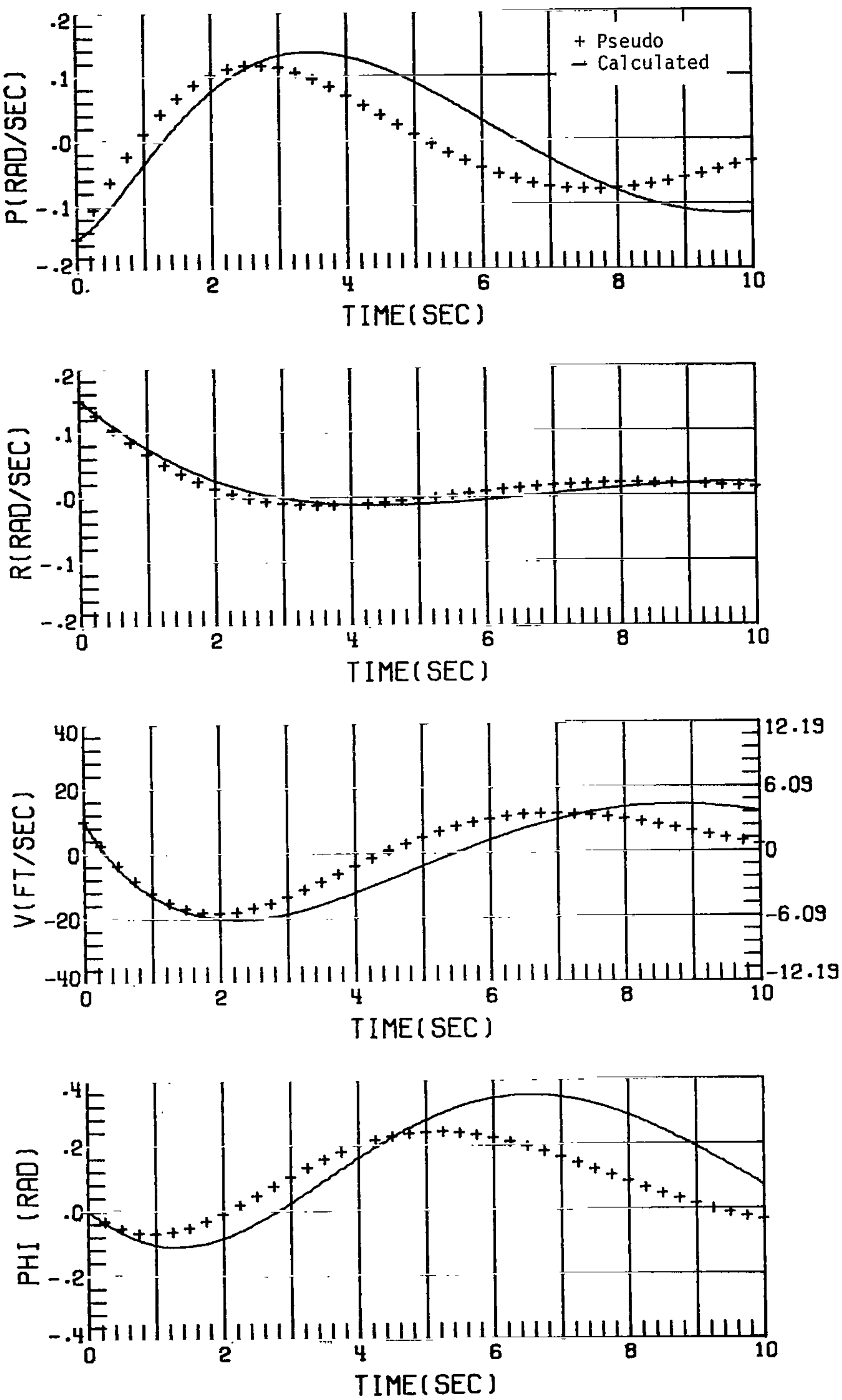
(b) Closeup of control panel on the program control console.

Figure 3.- Concluded.



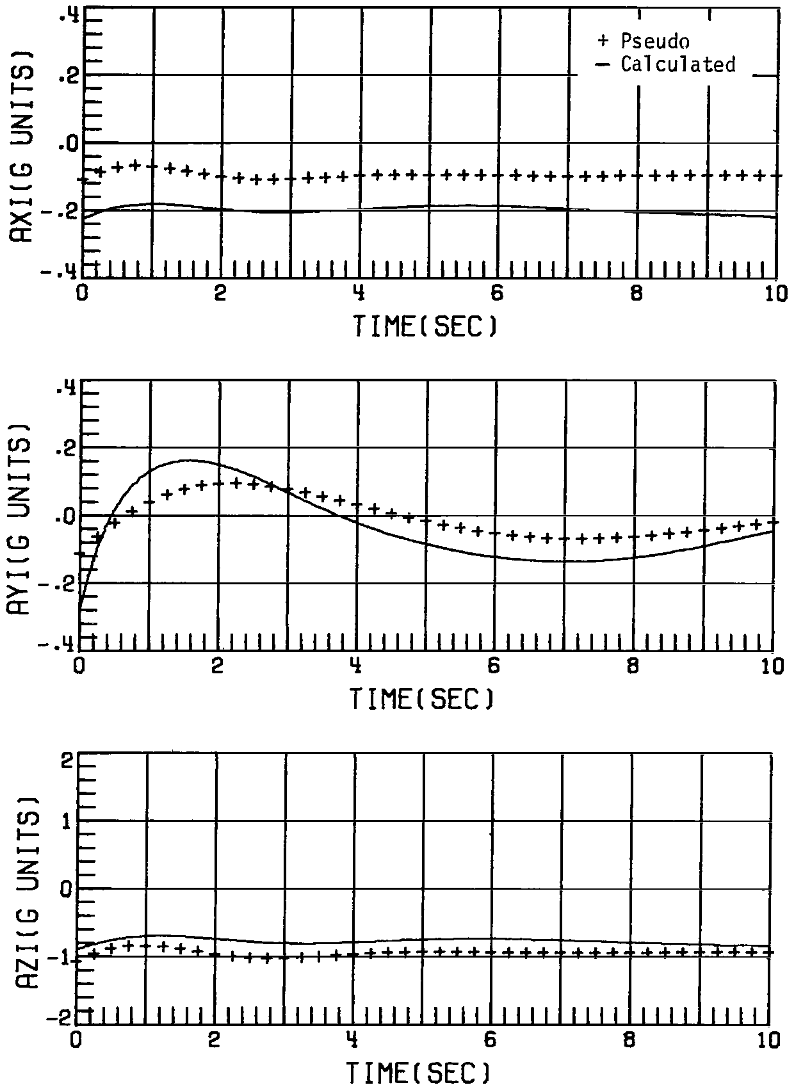
(a) Longitudinal motion.

Figure 4.- Graphic display of pseudo and calculated variables.



(b) Lateral motion.

Figure 4.- Continued.



(c) Accelerations.

Figure 4.- Concluded.

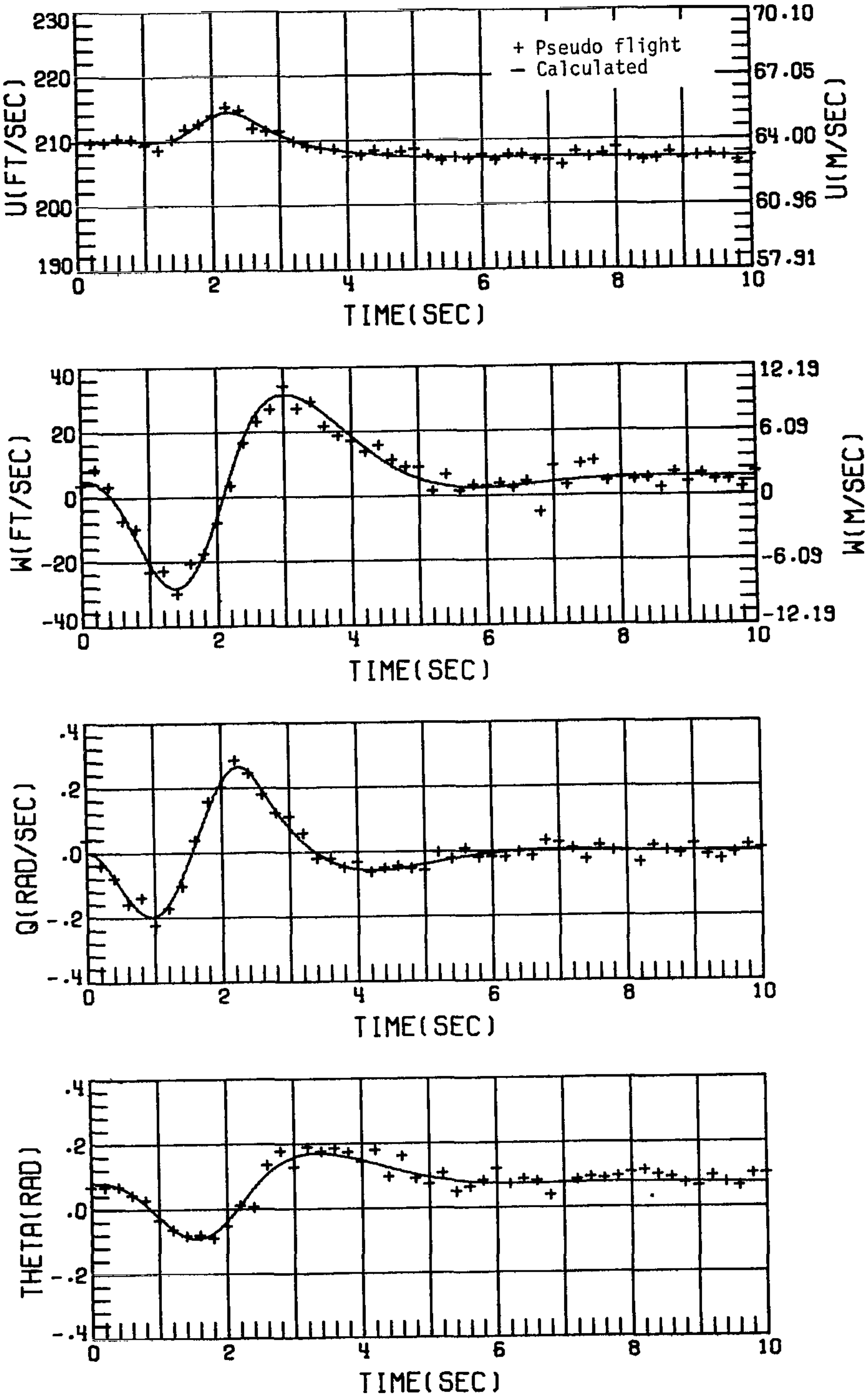


Figure 5.- Graphic display of pseudo flight and calculated (converged solution) longitudinal motion.

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