Process Semantics of General Inhibitor Nets[1]

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We define a causality semantics of Place/Transition nets with *weighted* inhibitor arcs (PTI-nets). We extend the standard approach to defining the partial order semantics of Place/Transition nets (PT-nets) based on the *process* semantics given through net unfolding and occurrence nets. To deal with inhibitor arcs at the level of occurrence nets activator arcs (and extra conditions) are used. The properties of the resulting activator occurrence nets are extensively investigated. It is then demonstrated how processes corresponding to step sequences of PTI-nets can be constructed algorithmically, and a non-algorithmic (axiomatic) characterisation is given of all those processes that can be obtained in this way. In addition, a general framework is established allowing to separately discuss behaviour, processes, causality, and their properties before proving that the resulting notions are mutually consistent for the various classes of Petri nets considered. This facilitates an efficient and uniform presentation of our results.

Key Words: theory of concurrency, Petri nets, weighted inhibitor arcs, causality semantics, processes, occurrence nets, step sequences.

1. INTRODUCTION

Petri nets are a formal model of concurrent computation that has been the subject of extensive development in the past few decades (see [8, 19] for a comprehensive overview of the results pertaining both to theory and application of Petri nets). In its most common formulation, a Petri net consists of *places*, or local states, and *transitions* effecting the change of local states. The latter is possible if, for a given transition, a specified set of local states is currently active, or *marked* in Petri net terminology. Such a model is what is usually referred to as *Place/Transition nets*, or PT-nets. Petri nets with inhibitor arcs (PTI-nets), where a transition's executability can also depend on some specific local states *not* being marked, is perhaps the most natural extension of the standard PT-net model. As stated in [18], 'Petri nets with inhibitor arcs are intuitively the most direct approach to increasing the modelling power of Petri nets'. PTI-nets are strictly more expressive than PT-nets; as they can simulate the computations of Turing machines, several important decision problems like reachability and liveness which are decidable for PT-nets are undecidable for PTI-nets [12].

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Unlike a standard Petri net, a Petri net with inhibitor arcs has essentially the possibility of testing whether a place is empty in the current marking (*zero testing*). This means that inhibitor arcs are well suited to model situations involving testing for a specific condition, rather than producing and consuming resources. Indeed, inhibitor arcs have been found to be particularly useful in areas such as communication protocols (see, e.g., [4]) and performance analysis (see, e.g., [7]). Despite their apparent usefulness, the theory of inhibitor nets has not yet received the level of attention it deserves, and it is our intention here to contribute towards rectifying this problem.

In this paper, we consider the general class of PTI-nets consisting of weighted PT-nets with weighted inhibitor arcs which can be used for testing whether a place does not contain more than a certain threshold number of tokens [1]. We are concerned with the development of a process semantics of general PTI-nets, based on net unfolding and occurrence nets.

The line of research presented here is a continuation of the work of [14] on elementary net systems with inhibitor arcs, which has been further developed in [17]. The key aspect of the adopted approach is to use the so-called stratified order structures to provide a causality semantics consistent with the operational semantics defined in terms of step sequences. Whereas for an elementary net system, an abstract causality semantics can be given in terms of partial orders alone, the presence of inhibitor arcs requires more information on the relationships between event occurrences.

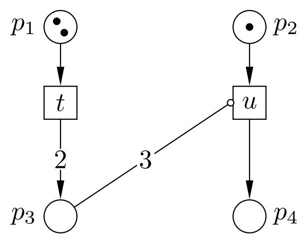

FIG. 1 Place/Transition net NI_{expl} with weighted inhibitor arc.

An example

We consider the most general class of inhibitor nets, for we allow both weighted arcs for consuming and producing tokens (the standard arcs), and weighted inhibitor arcs. To illustrate the role of the latter ones, let us consider the inhibitor net NI_{expl} with the two transitions, t and u, and four places, $p_1, \ldots, p_4$, shown in figure 1.

In addition to the weighted standard arcs, like that between transition t and place p_3 of weight 2 (which means that executing t leads to the addition of 2 tokens to p_3), there is an inhibitor arc between place p_3 and transition u of weight 3. This means that u can occur (is enabled) only if p_3 contains *at most* 3 tokens, in addition to the requirement represented by the arc with weight 1 from p_2 to u by which p_2 should contain at least one token which will be 'consumed' by u when it occurs. Executing u does not affect the tokens in p_3. Initially, both t and u are enabled and $\sigma_1 = u$, $\sigma_2 = ut$, $\sigma_3 = utt$, $\sigma_4 = t$, $\sigma_5 = tu$, $\sigma_6 = tut$, and $\sigma_7 = tt$ are the non-empty execution sequences of NI_{expl}.

The possibility to execute t is never affected by executing u. However, after the execution sequence (firing sequence) tt, transition u becomes disabled. This indicates that independence of transitions is no longer symmetric. In the *a priori* concurrency semantics of nets with inhibitor arcs as discussed in [6] and investigated in [14] and [17], t and u may also be executed simultaneously after executing t, since the inhibitor place p_3 of u holds less than 3 tokens *prior* to the occurrence of u. Thus also the step sequence $\{t\}\{t,u\}$ may be executed. Hence simultaneity of transition occurrences and absence of ordering are different notions.

Stratified order structures take care of these more involved relations between transition occurrences by providing next to a partial order a weak partial order. The former describes the standard causal relationships between the occurrences whereas the latter describes weak causal relationships as that described above: after the first occurrence of t, u *may precede* a next occurrence of t but not vice versa, and hence the step $\{t,u\}$ after t may be sequentialised to ut, but not to tu.

Causality semantics

For elementary net systems and PT-nets, an abstract partial order semantics follows immediately from their process semantics (see, e.g., [20], [2], [11]). Processes are constructed by unfolding the system according to a given run represented by a firing sequence. This leads to occurrence nets, which are (labelled) acyclic nets with non-branching places (conditions), since conflicts are resolved during the run. By abstracting from the conditions of an occurrence net, one obtains a (labelled) partial order which describes the causal relationships between the events (transition occurrences) in the given run: all labelled sequences which are linearisations of the partial order are firing sequences of the net and among them is the firing sequence on basis of which the process was constructed.

In order to obtain a causality semantics in terms of stratified order structures for nets with unweighted (i.e., zero-testing inhibitor arcs) also both [14] and [17] first develop a process semantics. Since in the *a priori* semantics not all concurrent runs of the system can be represented by a firing sequence, these processes are based on step sequences. (Consider again the net NI_{expl} in figure 1, with an additional inhibitor arc of weight 0 from p_4 to t. Now, neither tut nor ttu are firing sequences, although $\{t\}\{t,u\}$ is still a valid step sequence.) Given a step sequence of an elementary net system with inhibitor arcs, [14] unfolds the system into an occurrence net with additional arcs (activator arcs) to represent the inhibitor arcs. Testing if a place is empty (inhibitor arc) is in the unfolding represented by testing whether its complement condition (which can be assumed to exist) does hold using an activator arc. In the resulting activator occurrence net the conditions are again non-branching with respect to the normal arcs. Moreover, it is acyclic in a sense which includes the activator arcs ($\diamondsuit$–acyclic) and thus allows to extract a (labelled) stratified order structure which describes precisely the causality and weak causality relationships between the events in the given run. All step sequences which obey the constraints imposed by the stratified order structure are step sequences of the system and they include the step sequence on the basis of which the process was constructed.

To define a process semantics for unweighted PT-nets with unweighted inhibitor arcs, we investigated in [17] first the case that all inhibitor places are complemented (and thus bounded). In this case, the approaches of [2] and [14] can be combined and, again, from the resulting processes, labelled stratified order structures can be extracted which describe the causality between transition occurrences in the underlying concurrent run. To deal with inhibitor places which are not complemented, additional conditions (called z-conditions) were introduced 'on demand' during the construction of a process for a given step sequence. The presence of a z-condition signals an empty inhibitor place with the zero-testing represented by an activator arc. Since z-conditions may be branching (with respect to the normal arcs), this led to a new type of occurrence net with activator arcs. Still, also from these processes a labelled stratified order structure could be extracted describing precisely the causal relationships in the underlying run.

As already observed in [17], the process semantics and hence the causality semantics presented there could easily be generalised to PT-nets with weighted ordinary arcs. How to deal with weighted inhibitor arcs was however less obvious. In this paper, we demonstrate that in the case of complemented inhibitor places, the approach of [17] based on testing for occurrences of complement conditions can be easily adapted. For general PTI-nets however, we propose a completely new process semantics, again using extra conditions and activator arcs connected to these conditions. Together they represent the dependency between transition occurrences due to the presence of inhibitor arcs in the PTI-net. This is different from the role of the z-conditions in [17], and makes it possible to avoid references to the weights of inhibitor arcs, but rather to focus on the dependencies they give rise to. Consequently, in contrast to the unfolding in [17], the new construction has a 'local' flavour similar to the classical unfolding procedures discussed above. Moreover, it is no longer necessary to introduce a new type of occurrence nets with activator arcs. We describe how processes corresponding to step sequences of PTI-nets can be constructed and an axiomatic characterisation is given of the processes that can be obtained in this way. We also establish that the resulting semantics is fully consistent with the operational semantics of PTI-nets in terms of step sequences.

Our approach to defining causality semantics

Developing an abstract causality semantics for a class of Petri nets on basis of a process semantics requires going through several steps of defining various behavioural notions and relations between them. And, after looking at various proposals in the literature, it was revealing to observe that these steps do not depend on the kind of nets one is interested in. That is, there is a general pattern of proceeding, which only differs in technical (though non-trivial) aspects between different classes. In our presentation, we decided to take full advantage of this phenomenon, and we set out to develop a uniform framework for relating, in particular, behaviours, processes, and causality structures generated by nets. This decision proved to be a fruitful one, as we were able to boil down several interesting semantical characteristics (called the *aims*) to relatively few requirements (called the *properties*) which need to be established for a specific class of nets and/or behaviours to guarantee that the aims hold. The immediate advantage of this approach is that we obtain a clear separation of concerns when discussing different behavioural notions. Thus a second main contribution of this paper is the introduction of this semantical framework, and the demonstration how using it leads to an efficient and uniform presentation which avoids the listing of *ad hoc* intermediate results for each class of PTI-nets considered.

An outline of this paper

We first introduce several basic notions and concepts used throughout the paper. We then define the general semantical framework, after which the relationships between the different semantical objects we are interested in are clarified. In section 4, we recall the definitions and properties of executions and causal structures needed to deal with PT-nets and PTI-nets. Section 5 takes a closer look at the properties of occurrence nets with activator arcs. The following section explains how the standard process semantics can be seen as an instance of the general framework defined earlier on. After that we investigate PTI-nets with and without complemented inhibitor places and their semantics, showing that they are also an instance of the more general picture. Proofs of results omitted from the main body of the paper are included in the appendix.

This paper is largely self-contained, although it will be an advantage for the reader to be acquainted with the 'classical' process theory as presented in [2, 11, 20].

2. PRELIMINARIES

We use the standard mathematical notation. In particular, $\uplus$ denotes disjoint set union, $\mathbb{N}$ the set of natural numbers (including 0), and ∞ the first infinite ordinal. The set of all finite sequences over a set X is denoted by X^*; the empty sequence is denoted by ε and x^k is the sequence consisting of exactly k occurrences of an element $x \in X$. The powerset of a set X is denoted by $\mathbb{P}(X)$, and the cardinality of a finite set X is denoted by $|X|$. Throughout the paper we assume the existence of a universe $\mathcal{U}$ of atomic elements such that whenever $u, v \in \mathcal{U}$, then $u \notin v$.

Functions and relations

The standard $\circ$ notation for the composition of functions is used also also in the special case of functions $f : X \to \mathbb{P}(Y)$ and $g : Y \to \mathbb{P}(Z)$, for which $g \circ f : X \to \mathbb{P}(Z)$ is defined by

$$g \circ f(x) \stackrel{\text{df}}{=} \bigcup_{y \in f(x)} g(y) \ ,$$

for all $x \in X$. The restriction of a function $f : X \to Y$ to a set $Z \subseteq X$ is denoted by $f|_Z$. Unless specified explicitly, all functions are assumed to be total.

The composition of two binary relations $P \subseteq X \times Y$ and $Q \subseteq Y \times Z$ is given by

$$P \circ Q \stackrel{\text{df}}{=} \{(x, z) \mid \exists y \in Y : (x, y) \in P \wedge (y, z) \in Q\} \ .$$

As customary for binary relations, we will mostly use an infix notation and write xPy rather than $(x, y) \in P$. Moreover, $dom_P \stackrel{\text{df}}{=} \{x \mid (x, y) \in P\}$ and $codom_P \stackrel{\text{df}}{=} \{y \mid (x, y) \in P\}$. The restriction of a relation $P \subseteq X \times Y$ to a set $Z \subseteq X \times Y$ is denoted by $P|_Z$. By $id_X \stackrel{\text{df}}{=} \{(x, x) \mid x \in X\}$ we denote the identity relation on a set X. Relation $P \subseteq X \times X$ is reflexive if $id_X \subseteq P$; irreflexive if $id_X \cap P = \varnothing$; and transitive if $P \circ P \subseteq P$. The transitive closure of P is denoted by P^+, and the transitive and reflexive closure by $P^\star$. P is a partial order if it is acyclic (i.e., P^+ is irreflexive) and $P = P^+$ (so each partial order is irreflexive).

Multisets

A multiset (over a set X) is a function $\mathsf{m} : X \to \mathbb{N}$, and an *extended* multiset (over X) is a function $\mathsf{m} : X \to \mathbb{N} \cup \{\infty\}$. For two extended multisets m and m' over X, we denote $\mathsf{m} \leq \mathsf{m}'$ if $\mathsf{m}(x) \leq \mathsf{m}'(x)$ for all $x \in X$. (As usual, $n < \infty$ for all $n \in \mathbb{N}$.) Any subset of X may be viewed through its characteristic function as a multiset over X, and any multiset may always be considered as an extended multiset. We denote $x \in \mathsf{m}$ if $\mathsf{m}(x) \geq 1$ and $\mathsf{m}(x) \neq \infty$. The multiset $\mathbf{0}_X$ and the extended multiset $\mathbf{\Omega}_X$ are given respectively by $\mathbf{0}_X(x) \stackrel{\text{df}}{=} 0$ and $\mathbf{\Omega}_X(x) \stackrel{\text{df}}{=} \infty$, for all $x \in X$.

The sum of two multisets m and m' over X is given by $(\mathsf{m} + \mathsf{m}')(x) \stackrel{\text{df}}{=} \mathsf{m}(x) + \mathsf{m}'(x)$, the difference by $(\mathsf{m} - \mathsf{m}')(x) \stackrel{\text{df}}{=} \max\{0, \mathsf{m}(x) - \mathsf{m}'(x)\}$, the intersection by $(\mathsf{m} \cap \mathsf{m}')(x) \stackrel{\text{df}}{=} \min\{\mathsf{m}(x), \mathsf{m}'(x)\}$, and the multiplication of a multiset m by a natural number n by $(n \cdot \mathsf{m})(x) \stackrel{\text{df}}{=} n \cdot \mathsf{m}(x)$. A multiset m is finite if there are finitely many $x \in X$ such that $\mathsf{m}(x) \geq 1$. In such a case, the cardinality of m is defined as $|\mathsf{m}| \stackrel{\text{df}}{=} \sum_{x \in X} \mathsf{m}(x)$.

Labellings

A *labelling* for a set X is a function $\ell : X \to \mathcal{A}$, where $\mathcal{A}$ is a set of *labels*. Given a labelling $\ell : X \to \mathcal{A}$ and $a \in \mathcal{A}$, we say that $x \in X$ is a–labelled if $\ell(x) = a$. We can lift the labelling $\ell : X \to \mathcal{A}$ to a subset Y of X in two different ways. As usual, $\ell(Y)$ is the set of labels assigned by ℓ to Y, thus $\ell(Y) \stackrel{\text{df}}{=} \{a \in \mathcal{A} \mid \exists y \in Y : a = \ell(y)\}$. In addition, if Y is finite, then $\ell\langle Y \rangle$ is the multiset of labels assigned to the elements of Y, i.e., $\ell\langle Y \rangle$ is the multiset over $\mathcal{A}$ given by $\ell\langle Y \rangle(a) \stackrel{\text{df}}{=} |\ell^{-1}(a) \cap Y|$, for every $a \in \mathcal{A}$. For a sequence of sets $\sigma = X_1 \ldots X_n$ and a labelling ℓ for $X_1 \cup \ldots \cup X_n$ we write $\ell(\sigma) \stackrel{\text{df}}{=} \ell(X_1) \ldots \ell(X_n)$ and $\ell\langle \sigma \rangle \stackrel{\text{df}}{=} \ell\langle X_1 \rangle \ldots \ell\langle X_n \rangle$.

Let $X_1, \ldots, X_n$ be mutually disjoint sets, and let ℓ_i be a labelling for each X_i. Then $\ell_1 \cup \ldots \cup \ell_n$ is the labelling for $X_1 \cup \ldots \cup X_n$ defined by $\ell_1 \cup \ldots \cup \ell_n(x) \stackrel{\text{df}}{=} \ell_i(x)$ if $x \in X_i$ for some $1 \leq i \leq n$.

We will use the notion of a *labelled relational structure* (or *structure*) to refer to a tuple (X, P, ℓ) or (X, P, R, ℓ), where X is a set, $P, R \subseteq X \times X$, and ℓ is a labelling for X.

Petri nets

We now introduce the basic notion of a (Petri) net with weighted arcs which underlies all net models discussed later, and give its operational semantics in terms of step sequences. After that we introduce two extensions of this basic net notion, employing respectively inhibitor arcs and activator arcs.

A (weighted) *net* is a triple $N \stackrel{\text{df}}{=} (P, T, W)$ such that P and T are disjoint finite sets ($P, T \subseteq \mathcal{U}$), and $W : (T \times P) \cup (P \times T) \to \mathbb{N}$ is a multiset. The elements of P and T are respectively the *places* and *transitions*, and W is the *weight function* of N. In diagrams, places are drawn as circles, and transitions as rectangles. If $W(x, y) \geq 1$ for some $(x, y) \in (T \times P) \cup (P \times T)$, then (x, y) is an *arc* leading from x to y. As usual, arcs are annotated with their weight if this is 2 or more. We assume that, for every $t \in T$, there are places p and q such that $W(p, t) \geq 1$ and $W(t, q) \geq 1$ (i.e., nets are assumed to be *T-restricted*).

The *pre-* and *post-multiset* of a transition $t \in T$ are multisets of places, $\textsc{pre}_N(t)$ and $\textsc{post}_N(t)$, respectively given by $\textsc{pre}_N(t)(p) \stackrel{\text{df}}{=} W(p, t)$ and $\textsc{post}_N(t)(p) \stackrel{\text{df}}{=}$

$W(t,p)$, for all $p \in P$. Both notations extend to finite multisets of transitions U:

$$\text{PRE}_N(U) \stackrel{\text{df}}{=} \sum_{t \in U} U(t) \cdot \text{PRE}_N(t) \qquad \text{and} \qquad \text{POST}_N(U) \stackrel{\text{df}}{=} \sum_{t \in U} U(t) \cdot \text{POST}_N(t) \ .$$

For a place $p \in P$, we denote by $\text{PRE}_N(p)$ and $\text{POST}_N(p)$ the multisets of transitions respectively given by $\text{POST}_N(p)(t) \stackrel{\text{df}}{=} W(p,t)$ and $\text{PRE}_N(p)(t) \stackrel{\text{df}}{=} W(t,p)$, for all $t \in T$.

A *marking* of a net N is a multiset of places. Following the standard terminology, given a marking M of N and a place $p \in P$, we say that p is marked (under M) if $M(p) \geq 1$ and that $M(p)$ is the number of tokens in p. In diagrams, M will be represented by drawing in each place p exactly $M(p)$ tokens (small black dots).

Transitions represent actions which may occur at a given marking and then lead to a new marking. Here we define this dynamics in the more general terms of multisets of (simultaneously occurring) transitions.

A *step* is a finite multiset of transitions, $U : T \to \mathbb{N}$. It is *enabled* at a marking M if $M \geq \text{PRE}_N(U)$. Thus, in order for U to be enabled at M, for each place p, the number of tokens in p under M should at least be equal to the total number of tokens that are needed as an input to U, respecting the weights of the input arcs.

If U is enabled at M, then it can be *executed* leading to the marking $M' \stackrel{\text{df}}{=} M - \text{PRE}_N(U) + \text{POST}_N(U)$. This means that the execution of U 'consumes' from each place p exactly $W(p,t)$ tokens for each occurrence of a transition $t \in U$ that has p as an input place, and 'produces' in each place p exactly $W(t,p)$ tokens for each occurrence of a transition $t \in U$ with p as an output place. If the execution of U leads from M to M' we write $M[U\rangle M'$. Note that the empty step $\mathbf{0}_T$ is enabled at every marking of N, and that its execution has no effect on the marking, i.e., $M[\mathbf{0}_T\rangle M$ for all markings M of N.

A *step sequence* from a marking M to a marking M' is a possibly empty sequence $\sigma = U_1 \ldots U_n$ of non-empty steps U_i such that

$$M\,[U_1\rangle\, M_1 \,\cdots\, M_{n-1}\,[U_n\rangle\, M' \ ,$$

for some markings $M_1, \ldots, M_{n-1}$ of N. Moreover, the sequence of alternating markings and steps, $\mu = MU_1M_1 \ldots M_{n-1}U_nM'$, will be called a *mixed step sequence* of N from M to M'. If σ is the empty sequence ε, then $n = 0$ and $M = M'$. If σ is a step sequence from M to some M' we write $M[\sigma\rangle M'$ or $M[\sigma\rangle$, and say that M' is *reachable* from M.

The set of all markings reachable from M will be denoted by $[M\rangle$. Note that we always have $M \in [M\rangle$. If we want to make it clear which net we are dealing with, then we may add a subscript N and write $[\cdot\rangle_N$ rather than $[\cdot\rangle$.

In some cases, a net N has an implicit initial marking MIN_N. Then, knowing which transitions have been executed (and how many times) suffices to calculate the resulting marking. More precisely, if U is a multiset of transitions, then we denote by $\text{MAR}_N(U)$ the marking of N given by $(\text{MIN}_N + \text{POST}_N(U)) - \text{PRE}_N(U)$. It is then easy to see that $M[U_1 \ldots U_n\rangle M'$ implies $M' = \text{MAR}_N(U_1 + \cdots + U_n)$.

If each multiset in a step sequence $\sigma = U_1 \ldots U_n$ is a singleton, $U_i = \{x_i\}$ with $x_i \in T$, then the sequence $x_1 \ldots x_n$ is called a *firing sequence*. For ordinary Petri nets the reachability of markings does not depend on whether we use (general) step sequences or firing sequences; however, this may no longer hold if we also allow, e.g., inhibitor or activator arcs, described next.

Nets with inhibitor arcs

An *inhibitor net* is a net enriched with weighted *inhibitor arcs* leading from places to transitions. Formally, an inhibitor net NI is a tuple (P, T, W, I) such that $\text{UND}(NI) \stackrel{\text{df}}{=} (P, T, W)$ is a net (the *underlying* net of NI) and I — the inhibitor function — is an extended multiset over $P \times T$. If $I(p, t) = k \in \mathbb{N}$, then p is an *inhibitor place of* t, and this will imply that t can only be executed if p does not contain more than k tokens; in particular, if $k = 0$ then p must be empty. Moreover, $I(p, t) = \infty$ means that t can never be prevented from occurring by the presence of tokens in p. In diagrams, inhibitor arcs have small circles as arrowheads. An inhibitor arc from p to t is drawn only if its weight is different from ∞. Just like the normal arcs, inhibitor arcs are annotated with their weights. Now however, the weight 0 is not shown. A net (P, T, W) (without inhibitor arcs) can be considered as a special instance of an inhibitor net and identified with the inhibitor net $(P, T, W, \mathbf{\Omega}_{P \times T})$.

Let $NI = (P, T, W, I)$ be an inhibitor net. The various notations introduced above for transitions and places, are defined for NI through its underlying net $\text{UND}(NI)$. In addition, for every transition $t \in T$, $\text{INH}_{NI}(t)$ is the extended multiset of places given by $\text{INH}_{NI}(t)(p) \stackrel{\text{df}}{=} I(p, t)$ and, for a finite multiset U of transitions, $\text{INH}_{NI}(U)$ is the extended multiset of places given by

$$\text{INH}_{NI}(U)(p) \stackrel{\text{df}}{=} \min(\{\infty\} \cup \{\text{INH}_{NI}(t)(p) \mid t \in U\}) \; .$$

Steps and markings of NI are defined as for its underlying net $\text{UND}(NI)$. In NI, a step $U : T \to \mathbb{N}$ is *enabled* at a marking M if it is enabled at M in $\text{UND}(NI)$ and, in addition, $M \leq \text{INH}_{NI}(U)$. Thus, if a place p is an inhibitor place of some transition t occurring in U, then p must not contain more than $I(p, t)$ tokens. This definition of enabledness is based on an *a priori* condition: the inhibitor places of transitions occurring in a step should obey the inhibitor constraints before the step is executed.[2] Note that the empty step $\mathbf{0}_T$ is enabled at every marking of NI and that its execution has no effect. The notions of a step sequence, mixed step sequence and reachability are defined as for (ordinary) nets, using the modified notion of enabledness.

Nets with activator arcs

An *activator net* is a net enriched with weighted *activator arcs* leading from places to transitions, $NA \stackrel{\text{df}}{=} (P, T, W, Act)$ such that $\text{UND}(NA) \stackrel{\text{df}}{=} (P, T, W)$ is the *underlying* net of NA and Act is a multiset over $P \times T$. If $Act(p, t) > 0$, then p is an *activator place of* t, and this will imply that t can only be executed if p contains at least k tokens (the presence of the tokens is tested without the implication of them being consumed by t). Moreover, $Act(p, t) = 0$ means that t does not need any tokens in p to be enabled, unless $W(p, t) \geq 1$. In diagrams, activator arcs have small black dots as arrowheads, and are drawn only if their weights are positive. Just like the normal arcs, activator arcs are annotated with their weights if the latter are greater than 1. A net (P, T, W) (without activator arcs) can be considered as a special instance of an activator net and identified with the activator net $(P, T, W, \mathbf{0}_{P \times T})$.

[2] In the *a posteriori* approach [5], the inequality for enabledness is strengthened and becomes $M + \text{POST}_{NI}(U) \leq \text{INH}_{NI}(U)$.

Let $NA = (P, T, W, Act)$ be an activator net. The various notations introduced above for transitions and places, are defined for NA through its underlying net $\text{UND}(NA)$. In addition, for every transition $t \in T$, $\text{ACT}_{NA}(t)$ is the multiset of places given by $\text{ACT}_{NA}(t)(p) \stackrel{\text{df}}{=} Act(p, t)$ and, for a finite multiset U of transitions, $\text{ACT}_{NA}(U)$ is the multiset of places given by

$$\text{ACT}_{NA}(U)(p) \stackrel{\text{df}}{=} \max(\{0\} \cup \{\text{ACT}_{NA}(t)(p) \mid t \in U\}) .$$

Steps and markings of NA are defined as for its underlying net $\text{UND}(NA)$. In NA, a step $U : T \to \mathbb{N}$ is *enabled* at a marking M if it is enabled at M in $\text{UND}(NA)$ and, in addition, $M \geq \text{ACT}_{NA}(U)$. Thus, if a place p is an activator place of some $t \in U$, then $M(p) \geq Act(p, t)$. This definition of enabledness is again based on the *a priori* condition. (For alternative definitions see [5, 23].) Note that the empty step $\mathbf{0}_T$ is enabled at every marking of NA and that its execution has no effect. The notions of a step sequence, mixed step sequence and reachability are defined as for (ordinary) nets, using the modified notion of enabledness.

Labelled nets and marked nets

For each kind of net described above, we can consider *labelled* versions as well as *marked* versions, which amounts to adding an extra component to the tuple representing the net. In the former case, this component is a labelling for the places and transitions of the net, while in the latter a marking of the places, called the *initial* marking. All the notations relating to the structure and behaviour of labelled (marked) nets are inherited from the underlying unlabelled (resp. unmarked) nets. In diagrams, labels are given instead of the underlying elements.

Boundedness and complement places

A place p of a marked net N with initial marking M_0 is said to be *n–bounded*, where $n \in \mathbb{N}$, if $M(p) \leq n$ for every marking M reachable from M_0; it is *bounded* if it is n–bounded for some n; and otherwise it is *unbounded*. N is *safe* if all its places are 1-bounded.

A place q of a marked net N with initial marking M_0 is a *complement* of a place p of N if $q \neq p$, $\text{PRE}_N(p) = \text{POST}_N(q)$ and $\text{POST}_N(p) = \text{PRE}_N(q)$. In such a case, $\text{BND}_N(p) = \text{BND}_N(q) \stackrel{\text{df}}{=} M_0(p) + M_0(q)$ is a common bound for both p and q; moreover, $\text{BND}_N(p) = M(p) + M(q)$, for every marking M reachable from M_0.

3. THE SEMANTICAL FRAMEWORK

Aiming at a systematic presentation of the process and causality semantics for various types of Petri nets considered in this paper, we will use a common scheme the setup of which is pictured in figure 2. For a given Petri net model $\mathcal{PN}$, we will be working with the following semantical domains:

- $\mathcal{EX}$ are executions, such as step sequences, employed by the operational (behavioural) semantics of nets in $\mathcal{PN}$;
- $\mathcal{LAN}$ are labelled acyclic nets, such as occurrence nets, providing the structural description of abstract processes of nets in $\mathcal{PN}$, with each labelled net in $\mathcal{LAN}$ representing a single non-sequential history;

- $\mathcal{LEX}$ are labelled executions, such as labelled step sequences, employed by the operational semantics of nets in $\mathcal{LAN}$;
- $\mathcal{LCS}$ are labelled causal structures, such as labelled partial orders, defining an abstract causality semantics of nets in $\mathcal{PN}$.

The arrows in figure 2 indicate functions that will be instantiated later, and then used to define and relate the three views on semantics for the Petri net model $\mathcal{PN}$ captured respectively by $\mathcal{EX}$, $\mathcal{LAN}$ and $\mathcal{LCS}$. For each net model considered in this paper, it will be our aim to show that the different semantics agree in the sense that processes ($\mathcal{LAN}$) and causal structures ($\mathcal{LCS}$) describe relations between events consistent with the chosen operational semantics ($\mathcal{EX}$). This section will show how certain simple and natural conditions (called *properties*) guarantee such an agreement. As a result, we will later be in the position to focus solely on the definitions of the semantical domains and functions appearing in figure 2, and after establishing the properties in question, the desired results on the semantics will follow immediately.

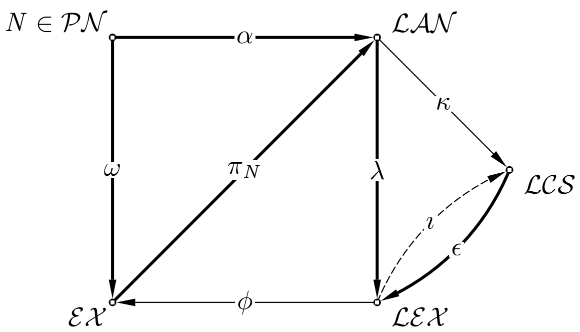

FIG. 2 The general setup for a Petri net N in $\mathcal{PN}$.

Let us now assume that a certain Petri net model $\mathcal{PN}$ has been fixed, and that the N in figure 2 is an arbitrary net from that model. We first consider the square-like part of the diagram (together with the diagonal), which essentially describes and relates two different ways in which a net in $\mathcal{PN}$ can be given a process semantics.

The function $\omega : \mathcal{PN} \to \mathbb{P}(\mathcal{EX})$ yields the non-empty set of executions of N, providing its *operational* semantics. The function $\alpha : \mathcal{PN} \to \mathbb{P}(\mathcal{LAN})$ associates with N a non-empty set of labelled acyclic nets (*processes*) from $\mathcal{LAN}$ satisfying certain *axioms*; a process is given an operational semantics through the function $\lambda : \mathcal{LAN} \to \mathbb{P}(\mathcal{LEX})$ which associates with it a non-empty set of *labelled* executions. A labelled execution can be interpreted as an ordinary execution (of the original net N) by *forgetting* some irrelevant information through the total function $\phi : \mathcal{LEX} \to \mathcal{EX}$. Finally, the partial function $\pi_N : \mathcal{EX} \to \mathbb{P}(\mathcal{LAN})$ defines, for each execution of N, a non-empty set of labelled acyclic nets which can be viewed as operationally defined processes of N. We thus have our first requirement.

Property 1. The functions ω, α, λ, ϕ and $\pi_N|_{\omega(N)}$ are total. Moreover, ω, α, λ and $\pi_N|_{\omega(N)}$ never return the empty set.

Two aims can now be formulated which, when fulfilled, guarantee that the axiomatic and behavioural process definition as well as the operational semantics of nets in $\mathcal{PN}$ are in agreement: the axiomatic processes of N (defined through α) coincide with the operational processes of N (defined through $\pi_N \circ \omega$); and the operational semantics of N (defined through ω) coincides with the operational semantics of the processes of N (defined through $\phi \circ \lambda \circ \alpha$). To prove these aims, we use a consistency property relating individual executions to individual processes: (i) any process defined from an execution ξ of N can also be defined axiomatically and then has ξ as one of its executions; and (ii) any labelled execution of a process LN of N can also be interpreted as an execution of N and then can be used to define LN operationally.

Property 2 (Consistency). For all $\xi \in \mathcal{EX}$ and $LN \in \mathcal{LAN}$,

$$\xi \in \omega(N) \wedge LN \in \pi_N(\xi) \text{ iff } LN \in \alpha(N) \wedge \xi \in \phi(\lambda(LN)) \, .$$

Provided that this property has been established for a given net model $\mathcal{PN}$, the two aims formulated above follow.

Aim 1. $\alpha = \pi_N \circ \omega$.

Proof. To show the ($\subseteq$) inclusion, suppose that $LN \in \alpha(N)$. Then, by property 1 for λ and ϕ, there exists $\xi \in \phi(\lambda(LN))$. Hence, by property 2, $\xi \in \omega(N)$ and $LN \in \pi_N(\xi)$. Thus $LN \in \pi_N(\omega(N))$. To show the ($\supseteq$) inclusion, suppose that $LN \in \pi_N(\omega(N))$. Then there exists $\xi \in \omega(N)$ such that $LN \in \pi_N(\xi)$. Hence, by property 2, $LN \in \alpha(N)$. ∎

Aim 2. $\omega = \phi \circ \lambda \circ \alpha$.

Proof. To show the ($\subseteq$) inclusion, suppose that $\xi \in \omega(N)$. Then, by property 1 for π_N, there exists $LN \in \pi_N(\xi)$. Hence, by property 2, $LN \in \alpha(N)$ and $\xi \in \phi(\lambda(LN))$. Thus $\xi \in \phi(\lambda(\alpha(N)))$. To show the ($\supseteq$) inclusion, suppose that $\xi \in \phi(\lambda(\alpha(N)))$. Then there exists $LN \in \alpha(N)$ such that $\xi \in \phi(\lambda(LN))$. Hence, by property 2, $\xi \in \omega(N)$. ∎

An immediate corollary of aims 1 and 2 is the consistency between the operational semantics of N and the operational semantics of its behaviourally defined processes.

Corollary 3.1. $\omega = \phi \circ \lambda \circ \pi_N \circ \omega$.

We now turn to the abstract causality semantics of processes which is represented by the triangle-like part on the right of the diagram in figure 2. By extracting from a labelled acyclic net the causal relationships between its labelled events one obtains an abstract representation of causality between events. This is formalised through a function $\kappa : \mathcal{LAN} \to \mathcal{LCS}$ which associates a labelled *causal* structure with each process in $\mathcal{LAN}$. To relate this abstract causality semantics to the operational semantics of processes, we use a total function $\epsilon : \mathcal{LCS} \to \mathbb{P}(\mathcal{LEX})$ and a partial function $\imath : \mathbb{P}(\mathcal{LEX}) \to \mathcal{LCS}$, which allow one to go back and forth between labelled causal structures and the corresponding labelled executions. Formally, we require

Property 3. The functions κ, ϵ and $\imath|_{\lambda(\mathcal{LAN})}$ are total. Moreover, ϵ never returns the empty set.

The function ϵ associates with each labelled causal structure a set of labelled *executions.* On ϵ we impose the restriction that the executions returned by ϵ should always contain enough information to uniquely reconstruct the original labelled causal structure. To formalise this requirement, we have the partial function $\imath$, which is defined for sets of labelled executions and yields labelled causal structures, typically through some kind of *intersection*. It is partially defined since it cannot associate a labelled causal structure to a set of labelled executions that do not have a common domain and labelling (and thus are unrelated).

Property 4 (Representation). $\imath \circ \epsilon = id_{\mathcal{LCS}}$.

Note that this implies that the domain of $\imath$ includes $\epsilon(\mathcal{LCS})$.

Clearly, the causality in a process of N (defined through κ) should coincide with the causality structure implied by its operational semantics (through $\imath \circ \lambda$). By taking care that the observational semantics for the structures in $\mathcal{LCS}$ fits with the operational semantics chosen for $\mathcal{LAN}$, such an aim can be achieved. Thus we require

Property 5 (Fitting). $\lambda = \epsilon \circ \kappa$.

and then we have

Aim 3. $\kappa = \imath \circ \lambda$.

Proof. By properties 4 and 5, $\kappa = id_{\mathcal{LCS}} \circ \kappa = \imath \circ \epsilon \circ \kappa = \imath \circ \lambda$. ∎

Finally, we can relate the operational semantics of the net N and the set of labelled causal structures associated with it, in effect joining together the two parts of the diagram in figure 2 considered so far separately.

COROLLARY 3.2. $\omega = \phi \circ \epsilon \circ \kappa \circ \alpha$.

Proof. By aim 2 and property 5, $\omega = \phi \circ \lambda \circ \alpha = \phi \circ \epsilon \circ \kappa \circ \alpha$. ∎

Aim 2 and corollaries 3.1 and 3.2 verify the consistency of the process and abstract causality semantics of the net N with its operational semantics given by the function ω (which captures the dynamics of the nets in $\mathcal{PN}$ and is in many instances given through, for example, the standard firing sequence or step sequence semantics).

To use the above setup in practice all we need to do is to establish properties 1 and 3, and check that the consistency, representation and fitting properties hold true (properties 2, 4 and 5). Having done so, the semantical aims follow from the above discussion.

4. EXECUTIONS AND CAUSAL STRUCTURES

In this section we will discuss the specific classes of executions, labelled executions, and labelled causal structures to be used in the rest of this paper. We thus instantiate $\mathcal{EX}$, $\mathcal{LEX}$, and $\mathcal{LCS}$ together with functions ϕ, $\imath$, and ϵ and we establish that these satisfy the requirements formulated in properties 1, 3, and 4.

4.1. Executions and labelled executions

We use two kinds of executions, step sequences ($\mathcal{STS}$) and firing sequences ($\mathcal{FS}$). A *step sequence* (over a set $X \subseteq \mathcal{U}$) is a finite — possibly empty — sequence of non-empty finite multisets (over X), while a *firing sequence* (over $X \subseteq \mathcal{U}$) is a finite sequence of elements (from X); i.e., a firing sequence over X is an element of X^*. Since we identify a finite sequence $x_1 \dots x_n$ with $\{x_1\} \dots \{x_n\}$, we have $\mathcal{FS} \subseteq \mathcal{STS}$.

We also use two kinds of labelled executions, labelled step sequences ($\mathcal{LSTS}$) and labelled firing sequences ($\mathcal{LFS}$). A *labelled step sequence* is a pair $\varpi \stackrel{\text{df}}{=} (\sigma, \ell)$, where $\sigma = X_1 \ldots X_n \in \mathcal{STS}$ is a step sequence consisting of mutually disjoint sets (rather than multisets) $X_i \subseteq \mathcal{U}$, and ℓ is a labelling for the set $X_1 \cup \ldots \cup X_n$ called the *domain* of ϖ. A *labelled firing sequence* is a labelled step sequence (σ, ℓ) such that $\sigma \in \mathcal{FS}$ is a firing sequence. Note that $\mathcal{LFS} \subseteq \mathcal{LSTS}$.

With each labelled step (firing) sequence $\varpi = (\sigma, \ell)$, where $\sigma = X_1 \ldots X_n$, we associate the step (firing) sequence $\phi(\varpi) \stackrel{\text{df}}{=} \ell\langle\sigma\rangle$, thus defining the function ϕ of figure 2 by forgetting the identity of the elements carrying the labels. (Note that ϕ is total and hence satisfies property 1.) Moreover, for $i \leq n$ and $x \in X_i$, we use $ind(\varpi, x) \stackrel{\text{df}}{=} i$ to denote the index of the unique set X_i in which x appears.

4.2. Labelled causal structures

We use two kinds of labelled causal structures, labelled partial orders ($\mathcal{LPO}$) and labelled stratified order structures ($\mathcal{LSOS}$).

A *labelled partially ordered set* (or *poset*) is a triple $lpo \stackrel{\text{df}}{=} (X, \prec, \ell)$, where X is a set (the domain of *lpo*), ℓ is a labelling for X, and $\prec \subseteq X \times X$ is a partial order. In this paper we will only be concerned with finite posets, i.e., posets with finite domains. To denote that $x = y$ or $x \prec y$, we write $x \preceq y$. The notation $x \not\leftrightarrow y$ indicates that x and y are distinct incomparable elements ($x \neq y \wedge x \not\prec y \wedge y \not\prec x$). *lpo* is *linear* if any two distinct elements of X are comparable ($\not\leftrightarrow = \varnothing$) and *stratified* [9] if $x \not\leftrightarrow y$ and $y \not\leftrightarrow z$ imply that $x \not\leftrightarrow z$ whenever $x \neq z$. A stratified poset $lpo = (X, \prec, \ell)$ can be identified with the labelled step sequence $(X_1 \ldots X_n, \ell)$, where the X_i's are the equivalence classes of the relation $\not\leftrightarrow \cup\, id_X$, with the property: $\prec = \bigcup_{i<j} X_i \times X_j$, and $\not\leftrightarrow = (\bigcup_i X_i \times X_i) \setminus id_X$. Similarly, a linear poset $lpo = (X, \prec, \ell)$ can be identified with the labelled firing sequence $(x_1 \ldots x_n, \ell)$, where $x_1 \ldots x_n$ is the enumeration of the elements of X with the property $\prec = \bigcup_{i<j} \{(x_i, x_j)\}$.

A poset *lpo* can be thought of as an abstract history of a concurrent system, where $\prec$ is interpreted as *causality*, and $\not\leftrightarrow$ as *independence*.

A *labelled stratified order structure* [10, 13] (or *so-structure*) is a structure $lsos \stackrel{\text{df}}{=} (X, \prec, \sqsubset, \ell)$, where X is a finite set (the domain of *lsos*), ℓ is a labelling for X, and $\prec$ and $\sqsubset$ are two binary relations over X such that for all $x, y, z \in X$,

$$\begin{array}{ccccll}
 & & & & x \not\sqsubset x & \text{C1} \\
 & & x \prec y & \Longrightarrow & x \sqsubset y & \text{C2} \\
x \sqsubset y \sqsubset z & \wedge & x \neq z & \Longrightarrow & x \sqsubset z & \text{C3} \\
x \sqsubset y \prec z & \vee & x \prec y \sqsubset z & \Longrightarrow & x \prec z & \text{C4}\,.
\end{array}$$

It is easily seen that $(X, \prec, \ell)$ is a poset and, furthermore, that $x \prec y$ implies $y \not\sqsubset x$. Moreover, if $(X, \prec, \ell)$ is a poset, then $(X, \prec, \prec, \ell)$ is an so-structure. Thus $\mathcal{LSOS}$ may be viewed as extending $\mathcal{LPO}$. In diagrams, $\prec$ is represented by solid arcs, and $\sqsubset$ by dashed arcs. We can omit arcs that can be deduced using C1-C4.

The first relation in an so-structure *lsos* should be interpreted as the standard *causality*, and the second relation as *weak causality*. While causality is an abstraction of the 'earlier than' relation, weak causality is a similar abstraction of the 'not later than' relation. For a detailed discussion of so-structures the reader is referred to [13].

Representation properties

We now instantiate the functions ϵ and $\imath$ relating labelled causal structures with labelled executions. First we establish a relationship between $\mathcal{LPO}$ and $\mathcal{LSTS}$ with $\mathcal{LFS}$ as a special case.

The set of *labelled step sequences* of a poset $lpo = (X, \prec, \ell)$ is the set $\epsilon_{\mathcal{LSTS}}(lpo)$ comprising all $\varpi \in \mathcal{LSTS}$ with domain X and labelling ℓ such that for all $x, y \in X$, $x \prec y$ implies $ind(\varpi, x) < ind(\varpi, y)$. In other words, $\epsilon_{\mathcal{LSTS}}(lpo)$ comprises all labelled step sequences (stratified posets) with the same domain and respecting the ordering $\prec$. Moreover, $\epsilon_{\mathcal{LFS}}(lpo) \stackrel{\mathrm{df}}{=} \epsilon_{\mathcal{LSTS}}(lpo) \cap \mathcal{LFS}$ consists of the *labelled firing sequences* or linearisations (linear posets) of lpo.

The *poset intersection* of a non-empty set $LSTS$ of labelled step sequences with the same domain X and labelling ℓ is defined as $\imath_{\mathcal{LPO}}(LSTS) \stackrel{\mathrm{df}}{=} (X, \prec, \ell)$, where $\prec$ is a binary relation on X such that for all $x, y \in X$, $x \prec y$ if $ind(\varpi, x) < ind(\varpi, y)$ for all $\varpi \in LSTS$. In other words, $\imath_{\mathcal{LPO}}(LSTS)$ intersects all the orderings on the set X implied by the elements of $LSTS$. It is easy to see that $\imath_{\mathcal{LPO}}(LSTS)$ is a poset. Moreover, every poset is completely determined by its labelled step sequences and, in fact, already by its labelled firing sequences ([21]).

FACT 4.1 (properties 3 and 4 for $\epsilon_{\mathcal{LFS}}$, $\epsilon_{\mathcal{LSTS}}$ and $\imath_{\mathcal{LPO}}$). *Let lpo be a poset.*

1. $\epsilon_{\mathcal{LFS}}(lpo) \neq \varnothing$ *and* $\epsilon_{\mathcal{LSTS}}(lpo) \neq \varnothing$.

2. $\imath_{\mathcal{LPO}}(\epsilon_{\mathcal{LFS}}(lpo)) = lpo$ *and* $\imath_{\mathcal{LPO}}(\epsilon_{\mathcal{LSTS}}(lpo)) = lpo$.

Next we consider representations of so-structures. The set of *labelled step sequences* of an so-structure $lsos = (X, \prec, \sqsubset, \ell)$ is the set $\epsilon(lsos)$ comprising all $\varpi \in \mathcal{LSTS}$ with domain X and labelling ℓ such that for all $x, y \in X$, $x \prec y$ implies $ind(\varpi, x) < ind(\varpi, y)$, and $x \sqsubset y$ implies $ind(\varpi, x) \leq ind(\varpi, y)$. In other words, $\epsilon(lsos)$ comprises all labelled step sequences with the same domain and respecting the orderings $\prec$ and $\sqsubset$, under the assumption that the latter allows simultaneity. Note that, if $lsos = (X, \prec, \prec, \ell)$, then $\epsilon(lsos) = \epsilon_{\mathcal{LSTS}}(lpo)$, where $lpo = (X, \prec, \ell)$.

The *so-structure intersection* of a non-empty set $LSTS$ of labelled step sequences with the same domain X and labelling ℓ is $\imath(LSTS) \stackrel{\mathrm{df}}{=} (X, \prec, \sqsubset, \ell)$, where $\prec$ and $\sqsubset$ are binary relations on X such that for all $x, y \in X$, $x \prec y$ if $ind(\varpi, x) < ind(\varpi, y)$ for all $\varpi \in LSTS$, and $x \sqsubset y$ if $ind(\varpi, x) \leq ind(\varpi, y)$ for all $\varpi \in LSTS$. It is easy to see that $\imath(LSTS)$ is an so-structure. Moreover, every so-structure is completely determined by its labelled step sequences ([14]).

FACT 4.2 (properties 3 and 4 for ϵ and $\imath$). *Let lsos be an so-structure.*

1. $\epsilon(lsos) \neq \varnothing$.

2. $\imath(\epsilon(lsos)) = lsos$.

By Szpilrajn's representation theorem (fact 4.1 for $\mathcal{LFS}$) each poset is already unambiguously identified by its labelled firing sequences (linearisations). A similar result does not hold for so-structures since these do not necessarily have linear order extensions and so one needs to consider labelled step sequences (stratified poset extensions) [15]. Consider, e.g., $lsos \stackrel{\mathrm{df}}{=} (\{a, b\}, \varnothing, \{(a, b), (b, a)\}, id_{\{a,b\}})$, which has $(\{a, b\}, id_{\{a,b\}})$ as its only labelled step sequence.

As the next proposition shows, the incomparability ($\not\leftrightarrow$) of two elements of an so-structure implies that they may be executed simultaneously. Moreover, if in addition one of the elements is not required to occur not later than the other one, it can actually be executed later on.

PROPOSITION 4.3. *Let $lsos = (X, \prec, \sqsubset, \ell)$ be an so-structure.*

1. *If x and y are distinct elements of X such that $\neg(x \prec y)$ and $\neg(y \prec x)$ then there is a labelled step sequence $(\sigma, \ell) \in \epsilon(lsos)$ such that x and y belong to the same step of σ.*
2. *If x and y are distinct elements of X such that $\neg(x \prec y)$ and $\neg(y \sqsubset x)$ then there is a labelled step sequence $(\sigma, \ell) \in \epsilon(lsos)$ such that y belongs to the step immediately following the step to which x belongs.*

Proof. See the appendix. ∎

Acyclicity and closure of labelled causal structures

When used as a tool for representing concurrent behaviours, labelled causal structures will be derived from locally defined information involving events which directly interact with one another. This local information is combined into a global relationship involving all the event occurrences; in particular, posets and so-structures can be built from local relationships using suitable closure operations.

For posets, the construction in question is nothing but the standard *transitive closure*. We say that a structure $rs = (X, \prec, \ell)$ is *acyclic* if $\prec^+$, the transitive closure of $\prec$, is irreflexive. Moreover the *transitive closure* of rs is $rs^+ \stackrel{\text{df}}{=} (X, \prec^+, \ell)$.

FACT 4.4 (Closure for posets). *Let $rs = (X, \prec, \ell)$ be a structure. Then rs^+ is a poset* iff *rs is acyclic. Moreover, for every poset lpo, it is the case that $lpo^+ = lpo$.*

For so-structures, we need slightly more complicated devices, developed in [14]. The $\diamondsuit$–closure is an operation which constructs an so-structure from local information given in the form of a structure with two relations. The $\diamondsuit$*–closure* of a structure $rs = (X, \prec, \sqsubset, \ell)$ is $rs^\diamondsuit \stackrel{\text{df}}{=} (X, \prec', \sqsubset', \ell)$, where

$$\prec' \stackrel{\text{df}}{=} (\prec \cup \sqsubset)^\star \circ \prec \circ (\prec \cup \sqsubset)^\star \quad \text{and} \quad \sqsubset' \stackrel{\text{df}}{=} (\prec \cup \sqsubset)^\star \setminus id_X .$$

We say that rs is $\diamondsuit$*–acyclic* if $\prec'$ is irreflexive. This property has a straightforward interpretation in operational terms, as it means that in a system history described by rs, there are no event occurrences $e_1, e_2, \ldots, e_k$ such that each e_i has occurred *before or simultaneously with* e_{i+1}, while e_k has occurred *before* e_1. It is also this property which characterises those cases when $rs^\diamondsuit$ is an so-structure.

FACT 4.5 (Closure for so-structures [14]). *Let $rs = (X, \prec, \sqsubset, \ell)$ be a structure. Then $rs^\diamondsuit$ is an so-structure* iff *rs is* $\diamondsuit$–acyclic. *Moreover, for every so-structure $lsos$, it is the case that $lsos^\diamondsuit = lsos$.*

Note that if $rs = (X, \prec, \varnothing, \ell)$ and $(X, \prec, \ell)$ is acyclic, then $rs^\diamondsuit = (X, \prec^+, \prec^+, \ell)$.

5. LABELLED ACYCLIC NETS

This section introduces two kinds of labelled acyclic nets which as instantiations of $\mathcal{LAN}$ in figure 2 will form the basis of the process semantics discussed later. We define functions κ and λ which relate these nets to the labelled causal structures and to the labelled executions of the previous section, and which satisfy the requirements of properties 1 and 3. Moreover, the fitting condition of property 5 is established. Thus, in each case we will have achieved our aim 3.

Labelled occurrence nets

For ordinary Petri nets, labelled occurrence nets are used to represent execution histories (see, e.g., [2, 3, 11, 20]). Such acyclic nets may be viewed as partial net unfoldings, with each transition representing an occurrence of a transition in the original net, and each place corresponding to the presence of a token on a place of the original net. Conflicts between transitions are resolved and thus places do not branch.

Definition 5.1 ($\mathcal{LAN}$ for PT-nets). *A* labelled occurrence net *(or* o-net*) is a labelled net* $ON \stackrel{\text{df}}{=} (B, E, R, \ell)$ *such that*

- $R \subseteq (B \times E) \cup (E \times B)$.[3]
- *For every* $b \in B$, $|\text{PRE}_{ON}(b)| \leq 1$ *and* $|\text{POST}_{ON}(b)| \leq 1$.
- *The structure* $rs_{ON} \stackrel{\text{df}}{=} (E, \prec_{loc}, \ell|_E)$ *is acyclic, where* $\prec_{loc} \stackrel{\text{df}}{=} (R \circ R)|_{E \times E}$.
- ℓ *is a labelling for* $B \cup E$.

The class of o-nets will be denoted by $\mathcal{LON}$.

The places of an o-net are called *conditions* ('Bedingungen' in German) and its transitions are called *events* ('Ereignisse' in German). In diagrams, we show only their labels.

The relation $\prec_{loc}$ in definition 5.1 represents the *local* information about causal relationships between the events. Since the structure rs_{ON} is acyclic, ON defines a poset $\kappa(ON) \stackrel{\text{df}}{=} rs^+_{ON} = (E, \prec^+_{loc}, \ell|_E)$ (see fact 4.4) which in turn provides a partial order description of the labelled event occurrences. Note also that κ is total and hence satisfies property 3. We refer to $\kappa(ON)$ as the poset *generated by* ON.

Executions of o-nets

A rich set of notions and results has been developed over the years for occurrence nets. In addition to providing a precise description of causal relationships between executed events, an o-net enjoys several specific behavioural properties which make tractable some hard verification problems, such as marking reachability. We now rephrase without proofs certain facts known from the literature, both to demonstrate how o-nets fit into our semantical template and to serve as the basis, or guide, for our subsequent dealing with labelled activator occurrence nets.

Let $ON = (B, E, R, \ell)$ be a fixed o-net, and $\kappa(ON) = (E, \prec_{ON}, \ell|_E)$ be the poset generated by ON. The default *initial* marking MIN_{ON} of ON consists of all conditions without incoming arcs, i.e., $\text{MIN}_{ON} \stackrel{\text{df}}{=} B \setminus codom_R$, while the default *final* marking MAX_{ON} of ON consists of all conditions without outgoing arcs, i.e., $\text{MAX}_{ON} \stackrel{\text{df}}{=} B \setminus dom_R$. The executions of ON are the standard step sequences or firing sequences leading from MIN_{ON} to MAX_{ON}. Since MIN_{ON} assigns at most one token to each condition, the weight function always returns 0 or 1, and ON is acyclic and without branching conditions, it follows that ON is safe and that in any step sequence from the initial marking it can execute a given event no more than once.

[3] We treat the weight function as a binary relation R in this case as it always returns 0 or 1.

FACT 5.2. *If $M_0E_1M_1 \ldots E_nM_n$ is a mixed step sequence of ON from the initial marking, then each M_i is a set, and the E_i's are disjoint sets.*

Two key verification problems for Petri nets and other concurrent system models are related to checking whether a given state can ever be reached from the initial one, and whether a (multi)set of actions can ever be executed. Though for general Petri nets both problems are hard, for occurrence nets they can be easily treated using two notions introduced next.

A *slice* of ON is a maximal (w.r.t. set inclusion) set $S \subseteq B$ of conditions which are causally unrelated, i.e., $(S \times S) \cap R^+ = \varnothing$; and a *configuration* is a set $D \subseteq E$ of events which comprises all their causal predecessors, i.e., $e \in D$ and $f \prec_{ON} e$ implies $f \in D$. We denote this respectively by $S \in \textsc{sl}(ON)$ and $D \in \textsc{cnf}(ON)$. Clearly, both $\textsc{min}_{ON}$ and $\textsc{max}_{ON}$ are slices of ON, and both $\varnothing$ and E are configurations. Moreover, $\textsc{min}_{ON} = \textsc{mar}_{ON}(\varnothing)$ and $\textsc{max}_{ON} = \textsc{mar}_{ON}(E)$, and this close relationship extends to other slices and configurations (see fact 5.4). One can also show that for any two configurations D and G, $\textsc{mar}_{ON}(D) = \textsc{mar}_{ON}(G)$ implies $D = G$.

FACT 5.3. *Let $\textsc{min}_{ON}[E_1 \ldots E_n\rangle_{ON} M$.*

1. *For every $i \leq n$, if $e \in E_i$ and $f \prec_{ON} e$ then $f \in E_1 \cup \ldots \cup E_{i-1}$.*
2. *$E_1 \cup \ldots \cup E_n$ is a configuration of ON.*
3. *$M = \textsc{max}_{ON}$ iff $E_1 \cup \ldots \cup E_n = E$.*

Thus any execution of ON from the initial marking amounts to executing a configuration of events. And, since any configuration of events can be executed from the initial marking, configurations are exactly those sets of events which can be executed from the initial marking of ON. The next result shows that slices are exactly those markings which can be reached from the initial marking of ON.

FACT 5.4. *Let $\mathcal{M} \stackrel{\text{df}}{=} [\textsc{min}_{ON}\rangle_{ON}$, and $\mathcal{M}'$ be the set of all markings $M \in \mathcal{M}$ such that $\textsc{max}_{ON} \in [M\rangle_{ON}$. Then $\textsc{sl}(ON) = \mathcal{M} = \mathcal{M}' = \textsc{mar}_{ON}(\textsc{cnf}(ON))$.*

The above result implies that the final marking of ON is always reachable from any marking reachable from the initial one. Essentially, this means that ON is deadlock-free until its final marking has been reached.

The name 'slice' is in part motivated by our next notion, which captures the way in which a member of $\textsc{sl}(ON)$ slices through the occurrence net, dividing it into two subnets. For a slice $S \in \textsc{sl}(ON)$, let $\textsc{preon}_{ON}(S) \stackrel{\text{df}}{=} (B', E', R', \ell')$ and $\textsc{poston}_{ON}(S) \stackrel{\text{df}}{=} (B'', E'', R'', \ell'')$ be nets given by:

$$
\begin{array}{lcllcl}
B' & \stackrel{\text{df}}{=} & \{b \in B \mid \exists c \in S : (b,c) \in R^\star\} & B'' & \stackrel{\text{df}}{=} & \{b \in B \mid \exists c \in S : (c,b) \in R^\star\} \\
E' & \stackrel{\text{df}}{=} & \{e \in E \mid \exists c \in S : (e,c) \in R^\star\} & E'' & \stackrel{\text{df}}{=} & \{e \in E \mid \exists c \in S : (c,e) \in R^\star\} \\
R' & \stackrel{\text{df}}{=} & R|_{(B' \times E') \cup (E' \times B')} & R'' & \stackrel{\text{df}}{=} & R|_{(B'' \times E'') \cup (E'' \times B'')} \\
\ell' & \stackrel{\text{df}}{=} & \ell|_{B' \cup E'} & \ell'' & \stackrel{\text{df}}{=} & \ell|_{B'' \cup E''} \,.
\end{array}
$$

Intuitively, $\textsc{preon}_{ON}(S)$ is the part of ON which has been executed to reach the slice S, and $\textsc{poston}_{ON}(S)$ that which can still be executed after S.

FACT 5.5. *Let S be a slice of ON. Moreover, let ON' and ON'' be respectively the nets $\textsc{preon}_{ON}(S)$ and $\textsc{poston}_{ON}(S)$.*

1. *ON' and ON'' are o-nets such that: $B = B' \cup B''$, $B' \cap B'' = S$, $E = E' \uplus E''$ and $R = R' \uplus R''$. Moreover, $\text{MIN}_{ON'} = \text{MIN}_{ON}$, $\text{MAX}_{ON'} = S = \text{MIN}_{ON''}$ and $\text{MAX}_{ON''} = \text{MAX}_{ON}$.*
2. *Each mixed step sequence of ON' from $\text{MIN}_{ON'}$ is also a mixed step sequence of ON.*
3. *Each mixed step sequence of ON from MIN_{ON} to some marking M is also a mixed step sequence of ON', if all its events belong to E' or $M \subseteq B'$.*
4. *Each mixed step sequence of ON'' from S is also a mixed step sequence of ON from S, and vice versa.*

Labelled executions of o-nets and posets

Now we are ready to define the labelled executions of an o-net by adding event labels to its executions. Again, let $ON = (B, E, R, \ell)$ be a fixed o-net.

Definition 5.6 (λ for o-nets). *The sets*

$$\begin{array}{lcl} \lambda_{\mathcal{LSTS}}(ON) & \stackrel{\text{df}}{=} & \{(\sigma, \ell|_E) \mid \text{MIN}_{ON}[\sigma\rangle_{ON}\text{MAX}_{ON}\} \\ \lambda_{\mathcal{LFS}}(ON) & \stackrel{\text{df}}{=} & \lambda_{\mathcal{LSTS}}(ON) \cap \mathcal{LFS} \end{array}$$

are respectively the labelled step sequences *and the* labelled firing sequences *of ON.*

From facts 5.2 and 5.3(3), it follows that $\lambda_{\mathcal{LSTS}}(ON) \subseteq \mathcal{LSTS}$ and $\lambda_{\mathcal{LFS}}(ON) \subseteq \mathcal{LFS}$. Hence definition 5.6 is sound. Furthermore, note that $\lambda_{\mathcal{LSTS}}(ON)$ is a non-empty set because $\text{MAX}_{ON} \in [\text{MIN}_{ON}\rangle_{ON}$ by fact 5.4. Since, as observed before, the reachability of a marking in an ordinary Petri net does not depend on whether we use step sequences or firing sequences, $\lambda_{\mathcal{LFS}}(ON)$ is also non-empty. Hence both $\lambda_{\mathcal{LSTS}}$ and $\lambda_{\mathcal{LFS}}$ satisfy property 1. We also note that all labelled step (firing) sequences of ON have the same domain and labelling, and so $\imath_{\mathcal{LPO}}|_{\lambda_{\mathcal{LFS}}(\mathcal{LON})}$ and $\imath_{\mathcal{LPO}}|_{\lambda_{\mathcal{LSTS}}(\mathcal{LON})}$ are total (property 3).

From facts 5.3(1,3), 5.4 and 5.5, it can be deduced that the operational semantics of ON defined through its labelled step sequences agrees with its partial order semantics captured by the poset $\kappa(ON)$. We therefore obtain the following, on the basis of our earlier discussion.

Fact 5.7 (property 5 (fitting) and aim 3 for o-nets).

1. *$\lambda_{\mathcal{LSTS}}(ON) = \epsilon_{\mathcal{LSTS}}(\kappa(ON))$ and $\lambda_{\mathcal{LFS}}(ON) = \epsilon_{\mathcal{LFS}}(\kappa(ON))$.*
2. *$\kappa(ON) = \imath_{\mathcal{LPO}}(\lambda_{\mathcal{LSTS}}(ON)) = \imath_{\mathcal{LPO}}(\lambda_{\mathcal{LFS}}(ON))$.*

Labelled activator occurrence nets

The presence of inhibitor arcs makes the standard unfolding procedure more complicated, due to the fact that local information regarding the lack of tokens in a place cannot be explicitly represented in an o-net. In [14] this problem is solved by using complement places and representing an (unweighted) inhibitor arc by an activator arc connected to a condition representing a complement place. The resulting nets are called activator occurrence nets.

Definition 5.8 ($\mathcal{LAN}$ for inhibitor nets). *A* labelled activator occurrence net *(or* ao-net*) is a labelled activator net $AON \stackrel{\text{df}}{=} (B, E, R, Act, \ell)$ such that*

- $R \subseteq (B \times E) \cup (E \times B)$ *and* $Act \subseteq B \times E$.[4]

- *For every* $b \in B$, $|\text{PRE}_{AON}(b)| \leq 1$ *and* $|\text{POST}_{AON}(b)| \leq 1$.

- *The structure* $rs_{AON} \stackrel{\text{df}}{=} (E, \prec_{loc}, \sqsubset_{loc}, \ell|_E)$ *is* $\diamondsuit$*–acyclic, where* $\prec_{loc}$ *and* $\sqsubset_{loc}$ *are relations respectively given by* $(R \circ R)|_{E\times E} \cup (R \circ Act)$ *and* $Act^{-1} \circ R$.

- ℓ *is a labelling for* $B \cup E$.

The class of ao-nets will be denoted by $\mathcal{LAON}$.

Let $AON = (B, E, R, Act, \ell)$ be an ao-net as in definition 5.8. Since rs_{AON} is $\diamondsuit$–acyclic, $(R \circ R)|_{E\times E}$ is acyclic in the usual sense, and so the labelled net underlying AON, $\text{UND}(AON) \stackrel{\text{df}}{=} (B, E, R, \ell)$, is an o-net.

Similarly as for o-nets, the relations $\prec_{loc}$ and $\sqsubset_{loc}$ represent the *local* information about the causal relationships between the events contained in AON. Figure 3 shows how $\prec_{loc}$ and $\sqsubset_{loc}$ are constructed from ordinary arcs and activator arcs. They define an so-structure which captures the relations between the occurrences of the labelled events.

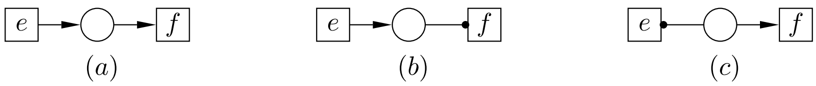

FIG. 3 (a,b) Two cases defining $e \prec_{loc} f$, and (c) one case defining $e \sqsubset_{loc} f$.

Definition 5.9 (κ for ao-nets). *The so-structure* generated *by* AON *is given by* $\kappa(AON) = (E, \prec_{AON}, \sqsubset_{AON}, \ell|_E) \stackrel{\text{df}}{=} rs_{AON}{}^{\diamondsuit}$.

Thus $\prec_{AON} = \Subset^\star \circ \prec_{loc} \circ \Subset^\star$ and $\sqsubset_{AON} = \Subset^\star \setminus id_X$, where $\Subset \stackrel{\text{df}}{=} \prec_{loc} \cup \sqsubset_{loc}$. Hence, since rs_{AON} is $\diamondsuit$–acyclic, definition 5.9 is sound, i.e., $\kappa(AON)$ is indeed an so-structure (see fact 4.5). Note that κ is total and thus satisfies property 3. Figure 4 shows an ao-net and the so-structure it generates.

We observe that $\prec_{AON}$ includes the partial order relation of the poset generated by $\text{UND}(AON)$. In fact, the definition of κ given here can be considered as a conservative extension of the previous definition of κ from $\mathcal{LON}$ to $\mathcal{LAON}$, as an ao-net AON without activator arcs can be identified with its underlying o-net. In such a case, we have $\kappa(AON) = (E, \prec, \prec, \ell|_E)$ where $\prec = ((R \circ R)|_{E\times E})^+$. Hence $\kappa(AON)$ is the so-structure determined by the poset $\kappa(\text{UND}(AON)) = (E, \prec, \ell|_E)$.

Executions of ao-nets

We have already mentioned that occurrence net are a model in which various verification questions, such as marking reachability, can be easily treated using the notions of a slice and configuration. We will now show how these concepts can be extended to activator occurrence nets.

Until the end of this section, let $AON = (B, E, R, Act, \ell)$ be a fixed ao-net, and ON be its underlying o-net. Moreover, let $\kappa(AON) = (E, \prec_{AON}, \sqsubset_{AON}, \ell|_E)$

[4]Again, the weight function R is treated as a binary relation which always returns 0 or 1, and as all activator arcs have weight 0 or 1, also Act may be viewed here as a binary relation.

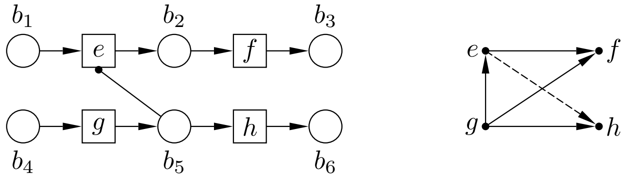

FIG. 4 An ao-net AON_0 where $\text{MIN}_{AON_0} = \{b_1, b_4\}$ and $\text{MAX}_{AON_0} = \{b_3, b_6\}$, and the so-structure it generates.

be the so-structure generated by AON, and $\kappa(ON) = (E, \prec_{ON}, \ell|_E)$ be the poset generated by ON. Recall that $\prec_{ON} = ((R \circ R)|_{E\times E})^+$.

The default *initial* marking MIN_{AON} and *final* marking MAX_{AON} of AON are respectively MIN_{ON} and MAX_{ON}. Thus each mixed step sequence of AON from the initial marking is a mixed step sequence of ON from the initial marking. The converse holds if the executed events obey the local constraints imposed by the activator arcs present in AON.

PROPOSITION 5.10. *$\mu \overset{\text{df}}{=} M_0E_1M_1 \ldots E_nM_n$ is a mixed step sequence of AON from the initial marking* iff *μ is a mixed step sequence of ON from the initial marking such that for every $i \leq n$ and $e \in E_i$, $f \prec_{loc} e$ implies $f \in E_1 \cup \ldots \cup E_{i-1}$, and $f \sqsubset_{loc} e$ implies $f \notin E_{i+1} \cup \ldots \cup E_n$.*

Proof. See the appendix. ▮

Hence, in view of fact 5.2, we obtain

PROPOSITION 5.11. *If $M_0E_1M_1 \ldots E_nM_n$ is a mixed step sequence of AON from the initial marking, then each M_i is a set, and the E_i's are disjoint sets.*

To characterise reachable markings and executable sets of events of AON, we will now extend the notions of a slice and configuration, which proved to work very well for o-nets. However, since the so-structure $\kappa(AON)$ has two ordering relations, we will have two different notions instead of just one defined previously.

A set $D \subseteq E$ is a *strong configuration* of AON, if $e \in D$ and $f \prec_{loc}^+ e$ implies $f \in D$. It is a *weak configuration*, if $e \in D$ and $f \Subset^+ e$ implies $f \in D$. We will denote this respectively by $D \in \text{SCNF}(AON)$ and $D \in \text{WCNF}(AON)$.

Since the ordering $\prec_{ON}$ is included in $\prec_{loc}^+$ which in turn is included in $\Subset^+$, we have $\text{CNF}(ON) \supseteq \text{SCNF}(AON) \supseteq \text{WCNF}(AON)$, and if $Act = \varnothing$ then both inclusions become equalities.

PROPOSITION 5.12. *Let $\text{MIN}_{AON}[E_1 \ldots E_n\rangle_{AON} M$.*

1. *$E_1 \cup \ldots \cup E_n$ is a strong configuration of AON.*
2. *$M = \text{MAX}_{AON}$* iff *$E_1 \cup \ldots \cup E_n = E$.*
3. *If $M = \text{MAX}_{AON}$ then, for every $i \leq n$ and $e \in E_i$, $f \prec_{AON} e$ implies $f \in E_1 \cup \ldots \cup E_{i-1}$, and $f \sqsubset_{AON} e$ implies $f \in E_1 \cup \ldots \cup E_i$.*

Proof. Follows from proposition 5.10 and fact 5.3(3). ▮

To introduce two kinds of slices for ao-nets, we first define two relations on the conditions of AON generalising the idea of causally related conditions in o-nets. Instead of simply using $R^+|_{B\times B} = R \circ ((R \circ R)|_{E\times E})^\star \circ R$ we now have

$\text{SLIN}(AON) \stackrel{\text{df}}{=} R\circ \prec_{loc}^{\star} \circ R$ and $\text{WLIN}(AON) \stackrel{\text{df}}{=} R\circ \Subset^{\star} \circ R$. Clearly, $R^{+}|_{B\times B}$ is included in $\text{SLIN}(AON)$ which in turn is included in $\text{WLIN}(AON)$, and if $Act = \varnothing$ then both inclusions become equalities.

For the ao-net AON_0 in figure 4, $\text{SLIN}(AON_0) = (\{b_4\}\times\{b_2,b_3,b_5,b_6\})\cup(\{b_1\}\times\{b_2,b_3\})\cup\{(b_2,b_3),(b_5,b_6)\}$ and $\text{WLIN}(AON_0) = \text{SLIN}(AON_0)\cup\{(b_1,b_6)\}$. Intuitively, $(b,c)\in\text{WLIN}(AON)$ means that b and c cannot be both marked at any marking reachable from MIN_{AON} from which MAX_{AON} is also reachable (referring to the ao-net in figure 4, b_1 and b_6 are such conditions because the only way to remove a token from b_1 is to execute e, and for this one needs a token in b_5; however, the token in b_5 has already been removed in order to produce a token in b_6). Moreover, $(b,c)\in\text{SLIN}(AON)$ means that b and c cannot be both marked at any marking reachable from MIN_{AON}, as in order to put a token in c there must have been a token in b which had to be consumed to mark c.

A *strong slice* of AON is a maximal (w.r.t. set inclusion) set $S\subseteq B$ of conditions which are incomparable w.r.t. $\text{SLIN}(AON)$, i.e., $(S\times S)\cap\text{SLIN}(AON) = \varnothing$; while a *weak slice* is a maximal set S of conditions which are incomparable w.r.t. $\text{WLIN}(AON)$, i.e., $(S\times S)\cap\text{WLIN}(AON) = \varnothing$. We denote this respectively by $S\in\text{SSL}(AON)$ and $S\in\text{WSL}(AON)$. For the ao-net AON_0 in figure 4, we have $\text{WSL}(AON_0) = \{\{b_1,b_4\},\{b_1,b_5\},\{b_2,b_5\},\{b_2,b_6\},\ \{b_3,b_5\},\{b_3,b_6\}\}$ and $\text{SSL}(AON_0) = \text{WSL}(AON_0)\cup\{(b_1,b_6)\}$.

PROPOSITION 5.13. *$\text{WSL}(AON)\subseteq\text{SSL}(AON)\subseteq\text{SL}(ON)$, and if $Act=\varnothing$ then both inclusions become equalities.*

Proof. See the appendix. ∎

We finally extend the notions of a net preceding and following a slice of an o-net. Let S be a slice of the o-net ON underlying AON. We define two nets with activator arcs, $AON' \stackrel{\text{df}}{=} (B',E',R',Act',\ell')$ and $AON'' \stackrel{\text{df}}{=} (B'',E'',R'',Act'',\ell'')$ so that $(B',E',R',\ell') = \text{PREON}_{ON}(S)$, $(B'',E'',R'',\ell'') = \text{POSTON}_{ON}(S)$, $Act' \stackrel{\text{df}}{=} Act|_{B'\times E'}$ and $Act'' \stackrel{\text{df}}{=} Act|_{B''\times E''}$. We will denote AON' and AON'' respectively by $\text{PREAON}_{AON}(S)$ and $\text{POSTAON}_{AON}(S)$.

Note that due to proposition 5.13, the last two notions are defined for every weak or strong slice of AON. Moreover, the structures $rs_{AON'} = (E',\prec'_{loc},\sqsubset'_{loc},\ell|_{E'})$ and $rs_{AON''} = (E'',\prec''_{loc},\sqsubset''_{loc},\ell|_{E''})$ are $\diamondsuit$–acyclic, because $\prec'_{loc}\cup\prec''_{loc}$ and $\sqsubset'_{loc}\cup\sqsubset''_{loc}$ are respectively included in $\prec_{loc}$ and $\sqsubset_{loc}$. Hence AON' and AON'' are both ao-nets.

What now follows is a series of results which re-establish (after some adjustments) the well-known facts about the behaviour of o-nets recalled earlier in this paper.

PROPOSITION 5.14. *Let S be a slice of ON. Moreover, let AON' and AON'' be respectively the ao-nets $\text{PREAON}_{AON}(S)$ and $\text{POSTAON}_{AON}(S)$.*

1. *If $S\in\text{SSL}(AON)$ then the following hold.*
   - (a) *$\text{ACT}_{AON}(e) = \text{ACT}_{AON'}(e)$, for every event $e\in E'$.*
   - (b) *Each mixed step sequence of AON' from $\text{MIN}_{AON'}$ is also a mixed step sequence of AON.*
   - (c) *Each mixed step sequence of AON from MIN_{AON} to some marking M is also a mixed step sequence of AON', if all its events belong to E' or $M\subseteq B'$.*

2. If $S \in \text{WSL}(AON)$ then the following hold.

(a) $\text{ACT}_{AON}(e) = \text{ACT}_{AON''}(e)$, for every event $e \in E''$.

(b) Each mixed step sequence of AON'' from S is also a mixed step sequence of AON from S, and vice versa.

Proof. (1) Suppose that $e \in E'$ and $d \in \text{ACT}_{AON}(e) \cap (B'' \backslash S)$. Then there are $b, c \in S$ and $f \in E''$ such that $eR^\star c$ and $bR^\star fRd$. Hence $(b, c) \in \text{SLIN}(AON)$, a contradiction. Parts (1b) and (1c) follow from fact 5.5(2,3) and part (1a).

(2) Suppose that $e \in E''$ and $d \in \text{ACT}_{AON}(e) \cap (B' \backslash S)$. Then there are $b, c \in S$ and $f \in E'$ such that $bR^\star e$ and $dRfR^\star c$. Hence $(b, c) \in \text{WLIN}(AON)$, a contradiction. Part (2b) follows from fact 5.5(4) and part (2a). ∎

As for o-nets, slices correspond to reachable markings and, intuitively, the ao-net $\text{PREAON}_{AON}(S)$ is the part of AON which has been executed to reach S, and $\text{POSTAON}_{AON}(S)$ that which can still be executed after S. However, if S is not a weak slice (i.e., $S \notin \text{WSL}(AON)$), then MAX_{AON} is not reachable from S in $\text{POSTAON}_{AON}(S)$. Consider, for instance, the strong slice $\{b_1, b_6\}$ of AON_0 in figure 4. To reach $\{b_1, b_6\}$ the weak causality implied by the activator arc is ignored and condition b_5 can no longer be marked despite the fact that the event which has to test it has not yet occurred.

PROPOSITION 5.15. *Let $\text{MIN}_{AON}[E_1 \ldots E_n\rangle_{AON} M$. Then the following statements are equivalent:*

1. $\text{MAX}_{AON} \in [M\rangle_{AON}$.

2. $E_1 \cup \ldots \cup E_n \in \text{WCNF}(AON)$.

3. $M \in \text{WSL}(AON)$.

Proof. See the appendix. ∎

Hence since the initial marking is a weak slice we obtain

COROLLARY 5.16. $\text{MAX}_{AON} \in [\text{MIN}_{AON}\rangle_{AON}$.

Thus to describe the executions of AON (from MIN_{AON} to MAX_{AON}), we have to use the weak slices; clearly, both MIN_{AON} and MAX_{AON} are weak slices. The strong slices of AON coincide with the markings reachable from MIN_{AON} in AON, and the weak slices with those from which in addition MAX_{AON} is reachable.

PROPOSITION 5.17. *Let $\mathcal{M} \overset{\text{df}}{=} [\text{MIN}_{AON}\rangle_{AON}$, and $\mathcal{M}'$ be the set of all markings $M \in \mathcal{M}$ such that $\text{MAX}_{AON} \in [M\rangle_{AON}$.*

1. $\text{SSL}(AON) = \mathcal{M} = \text{MAR}_{AON}(\text{SCNF}(AON))$.

2. $\text{WSL}(AON) = \mathcal{M}' = \text{MAR}_{AON}(\text{WCNF}(AON))$.

Proof. (1) The inclusion $\text{SSL}(AON) \subseteq \mathcal{M}$ holds by proposition 5.14(1c) and corollary 5.16. The $\mathcal{M} \subseteq \text{MAR}_{AON}(\text{SCNF}(AON))$ inclusion follows from proposition 5.10. To show $\text{MAR}_{AON}(\text{SCNF}(AON)) \subseteq \text{SSL}(AON)$, suppose $D \in \text{SCNF}(AON)$ and $b, c \in \text{MAR}_{AON}(D)$ are such that $(b, c) \in \text{SLIN}(AON)$. Then there are $e, f \in E$ such that $bRe{\prec}^\star_{loc} fRc$. Hence $f \in D$ and $e \notin D$, contradicting $D \in \text{SCNF}(AON)$.

(2) Follows from part (1), the fact that $\text{WSL}(AON)$ and $\text{WCNF}(AON)$ are respectively included in $\text{SSL}(AON)$ and $\text{SCNF}(AON)$, and proposition 5.15. ∎

Labelled executions of ao-nets and so-structures

By adding event labels to the executions of an ao-net we obtain its labelled executions.

Definition 5.18 (λ for ao-nets). *The set of* labelled step sequences *of AON is given by* $\lambda(AON) \stackrel{\text{df}}{=} \{(\sigma, \ell|_E) \mid \text{MIN}_{AON}[\sigma\rangle_{AON}\text{MAX}_{AON}\}$.

The soundness of the above definition, i.e., that $\lambda(AON) \subseteq \mathcal{LSTS}$ follows from proposition 5.11. By corollary 5.16, $\lambda(AON)$ is a non-empty set, and so $\lambda(AON)$ satisfies property 1. We also note that the labelled step sequences in $\lambda(AON)$ all have the same domain and labelling, and so $\imath|_{\lambda(\mathcal{LAON})}$ is total (property 3). We observe that the definition of λ is an extension of the definition of $\lambda_{\mathcal{LSTS}}$ for o-nets since an o-net can be considered as an ao-net without activator arcs and $\lambda(AON) = \lambda_{\mathcal{LSTS}}(\text{UND}(AON))$ if AON has no activator arcs.

Proposition 5.19 (property 5: fitting for ao-nets). *$\lambda(AON) = \epsilon(\kappa(AON))$.*

Proof. The $(\subseteq)$ inclusion follows from proposition 5.12(3), while the $(\supseteq)$ inclusion follows from $\prec_{ON} \subseteq \prec_{AON}$, fact 5.7 and proposition 5.10. ∎

We have therefore established

Theorem 5.20 (aim 3 for ao-nets). *$\kappa(AON) = \imath(\lambda(AON))$.*

The labelled step sequences of AON have a causality interpretation in terms of the partial order and the weak partial order provided by $\kappa(AON)$. In fact, a single partial order (as defined by an occurrence net) is insufficient, as it cannot fully express the relationship between simultaneous events if they cannot be sequentialised. For example, in figure 4 we have that $\{g\}\{e,h\}\{f\}$ and $\{g\}\{e\}\{h\}\{f\}$ are step sequences leading from MIN_{AON_0} to MAX_{AON_0}, but $\{g\}\{h\}\{e\}\{f\}$ cannot be executed, despite the fact that e and h are not related by the usual partial ordering.

6. PROCESS SEMANTICS OF PT-NETS

In this section we provide a rephrasing of the process semantics of [2, 11] for the case of general, possibly non-safe, finite PT-nets and show how this semantics fits into our framework. The processes used come from $\mathcal{LON}$ and for each PT-net, its associated o-nets can be defined in two different ways: (i) operational, through unfoldings based on step sequences; and (ii) axiomatic, from the structure of the net. In both cases, the resulting processes are the same. That is, we have consistency (property 2). Thus, together with what has been established already in the previous sections, the process and causality semantics of PT-nets fulfils aims 1 and 2, and their corollaries.

A *Place/Transition net* (or *PT-net*) is any marked net $N = (P, T, W, M_0)$, which will be fixed for the rest of this section.

Definition 6.1 (ω for PT-nets). *The set* $\omega_{\mathcal{STS}}(N) \stackrel{\text{df}}{=} \{\sigma \mid M_0[\sigma\rangle_N\}$ *comprising all step sequences starting from the initial marking M_0 of N, is the set of* step sequences *of N. Moreover, the set of* firing sequences *of N is the set* $\omega_{\mathcal{FS}}(N) \stackrel{\text{df}}{=} \omega_{\mathcal{STS}}(N) \cap \mathcal{FS}$.

Since $\varepsilon \in \omega_{\mathcal{FS}}(N)$, it follows that both $\omega_{\mathcal{FS}}$ and $\omega_{\mathcal{STS}}$ satisfy property 1.

First we give the operational definition of the processes of N which is based on its step sequences.

Definition 6.2 (π_N for PT-nets). *Let $\sigma = U_1 \dots U_n$ be a step sequence of N. A* process generated by σ *is the last labelled net in a sequence $N_0, \dots, N_n$, where for $0 \le k \le n$,*

$$N_k = (B_k, E_k, R_k, \ell_k) \stackrel{\mathrm{df}}{=} \left(\biguplus_{i=0}^{k} B^i, \biguplus_{i=0}^{k} E^i, \biguplus_{i=0}^{k} R^i, \biguplus_{i=0}^{k} \ell^i \right)$$

is constructed in the following way (in this, and other similar definitions presented later on, it is assumed that the sets of conditions, events and arcs do not contain any elements other than those specified explicitly).

- *For each $0 \le i \le n$, $\ell^i : B^i \cup E^i \to P \cup T$ is a labelling defined below.*
- *$E^0 = \varnothing$ and for $1 \le i \le n$, E^i comprises a distinct event for each transition occurrence in U_i. The event corresponding to the j-th occurrence of t in U_i is t–labelled and denoted by $t^{i,j}$.*
- *B^0 comprises a distinct condition for each place occurrence in M_0. The condition corresponding to the j-th occurrence of s in M_0 is s–labelled and denoted by s^j.*
- *For $1 \le i \le n$ and for every $e \in E^i$, B^i comprises a distinct condition for each place occurrence in $\textsc{post}_N(\ell_i(e))$. The condition corresponding to the j-th occurrence of p in $\textsc{post}_N(\ell_i(e))$ is p–labelled and denoted by $p^{e,j}$.*
- *$R^0 = \varnothing$, and for $1 \le i \le n$ and every $e \in E^i$:*
  - *We add an arc $(e, p^{e,j})$ to R^i for each $p^{e,j} \in B^i$.*
  - *We choose a disjoint (i.e., $B_f \cap B_g = \varnothing$ whenever $f \ne g$) set of conditions $B_e \subseteq B_{i-1} \backslash dom_{R_{i-1}}$ such that $\ell_i \langle B_e \rangle = \textsc{pre}_N(\ell_i(e))$ and add an arc (b, e) to R^i for each $b \in B_e$.*

We will denote the set of processes generated by σ by $\pi_N(\sigma)$.

Note that the last part of definition 6.2 is the only difference with the operational definition of processes for safe PT-nets. For such nets, there is always only one candidate set of conditions B_e and hence the processes generated by a step sequence are all isomorphic.

It is straightforward to check that, for every step sequence σ of N, all processes generated by σ satisfy definition 5.1 and hence are o-nets. Moreover, $\pi_N|_{\omega\mathcal{STS}(N)}$ and $\pi_N|_{\omega\mathcal{FS}(N)}$ are total and never return the empty set. Thus property 1 is satisfied in both cases.

Any process generated by some step sequence σ of N will have σ as an associated step sequence, i.e., it has a labelled step sequence ϖ such that $\sigma = \phi(\varpi)$. This follows from the observation that the successive addition of sets of events in definition 6.2 to construct the process actually defines an execution of the process.

Fact 6.3. *Assuming the notation as in definition 6.2, let $\textsc{max}_i \stackrel{\mathrm{df}}{=} B_i \backslash dom_{R_i}$, for every $0 \le i \le n$. Then $\textsc{max}_0 E^1 \textsc{max}_1 \dots E^n \textsc{max}_n$ is a mixed step sequence of the o-net N_n from its default initial marking to its default final marking.*

Corollary 6.4. *The following hold.*

1. *If* $\sigma \in \omega_{\mathcal{STS}}(N)$ *and* $ON \in \pi_N(\sigma)$*, then* $\sigma \in \phi(\lambda_{\mathcal{LSTS}}(ON))$*.*

2. *If* $\sigma \in \omega_{\mathcal{FS}}(N)$ *and* $ON \in \pi_N(\sigma)$*, then* $\sigma \in \phi(\lambda_{\mathcal{LFS}}(ON))$*.*

Next we give an axiomatic definition of processes based on the structure of the PT-net.

DEFINITION 6.5 (α for PT-nets). *A* process *of* N *is an o-net* $ON = (B, E, R, \ell)$ *satisfying the following:*

- ℓ *is a labelling function for* $B \cup E$ *such that* $\ell(B) \subseteq P$ *and* $\ell(E) \subseteq T$*.*
- *For all* $e \in E$*,* $\text{PRE}_N(\ell(e)) = \ell\langle \text{PRE}_{ON}(e)\rangle$ *and* $\text{POST}_N(\ell(e)) = \ell\langle \text{POST}_{ON}(e)\rangle$*.*
- $M_0 = \ell\langle \text{MIN}_{ON}\rangle$*.*

We will denote the set of processes of N *by* $\alpha(N)$*.*

Every process generated by a step sequence of N satisfies definition 6.5 and so we have that $\pi_N(\sigma) \subseteq \alpha(N)$ for all $\sigma \in \omega_{\mathcal{STS}}(N)$. Consequently, also α satisfies property 1.

Since in a process of N the neighbourhood relations of the transitions of N are faithfully reflected, its (mixed) step sequences correspond after labelling to those of N.

FACT 6.6. *Let* $ON \in \alpha(N)$ *be a process of* N *and* ξ *be a (mixed) step sequence of* ON *from* MIN_{ON}*. Then* $\ell\langle\xi\rangle$ *is a (mixed) step sequence of* N *from the initial marking* M_0*, where* ℓ *is the labelling of* ON*.*

COROLLARY 6.7. $\phi(\lambda_{\mathcal{LSTS}}(ON)) \subseteq \omega_{\mathcal{STS}}(N)$ *and* $\phi(\lambda_{\mathcal{LFS}}(ON)) \subseteq \omega_{\mathcal{FS}}(N)$*, for every process* $ON \in \alpha(N)$*.*

Moreover (see facts 5.5 and 5.4), the part of a process ON of N executed to reach a marking (i.e., a slice) S of ON is $\text{PREON}_{ON}(S)$ which is the 'prefix' of ON upto S. Clearly, $\text{PREON}_{ON}(S)$ satisfies definition 6.5 and hence is itself also a process of N.

FACT 6.8. *Let* $ON \in \alpha(N)$ *and let* $S \in \text{SL}(ON)$*. Then* $\text{PREON}_{ON}(S) \in \alpha(N)$*.*

On the other hand, given a labelled step sequence of a process of N, its associated step sequence, which by corollary 6.7 is a step sequence of N, can be used to construct the process stepwise in accordance with definition 6.2.

FACT 6.9. *Let* $ON \in \alpha(N)$ *and* $D_0F_1D_1 \ldots F_nD_n$ *be a mixed step sequence from* MIN_{ON} *to* MAX_{ON}*. Then there is a run of the construction described in definition 6.2, generating* ON*. Moreover, referring to the notation in definition 6.2,* $F_i = E^i$ *for every* $1 \leq i \leq n$*, and* $D_i = B_i \setminus dom_{R_i}$ *for every* $0 \leq i \leq n$*.*

COROLLARY 6.10. *Let* $ON \in \alpha(N)$ *be a process of* N *and* $\sigma \in \phi(\lambda_{\mathcal{LSTS}}(ON))$*. Then* $ON \in \pi_N(\sigma)$*.*

Thus we now have

FACT 6.11 (property 2: consistency for PT-nets). *For every step sequence* σ*, every firing sequence* σ' *and every o-net* ON*,*

1. $\sigma \in \omega_{\mathcal{STS}}(N) \wedge ON \in \pi_N(\sigma)$ iff $ON \in \alpha(N) \wedge \sigma \in \phi(\lambda_{\mathcal{LSTS}}(ON))$*.*

2. $\sigma' \in \omega_{\mathcal{FS}}(N) \wedge ON \in \pi_N(\sigma')$ iff $ON \in \alpha(N) \wedge \sigma' \in \phi(\lambda_{\mathcal{LFS}}(ON))$.

Hence for PT-nets the remaining aims are fulfilled:

FACT 6.12 (aims 1 and 2 for PT-nets).

$$\begin{array}{lcl} \alpha(N) & = & \pi_N(\omega_{\mathcal{STS}}(N)) = \pi_N(\omega_{\mathcal{FS}}(N)) \\ \omega_{\mathcal{STS}}(N) & = & \phi(\lambda_{\mathcal{LSTS}}(\alpha(N))) \\ \omega_{\mathcal{FS}}(N) & = & \phi(\lambda_{\mathcal{LFS}}(\alpha(N))) \ . \end{array}$$

Thus the operationally and axiomatically defined processes coincide, the operational semantics of a PT-net corresponds with the operational semantics of its processes, and through its processes the abstract causal relationship between transition occurrences can be defined (aim 3 for o-nets, fact 5.7(2)).

7. PT-NETS WITH INHIBITOR ARCS

In this section we formally introduce Place/Transition nets with inhibitor arcs and define three specific subclasses of such nets.

A *PT-net with inhibitor arcs* (or *PTI-net*) is a marked inhibitor net $NI \stackrel{\mathrm{df}}{=} (P, T, W, I, M_0)$, which is fixed for the rest of this subsection.

DEFINITION 7.1 (ω for PTI-nets). *The set of* step sequences *of a PTI-net NI is the set* $\omega_{\mathcal{STS}}(NI) \stackrel{\mathrm{df}}{=} \{\sigma \mid M_0[\sigma\rangle_{NI}\}$ *comprising all step sequences starting from the initial marking* M_0 *of NI. Moreover,* $\omega_{\mathcal{FS}}(NI) \stackrel{\mathrm{df}}{=} \omega_{\mathcal{STS}}(NI) \cap \mathcal{FS}$ *is the set of* firing sequences *of NI.*

Let $\textsc{und}(NI) \stackrel{\mathrm{df}}{=} (P, T, W, M_0)$ be the Place/Transition-net underlying NI. Note that $\omega_{\mathcal{STS}}(NI) \subseteq \omega_{\mathcal{STS}}(\textsc{und}(NI))$, and that if $I = \mathbf{\Omega}_{P \times T}$, then we are actually dealing with a PT-net, and NI is fully described by $\textsc{und}(NI)$ and may be specified in the form (P, T, W, M_0). In such a case we have $\omega_{\mathcal{STS}}(NI) = \omega_{\mathcal{STS}}(\textsc{und}(NI))$ and thus also $\omega_{\mathcal{FS}}(NI) = \omega_{\mathcal{FS}}(\textsc{und}(NI))$.

Since the empty sequence ε is always a step sequence of NI, $\omega_{\mathcal{STS}}$ is total (defined for every PTI-net) and never returns the empty set. Thus also in the case of PTI-nets, $\omega_{\mathcal{STS}}$ (as well as $\omega_{\mathcal{FS}}$) satisfies property 1.

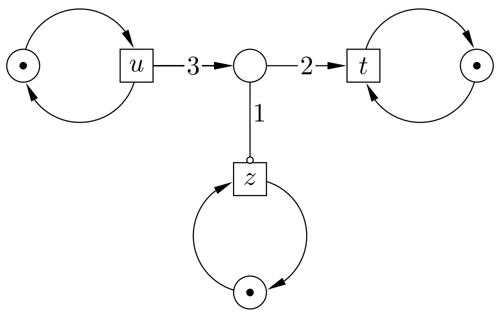

FIG. 5 PTI-net which is also a PTDI-net.

PTCI-nets, PTDI-nets, and PTSI-nets

A *PT-net with complemented inhibitor places* (or *PTCI-net*) is a PTI-net in which every inhibitor place p has a designated complement place denoted by p^{cpl}. In any PT-net, places which have a complement are bounded and the token count on a place and its complement is the same in every reachable marking. Thus PTCI-nets have bounded inhibitor places and an inhibitor place p of a PTCI-net NI contains no more than k tokens *iff* its complement p^{cpl} contains at least $\text{BND}_{NI}(p)-k$ tokens. Figure 7 shows a PTCI-net.

A *PT-net with dominated inhibitor places* (or *PTDI-net*) is a PTDI-net in which — independently of the current marking — transitions which output to an inhibitor place cannot occur immediately before a transition which tests this place by means of an inhibitor arc, and transitions which take input tokens from an inhibitor place cannot occur simultaneously with transitions testing the inhibitor place. Formally, $NI \overset{\text{df}}{=} (P, T, W, I, M_0)$ is a PTDI-net if whenever p is an inhibitor place of a transition $z \in T$ then $W(u,p) > I(p,z)$ and $W(p,t) > I(p,z)$, for all $u \in \text{PRE}_{NI}(p)$ and $t \in \text{POST}_{NI}(p)$. The PTI-net of figure 5 is a PTDI-net. Thus, as implied by the definition, the occurrences of u and t are always related in the same manner to those of z. More precisely, t and z can never be executed in a single step, and the occurrence of a step $\{u,z\}$ implies that this occurrence of z cannot be executed later than that of u, since first some of the tokens deposited in the inhibitor place have to be removed (by t) in order to enable z.

PTDI-nets are a generalisation of what is usually referred to in the literature as *inhibitor nets*. These are nets in which inhibitor arcs are only used to test whether a place is empty or not. In our set-up, a *PT-net with simple inhibitor places* (or *PTSI-net*) is a PTI-net $NI = (P, T, W, I, M_0)$ in which I always returns 0 or ∞.

PTDI-nets are a genuine generalisation of inhibitor nets in the sense that not every PTDI-net can be modelled as a PTSI-net with the same set of firing sequences and hence certainly not with the same set of step sequences.

PROPOSITION 7.2. *There is no PTSI-net with the same set of firing sequences as the PTDI-net of figure 5.*

Proof. See the appendix. ∎

On the other hand, since PTSI-nets may have unbounded inhibitor places, not every PTSI-net can be simulated by a PTCI net with the same set of firing sequences.

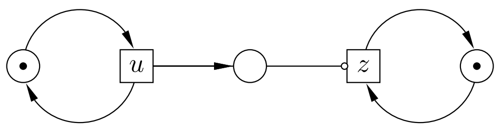

FIG. 6 PTSI-net which cannot be modelled by a PTCI-net.

PROPOSITION 7.3. *There is no PTCI-net with the same set of firing sequences as the PTSI-net of figure 6.*

Proof. Suppose that $NI = (P, T, W, I, M_0)$ is a PTCI-net with the same set of firing sequences Δ as the PTSI-net of figure 6. Thus NI has transitions u and z.

Since $u^\star \subseteq \Delta$ and every inhibitor place of z has a complement, executing u has no effect on the marking of the inhibitor places of z. Since $u \in \Delta$, $z \in \Delta$, but not $uz \in \Delta$, it must then be the case that there is a place $p \in P$ such that $\text{PRE}_{NI}(z)(p) > 0$ and $\text{PRE}_{NI}(u)(p) - \text{POST}_{NI}(u)(p) > 0$. However, again by $u^\star \subseteq \Delta$, this yields a contradiction with $\text{PRE}_{NI}(u) \leq \text{POST}_{NI}(u)$. ∎

Every PTCI-net can be converted into a PT-net which has the same set of firing sequences. This follows immediately from the observation that in a PTCI-net NI an inhibitor arc from a place p to a transition t with weight k can be replaced by two ordinary arcs, each with weight $\text{BND}_{NI}(p) - k$ from the complement p^{cpl} of p to t and from t to p^{cpl}. Thus testing whether there are no more than k tokens in p is replaced by testing whether its complement contains at least $\text{BND}_{NI}(p) - k$ tokens. After removing in this way all inhibitor arcs a PT-net results which has the same firing sequences as NI. Hence as far as firing sequences are concerned every PTCI-net can be simulated by a PTDI-net (without inhibitor arcs). However, this does not work when step sequences are considered. Whereas the arcs replacing an inhibitor arc can be viewed as consuming and producing tokens, an inhibitor arc only tests without consuming. For instance the PTCI-net of figure 7 allows a step sequence $\{u\}\{u,z\}$ whereas the net resulting from the construction described above, cannot execute u and z simultaneously after u has occurred. We show next, that for the PTCI-net of figure 7 there does not exist a PTDI-net with the same set of step sequences. Note that this PTCI-net has only one inhibitor arc with weight 1 and may be considered as a 'smallest' counterexample, since any PTCI-net which has only inhibitor arcs with weight 0 is a PTSI-net and hence also a PTDI-net.

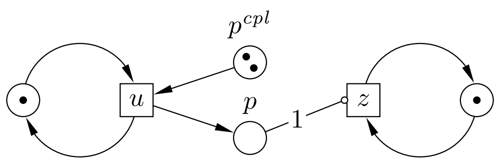

FIG. 7 PTCI-net which cannot be modelled by a PTDI-net.

PROPOSITION 7.4. *There is no PTDI-net with the same set of step sequences as the PTCI-net of figure 7.*

Proof. Suppose that $NI = (P, T, W, I, M_0)$ is a PTDI-net with the same set of step sequences Δ as the PTCI-net of figure 7. Thus NI has u and z among its transitions. Since $\{u\}\{z\} \in \Delta$, $\{u,z\} \in \Delta$, and NI is a PTDI-net, executing u has no effect on the inhibitor places of z. Then from $\{u\}\{u,z\} \in \Delta$ and $\{z\} \in \Delta$, it follows that it is possible to execute $\{u\}\{u\}\{z\}$ from M_0 in NI. However, $\{u\}\{u\}\{z\} \notin \Delta$. ∎

8. PROCESS SEMANTICS OF PTCI-NETS

In order to obtain a process semantics for PTCI-nets without weights, we combined in [17] the process semantics for non-safe PT-nets (see section 6) with the process semantics from [14] for elementary net systems (safe PT-nets) with inhibitor arcs. In this section we extend this work to the full class of PTCI-nets and fit it into the semantical framework, thus extending and systematizing our previous results and formulating an abstract causality semantics for PTCI-nets.

In the processes of PT-nets, the *presence* of tokens is represented by conditions, but their *absence* cannot be tested. The idea of [14] is now that an inhibitor arc which tests whether a place is empty, can be simulated by an activator arc which tests whether its complement place is not empty. To apply this idea in the non-safe case (as explored in [17]), the inhibitor places should be bounded and have complement places. However, in contrast to the safe case, complement places cannot just be added for the bounded inhibitor places, since this may lead to new processes. Hence in general this approach cannot be applied to PTI-nets, not even when they are bounded. But for PTCI-nets, in which every inhibitor place comes with a complement place (and thus is bounded), one can use this approach. Let $NCI = (P, T, W, I, M_0)$ be a PTCI-net, fixed for the rest of this section.

First we provide the operational definition which takes a step sequence and constructs a corresponding ao-net essentially as done for PT-nets but now adding on the way activator arcs to complement places (with the number of activator arcs to be added determined by the bound of the inhibitor place and the weight of the inhibitor arc).

Definition 8.1 (π_N for PTCI-nets). *Let $\sigma = U_1 \dots U_n$ be a step sequence of NCI. A* complement activator process *(or* ca-process*) generated by σ is the last labelled activator net in a sequence $N_0, \dots, N_n$, where for $0 \le k \le n$,*

$$N_k = (B_k, E_k, R_k, Act_k, \ell_k) \stackrel{\mathrm{df}}{=} \left(\biguplus_{i=0}^{k} B^i, \biguplus_{i=0}^{k} E^i, \biguplus_{i=0}^{k} R^i, \biguplus_{i=0}^{k} Act^i, \biguplus_{i=0}^{k} \ell^i \right)$$

is constructed as in definition 6.2, except for the activator arcs Act^i, which are defined in the following way.

- *$Act^0 = \varnothing$, and for $1 \le k \le n$ and every $e \in E^k$, if p is an inhibitor place of $\ell_k(e)$ then we choose a set A_e of exactly $\mathrm{BND}_{NCI}(p) - \mathrm{INH}_{NCI}(\ell_k(e))(p)$ conditions in $B_{k-1} \setminus dom_{R_{k-1}}$ labelled by p^{cpl}. After that we add an activator arc (b, e) to Act^i for each $b \in A_e$.*

We will denote the set of ca-processes generated by σ by $\pi^{cpl}_{NCI}(\sigma)$.

Figure 8 shows a PTCI-net NCI and illustrates the generation of a ca-process for the step sequence $\sigma = \{w, w\}\{t\}\{u, u\}\{w, w\}\{t\}\{t\}$. Note that $\mathrm{BND}_{NI}(q) = \mathrm{BND}_{NI}(r) = 2$ and $r = q^{cpl}$. The vertical lines indicate the stages (from left to right) in which the net has been derived.

Note that in definition 8.1 it may happen that $A_e \cap A_f \neq \varnothing$ for $e \neq f$. As the next proposition 8.2 shows, the required sets A_e can always be found and thus definition 8.1 is sound.

Proposition 8.2. *Assuming the notation as in definition 8.1, let $\ell \stackrel{\mathrm{df}}{=} \ell_n$ and $\mathrm{MAX}_i \stackrel{\mathrm{df}}{=} B_i \setminus dom_{R_i}$, for every $0 \le i \le n$. Moreover, let $1 \le k \le n$, $e \in E^k$ and p be an inhibitor place of $\ell(e)$. Then*

$$|\ell^{-1}(p^{cpl}) \cap \mathrm{MAX}_{k-1}| \ge \mathrm{BND}_{NCI}(p) - \mathrm{INH}_{NCI}(\ell(e))(p) \ .$$

Proof. Let $N \stackrel{\mathrm{df}}{=} \mathrm{UND}(NCI)$, $AON \stackrel{\mathrm{df}}{=} N_n$ and $ON \stackrel{\mathrm{df}}{=} \mathrm{UND}(AON)$. We observe that $\sigma \in \omega_{\mathcal{STS}}(N)$ and $ON \in \pi_N(\sigma)$, which follows directly from the definitions. Thus, by fact 6.11(1), $ON \in \alpha(N)$. Consequently, by facts 6.3 and 6.6, $\mu \stackrel{\mathrm{df}}{=} \ell\langle \mathrm{MAX}_0 \rangle U_1 \ell\langle \mathrm{MAX}_1 \rangle \dots U_n \ell\langle \mathrm{MAX}_n \rangle$ is a mixed step sequence of N from M_0. Since $\sigma = U_1 \dots U_n$ is a step sequence of NCI, this implies that μ is also a mixed

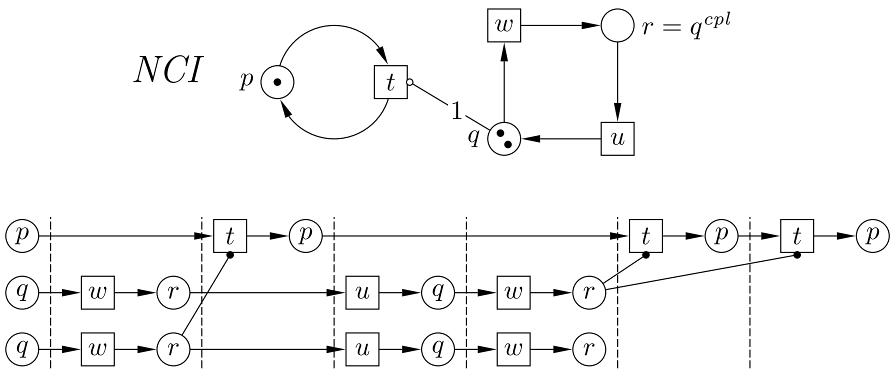

FIG. 8 A PTCI-net and a ca-process generated by $\{w, w\}\{t\}\{u, u\}\{w, w\}\{t\}\{t\}$.

step sequence of *NCI* from M_0. Thus $\ell\langle \text{MAX}_{k-1}\rangle(p) \leq \text{INH}_{NCI}(\ell(e))(p)$, and so $\ell\langle \text{MAX}_{k-1}\rangle(p^{cpl}) \geq \text{BND}_{NCI}(p) - \text{INH}_{NCI}(\ell(e))(p)$. ∎

It is fairly easy to check that, for every step sequence σ of *NCI*, all ca-processes generated by σ are ao-nets.

PROPOSITION 8.3. *Let $\sigma \in \omega_{\mathcal{STS}}(NCI)$. Then $\pi^{cpl}_{NCI}(\sigma) \subseteq \mathcal{LAON}$.*

Proof. Let $AON \stackrel{\text{df}}{=} N_n$ be as in definition 8.1. Then $\text{UND}(AON)$ is an o-net. Hence it suffices to observe that rs_{AON} is a $\diamondsuit$–acyclic structure as, by construction, $e \prec_{loc} f$ implies $i < j$, and $e \sqsubset_{loc} f$ implies $i \leq j$, for all $e \in E^i$ and $f \in E^j$. ∎

Since also $\pi^{cpl}_{NCI}|_{\omega_{\mathcal{STS}}(NCI)}$ is total and never returns the empty set, property 1 is satisfied. We now propose the following axiomatic definition for the ca-processes of a PTCI-net.

DEFINITION 8.4 (α for PTCI-nets). *A* complement activator process *(or* ca-process*) of NCI is an ao-net $AON = (B, E, R, Act, \ell)$ such that $\text{UND}(AON)$ is a process of $\text{UND}(NCI)$ and, moreover, if $e \in E$ and p is an inhibitor place of $\ell(e)$ then*

$$|\ell^{-1}(p^{cpl}) \cap \text{ACT}_{AON}(e)| = \text{BND}_{NCI}(p) - \text{INH}_{NCI}(\ell(e))(p) \,. \tag{8.1}$$

We will denote the set of ca-processes of NCI by $\alpha^{cpl}(NCI)$.

Intuitively, the last condition means that if event e is enabled then there are enough tokens in p^{cpl} to ensure that p does not inhibit transition $\ell(e)$.

Figure 9 shows three ca-processes in $\alpha^{cpl}(NCI)$ for the PTCI-net *NCI* of figure 8. Notice that AON_3 is isomorphic to the ca-process generated by σ in figure 8. In fact, every ca-process generated by a step sequence of *NCI* satisfies definition 8.4 and thus is a ca-process of *NCI*.

PROPOSITION 8.5. *Let $\sigma \in \omega_{\mathcal{STS}}(NCI)$. Then $\pi^{cpl}_{NCI}(\sigma) \subseteq \alpha^{cpl}(NCI)$.*

Proof. Assume the notation from definition 8.1 and denote $AON \stackrel{\text{df}}{=} N_n$. We first observe that $ON \stackrel{\text{df}}{=} (B_n, E_n, R_n, \ell_n) \in \pi_{\text{UND}(NCI)}(\sigma)$, which follows directly from the definitions and thus, by fact 6.11(1), $ON \in \alpha(\text{UND}(NCI))$. Moreover, by proposition 8.3, AON is an ao-net and, by construction, the condition (8.1) in definition 8.4 is satisfied. Hence $AON \in \alpha^{cpl}(NCI)$. ∎

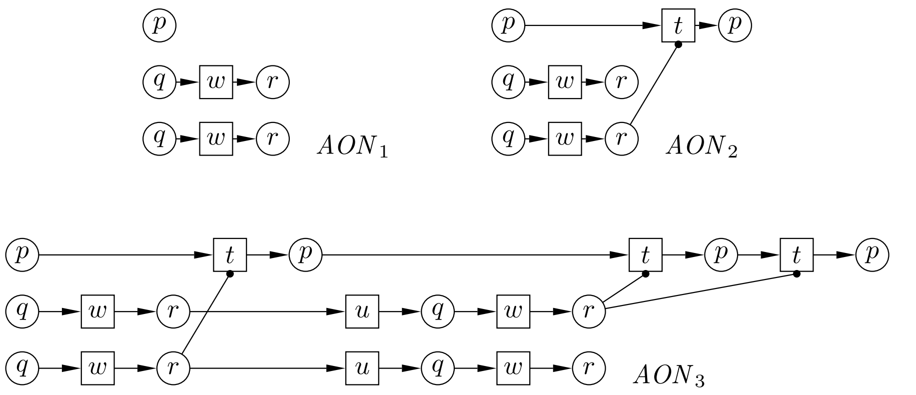

FIG. 9 Three ca-processes in $\alpha^{cpl}(NCI)$.

Consequently, also α^{cpl} is total and never returns the empty set. Since, by definition, $\alpha^{cpl}(NCI) \subseteq \mathcal{LAON}$, property 1 is satisfied.

Properties of the ca-processes of PTCI-nets

In definition 8.1 (which is based on definition 6.2), the successive addition of sets of events describes an execution of the resulting ca-process considered as a net with activator arcs.

PROPOSITION 8.6. *Assuming the notation as in definition 8.1, let* $\text{MAX}_i \stackrel{\text{df}}{=} B_i \backslash dom_{R_i}$, *for every* $0 \leq i \leq n$. *Then* $\mu \stackrel{\text{df}}{=} \text{MAX}_0 E^1 \text{MAX}_1 \ldots E^n \text{MAX}_n$ *is a mixed step sequence of the ao-net* N_n *from its default initial marking to its default final marking.*

Proof. By fact 6.3, μ is a mixed step sequence of $\text{UND}(N_n)$ from $\text{MIN}_{\text{UND}(N_n)} = \text{MIN}_{N_n}$ to $\text{MAX}_{\text{UND}(N_n)} = \text{MAX}_{N_n}$. Moreover, $\text{ACT}_{N_n}(e) \subseteq \text{MAX}_{k-1}$, for all $1 \leq k \leq n$ and $e \in E^k$. Hence μ is a mixed step sequence of N_n. ∎

As a consequence, any ca-process generated by a step sequence σ of *NCI* will have a labelled step sequence corresponding to σ (after forgetting about the identities of the underlying events through the function ϕ).

COROLLARY 8.7. *If* $\sigma \in \omega_{\mathcal{STS}}(NCI)$ *and* $AON \in \pi^{cpl}_{NCI}(\sigma)$, *then it is the case that* $\sigma \in \phi(\lambda(AON))$.

PROPOSITION 8.8. *Let* $AON \in \alpha^{cpl}(NCI)$ *and let* ξ *be a (mixed) step sequence of AON from* MIN_{AON}. *Then* $\ell\langle\xi\rangle$ *is a (mixed) step sequence of NCI from* M_0, *where* ℓ *is the labelling of AON.*

Proof. It suffices to show the result for $\xi \stackrel{\text{df}}{=} B_0 E_1 B_1 \ldots E_n B_n$, i.e., a mixed step sequence. Let $N \stackrel{\text{df}}{=} \text{UND}(NCI)$ and $ON \stackrel{\text{df}}{=} \text{UND}(AON)$.

Since $ON \in \alpha(N)$ we have, by fact 6.6, that $\ell\langle\xi\rangle$ is a mixed step sequence of N. Thus it suffices to show that if $e \in E_i$ and p is an inhibitor place of $\ell(e)$, then $\ell\langle B_{i-1}\rangle(p) \leq \text{INH}_{NCI}(\ell(e))(p)$. The latter is equivalent, by $\ell\langle B_{i-1}\rangle \in [M_0\rangle_N$, to showing that $\ell\langle B_{i-1}\rangle(p^{cpl}) \geq \text{BND}_{NCI}(p) - \text{INH}_{NCI}(\ell(e))(p)$. This, in turn, follows from the fact that e is enabled and the condition (8.1) in definition 8.4. ∎

Corollary 8.9. *If $AON \in \alpha^{cpl}(NCI)$, then $\phi(\lambda(AON)) \subseteq \omega_{\mathcal{STS}}(NCI)$.*

By propositions 5.17(1) and 5.14(1), the part of a ca-process AON of NCI executed to reach a marking (i.e., a strong slice) S is $\text{PREAON}_{AON}(S)$. This 'prefix'of AON can be shown to be also a ca-process of NCI.

Proposition 8.10. *Let $AON \in \alpha^{cpl}(NCI)$, and let $S \in \text{SSL}(AON)$ be a strong slice of AON. Then $\text{PREAON}_{AON}(S) \in \alpha^{cpl}(NCI)$.*

Proof. Let $AON' \stackrel{\text{df}}{=} \text{PREAON}_{AON}(S)$. By proposition 5.14(1a), we have that for all events e in AON', $\text{ACT}_{AON}(e) = \text{ACT}_{AON'}(e)$. Thus the condition (8.1) in definition 8.4 holds for AON', as it held for AON. By proposition 5.13, $\text{SSL}(AON) \subseteq \text{SL}(\text{UND}(AON))$. Moreover, by fact 6.8, $\text{PREON}_{\text{UND}(AON)}(S) \in \alpha(\text{UND}(NCI))$. Since $\text{PREON}_{\text{UND}(AON)}(S) = \text{UND}(AON')$, it follows that $AON' \in \alpha^{cpl}(NCI)$. ∎

Furthermore, given a labelled step sequence of a ca-process of NCI its associated step sequence is one of the generators of that process.

Proposition 8.11. *Let $AON \in \alpha^{cpl}(NCI)$ be a ca-process of NCI, and let $\sigma \in \phi(\lambda(AON))$. Then $AON \in \pi^{cpl}_{NCI}(\sigma)$.*

Proof. Let $ON \stackrel{\text{df}}{=} \text{UND}(AON)$, $N \stackrel{\text{df}}{=} \text{UND}(NCI)$, and $\mu \stackrel{\text{df}}{=} D_0F_1D_1 \dots F_nD_n$ be a mixed step sequence of AON from MIN_{AON} to MAX_{AON} such that $\sigma = \ell\langle F_1 \dots F_k\rangle$, where ℓ is the labelling of AON.

Clearly, μ is also a mixed step sequence of ON from MIN_{ON} to MAX_{ON}. Since $ON \in \alpha(N)$, we know from fact 6.9, that there is a run of the construction described in definition 6.2, generating ON. Moreover, referring to the notation in definition 6.2, $F_i = E^i$ for every $1 \leq i \leq n$, and $D_i = B_i \setminus dom_{R_i}$ for every $0 \leq i \leq n$. Hence, by definition 8.1, we can re-run this construction, adding at each stage the sets Act^k, and resulting in AON, provided that for every $1 \leq i \leq n$ and $e \in F_i$, $b \in \text{ACT}_{AON}(e)$ implies $b \in B_i \setminus dom_{R_i}$. This can be shown as follows. Suppose that $e \in F_i$ and $b \in \text{ACT}_{AON}(e)$ are such that $b \notin B_i \setminus dom_{R_i} = D_{i-1}$. Then there must be f such that one of the following holds: $f \in \text{PRE}_{AON}(b)$ and $f \notin F_1 \cup \ldots \cup F_{i-1}$ or $f \in \text{POST}_{AON}(b)$ and $f \in F_1 \cup \ldots \cup F_{i-1}$. In either case, we obtain a contradiction with proposition 5.12(3). ∎

Thus every ca-process of NCI can be generated by a step sequence of NCI and we now have

Proposition 8.12 (property 2: consistency for PTCI-nets). *For every step sequence σ and every ao-net AON,*

$$\sigma \in \omega_{\mathcal{STS}}(NCI) \wedge AON \in \pi^{cpl}_{NCI}(\sigma) \text{ iff } AON \in \alpha^{cpl}(NCI) \wedge \sigma \in \phi(\lambda(AON)) .$$

Hence also for PTCI-nets the remaining aims are fulfilled:

Theorem 8.13 (aims 1 and 2 for PTCI-nets).

$$\begin{array}{rcl} \alpha^{cpl}(NCI) & = & \pi^{cpl}_{NCI}(\omega_{\mathcal{STS}}(NCI)) \\ \omega_{\mathcal{STS}}(NCI) & = & \phi(\lambda(\alpha^{cpl}(NCI))) . \end{array}$$

9. PROCESS SEMANTICS OF GENERAL PTI-NETS

We now turn to defining a process semantics for general PTI-nets. Since inhibitor places do not necessarily have complements, a new feature is needed to represent the test that an inhibitor place does not contain too many tokens. Our proposal is to add 'on demand' new artificial conditions with activator arcs to represent the testing by inhibitor arcs. Moreover, if a transition has an inhibitor place which is input or output to some other transition, then occurrences of these two transitions may have a causal relationship which should be faithfully reflected by the neighbourhood of the new condition.

Let $NI = (P, T, W, I, M_0)$ be a PTI-net fixed for the rest of this section. If $p \in P$ and $t, w \in T$ are such that $\text{INH}_{NI}(t)(p) \neq \infty$ and $\text{PRE}_{NI}(w)(p) + \text{POST}_{NI}(w)(p) \neq 0$, then we write $w \overset{p}{\multimap} t$, and $w \multimap t$ if there is at least one p such that $w \overset{p}{\multimap} t$. Similarly, for an ao-net $AON = (B, E, R, Act, \ell)$, if $b \in B$ and $e, f \in E$ are such that $\text{ACT}_{AON}(e)(b) \neq 0$ and $\text{PRE}_{AON}(f)(b) + \text{POST}_{AON}(f)(b) \neq 0$, then we denote $f \overset{b}{\multimap\!\!\bullet}\, e$, or simply $f \multimap\!\!\bullet\, e$. The main idea behind the process construction presented next is to ensure that whenever $w \multimap t$, any two occurrences, f of w and e of t, are adjacent to a common condition so that $f \multimap\!\!\bullet\, e$, and thus are related in the corresponding causal structure. Note that this resembles the technique used in [22] to define a process semantics of PT-nets, where the construction always makes occurrences of transitions adjacent to a common place causally dependent.

First we define the operational process semantics and demonstrate how to construct an ao-net for a given step sequence of NI. Again, the construction follows the pattern established for PT-nets, but now new conditions — labelled by the special symbol $\curlywedge$ — may have to be added on the way.

Definition 9.1 (π_N for PTI-nets). *Let $\sigma = U_1 \dots U_n$ be a step sequence of NI. An* activator process *(or* a-process*) generated by σ is the last labelled activator net in a sequence $N_0, \dots, N_n$, where for $0 \leq k \leq n$,*

$$N_k = (B_k \uplus \widetilde{B}_k, E_k, R_k, Act_k, \ell_k) \overset{\text{df}}{=} \left(\biguplus_{i=0}^{k} B^i \uplus \biguplus_{i=0}^{k} \widetilde{B}^i, \biguplus_{i=0}^{k} E^i, \biguplus_{i=0}^{k} R^i, \biguplus_{i=0}^{k} Act^i, \biguplus_{i=0}^{k} \ell^i \right)$$

is constructed as in definition 6.2, except that $\widetilde{B}^0 = Act^0 \overset{\text{df}}{=} \varnothing$ and, for $k = 1, \dots, n$:

- *ℓ^k is extended to a labelling of $B^k \cup \widetilde{B}^k \cup E^k$, by $\ell^k(b) \overset{\text{df}}{=} \curlywedge$ for all $b \in \widetilde{B}^k$.*
- *If $e \in E^k$ and $f \in E^j$ (for $j < k$) are such that $\ell(f) \multimap \ell(e)$ then we create exactly one condition $b \in \widetilde{B}^k$ and add two arcs: $(f, b) \in R^k$ and $(b, e) \in Act^k$.*
- *If $f \in E^k$ and $e \in E^j$ (for $j \leq k$) are such that $\ell(f) \multimap \ell(e)$ then we create exactly one condition $b \in \widetilde{B}^k$ and add two arcs: $(b, f) \in R^k$ and $(b, e) \in Act^k$.*

We will denote the set of a-processes generated by σ by $\pi_{NI}(\sigma)$.

We observe that if NI has no inhibitor arcs (i.e., $I = \mathbf{\Omega}_{P \times T}$ and so NI is in fact a PT-net), then the a-processes of NI generated by a step sequence σ are exactly the processes of NI generated by σ according to definition 6.2. Thus the function π_N for PTI-nets defined here is a conservative extension of π_N defined for PT-nets.

Definition 9.1 is illustrated in figure 10 for a PTI-net and one of its step sequences, $\sigma \overset{\text{df}}{=} \{w\}\{t\}\{t, u\}$. Note that this PTI-net is a PTDI-net but not a PTCI-net. As before, the stages are shown in which the nodes and connections were

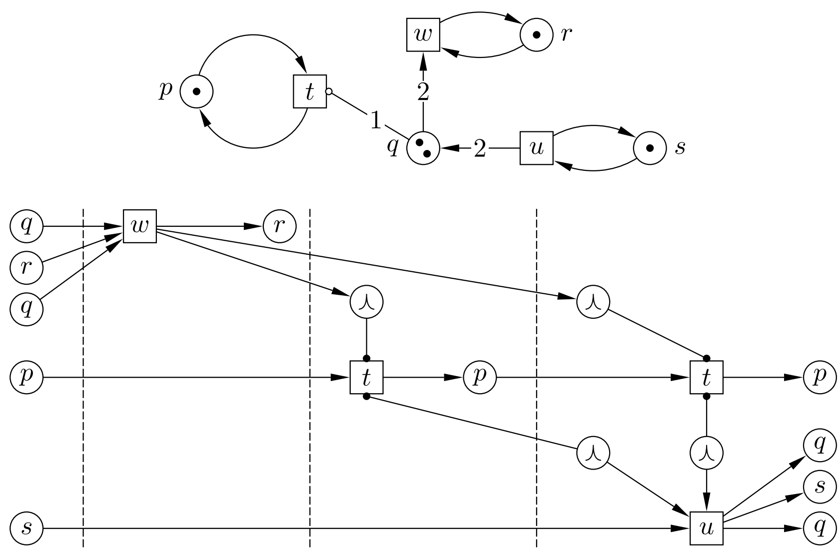

FIG. 10 A PTI-net and an a-process generated by $\{w\}\{t\}\{t,u\}$.

generated. The resulting process has $E_3 = E^1 \uplus E^2 \uplus E^3$ as its set of events for which we let $E^1 \stackrel{\text{df}}{=} \{e_w\}$ with $\ell_3(e_w) = w$, $E^2 \stackrel{\text{df}}{=} \{e_{t,1}\}$ with $\ell_3(e_{t,1}) = t$, and $E^3 \stackrel{\text{df}}{=} \{e_{t,2}, e_u\}$ with $\ell_3(e_{t,2}) = t$ and $\ell_3(e_u) = u$. Since in the step sequence σ, the occurrence of w precedes the first occurrence of t and $w \multimap t$, a $\curlywedge$–labelled condition is created such that $e_w \!-\!\!\bullet\, e_{t,1}$ in the a-process being created. Adding E^3 to the a-process under construction leads to three more $\curlywedge$–labelled conditions: for w and the second occurrence of t, a $\curlywedge$–labelled condition is created so that $e_w \!-\!\!\bullet\, e_{t,2}$; and since in the PTI-net we have $u \multimap t$, two $\curlywedge$–labelled conditions are created so that $e_{t,1} \!-\!\!\bullet\, e_u$ and $e_{t,2} \!-\!\!\bullet\, e_u$.

In the construction of definition 9.1, whenever an event f is introduced before an event e and $\ell(f) \multimap \ell(e)$, then this will *always* lead to $f \prec e$ in the generated so-structure. Similarly, whenever an event e is introduced not later than an event f and $\ell(f) \multimap \ell(e)$, then this will *always* lead to $e \sqsubset f$. Whether or not it is necessary to enforce these relations depends, in general, on the current number of tokens in the inhibitor places p for which $\ell(f) \stackrel{p}{\multimap} l(e)$. Thus, as we will demonstrate later, in case of PTDI-nets it can never be avoided. Moreover, the uniform strategy based on 'local' structural relationships as adopted in definition 9.1 leads to a process semantics and an abstract causality semantics which fulfil the aims of our set-up and thus are in agreement with the operational semantics of PTI-nets. In addition, the causality semantics for PTCI-nets which are also PTDI-nets is the same whether it is based on the ca-processes defined in section 8 or on the a-processes from this section.

The a-processes generated by the step sequences of NI are indeed ao-nets.

PROPOSITION 9.2. *Let* $\sigma \in \omega_{\mathcal{STS}}(NI)$. *Then* $\pi_{NI}(\sigma) \subseteq \mathcal{LAON}$.

Proof. Let $AON \stackrel{\text{df}}{=} N_n$ be as in definition 9.1. Then $\text{UND}(AON)$ is an o-net. Hence it suffices to observe that rs_{AON} is a $\diamondsuit$–acyclic structure as, by construction, $e \prec_{loc} f$ implies $i < j$, and $e \sqsubset_{loc} f$ implies $i \leq j$, for all $e \in E^i$ and $f \in E^j$. ∎

Clearly, $\pi_{NI}|_{\omega_{STS}(NI)}$ is total and never returns the empty set. Hence property 1 is satisfied. In the next step we give an axiomatic definition for the notion of an a-process.

DEFINITION 9.3 (α for PTI-nets). *An* activator process *(or* a-process*) of NI is an ao-net* $AON = (B \uplus \widetilde{B}, E, R, Act, \ell)$ *satisfying the following:*

1. $\ell(B) \subseteq P$ *and* $\ell(E) \subseteq T$.
2. *The conditions in* $\widetilde{B} = dom_{Act}$ *are labelled by the special symbol* $\curlywedge$.
3. $M_0 = \ell\langle \text{MIN}_{AON} \cap B \rangle$.
4. *For all* $e \in E$,
   $\text{PRE}_{NI}(\ell(e)) = \ell\langle \text{PRE}_{AON}(e) \cap B \rangle$ *and* $\text{POST}_{NI}(\ell(e)) = \ell\langle \text{POST}_{AON}(e) \cap B \rangle$.
5. *For all* $b \in \widetilde{B}$, *there are unique* $g, h \in E$ *such that*
   $\text{PRE}_{AON}(b) + \text{POST}_{AON}(b) = \{g\}$, $b \in \text{ACT}_{AON}(h)$ *and* $\ell(g) \multimap \ell(h)$.
6. *For all* $e, f \in E$,
   if $\ell(f) \multimap \ell(e)$ *then there is exactly one* $c \in \widetilde{B}$ *such that* $f \overset{c}{-\!\!\bullet} e$.
7. *For all* $e \in E$ *and* $S \in \text{SSL}(AON)$,
   if $\text{PRE}_{AON}(e) \cup \text{ACT}_{AON}(e) \subseteq S$ *then* $\ell\langle S \cap B \rangle \leq \text{INH}_{NI}(\ell(e))$.

We will denote the set of a-processes of NI by $\alpha(NI)$.

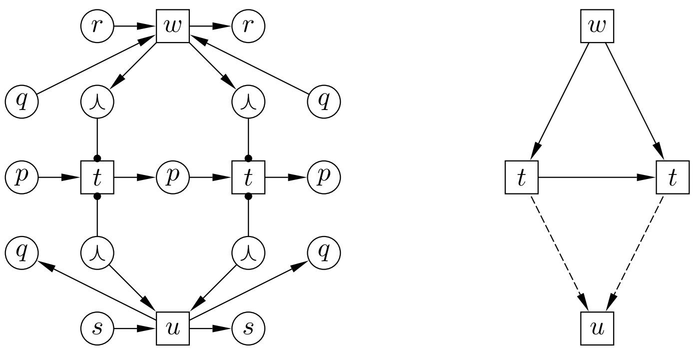

FIG. 11 An a-process in $\alpha(NI)$ and the generated local causality structure.

Figure 11 shows an a-process for the PTI-net of figure 10. Notice that the processes in figure 10 and figure 11 are isomorphic.

In what follows, if NA is a labelled activator net with the special symbol $\curlywedge$ as one of its labels, then $NA^{\curlywedge}$ denotes NA with all those $\curlywedge$–labelled places deleted which are not activators for any transition (together with the ordinary arcs connected to them) and for a multiset of places M, $M^{\curlywedge}$ is M with all instances of $\curlywedge$–labelled places deleted. For an a-process AON of NI, we have $AON^{\curlywedge} = AON$ by definition 9.3(2). Furthermore, $\text{UND}(AON)$ has no activator arcs and is an o-net possibly with $\curlywedge$–labelled conditions which are all removed in $\text{UND}(AON)^{\curlywedge}$. Thus in general, $\text{UND}(AON)^{\curlywedge} \neq \text{UND}(AON) = \text{UND}(AON^{\curlywedge})$.

Definition 9.3(1,3,4) corresponds to the requirements of definition 6.5 and guarantees that $\text{UND}(AON)^{\curlywedge}$ is a process of $\text{UND}(NI)$. Definition 9.3(5) describes the immediate neighbourhood of the $\curlywedge$–labelled conditions. Each such condition has exactly one event to which it is connected by an ordinary arc, and one for which it acts as an activator place (while respecting the requirement that AON should be an ao-net). Moreover this neighbourhood has to correspond to an inhibitor arc in NI. Conversely, definition 9.3(6) requires that whenever events in AON represent transitions related through an inhibitor place, there should be a $\curlywedge$–labelled condition relating these events. Finally, definition 9.3(7) refers to proposition 5.17, and requires that the strong slices of AON (i.e., markings reachable from MIN_{AON}) properly reflect the inhibitor constraints present in NI: an event can only occur at a slice if there are not too many conditions corresponding to tokens in the inhibitor places of its counterpart in NI.

Every a-process generated by a step sequence of NI satisfies definition 9.3 and so we have

PROPOSITION 9.4. *Let $\sigma \in \omega_{\mathcal{STS}}(NI)$. Then $\pi_{NI}(\sigma) \subseteq \alpha(NI)$.*

Proof. See the appendix. ∎

Consequently, also α is total and never returns the empty set. Since, by definition $\alpha(NI) \subseteq \mathcal{LAON}$, property 1 is satisfied.

Properties of the a-processes of PTI-nets

The successive addition of sets of events as described in definition 9.1 corresponds to an execution of the resulting a-process (as a net with activator arcs).

PROPOSITION 9.5. *Assuming the notation as in definition 9.1, let $\text{MAX}_i \overset{\text{df}}{=} (B_i \cup \widetilde{B}_i) \setminus dom_{R_i}$, for every $0 \leq i \leq n$. Then $\mu \overset{\text{df}}{=} \text{MAX}_0 E^1 \text{MAX}_1 \dots E^n \text{MAX}_n$ is a mixed step sequence of the ao-net $AON \overset{\text{df}}{=} N_n$ from its default initial marking to its default final marking.*

Proof. By fact 6.3, $\text{MAX}_0^{\curlywedge} E^1 \text{MAX}_1^{\curlywedge} \dots E^n \text{MAX}_n^{\curlywedge}$ is a mixed step sequence of $ON \overset{\text{df}}{=} \text{UND}(AON)^{\curlywedge}$ such that $\text{MAX}_0^{\curlywedge} = \text{MIN}_{ON}$ and $\text{MAX}_n^{\curlywedge} = \text{MAX}_{ON}$. Moreover, for every $b \in \widetilde{B}_n$, if $b \in \text{PRE}_{AON}(e)$ for some event $e \in E_n$, then $b \in \text{MIN}_{AON}$. Hence, to show that μ is a mixed step sequence of AON, it suffices to prove that $\text{ACT}_{AON}(e) \subseteq \text{MAX}_{k-1}$, for all $1 \leq k \leq n$ and $e \in E^k$.

Suppose that $e \in E^k$ and $b \in \text{ACT}_{AON}(e)$. Then, by definition 9.1, there is exactly one f such that one of the following holds: $j < k$ and $\text{PRE}_{AON}(b) = \{f\}$ and $\text{POST}_{AON}(b) = \varnothing$, or $j \geq k$ and $\text{POST}_{AON}(b) = \{f\}$ and $\text{PRE}_{AON}(b) = \varnothing$, where j satisfies $f \in E^j$. In either case, $b \in \text{MAX}_{k-1}$. ∎

COROLLARY 9.6. *If $\sigma \in \omega_{\mathcal{STS}}(NI)$ and $AON \in \pi_{NI}(\sigma)$, then $\sigma \in \phi(\lambda(AON))$.*

In the a-processes of NI the neighbourhood relations of the transitions are reflected including a representation of inhibitor places. This makes it possible to show that each of their step sequences represents a step sequence of NI.

PROPOSITION 9.7. *Let $AON \in \alpha(NCI)$, and let $\mu \overset{\text{df}}{=} B_0 E_1 B_1 \dots E_n B_n$ be a mixed step sequence of AON from MIN_{AON}. Then $\ell\langle\mu'\rangle$ is a mixed step sequence of NI from M_0, where $\mu' \overset{\text{df}}{=} B_0^{\curlywedge} E_1 B_1^{\curlywedge} \dots E_n B_n^{\curlywedge}$ and ℓ is the labelling of AON.*

Proof. Let $ON \overset{\text{df}}{=} \text{UND}(AON)^{\curlywedge}$ and $N \overset{\text{df}}{=} \text{UND}(NI)$. Clearly, $ON \in \alpha(N)$ and μ' is a mixed step sequence of ON from MIN_{ON}. Hence, by fact 6.6, $\ell\langle\mu'\rangle$

is a mixed step sequence of N from M_0. Thus it suffices to show that if $e \in E_i$ and p is an inhibitor place of $\ell(e)$, then $\ell\langle B_{i-1}\rangle(p) \leq \textsc{inh}_{NCI}(\ell(e))(p)$. This, however, follows from $B_{i-1} \in \textsc{ssl}(AON)$ (which holds due to proposition 5.17(1)), and definition 9.3(7). ∎

Corollary 9.8. *If $AON \in \alpha(NI)$, then $\phi(\lambda(AON)) \subseteq \omega_{\mathcal{STS}}(NI)$.*

Again, as for ca-processes of PTCI-nets, it can be shown that the 'prefixes' of an a-process AON of NI executed to reach a marking (a strong slice) S of AON are also a-processes of NI. Now, however, there may be $\curlywedge$–labelled conditions which are activator places for 'later' events and thus lead to a violation of the definition of an a-process. Hence, rather than $\textsc{preaon}_{AON}(S)$, it will be $\textsc{preaon}_{AON}(S)^{\curlywedge}$ which is an a-process of NI.

Proposition 9.9. *Let $AON \in \alpha(NI)$ and let $S \in \textsc{ssl}(AON)$ be a strong slice of AON. Then $\textsc{preaon}_{AON}(S)^{\curlywedge} \in \alpha(NI)$.*

Proof. See the appendix. ∎

On the other hand, given a labelled step sequence of an a-process AON of NI, its associated step sequence is one of the generators of AON.

Proposition 9.10. *Let $AON \in \alpha(NI)$ and let $\sigma \in \phi(\lambda(AON))$. Then $AON \in \pi_{NI}(\sigma)$.*

Proof. See the appendix. ∎

Consistency of the execution based process semantics and the axiomatic process semantics of NI now follows from propositions 9.4 and 9.10, as well as corollaries 9.6 and 9.8.

Proposition 9.11 (property 2: consistency for PTI-nets). *For every step sequence σ and every ao-net AON,*

$$\sigma \in \omega_{\mathcal{STS}}(NI) \wedge AON \in \pi_{NI}(\sigma) \text{ iff } AON \in \alpha(NI) \ \wedge\ \sigma \in \phi(\lambda(AON)) \ .$$

Consequently, also the remaining aims for PTI-nets are fulfilled and we may conclude that the two proposed process semantics are in full agreement with the the operational semantics of PTI-nets.

Theorem 9.12 (aims 1 and 2 for PTI-nets). *For every PTI-net NI,*

$$\begin{array}{lcl} \alpha(NI) & = & \pi_{NI}(\omega_{\mathcal{STS}}(NI)) \\ \omega_{\mathcal{STS}}(NI) & = & \phi(\lambda(\alpha(NI))) \ . \end{array}$$

The construction of a-processes for general PTI-nets uses constraints introduced through 'artificial' $\curlywedge$–labelled conditions, which do not have direct counterparts in the original PTI-net, but rather represent dynamic relationships between the executed transitions. The question therefore arises whether such a technique does not introduce too many constraints in the causality structures generated by a-processes. That this is indeed possible can be observed by taking the PTI-net NI and one of its a-processes AON shown in figure 12 (it can be generated, e.g., from the step sequence $\{u,t\}\{z\}$). One may easily verify that we can safely delete one of the activator arcs (but not both), which leads to another a-process generating weaker constraints than AON.

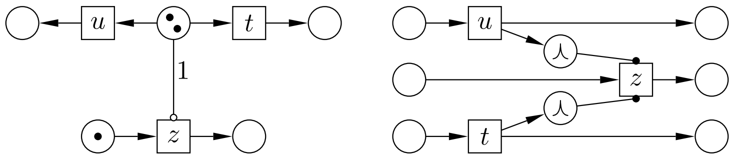

FIG. 12 PTI-net and its a-process whose so-structure can be weakened.

Having said that, it turns out that PTDI-nets are special in that the proposed semantics is minimal, in the sense that making the causal structure more relaxed, by removing some of the activator arcs, leads to inconsistency with the semantics of the underlying PTDI-net.

PROPOSITION 9.13. *Let NI be a PTDI-net and* $AON = (B, E, R, Act, \ell)$ *be one of its a-processes. Moreover, let* $AON' = (B, E, R, Act', \ell)$ *be an ao-net such that* $Act' \subseteq Act$ *and* $\kappa(AON') \neq \kappa(AON)$*. Then* $\phi(\lambda(AON'))\backslash\omega(NI) \neq \varnothing$*.*

Proof. See the appendix. ∎

Thus, in particular, for all the standard inhibitor nets (PTSI-nets) the proposed semantics introduces a minimal number of constraints.

We finally address the issue of having two different process semantics for PTCI-nets, which in general may lead to different causality semantics. Consider, for example, the PTCI-net in figure 8, and one of its step sequences $\{w, w\}\{t\}$. It is not difficult to see that the so-structure generated by the a-process of this step sequence using the second semantics can never be generated by that based on complement places (basically, t can only be related to one occurrence of w in this case).

Although, in general, the semantics are different, for PTCI-nets which are PTDI-nets processes derived in either way lead to the same causality structures.

PROPOSITION 9.14. *Let NI be a PTI-net which is both a PTCI-net and PTDI-net. Then* $\kappa(\alpha^{cpl}(NI)) = \kappa(\alpha(NI))$*.*

Proof. See the appendix. ∎

Thus, in particular, for all the standard inhibitor nets (PTSI-nets) with complemented inhibitor places the two semantics are in essence the same.

10. CONCLUSIONS

The central contribution of this paper is a proposal for a process semantics for PT-nets with inhibitor arcs while assuming an *a priori* operational semantics. Our investigation has been conducted within a general framework for dealing with process semantics of Petri nets, also proposed here. In essence, the investigation of the relationship between nets and their processes is separated from the investigation of the causality within these processes, with an operational/observational interpretation in terms of executions as the bridge between them.

There are at least two potential applications of these results: first, they can be useful in the development of model checking algorithms for PTI-nets based on unfoldings; second, they can be used as a basis for obtaining a causality semantics for PT-nets with priorities, extending the results obtained for the elementary net systems with priorities in [16].

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APPENDIX A
PROOFS OMITTED FROM THE MAIN TEXT

Proof of proposition 4.3

(1) Let $Z \stackrel{\mathrm{df}}{=} \{z \mid x \sqsubset z \sqsubset x \vee y \sqsubset z \sqsubset y\} \cup \{x, y\}$ be the set consisting of all elements which have to occur simultaneously with x or y. Moreover, let $X_0 \stackrel{\mathrm{df}}{=} \{w \in X \backslash Z \mid w \sqsubset x \vee w \sqsubset y\}$ consist of all remaining elements which cannot occur later than x or later than y and let $X_1 \stackrel{\mathrm{df}}{=} X \backslash (Z \cup X_0)$ comprise all other elements. By applying the conditions (C2–C4), we obtain:

- $\prec \cap (Z \times Z) = \varnothing$, since otherwise $x \prec x$ or $y \prec y$ or $x \prec y$ or $y \prec x$. (i)
- If $z \in Z$ and $w \in X_0$ then $\neg(z \sqsubset w)$, since otherwise $w \in Z$. (ii)
- If $z \in X_1$ and $w \in Z \cup X_0$ then $\neg(z \sqsubset w)$, since otherwise $z \in X_0$. (iii)

Consider the so-structure $lsos_i \stackrel{\mathrm{df}}{=} (X_i, \prec|_{X_i \times X_i}, \sqsubset|_{X_i \times X_i}, \ell|_{X_i})$, for $i = 0, 1$. From fact 4.2(1), there are labelled step sequences $(\sigma_i, \ell|_{X_i}) \in \epsilon(lsos_i)$, for $i = 0, 1$. It is easy to see that (i)—(iii) imply that $(\sigma_0 Z \sigma_1, \ell) \in \epsilon(lsos)$.

(2) Let $Y \stackrel{\mathrm{df}}{=} \{w \mid w \sqsubset y \sqsubset w\} \cup \{y\}$ be the set consisting of all elements which have to occur simultaneously with y, and let $Z \stackrel{\mathrm{df}}{=} \{z \in X \backslash Y \mid x \sqsubset z \sqsubset x\} \cup \{z \in X \backslash Y \mid x \sqsubset z \sqsubset y\} \cup \{x\}$ be the set consisting of all elements which have to occur simultaneously with x or not later than y but not before x. Moreover, let $X_0 \stackrel{\mathrm{df}}{=} \{w \in X \backslash (Z \cup Y) \mid w \sqsubset x \vee w \sqsubset y\}$ consist of all remaining elements which cannot occur later than x or later than y, and let $X_1 \stackrel{\mathrm{df}}{=} X \backslash (Z \cup Y \cup X_0)$ comprise all other elements. By applying the conditions (C2–C4), we obtain:

- $\prec \cap (Y \times Y) = \varnothing$, since otherwise $y \prec y$. (iv)
- $\prec \cap (Z \times Z) = \varnothing$, since otherwise $x \prec x$ or $x \prec y$. (v)
- If $z \in Z$ and $w \in Y$ then $\neg(w \sqsubset z)$, since otherwise $y \sqsubset x$ or $z \in Y$. (vi)
- If $z \in Z$ and $w \in X_0$ then $\neg(z \sqsubset w)$, since otherwise $w \in Z$. (vii)
- If $z \in Y$ and $w \in X_0$ then $\neg(z \sqsubset w)$, since otherwise $w \in Y$ or $y \sqsubset x$. $(viii)$
- If $z \in X_1$ and $w \in Z \cup Y \cup X_0$ then $\neg(z \sqsubset w)$, since otherwise $z \in X_0 \cup Y \cup Z$. (ix)

Consider the so-structure $lsos_i \stackrel{\text{df}}{=} (X_i, \prec|_{X_i \times X_i}, \sqsubset|_{X_i \times X_i}, \ell|_{X_i})$, for $i = 0, 1$. From fact 4.2(1), there are labelled step sequences $(\sigma_i, \ell|_{X_i}) \in \epsilon(lsos_i)$, for $i = 0, 1$. It is easy to see that (iv)—(ix) imply that $(\sigma_0 Z Y \sigma_1, \ell) \in \epsilon(lsos)$.

Proof of proposition 5.10

($\Longrightarrow$) Clearly, μ is a mixed step sequence of ON from the initial marking. Suppose that $e \in E_i$ and $f \prec_{loc} e$. Then there is $b \in B$ such that $f R b (R \cup Act) e$. Hence $b \notin \text{MIN}_{AON}$ and, by e being enabled at M_{i-1}, $b \in M_{i-1}$. Thus $f \in E_1 \cup \ldots \cup E_{i-1}$ as $|\text{PRE}_{AON}(b)| \leq 1$.

Suppose now that $e \in E_i$, $f \in E_j$ $(j > i)$ and $f \sqsubset_{loc} e$. Then there is $b \in B$ such that bRe and $bActf$. Hence $b \in M_{i-1}$ (by bRe) and $b \notin M_i \cup \ldots \cup M_n$ (by fact 5.3(1)). Thus $b \notin M_{j-1}$, contradicting $bActf$ and f being enabled at M_{j-1}.

($\Longleftarrow$) It suffices to show that for every $i \leq n$ and $e \in E_i$, if $b \in \text{ACT}_{AON}(e)$ then $b \in M_{i-1}$. This, in turn, follows if fRb implies $f \in E_1 \cup \ldots \cup E_{i-1}$, and bRg implies $g \notin E_1 \cup \ldots \cup E_{i-1}$. And the last two properties follow immediately from the assumptions we made.

Proof of proposition 5.13

The second part clearly holds, so we first show that $\text{SSL}(AON) \subseteq \text{SL}(ON)$. Let $S \in \text{SSL}(AON)$. Thus $(S \times S) \cap R^+ = \varnothing$ and so to prove that $S \in \text{SL}(ON)$ it suffices to show that no condition can be added to S without destroying this property.

Suppose that there is $b \in B \backslash S$ such that $((\{b\} \times S) \cup (S \times \{b\})) \cap R^+ = \varnothing$. Since $b \notin S$ and S as a strong slice of AON is maximal w.r.t. set inclusion, we may consider the following two cases.

Case 1: $(S \times \{b\}) \cap \text{SLIN}(AON) \neq \varnothing$. Since $(S \times \{b\}) \cap R^+ = \varnothing$, there must be $c \in S$, $c' \in B$ and $e_1, \ldots, e_k \in E$ such that: $k \geq 1$, $(c, c') \in \text{SLIN}(AON)$ and $c' Act\, e_1 R \circ R e_2 \ldots e_{k-1} R \circ R e_k R b$. Moreover, we can choose these elements in such a way that k is maximal (this is possible since E is finite and R^+ acyclic). Let $d \in B$ be such that dRe_1. Since dR^+b, we have that $d \notin S$. Because $S \in \text{SSL}(AON)$ this implies that $((S \times \{d\}) \cup (\{d\} \times S)) \cap \text{SLIN}(AON) \neq \varnothing$. If $(S \times \{d\}) \cap \text{SLIN}(AON) \neq \varnothing$ then there is a path from S to b which passes through d and ends with a sequence of arcs in R^+ which is longer than k, in contradiction with the maximality of k. Thus it must be the case that $(\{d\} \times S) \cap \text{SLIN}(AON) \neq \varnothing$. Hence, by $|\text{POST}_{AON}(d)| \leq 1$, there are $f \in E$ and $d' \in S$ such that $e_1 \prec_{loc}^{\star} fRd'$. Since $(c, c') \in \text{SLIN}(AON)$ and $c' Act\, e_1$, this means that $(c, d') \in \text{SLIN}(AON)$, a contradiction with $S \in \text{SSL}(AON)$.

Case 2: $(\{b\} \times S) \cap \text{SLIN}(AON) \neq \varnothing$. Since $(\{b\} \times S) \cap R^+ = \varnothing$, there must be $d, c' \in B$, $c \in S$ and $e_1, \ldots, e_k \in E$ such that the following are satisfied: $k \geq 2$,

$bRe_1R\circ Re_2\dots e_{k-2}R\circ Re_{k-1}RdAct\, e_kRc'$ and $(c',c)\in \textsc{slin}(AON)$. Moreover, we can choose these elements in such a way that k is maximal. Since bR^+d, we have that $d\notin S$. We now observe that if $(\{d\}\times S)\cap\textsc{slin}(AON)\neq\varnothing$, then we obtain a contradiction with the maximality of k. If $(S\times\{d\})\cap\textsc{slin}(AON)\neq\varnothing$, then, by $|\textsc{pre}_{AON}(d)|\leq 1$, there are $d'\in S$ and $f\in E$ such that $d'Rf\prec_{loc}^{\star} e_{k-1}$. But this means that $(d,c)\in\textsc{slin}(AON)$, a contradiction.

We have shown that $\textsc{ssl}(AON)\subseteq\textsc{sl}(ON)$. To prove $\textsc{wsl}(AON)\subseteq\textsc{ssl}(AON)$, let $S\in\textsc{wsl}(AON)$. Then, clearly, $\textsc{slin}(AON)\cap(S\times S)=\varnothing$. Thus, to show that $S\in\textsc{ssl}(AON)$ it suffices to show the maximality of S w.r.t. $\textsc{slin}(AON)$.

Suppose that there is $b\in B\setminus S$ such that $((\{b\}\times S)\cup(S\times\{b\}))\cap\textsc{slin}(AON)=\varnothing$. Since $b\notin S$, we may consider the following two cases.

Case 1: $(S\times\{b\})\cap\textsc{wlin}(AON)\neq\varnothing$. Since $(S\times\{b\})\cap\textsc{slin}(AON)=\varnothing$, there must be $c\in S$, $c'\in B$ and $e,e',e_1,\dots,e_k\in E$ such that: $k\geq 1$ and $cRe\Subset^{\star} e'Act^{-1}c'Re_1\prec_{loc} e_2\dots e_{k-1}\prec_{loc} e_kRb$. Moreover, we can choose these elements in such a way that k is maximal (this is possible since E is finite and $\prec_{loc}$ is acyclic). Since $(c',b)\in\textsc{slin}(AON)$, we have that $c'\notin S$. We now observe that if $(S\times\{c'\})\cap\textsc{wlin}(AON)\neq\varnothing$ then we obtain a contradiction with the maximality of k. Thus it must be the case that $(\{c'\}\times S)\cap\textsc{wlin}(AON)\neq\varnothing$. Hence, by $|\textsc{post}_{AON}(c')|\leq 1$, there are $f\in E$ and $d'\in S$ such that $e_1\Subset^{\star} fRd'$. Since $(c,c')\in\textsc{wlin}(AON)$ and $c'Act^{-1}e_1$, this means that $(c,d')\in\textsc{wlin}(AON)$, a contradiction with $S\in\textsc{wsl}(AON)$.

Case 2: $(\{b\}\times S)\cap\textsc{wlin}(AON)\neq\varnothing$. Since $(\{b\}\times S)\cap\textsc{slin}(AON)=\varnothing$, there must be $c\in S$ and $e_1,\dots,e_k,e,e'\in E$ such that: $k\geq 1$, $bRe_1\prec_{loc} e_2\dots e_{k-1}\prec_{loc} e_k(\sqsubset_{loc}\setminus\prec_{loc})e\Subset^{\star} e'Rc$. Moreover, we can choose these elements in such a way that k is maximal. Let $d\in B$ be such that $(e_k,d)\in R$. Since $(b,d)\in\textsc{slin}(AON)$, we have that $d\notin S$. We now observe that if $(\{d\}\times S)\cap\textsc{wlin}(AON)\neq\varnothing$, then we obtain a contradiction with the maximality of k. If $(S\times\{d\})\cap\textsc{wlin}(AON)\neq\varnothing$, then, by $|\textsc{pre}_{AON}(d)|\leq 1$, there are $d'\in S$ and $f\in E$ such that $d'Rf\Subset^{\star} e_k$. But this means that $(d',c)\in\textsc{wlin}(AON)$, a contradiction.

Proof of proposition 5.15

(1) $\Rightarrow$ (2) Follows from proposition 5.12(3).

(2) $\Rightarrow$ (3) Suppose that there are $b,c\in M$ such that $(b,c)\in\textsc{wlin}(AON)$. Then there are $e_1,\dots,e_k\in E$ such that: $k\geq 1$ and $bRe_1\Subset e_2\dots e_{k-1}\Subset e_kRc$. The latter means that $e_k\in E'=E_1\cup\dots\cup E_n$. Hence, since $e_{k-1}\Subset e_k$ and $E'\in\textsc{wcnf}(AON)$, we obtain $e_{k-1}\in E'$. By applying the same argument $k-1$ times we obtain that $e_1\in E'$. But this means that $b\notin M$, a contradiction.

(3) $\Rightarrow$ (1) Suppose that $M\neq\textsc{max}_{AON}$. It suffices to show that there is a set of events $\widetilde{E}\neq\varnothing$ such that $M[\widetilde{E}\rangle_{AON}\widetilde{M}$ and $\widetilde{M}\in\textsc{wsl}(AON)$.

Let $AON'=(B',E',R',Act',\ell')$ be the ao-net $\textsc{preaon}_{AON}(M)$ and, moreover, let $AON''=(B'',E'',R'',Act'',\ell'')$ be $\textsc{postaon}_{AON}(M)$. We first define an infinite sequence of sets $E^i\subseteq E''$:

$$\begin{array}{lcl} E^0 & \stackrel{\mathrm{df}}{=} & \{e\in E''\mid \textsc{pre}_{AON}(e)\cup\textsc{act}_{AON}(e)\subseteq M\}\\ E^{i+1} & \stackrel{\mathrm{df}}{=} & \{e\in E^i\mid \forall f\in E''\setminus E^i:\textsc{act}_{AON}(f)\cap\textsc{pre}_{AON}(e)=\varnothing\}\,, \end{array}$$

where $i\geq 0$. Let $\widetilde{E}$ be defined as the intersection of all the E^i's. Clearly $E^{i+1}\subseteq E^i\subseteq E''$, and so, since E'' is finite, there is $k\leq|E''|$ such that $\widetilde{E}=E^k$. Moreover, since $\widetilde{E} \subseteq E^0$, $\widetilde{E}$ is enabled at M, and so there is $\widetilde{M}$ such that $M[\widetilde{E}\rangle_{AON}\widetilde{M}$. What we need to show is that $\widetilde{E} \neq \varnothing$ and $\widetilde{M} \in \textsc{wsl}(AON)$.

Let E_{min} be the set of all events in E'' which are minimal w.r.t. $\prec_{AON}|_{E''\times E''}$. Since $M \neq \textsc{max}_{AON}$, we have $E'' \neq \varnothing$, and so since $\prec_{AON}$ is a partial order, we have $E_{min} \neq \varnothing$. We then observe that $E_{min} \subseteq E^0$, which follows from proposition 5.14(2a). Suppose now that $E_{min} \subseteq E^i$ and $e \in E_{min}\backslash E^{i+1}$. Then, there is $f \in E''$ such that $\textsc{act}_{AON}(f) \cap \textsc{pre}_{AON}(e) \neq \varnothing$ and $f \notin E^i$. Hence $f \notin E_{min}$ and there is $g \in E_{min}$ such that $g \prec_{AON} f$. Thus $g \prec_{AON} e$, a contradiction with the minimality of e. Therefore, we obtained that $E_{min} \subseteq E^i$, for all $i \geq 0$, and so $E_{min} \subseteq \widetilde{E}$. Suppose that $\widetilde{E} \neq E_{min}$. Then there are $e \in \widetilde{E}\backslash E_{min}$ and $f \in E_{min}$ such that $f \prec_{loc} e$ implying that $\textsc{post}_{AON}(f) \cap M \neq \varnothing$, a contradiction.

Hence $\widetilde{E} = E_{min} \neq \varnothing$. What we still need to show is that $\widetilde{M} \in \textsc{wsl}(AON)$. To the contrary, suppose that there are $b, c \in \widetilde{M}$ and $e, f \in E$ such that $bRe \Subset^\star fRc$. Thus, since $M \in \textsc{wsl}(AON)$, one of the following three cases holds.

Case 1: $b \in \widetilde{M}\backslash M$ and $c \in \widetilde{M} \cap M$. Then there is $g \in \widetilde{E}$ such that gRb. Let d be any condition such that dRg. We have $d \in M$ and $g(\prec_{loc} \circ \Subset^\star)f$. Hence $(d, c) \in \textsc{wlin}(AON)$, contradicting $M \in \textsc{wsl}(AON)$.

Case 2: $b \in \widetilde{M}\backslash M$ and $c \in \widetilde{M}\backslash M$. Then there is $g \in \widetilde{E} = E_{min}$ such that gRb. Moreover, $f \in \widetilde{E} = E_{min}$ as we have $c \in \widetilde{M}\backslash M$, fRc and $|\textsc{pre}_{AON}(c)| \leq 1$. Hence $g \prec_{loc} e$ and so $g \prec_{AON} e \sqsubset_{AON} f$, meaning that $f \notin E_{min}$, a contradiction.

Case 3: $b \in \widetilde{M} \cap M$ and $c \in \widetilde{M}\backslash M$. Then there are $e_1, \ldots, e_k \in E$ such that $e_1 = e$, $e_k = f$ and $e_i \Subset e_{i+1}$, for all $i < k$.
We first observe that $e_1 \in E''$ as $b \in M$ and bRe. Suppose now that there is $i \leq k$ such that $e_i \in E'$, and let us choose the smallest such i. We have $i \geq 2$ as $e_1 \in E''$. Hence $\textsc{post}_{AON}(e_{i-1}) \cap \textsc{pre}_{AON}(e_i) \neq \varnothing$, or $\textsc{post}_{AON}(e_{i-1}) \cap \textsc{act}_{AON}(e_i) \neq \varnothing$, or $\textsc{act}_{AON}(e_{i-1}) \cap \textsc{pre}_{AON}(e_i) \neq \varnothing$. We then obtain respectively a contradiction with $e_i \in E' \wedge e_{i-1} \in E''$, or proposition 5.14(1a), or proposition 5.14(2a). Thus $e_1, \ldots, e_k \in E''$. Consequently, as $e_1 \sqsubset_{AON} \cdots \sqsubset_{AON} e_k$ and $e_k = f \in E_{min}$, we have $e_{k-1}, \ldots, e_1 \in E_{min}$. Hence $e = e_1 \in E_{min} = \widetilde{E}$, and so, by bRe, we obtained that $b \notin \widetilde{M}$, a contradiction.

Proof of proposition 7.2

Suppose that $NI = (P, T, W, I, M_0)$ is a PTSI-net with the same set of firing sequences Θ as the PTDI-net of figure 5. Thus NI has z, u, and t among its transitions. Let R be the set of inhibitor places of z in NI. Since $z \in \Theta$ and $uz \notin \Theta$, there is a place $r \in R$ such that $\textsc{post}_{NI}(u)(r) > 0$, or there is a place $p \in P$ such that $\textsc{pre}_{NI}(z)(p) > 0$ and $\textsc{pre}_{NI}(u)(p) - \textsc{post}_{NI}(u)(p) > 0$. However, since $u^* \subseteq \Theta$ it must be the case that $\textsc{pre}_{NI}(u) \leq \textsc{post}_{NI}(u)$, and so there must be an $r \in R$ such that $\textsc{post}_{NI}(u)(r) > 0$. Note that this implies that $R \neq \varnothing$.

Suppose now that there is $\theta \in \Theta$ such that the marking M reached through θ satisfies $M|_R = \mathbf{0}_R$ and $M(p) < W(p, z)$, for some $p \in P$ (in other words, z is disabled due to the lack of tokens in an input place p, and not by one of its inhibitor places being marked). Then, since there is $k \geq 1$ such that $\theta t^k z \in \Theta$, $\textsc{pre}_{NI}(t)|_R = \textsc{post}_{NI}(t)|_R = \mathbf{0}_R$. But now $utz \in \Theta$ implies that $\textsc{post}_{NI}(u)|_R = \mathbf{0}_R$ in contradiction with $\textsc{post}_{NI}(u)(r) > 0$, for some $r \in R$.

Thus the enabledness of z for a marking M reachable from M_0 can be tied to the lack of tokens on its inhibitor places R; more precisely, we obtained that z is enabled at a marking M reachable from M_0 *iff* $M|_R = \mathbf{0}_R$.

Let $R = \{r_1, \ldots, r_m\}$ for some $m \geq 1$ and let $k_i \stackrel{\text{df}}{=} W(u, r_i) - W(r_i, u)$ and $l_i \stackrel{\text{df}}{=} W(r_i, t) - W(t, r_i)$ for every $i \leq m$. Consider all sequences of the form $u^k t^l$, for $k, l \geq 0$.

Looking at the PTDI-net of figure 5, it is easy to see that $u^k t^l \in \Theta$ *iff* $3k - 2l \geq 0$. Moreover, $u^k t^l z \in \Theta$ *iff* $3k - 2l \leq 1$. On the other hand, by $z \in \Theta$ and what we have already proved, $u^k t^l z$ is a firing sequence of NI *iff* $k_i k - l_i l = 0$, for every $i \leq m$. As a result, we obtained that the following two systems of linear constraints

$$\left\{ \begin{array}{cccc} 3k & - & 2l & \geq & 0 \\ 3k & - & 2l & \leq & 1 \end{array} \right. \qquad\qquad \left\{ \begin{array}{ccccc} 3k & - & 2l & \geq & 0 \\ k_1 k & - & l_1 l & = & 0 \\ \ldots & & & & \\ k_m k & - & l_m l & = & 0 \end{array} \right.$$

have exactly the same solutions in non-negative integers k and l. This, however, is impossible as we show next.

Suppose first that $k_i = 0$ for some $i \leq m$. Then $l_i = 0$, otherwise the solution on l for the second system would have always to be 0, despite the fact that $(1, 1)$ is a solution of the first system. Thus any equation $0k - l_i l = 0$ can be discarded as not contributing any constraints. Similarly, we can assume that $l_i \neq 0$ for all $i \leq m$. If for some i, $k_i \neq 0 \neq l_i$, then all solutions of the second system must lie on a single line. But the pairs $(0, 0)$, $(1, 1)$ and $(2, 3)$ which are solutions of the first system are not co-linear.

Proof of proposition 9.4

Assume the notation from definition 9.1. Moreover, let $N \stackrel{\text{df}}{=} \textsc{und}(NI)$, $AON \stackrel{\text{df}}{=} N_n$ and $ON \stackrel{\text{df}}{=} \textsc{und}(AON)^{\curlywedge}$. We first observe that $ON \in \pi_N(\sigma)$, which follows directly from definition 9.1. Hence, by fact 6.11, $ON \in \alpha(N)$ and so AON satisfies definition 9.3(1,3,4). Conditions in definition 9.3(2,5,6) are guaranteed by the construction of AON. Hence, to complete the proof of $AON \in \alpha(NI)$, we need to show definition 9.3(7).

In what follows, for every event e of AON, we let $\#e$ be the i such that $e \in E^i$. Moreover, $\textsc{max}_i \stackrel{\text{df}}{=} (B_i \cup \widetilde{B}_i) \backslash dom_{R_i}$, for every $0 \leq i \leq n$.

Let $e \in E_n$ and $S \in \textsc{ssl}(AON)$ be such that $\textsc{act}_{AON}(e) \cup \textsc{pre}_{AON}(e) \subseteq S$. We have to prove that $\ell\langle S^{\curlywedge} \rangle \leq \textsc{inh}_{NI}(\ell(e))$. What we will show is that $b \in S \backslash \textsc{max}_{\#e-1}$ and $p \in \textsc{inh}_{NI}(\ell(e))$ and $\ell(b) = p$ leads to a contradiction, and so $\ell\langle S^{\curlywedge} \rangle \leq \ell\langle \textsc{max}_{\#e-1} \rangle \leq \textsc{inh}_{NI}(\ell(e))$. We consider the following six cases.

Case 1: $\textsc{pre}_{AON}(b) = \textsc{post}_{AON}(b) = \varnothing$. Then $b \in \textsc{max}_{\#e-1}$, a contradiction.

Case 2: $\textsc{post}_{AON}(b) = \{f\}$ and $\#f < \#e$. Then $(b, c) \in \textsc{slin}(AON)$, where $f \overset{c}{\multimap} e$, a contradiction as $c \in S \in \textsc{ssl}(AON)$ and $\textsc{act}_{AON}(e) \subseteq S$.

Case 3: $\textsc{pre}_{AON}(b) = \{f\}$ and $\#e \leq \#f$. Then $(c, b) \in \textsc{slin}(AON)$, where $f \overset{c}{\multimap} e$, and we obtain a contradiction similarly as above.

Case 4: $\textsc{post}_{AON}(b) = \{g\}$, $\textsc{pre}_{AON}(b) = \varnothing$ and $\#e \leq \#g$.

Case 5: $\textsc{pre}_{AON}(b) = \{f\}$, $\textsc{post}_{AON}(b) = \varnothing$ and $\#f < \#e$.

Case 6: $\textsc{pre}_{AON}(b) = \{f\}$, $\textsc{post}_{AON}(b) = \{g\}$ and $\#f < \#e \leq \#g$.

In the last three cases, we obtain $b \in \textsc{max}_{\#e-1}$, which yields a contradiction.

Proof of proposition 9.9

Let $AON \stackrel{\text{df}}{=} (B \uplus \widetilde{B}, E, R, Act, \ell)$ and $AON' \stackrel{\text{df}}{=} \textsc{preaon}_{AON}(S)^{\curlywedge}$. It is immediate that AON' satisfies definition 9.3(1-5). Consider events e and f of AON' such that $\ell(f) \multimap \ell(e)$ and let c be the unique condition in $\widetilde{B}$ such that $f \overset{c}{\multimap\!\!\bullet} e$ in AON. By proposition 5.14(1a), also $f \overset{c}{\multimap\!\!\bullet} e$ in $\text{PREAON}_{AON}(S)$ and hence in AON'. Thus definition 9.3(6) holds for AON'.

To show definition 9.3(7) for AON', suppose that $S' \in \text{SSL}(AON')$ and event e of AON' are such that $\text{PRE}_{AON'}(e) \cup \text{ACT}_{AON'}(e) \subseteq S'$. If there is $S'' \in \text{SSL}(AON)$ such that $S' \subseteq S''$, then we are done since definition 9.3(7) holds for AON. Suppose therefore that such an S'' does not exist. This means that there are $b, b' \in S'$ such that $(b, b') \in \text{SLIN}(AON')$. Hence there are events $e_1, \ldots, e_k$ and conditions $b_1, \ldots, b_k = b'$ such that bRe_1, e_iRb_i (for $1 \leq i \leq k$), and $b_i(R \cup Act)e_{i+1}$ (for $1 \leq i \leq k-1$). Not all e_i's and b_i's belong to AON'; otherwise we would have had $(b, b') \in \text{SLIN}(AON')$. We now observe that if e_i is not in AON' then the same is true of b_i (which follows from $\text{PRE}_{AON'}(c) = \text{PRE}_{AON}(c)$, for all conditions c of AON'), and if b_i ($i < k$) is not in AON' then the same is true of e_{i+1} (which follows if $b_i \in B$ from $\text{PRE}_{\text{PREAON}_{AON}(S)}(e) = \text{PRE}_{AON}(e)$, for all events e of AON', and if $b_i \in \widetilde{B}$ from proposition 5.14(1a) by which $\text{ACT}_{\text{PREAON}_{AON}(S)}(e) = \text{ACT}_{AON}(e)$, for every event $e \in E'$). Hence e_k is not in AON', and so b' is also not in AON', a contradiction.

Proof of proposition 9.10

By corollary 9.8, we have $\sigma \in \omega_{STS}(NI)$. Since $\text{UND}(AON)^{\curlywedge} \in \alpha(\text{UND}(NI))$ and $\sigma \in \phi(\lambda(AON)) \subseteq \phi(\lambda(\text{UND}(AON)^{\curlywedge}))$, there is by corollary 6.10, a run of the construction described in definition 6.2, generating $\text{UND}(AON)^{\curlywedge}$. We can then re-run this construction of $\text{UND}(AON)^{\curlywedge}$, adding at each stage sets Act^k and $\widetilde{B}^k$, as well as adjusting R^k and ℓ^k, as prescribed in definition 9.1, which is a deterministic construction. Let AON' be the resulting a-process. All we need to show is that AON and AON' are the same (isomorphic). If this is not the case then, for some events e and f, either $e \prec_{loc} f$ in AON, and $f \sqsubset'_{loc} e$ in AON' or $e \sqsubset_{loc} f$ in AON and $f \prec'_{loc} e$ in AON'. We now observe that $e \prec_{loc} f$ means that e occurs before f in the underlying step sequence of σ. But this means that, when re-running the construction, we could never create $f \sqsubset'_{loc} e$. Similarly, if $e \sqsubset_{loc} f$ in AON, then e occurs not later than f in the underlying step sequence of σ and so we would never create $f \prec'_{loc} e$.

Proof of proposition 9.13

Since $\kappa(AON) \neq \kappa(AON')$, there are ao-nets $AON_i = (B, E, R, Act_i, \ell)$ (for $i = 0, 1$), events $e, f \in E$ and a condition $b \in B$ such that: $Act \supseteq Act_0$, $(b, e) \in Act_0$, $Act_1 = Act_0 \setminus \{(b, e)\} \supseteq Act'$, $f \overset{b}{\multimap\!\!\bullet} e$, and the relationship between f and e in $\kappa(AON_0)$ is different (stronger) than that in $\kappa(AON_1)$. The latter means that one of the following holds:

- $\text{POST}_{AON}(b) = \{f\}$, $\neg(e \prec_{AON_0} f)$ and $\neg(e \sqsubset_{AON_1} f)$. $\quad (i)$
- $\text{PRE}_{AON}(b) = \{f\}$, $f \prec_{AON_0} e$ and $\neg(f \prec_{AON_1} e)$. $\quad (ii)$
- $\text{POST}_{AON}(b) = \{f\}$, $e \prec_{AON_0} f$ and $\neg(e \prec_{AON_1} f)$. $\quad (iii)$

In case of (i) and (iii), we have $e \sqsubset_{AON_0} f$. Furthermore, $f \prec_{AON_1} e$ cannot hold, since $f \prec_{AON_1} e$ implies that $f \prec_{AON_0} e$ and so $e \sqsubset_{AON_0} f \prec_{AON_0} e$ which by C4 in the definition of so-structures yields $e \prec_{AON_0} e$ in contradiction with the irreflexivity of $\prec_{AON_0}$. Similarly, in case of (ii), $e \sqsubset_{AON_1} f$ does not hold since otherwise $f \prec_{AON_0} e \sqsubset_{AON_0} f$. Therefore, when (i) or (ii) holds, we have both $\neg(f \prec_{AON_1} e)$ and $\neg(e \sqsubset_{AON_1} f)$ and thus also $\neg(e \prec_{AON_1} f)$. Hence, by proposition 4.3, we can find two labelled step sequences, $\varpi = (\sigma, \ell)$ and $\varpi' = (\sigma', \ell)$, both belonging to $\epsilon(\kappa(AON_1)) \subseteq \epsilon(\kappa(AON'))$ and such that e and f are in the same set in σ, and e is in the set immediately following that to which f belongs in σ'. Recall that $\epsilon \circ \kappa = \lambda$ by the fitting property 5.19 for ao-nets. Now observe that $\phi(\varpi) = \ell\langle\sigma\rangle$ and $\phi(\varpi') = \ell\langle\sigma'\rangle$ cannot both be valid step sequences of NI due to the definition of PTDI-nets. Since AON is an a-process of NI, $f {-\!\bullet}\, e$ in AON implies that $\ell(f) \multimap \ell(e)$ in NI and hence either $\ell(f)$ and $\ell(e)$ can occur in one step or they can be executed consecutively, but not both. Hence we obtained a contradiction.

As to the case (iii), it can never be satisfied. Indeed, let $\gamma = e_1 b_1 \ldots b_{k-1} e_k$ be any sequence of nodes establishing the relationship $e \prec_{AON_0} f$. This means that: $e_1 = e$, $e_k = f$, and for $1 \leq i \leq k-1$, the following hold:

- $e_i R b_i (R \cup Act_0) e_{i+1}$. $\qquad (iv)$
  or
  $e_i Act_0^{-1} b_i R e_{i+1}$. $\qquad (v)$
- There is $1 \leq i_0 \leq k-1$ such that (iv) holds for $i = i_0$. $\qquad (vi)$

If $b \notin \{b_1, \ldots, b_{k-1}\}$ then, clearly, γ is a sequence of nodes establishing $e \prec_{AON_1} f$, a contradiction. So we have $b = b_j$ for some $1 \leq j \leq k-1$. Then $j \neq i_0$ since $\text{PRE}_{AON_0}(b) = \varnothing$. Hence we have $1 \leq i_0 < j$ or $j < i_0 \leq k-1$. In the former case, by definition 9.3(5) and $(b, e) \in Act_0$, we have that $e_j = e$, and hence by (vi), $e \prec_{AON_0} e$, a contradiction. In the latter case, by $\text{POST}_{AON_0}(b) = \{f\}$, we have that $e_{j+1} = f$, and hence by (vi), $f \prec_{AON_0} f$, a contradiction.

Proof of proposition 9.14

($\subseteq$) Let $AON = (B, E, R, Act, \ell) \in \alpha^{cpl}(NI)$ be a ca-process of NI. We transform AON into an a-process $AON' \in \alpha(NI)$ for which $\kappa(AON') = \kappa(AON)$. This is done by removing the original activator arcs in AON and adding $\curlywedge$–labelled places $\widetilde{B}$ with new activator arcs Act'. First, $\widetilde{B} = Act' \stackrel{\text{df}}{=} \varnothing$ and $R' \stackrel{\text{df}}{=} R$. Then, for all $e, f \in E$ such that $\ell(f) \multimap \ell(e)$ we create exactly one condition $b \in \widetilde{B}$, add the arc (b, e) to Act', and consider two cases:

Case 1: $f \prec_{AON} e$. Then we add (f, b) to R.

Case 2: $e \sqsubset_{AON} f$. Then we add (b, f) to R.

Set $AON' \stackrel{\text{df}}{=} (B \uplus \widetilde{B}, E, R', Act', \ell')$, where ℓ' is the labelling ℓ extended to be a labelling of $B \cup \widetilde{B} \cup E$ so that $\ell(b) \stackrel{\text{df}}{=} \curlywedge$, for all $b \in \widetilde{B}$.

We now observe that case 1 or 2 always holds whenever $\ell(f) \multimap \ell(e)$, and so definition 9.3(6) is satisfied. Indeed, suppose that $\ell(f) \stackrel{p}{\multimap} l(e)$, and let $D \stackrel{\text{df}}{=} \text{PRE}_{AON}(f)$ if $p \in \text{PRE}_{NI}(\ell(f))$, and $D \stackrel{\text{df}}{=} \text{POST}_{AON}(f)$ otherwise. Thus $|D \cap \ell^{-1}(p)| = \text{PRE}_{NI}(\ell(f))(p)$ or $|D \cap \ell^{-1}(p)| = \text{POST}_{NI}(\ell(f))(p)$, respectively. Suppose that there is $S \in \text{SSL}(AON)$ such that $D \cup \text{ACT}_{AON}(e) \subseteq S$. By the condition (8.1) in definition 8.4, $|S \cap \ell^{-1}(p^{cpl})| \geq \text{BND}_{NI}(p) - \text{INH}_{NI}(\ell(e))(p)$. On the other hand, by the definition of PTDI-nets, $|S \cap \ell^{-1}(p)| \geq |D \cap \ell^{-1}(p)| > \text{INH}_{NI}(\ell(e))(p)$. Thus $|S \cap \ell^{-1}(p^{cpl})| + |S \cap \ell^{-1}(p)| > \text{BND}_{NI}(p)$. However, $\ell\langle S\rangle \in [M_0\rangle$ by proposition 5.17(1) and proposition 8.8, and we thus have a contradiction with the fact that p^{cpl} is a complement of p. Consequently, such an S does not exist, and so $\text{SLIN}(AON) \cap ((D \cup \text{ACT}_{AON}(e)) \times (D \cup \text{ACT}_{AON}(e))) \neq \varnothing$.

It is clear that $\text{SLIN}(AON) \cap (D \times D) = \varnothing$ as well as $\text{SLIN}(AON) \cap (\text{ACT}_{AON}(e) \times \text{ACT}_{AON}(e)) = \varnothing$. This follows from proposition 5.17(1) and the fact that both e and f can be executed in some step sequence from MIN_{AON} (follows from corollary 5.16 and proposition 5.12(2)). Thus there is a pair $(b, b') \in \text{SLIN}(AON)$ belonging to $D \times \text{ACT}_{AON}(e)$ or $\text{ACT}_{AON}(e) \times D$. As one can check, in the former case $f \prec_{AON} e$, and in the latter one $e \sqsubset_{AON} f$. Hence we have shown that case 1 or 2 always holds.

We have $\prec'_{loc} \subseteq \prec_{AON}$ and $\sqsubset'_{loc} \subseteq \sqsubset_{AON}$ and $rs_{AON'}$ is $\diamondsuit$–acyclic. Moreover, $\prec_{loc} \subseteq \prec'_{loc}$, and so $\sqsubset_{loc} \subseteq \sqsubset'_{loc}$. Therefore AON' is an ao-net such that $\kappa(AON') = \kappa(AON)$. To show that $AON' \in \alpha(NI)$, we still need to prove definition 9.3(7).

Suppose that $e \in E$ and $\text{ACT}_{AON}(e) \cup \text{PRE}_{AON}(e) \subseteq S \in \text{SSL}(AON')$. By proposition 5.17(1), there is $G \in \text{SCNF}(AON')$ such that $S = \text{MAR}_{AON'}(G)$. We now observe that $G \in \text{SCNF}(AON)$. Indeed, this follows from the fact that $\prec_{loc} \subseteq \prec'_{loc}$ and $G \in \text{SCNF}(AON')$. Hence $S^{\curlywedge} = \text{MAR}_{AON}(G) \in \text{SSL}(AON)$. Thus, by proposition 8.8, $\ell\langle S^{\curlywedge}\rangle \in [M_0\rangle_{NI}$. Consider $p \in \text{INH}_{NI}(\ell(e))$. We have $|\ell^{-1}(p^{cpl}) \cap S^{\curlywedge}| \geq \text{BND}_{NI}(p) - \text{INH}_{NI}(\ell(e))(p)$, by $\text{ACT}_{AON}(e) \subseteq S$. Moreover, $|\ell^{-1}(p^{cpl}) \cap S^{\curlywedge}| + |\ell^{-1}(p) \cap S^{\curlywedge}| = \text{BND}_{NI}(p)$, by $\ell\langle S^{\curlywedge}\rangle \in [M_0\rangle_{NI}$. Hence $|\ell^{-1}(p) \cap S^{\curlywedge}| \leq \text{INH}_{NI}(\ell(e))(p)$, and so $\ell\langle S^{\curlywedge}\rangle \leq \text{INH}_{NI}(\ell(e))$.

($\supseteq$) Let $AON = (B \cup \widetilde{B}, E, R, Act, \ell) \in \alpha(NI)$ be an a-process of NI. We transform AON into a ca-process $AON' \in \alpha^{cpl}(NI)$, for which $\kappa(AON') = \kappa(AON)$, by adding activator arcs Act' to $\text{UND}(AON)^{\curlywedge}$. At the beginning, $Act' \overset{\text{df}}{=} \varnothing$.

Let $\mu = B_0E_1B_1 \ldots E_nB_n$ be a fixed mixed step sequence of AON such that $B_0 = \text{MIN}_{AON}$, $B_n = \text{MAX}_{AON}$ and $E = E_1 \uplus \ldots \uplus E_n$ (such a μ exists, by corollary 5.16 and propositions 5.11 and 5.12(2)). For every $e \in E$, we denote by $\#e$ the i such that $e \in E_i$.

Consider all pairs, $e \in E$ and $p \in P$, such that $p \in \text{INH}_{NI}(\ell(e))$. By proposition 9.7, $\ell\langle B^{\curlywedge}_{\#e-1}\rangle \in [M_0\rangle_{NI}$ and $\ell\langle B^{\curlywedge}_{\#e-1}\rangle(p) \leq \text{INH}_{NI}(\ell(e))(p)$. Hence, since p^{cpl} is a complement of p, $\ell\langle B^{\curlywedge}_{\#e-1}\rangle(p^{cpl}) \geq \text{BND}_{NI}(p) - \text{INH}_{NI}(\ell(e))(p)$. Therefore, we can choose a subset $B' \subseteq \ell^{-1}(p^{cpl}) \cap B_{\#e-1}$ such that $|B'| = \text{BND}_{NI}(p) - \text{INH}_{NI}(\ell(e))(p)$. We then add $B' \times \{e\}$ to Act'.

Let $AON' \overset{\text{df}}{=} (B, E, R', Act', \ell')$ with $R' \overset{\text{df}}{=} R|_{(B \times E) \cup (E \times B)}$ and $\ell' \overset{\text{df}}{=} \ell|_{B \cup E}$. Suppose that $f \prec'_{loc} e$. Then $\#f < \#e$, and so, by proposition 5.19, $\neg(e \sqsubset_{AON} f)$. Consequently, by definition 9.3(6), $f \prec_{AON} e$. Similarly, we can show that $e \sqsubset'_{loc} f$ implies $e \sqsubset_{AON} f$. Hence $rs_{AON'}$ is $\diamondsuit$–acyclic, and so AON' is an ao-net. Moreover, by definition 9.1 and theorem 9.12, $\text{UND}(AON') = \text{UND}(AON)^{\curlywedge} \in \alpha(\text{UND}(NI))$. Hence, since the condition (8.1) in definition 8.4 holds by construction, we have $AON' \in \alpha^{cpl}(NI)$.

Thus once we have shown that $\prec_{loc} \subseteq \prec_{AON'}$ and $\sqsubset_{loc} \subseteq \sqsubset_{AON'}$, then $\kappa(AON') = \kappa(AON)$ follows and we are done. We consider the following two cases.

Case 1: $f \prec_{loc} e$ because $\text{POST}_{AON}(f) \cap \text{ACT}_{AON}(e) \neq \varnothing$ on account of $p \in P$ such that $\ell(f) \overset{p}{-\!\circ}\, l(e)$. Let $D \overset{\text{df}}{=} \text{PRE}_{AON}(f)$ and $i = \#f - 1$ if $p \in \text{PRE}_{NI}(\ell(f))$, and $D \overset{\text{df}}{=} \text{POST}_{AON}(f)$ and $i = \#f$ otherwise. Furthermore, let $j = \#e - 1$. We have $\#f < \#e$ and so $i \leq j$. Moreover, $D \subseteq B_i$ and $|D \cap \ell^{-1}(p)| > \text{INH}_{NI}(\ell(e))(p)$ by the definition of PTDI-nets. Observe that as before, since $p \in \text{INH}_{NI}(\ell(e))$, there exists a subset $B' \subseteq \ell^{-1}(p^{cpl}) \cap B_j$ such that $|B'| = \text{BND}_{NI}(p) - \text{INH}_{NI}(\ell(e))(p)$ and moreover $B' \times \{e\} \subseteq Act'$. Thus $|(D \cap \ell^{-1}(p)) \cup B'| > \text{BND}_{NI}(p)$, and so there are $c \in D$ and $d \in B'$ such that $(c, d) \in R^+$; otherwise there would be a marking M reachable from M_0 in $\text{UND}(NI)$ such that $M(p) + M(p^{cpl}) \geq |D \cup B'| > \text{BND}_{NI}(p)$, a contradiction. If $D = \text{POST}_{NI}(\ell(f))$ then we get immediately that $e \prec_{AON'} f$.

If $D = \textsc{pre}_{NI}(\ell(e))$ then the same conclusion can be drawn after observing that $\textsc{post}_{AON}(c) = \{f\}$.

Case 2: $e \sqsubset_{loc} f$ because $\textsc{pre}_{AON}(f) \cap \textsc{act}_{AON}(e) \neq \varnothing$ on account of $p \in P$ such that $\ell(f) \overset{p}{—\!\circ}\, l(e)$. Let $D \overset{\text{df}}{=} \textsc{pre}_{AON}(f)$ and $i = \#f - 1$ if $p \in \textsc{pre}_{NI}(\ell(f))$, and $D \overset{\text{df}}{=} \textsc{post}_{AON}(f)$ and $i = \#f$ otherwise. Furthermore, let $j = \#e - 1$. We have $\#e \leq \#f$ and so $j \leq i$. As in case 1, $D \subseteq B_i$ and $|D \cap \ell^{-1}(p)| > \textsc{inh}_{NI}(\ell(e))(p)$. Again, there exists a subset $B' \subseteq \ell^{-1}(p^{cpl}) \cap B_j$ such that $|B'| = \textsc{bnd}_{NI}(p) - \textsc{inh}_{NI}(\ell(e))(p)$ and moreover $B' \times \{e\} \subseteq Act'$. Hence, similarly as in case 1, there are $c \in B'$ and $d \in D$ such that $(c, d) \in R^+$. If $D = \textsc{pre}_{NI}(\ell(f))$ then we get immediately that $e \sqsubset_{AON'} f$. If $D = \textsc{post}_{NI}(\ell(f))$ then the same conclusion can be drawn after observing that $\textsc{pre}_{AON}(d) = \{f\}$.