Product Form Solution for Generalized Stochastic Petri Nets

Gianfranco Balbo, Steven C. Bruell, and Matteo Sereno

Abstract—In this paper, we show the structural characteristics that a particular class of Generalized Stochastic Petri Nets must exhibit in order for their stationary probabilities to have a product-form. Sufficient conditions for identifying such a class are derived and proven with the development of a series of transformations that can also be used to construct, for any GSPN of the class, an equivalent SPN. These resulting SPNs represent the structures that can be analyzed with standard methods for Product-Form SPNs to establish whether the original GSPNs have Product-Form solutions and to compute their performance indices with effective approaches based on computationally efficient algorithms that avoid the generation of their underlying state spaces.

Index Terms—Generalized stochastic Petri nets, product form solution.

1 Introduction

Stochastic Petri Nets (SPNs) are a powerful tool for modeling and evaluating the performance of systems involving concurrency, nondeterminism, and synchronization. They are equivalent to continuous-time Markov chains and their steady-state analysis can thus be expressed as the solution of a linear system of equilibrium equations, one for each possible marking in the corresponding state space. The major problem in the computation of performance measures for SPNs is that the size of their reachability set increases exponentially both with the number of tokens in the initial marking and with the number of places in the net. As a consequence, except for special classes of nets, the size of the reachability set and the time complexity of the solution procedure preclude the exact numerical evaluation of many interesting models.

In order to overcome this problem, a class of SPNs has been discovered [9], [10] that is characterized by the fact that the stationary probability distribution of any net in this class can be factored into a product of terms, one term per place in the net. Nets possessing this property are called *Product-Form Stochastic Petri Nets* (PF-SPNs) and are easily identified by the structural criteria proposed by Coleman et al. [6], [9], [10]. Moreover, the product-form solution for this class of nets closely resembles that of a class of Queueing Networks (QN) [3], [7], [11] for which efficient computational algorithms have been derived. This similarity has led to the development of analogous algorithms for this class of SPNs as well [6], [13], [14].

In this paper, we show that a class of Generalized Stochastic Petri Nets (GSPNs) also possess a product-form solution. We start with a GSPN model that obeys the same structural criteria used to identify PF-SPNs. By introducing some additional restrictions and by employing a series of transformation steps, that convert the original GSPN into an equivalent GSPN and then to an equivalent PF-SPN, we establish our desired objective.

The main point of the transformation is the derivation of the so-called *routing process* for the PF-GSPNs. Similar to what has been done in [9], [10], the routing process can be considered the starting point for the produc-form analysis since it is from the routing process that we can derive the *traffic equations* of the GSPN.

The transformation steps used to prove our result for GSPN models are synthesized into an algorithm that constructs the routing matrix for the traffic equations and the incidence matrix of the equivalent SPN model that satisfies the standard criteria for PF-SPNs.

The balance of this paper is outlined as follows: Section 2 briefly reviews pertinent notation and definitions used in describing GSPNs. Section 3 reviews the structural conditions that are required for an SPN to have a product-form solution. We show in Section 4 that, in the case of GSPNs, additional constraints are needed for the produc-form analysis. In Section 5, we present the transformation steps that convert the original GSPN into an equivalent SPN that has a PFS. In this section, we first investigate the case of PF-GSPNs with only free-choice conflicts among immediate transitions and then we present the analysis for the case when immediate transitions can be in nonfree choice conflicts. Finally, Section 7 provides some concluding remarks.

2 Definitions and Notations

In this section, we review the basic concepts and notation that we use throughout this paper. More comprehensive presentations of Petri net concepts can be found, for instance, in [12], [15]. A presentation of concepts related to Generalized Stochastic Petri Nets can be found in [1], [2].

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A *Generalized Stochastic Petri Net* can be defined as an 8-tuple

$$GSPN = (\mathcal{P}, \mathcal{T}, I(\cdot,\cdot), O(\cdot,\cdot), H(\cdot,\cdot), \Pi(\cdot), w(\cdot), \boldsymbol{m}_0),$$

where $\mathcal{P}$ is the set of *places* and $\mathcal{T}$ is the set of *transitions* (timed and immediate). The function $I(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \mathbb{N}$ is the input function, $O(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \mathbb{N}$ is the output function, and $H(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \mathbb{N}$ is the inhibition function. A net is said to be *ordinary* if $I(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \{0, 1\}$, $O(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \{0, 1\}$, and $H(t_j, p_i) : \mathcal{T} \times \mathcal{P} \rightarrow \{0, 1\}$, i.e., all the arcs have weight equal to 1. The function $\Pi(\cdot) : \mathcal{T} \rightarrow \mathbb{N}$ specifies the priority levels associated with the transitions of the net. For transitions with priority zero, delays are exponentially distributed random variables; such transitions are consequently referred to as *timed*. For transitions with priority $n \geq 1$, delays are deterministically zero; such transitions are referred to as *n-immediate*. In this paper, we only consider two priority levels; hence, transitions are simply referred to as timed and immediate. The function $w(\cdot) : \mathcal{T} \rightarrow \mathbb{R}$ specifies the stochastic component of a GSPN model. In particular, it maps transitions into real positive numbers. The quantity $w(t_j)$ is called the "rate" of transition t_j if t_j is timed, and the "weight" of transition t_j if t_j is immediate. The initial marking of a GSPN is denoted by m_0.

In the graphical representation of GSPNs, transitions are drawn as bars or white boxes depending on whether they are immediate or timed. When describing the behavior of a GSPN, we often use small ts to refer to immediate transitions and capital Ts for timed ones. Finally, a marked place is a place that contains (at least) one token drawn as a black dot. A distribution of tokens over the places of the net identifies one possible marking.

For a given transition $t_j \in \mathcal{T}$, its *preset*, *post-set*, and *inhibition set* are given by ${}^\bullet t_j = \{p_i \mid I(t_j, p_i) > 0\}$, $t_j^\bullet = \{p_i \mid O(t_j, p_i) > 0\}$, and ${}^\circ t_j = \{p_i \mid H(t_j, p_i) > 0\}$, respectively. In a similar manner, we can define the *preset* and *postset* of a given place.

For any transition t_j, using the weighted flow relation, we can define the *input vector* $\boldsymbol{i}(t_j) = [I(t_j, p_1), I(t_j, p_2), \ldots, I(t_j, p_{|\mathcal{P}|})]$, the *output vector* $\boldsymbol{o}(t_j) = [O(t_j, p_1), O(t_j, p_2), \ldots, O(t_j, p_{|\mathcal{P}|})]$, and the *inhibition vector* $\boldsymbol{h}(t_j) = [H(t_j, p_1), H(t_j, p_2), \ldots, H(t_j, p_{|\mathcal{P}|})]$. From the weighted flow relation, we can also define the *incidence matrix* $\boldsymbol{C}$ with entries $\boldsymbol{C}[i, j] = O(t_j, p_i) - I(t_j, p_i)$.

A transition t_j is *enabled* in a marking $\boldsymbol{m}$ iff $\boldsymbol{m} \geq \boldsymbol{i}(t_j)$, $\boldsymbol{m} < \boldsymbol{h}(t_j)$ and if no other transition t_h with priority higher than that of t_j exists such that $\boldsymbol{m} \geq \boldsymbol{i}(t_h)$ and $\boldsymbol{m} < \boldsymbol{h}(t_h)$.[1]

Being enabled, t_j may *occur* (or *fire*) yielding a new marking $\boldsymbol{m}' = \boldsymbol{m} + \boldsymbol{C}[\mathcal{P}, j]$, and this is denoted by $\boldsymbol{m} \xrightarrow{t_j} \boldsymbol{m}'$. The set of all the markings reachable from $\boldsymbol{m}_0$ is called the *reachability set*, and is denoted by $\mathrm{RS}(\boldsymbol{m}_0)$.

Markings that only enable timed transitions are said to be *tangible*, whereas markings that enable at least one immediate transition are said to be *vanishing*. When a vanishing marking is entered, the weights of the enabled immediate transitions are used to probabilistically select the (immediate) transition to fire. The time spent in any vanishing marking is deterministically equal to zero. When a tangible marking is entered, the rates of the transitions are used to probabilistically select one timed transition to fire. From the values of the transition rates (weights), it is possible to compute the probability that a given enabled timed (immediate) transition, say t_j, fires in a tangible (vanishing) marking $\boldsymbol{m}$:

$$\mathrm{p}\{\, t_j \mid \boldsymbol{m} \,\} = \frac{w(t_j)}{\sum\limits_{t_l \text{ enabled in } \boldsymbol{m}} w(t_l)}. \tag{1}$$

When all the transitions are timed, the class of models is simply called *Stochastic Petri Nets* (SPNs). In this case, the function $\Pi(\cdot)$ disappears from the definition and an SPN can be defined as a 7-tuple, i.e.,

$$SPN = (\mathcal{P}, \mathcal{T}, I(\cdot,\cdot), O(\cdot,\cdot), H(\cdot,\cdot), w(\cdot), \boldsymbol{m}_0).$$

Structural properties of Petri nets can be derived from the information provided by the incidence matrix. A T-*semiflow* $\boldsymbol{x}$ is a vector of nonnegative integers such that $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$. A P-*semiflow* $\boldsymbol{s}$ is a vector of nonnegative integers such that $\boldsymbol{s} \cdot \boldsymbol{C} = \boldsymbol{0}$.

The set $||\boldsymbol{x}|| = \{t \in \mathcal{T} | x_t > 0\}$ (respectively, $||\boldsymbol{s}|| = \{p \in \mathcal{P} | s_p > 0\}$) is called the *support* of $\boldsymbol{x}$ (respectively, the *support* of $\boldsymbol{s}$). A semiflow $\boldsymbol{y}$ (T- or P-semiflow) is called *minimal* if no semiflow $\boldsymbol{y}'$ exists whose support is a proper subset of the support of $\boldsymbol{y}$. T-semiflows are related with liveness properties, in particular a necessary, but not sufficient, condition for liveness is that all the transitions are covered by T-semiflows, i.e., belong to at least one T-semiflow. P-semiflows are related with boundedness properties, in particular, when all the places are covered by P-semiflows the net is said to be bounded, i.e., for any initial marking $\boldsymbol{m}_0$ the reachability set is finite. In this case, for any place $p_i \in \mathcal{P}$, it is possible to derive a bound for the number of tokens that can be contained in p_i (see [2] for details). In this paper, we consider only nets that are covered by T and P-semiflows.

Conflict relations in GSPN. In the analysis that we propose in this paper, we will use concepts related with conflicts among immediate transitions. A more comprehensive presentation and discussion of these definitions can be found in [1]. Here, we only introduce the concepts that are necessary to understand the rest of the paper.

A transition t_i is in *structural conflict* relation with transition t_j (denoted t_i **SC** t_j) iff ${}^\bullet t_i \cap {}^\bullet t_j \neq \emptyset$ **or** $t_i^\bullet \cap {}^\circ t_j \neq \emptyset$. A transition t_i is in *free-choice structural conflict* relation with transition t_j iff $\boldsymbol{i}(t_i) = \boldsymbol{i}(t_j)$ **and** $\boldsymbol{h}(t_i) = \boldsymbol{h}(t_j)$.

Another notion that we use is that of *extended conflict set* of a given transition t_j (denoted $\mathbf{ECS}(t_j)$). The **ECS** of t_j includes all the transitions that are in structural conflict with t_j as well as those whose firing may disable t_j (indirect conflict).

For the sake of simplicity and without loss of generality, all our discussion will be restricted to conflict sets composed of transitions with the same priority levels. In fact, low priority transitions included in free-choice conflict sets together with higher priority transitions are dead (they are never allowed to fire). This same observation can be used to show that the transformation algorithms developed

1. In Petri nets with priorities, the concept of *concession* has been introduced (e.g., see [1]) to capture the fact that tokens in the input and inhibitor sets of a transition satisfy the usual firing conditions, while a transition is *enabled* only if it has concession and no other transitions of higher priorities have concession in the same marking.

in this paper can be easily generalized to account for more complex situations deriving from the presence of transitions of different priority levels in nonfree choice conflict sets, since our way of dealing with these situations (as we shall see later in the paper) is that of decomposing these conflicts into a series of free-choice-like conflict sets.

3 Product-Form Results for SPNs

In this section, we review the basic concepts of the class of Stochastic Petri Nets that have a *Product-Form Solution* (PFS). The PFS for SPN criterion considered here is that proposed by Henderson et al.; more comprehensive presentations of the results related to this topic can be found in the references [4], [6], [9], [10].

The key for identifying these SPNs (PF-SPNs) is to consider the input and output vectors of the transitions to be states of a Markov Chain. This Markov Chain has been called the *routing process* [9]. We now review the basic definitions underlying the PF-SPN criterion.

Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_h$ denote the minimal T-semiflows found from the incidence matrix. The following definitions and assumptions are essential to the analysis that will be briefly presented in this section.

Definition 1 (Closed set of transitions [4]). *For $\mathcal{T}' \subseteq \mathcal{T}$, let $\mathcal{K}(\mathcal{T}')$ be the set of input and output vectors for transitions in $\mathcal{T}'$; formally, $\mathcal{K}(\mathcal{T}')$ is represented by the following expression:*

$$\mathcal{K}(\mathcal{T}') = \bigcup_{t \,\in\, \mathcal{T}'} \{\boldsymbol{i}(t) \cup \boldsymbol{o}(t)\}.$$

The subset of transitions $\mathcal{T}'$ is said to be closed if, for any $\boldsymbol{l} \in \mathcal{K}(\mathcal{T}')$, there exist $t_i, t_j \in \mathcal{T}'$ such that $\boldsymbol{l} = \boldsymbol{i}(t_i)$ and $\boldsymbol{l} = \boldsymbol{o}(t_j)$; i.e., each output vector is also an input vector for some transition in $\mathcal{T}'$, and vice versa each input vector is also an output vector.

When Definition 1 applies to the support of a T-semiflow ($||\boldsymbol{x}||$), we can state that such a T-semiflow $\boldsymbol{x}$ is a *closed support* T-semiflow. Definition 1 represents a criterion for identifying the closed support T-semiflows of an SPN. From the definition of closure, we can observe that if an initial marking is given for the system in which one of the transitions covered by a closed support T-semiflow may fire, then all the transitions covered by the same T-semiflow may fire infinitely often.

The following definition is the key to identifying (at the structural level) PF-SPNs.

Definition 2 (Structural Constraints [4]). *An SPN $(\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), w(.), \boldsymbol{m}_0)$ is said to be closed iff $\forall\, t \in \mathcal{T}$ there exists a minimal T-semiflow $\boldsymbol{x}$ such that $t \in ||\boldsymbol{x}||$, and $||\boldsymbol{x}||$ is a closed set of transitions.*

It has been proven (see [4]) that the closure property ensures that the SPN is *structurally live*. That is, if the initial marking of the closed SPN is such that at least one transition in each closed support T-semiflow is enabled, then the SPN is live. When an SPN satisfies the previous definition, the closed T-semiflows have a "special" structure:

Lemma 1 ([4]). *If $\boldsymbol{x}$ is a closed support T-semiflow, then $x_i \in \{0, 1\}$, i.e., the ith component of $\boldsymbol{x}$ can only be either 0 or 1.*

Routing Process and Closed T**-semiflows.** Definition 2 says that an SPN is closed if all its transitions are covered by closed support minimal T-semiflows. Among the minimal closed support T-semiflows we can identify a relation that can be used to derive the PFS. In the following, we denote by $\mathcal{X}_{cd}$ the set of closed support minimal T-semiflows of a net.

Definition 3 [Freely related T**-semiflows].** *Let $\mathcal{N} = (\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), w(.), \boldsymbol{m}_0)$ be a closed SPN and $\boldsymbol{x}'$, $\boldsymbol{x}''$ be two different minimal closed support T-semiflows of $\mathcal{N}$. $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are said to be freely related, denoted as $(\boldsymbol{x}', \boldsymbol{x}'') \in FR$, if there exist $t' \in ||\boldsymbol{x}'||$ and $t'' \in ||\boldsymbol{x}''||$ such that $\boldsymbol{i}(t') = \boldsymbol{i}(t'')$. The relation FR^* is the transitive closure of FR.*

It is easy to see that the relation FR^* yields a partitioning of the set of minimal closed support T-semiflows of $\mathcal{N}$ into equivalence classes that we denote by $[\boldsymbol{x}]$.[2] Since (by definition) any transition t_j cannot be part of the supports of closed T-semiflows that belong to different members of the partition FR^*, then the relation FR^* induces a partition among the transitions of the SPN. We denote by $[t_j]$ the corresponding equivalence classes, i.e.,

$$[t_j] = \left\{ \bigcup_{\mathbf{x_h} \in [\mathbf{x}]} ||\boldsymbol{x_h}|| : t_j \in ||\boldsymbol{x}|| \text{ and } \boldsymbol{x} \text{ is a min. closed } T\text{-semiflow} \right\}.$$

Using Definition 3, we can denote the routing process $\boldsymbol{y} = (\boldsymbol{y}(z), z \geq 0)$ as a Markov chain whose state space $\mathcal{S} = \{\boldsymbol{i}(t_j), t_j \in \mathcal{T}\}$ and whose transition rates are $q(\boldsymbol{i}(t_j), \boldsymbol{i}(t_m)) = \mu(\boldsymbol{i}(t_j)) P(\boldsymbol{i}(t_j), \boldsymbol{i}(t_m))$, with

$$\mu(\boldsymbol{i}(t_j)) = \sum_{n: \boldsymbol{i}(t_n) \,=\, \boldsymbol{i}(t_j)} \mu_n.$$

The term $P(\boldsymbol{i}(t_j), \boldsymbol{i}(t_m))$ is an element of a decomposable transition probability matrix that represents the probability that, if transition t_j fires in a given marking $\boldsymbol{m}$, the next marking is $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{i}(t_j) + \boldsymbol{o}(t_j) = \boldsymbol{m} - \boldsymbol{i}(t_j) + \boldsymbol{i}(t_m)$, where transition t_m becomes enabled and that is defined in the following manner

$$P(\boldsymbol{i}(t_j), \boldsymbol{i}(t_m)) = \begin{cases} \dfrac{\mu_j}{\sum_{n: \boldsymbol{i}(t_n) = \boldsymbol{i}(t_j)} \mu_n} & \text{if } [t_j] = [t_m] \text{ and } \boldsymbol{o}(t_j) = \boldsymbol{i}(t_m) \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

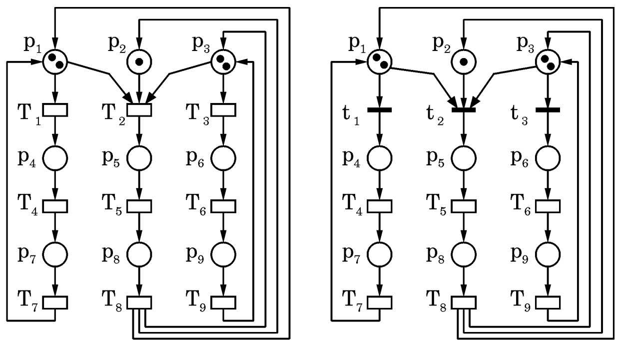
The *global balance equations* for the routing process $\boldsymbol{y}$ are:

$$v(\boldsymbol{i}(t_j)) = \sum_{t_m \,\in\, \mathcal{T}} v(\boldsymbol{i}(t_m)) P(\boldsymbol{i}(t_m), \boldsymbol{i}(t_j)), \quad \forall\, t_j \in \mathcal{T}, \quad (3)$$

that can be interpreted as the *traffic equations* for the SPN and which can be observed to be partitioned into separate systems of linear equations, one for each FR^* class. The solution of the traffic equations is the vector of visit-ratios.[3]

2. $[\boldsymbol{x}] = \{\boldsymbol{x_h} : \boldsymbol{x_h}$ is a minimal closed T-semiflow and $(\boldsymbol{x}, \boldsymbol{x_h}) \in FR^*\}$.

3. The term $v(\boldsymbol{i}(t_j))$ is called a visit ratio since it can be interpreted as the mean number of times the Markov chain enters state $\boldsymbol{i}(t_j)$ between two subsequent arrivals into an arbitrary selected reference state (of the same chain).



(a) (b)

Fig. 1. (a) A closed live SPN and (b) a closed GSPN with transient behavior.

Boucherie and Sereno proved that a necessary and sufficient condition for an SPN to have a solution for the traffic equations is that Definition 2 holds [4]. This is the first step in showing that such closed SPNs possess a product-form solution. In the analysis of PF-SPNs, the routing process plays a crucial role. The result proven in [4] establishes a relation between the routing process and the structure of the T-semiflows of the SPN. In this manner, the existence of a solution for the traffic equations of an SPN can be checked by using structural information, i.e., the T-semiflows.

4 From SPNs to GSPNs

In this section, we show that the "structural" criterion that derives from Definition 2 cannot be directly applied for the analysis of PF-GSPNs. The nets depicted in Fig. 1 outline one of the problems that may arise in GSPNs.

The SPN of Fig. 1a satisfies Definition 1 and, hence, we can state that there exists a solution for its traffic equations. One could expect that, by changing timed transitions T_1, T_2, and T_3 into immediate transitions (Fig. 1b), the structural criterion for the existence of a solution for the traffic equations can still be applied. Unfortunately, this is not true because immediate transitions in GSPNs are not just "timed transitions with null firing time," but transitions that have priority over the timed one. Because of this priority structure, the GSPN of Fig. 1b is not live. In fact, the initial marking of this net cannot be reproduced (is not a home state) and, as soon as p_1 (p_3) becomes empty because of two firings of transition t_1 (t_3), the token in p_2 becomes trapped and transitions t_2, T_5, and T_8 will not have more chances of firing, thus becoming not-live. Hence, the traffic equations for nets of this type cannot be derived through a simple extension of the invariant analysis developed for SPNs.

The structural component of a GSPN is represented by the untimed Petri net that captures its qualitative behavior disregarding time considerations. The Petri net that underlies a GSPN model is a Petri net with priorities and inhibitor arcs. Both these extensions have the effect of restricting the qualitative behavior of the model by making unfeasible certain evolutions of the net and, thus, preventing the net from reaching certain states. These constraints are not reflected at the level of the incidence-matrix, so that invariant properties that are derived from the analysis of such a matrix neglect some of the important features of these models. Focusing our attention on T-semiflows, we may recall that the results obtained from the analysis of the incidence matrix identify sequences of transitions that may bring the net back to its initial state, if firable. For basic Petri nets, the "firability" of a T-semiflow depends on the initial marking that is chosen for the net; in the case of the structural components of GSPN models, certain T-semiflows may become unfirable also because of the existence of priorities and inhibitor arcs. Disregarding these considerations, we may also identify *closed support* T-semiflows in the case of GSPNs by testing if Definition 1 applies to the sets of (timed and immediate) transitions that represent their supports. We can thus rephrase Definition 2 introduced for characterizing at the structural level closed SPNs, by saying that a closed GSPN is defined in the following way:

Definition 4 (Structural Constraints). *A GSPN* $(\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), H(.,.), \Pi(.), w(.), \boldsymbol{m}_0)$ *is said to be closed iff* $\forall\, t \in \mathcal{T}$ *there exists a minimal* T-semiflow $\boldsymbol{x}$ *such that* $t \in ||\boldsymbol{x}||$, *and* $||\boldsymbol{x}||$ *is a closed set.*

As we observed during the discussion of example of Fig. 1b, the closure property of a GSPN is not sufficient to ensure its liveness. The GSPN of Fig. 1b satisfies Definition 4. All the transitions of this GSPN are covered by closed support minimal T-semiflows; nevertheless, this GSPN has a transient behavior that it is mainly due to the priority structure. This observation is sufficient to conclude that the closure property cannot be used as the starting point for proving the existence of a produc-form solution. It follows that we need to introduce some additional constraints that avoid the cases similar to those found for the GSPN of Fig. 1b.

To this aim we introduce the following definition that captures the above characteristics.[4]

Definition 5. *A GSPN* $(\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), H(.,.), \Pi(.), w(.), \boldsymbol{m}_0)$ *is said to be a* free-killing-conflict *if any extended conflict set* $\mathbf{ECS}(t_i)$, *that involves immediate transitions having the same priority level has the following property:*

$$\forall t_{i_a}, t_{i_b} \in \mathbf{ECS}(t_i) \quad {}^\bullet t_{i_a} \cap {}^\bullet t_{i_b} \neq \emptyset \ \textbf{ or } \ t_{i_a}^\bullet \cap {}^\circ t_{i_b} \neq \emptyset \ \textbf{ or } \ {}^\circ t_{i_a} \cap t_{i_b}^\bullet \neq \emptyset. \tag{4}$$

The meaning of this definition is that when an immediate transition t_i fires, its effect is that of disabling all the other (enabled) immediate transitions of its same $\mathsf{ECS}(t_i)$.

In Section A of the appendix, we prove that a GSPN satisfying Definitions 4 and 5 is structurally live. The proof of this result is similar to that of PFS-SPNs (see [4], [8]).

5 Product-Form Solution for GSPNs

To show that, for GSPNs satisfying the constraints summarized in the following, it is possible to derive the routing process and to find a solution for the traffic equations that are the basis for the derivation of a

4. As we mentioned in the previous part of the paper, for the sake of simplicity, we only consider ordinary Petri nets. A more complex definition can be provided for nonordinary nets.

product-form expression for the stationary probability distribution, we consider GSPNs that have the following properties:

1. they are ordinary nets,
2. all the immediate transitions have the same priority,
3. they satisfy Definition 4, and
4. they satisfy Definition 5.

The first two restrictions are introduced for the sake of simplifying the derivation of the results presented in this paper. The third restriction derives from the PF-SPN analysis and it is one of the crucial points of our investigations. The fourth restriction is a sufficient condition to ensure the structural liveness (and, thus, the steady-state behavior) of the studied model.

5.1 Free-Choice Closed GSPNs

In this section, we investigate GSPNs satisfying Definition 4 where the conflicts among immediate transitions are *only* of the free-choice type.[5] This is the simplest case of GSPNs with PFS, but it allows us to illustrate, in a simple manner, several concepts that we will generalize in the case of GSPNs with nonfree-choice conflicts among immediate transitions. It is easy to see that, when all the conflicts among immediate transitions are free-choice, the GSPN satisfies Definition 4. Let $FC(T_j)$ denote the set of immediate transitions that are in free-choice conflict as the result of the firing of timed transition T_j. This set can be defined as follows:

$$FC(T_j) = \{t_i \mid \boldsymbol{o}(T_j) = \boldsymbol{i}(t_i)\}.$$

The closure property ensures that at least one of the immediate transitions enabled upon firing of timed transition T_j has an input vector that matches the output vector of T_j; on the other hand, the free-choice property of the conflict ensures that all the immediate transitions enabled upon firing of timed transition T_j have the same input vector. In this case, the GSPN can be easily transformed into an equivalent model where the conflict among immediate transitions is removed. We perform this transformation by using the concept of fusing timed and immediate transitions originally introduced in [5]. In order to transform the original GSPN model into one that is equivalent, we identify all the timed transitions that enable immediate transitions in free-choice conflict and, then, for any timed transition T_j of this type, we "fuse" T_j with each immediate free-choice transition in $FC(T_j)$. More precisely, for each transition $t_i \in FC(T_j)$, we construct a new transition with the name of transitions T_j and t_i appended to form "$T_j t_i$." The input and output vectors of $T_j t_i$ are inherited from the corresponding vectors of T_j and t_i, i.e.,

$$\boldsymbol{i}(T_j t_i) = \boldsymbol{i}(T_j) \text{ and } \boldsymbol{o}(T_j t_i) = \boldsymbol{o}(t_i). \quad (5)$$

The firing rate of this newly formed transition is obtained from the product of the firing rate of the original timed

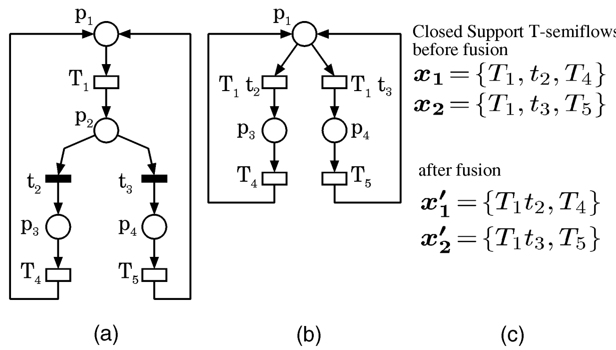


Fig. 2. Effect of the fusion on a "Case a" free choice conflict: (a) GSPN before the fusion, (b) GSPN after the fusion, and (c) closed T-semiflows before and after the fusion.

transition T_j, $w(T_j)$, and of the probability, $p(t_i)$, that transition t_i fires when it is enabled in the free-choice conflict set $FC(T)$, i.e.,

$$p(t_i) = \frac{w(t_i)}{\sum_{t_l \in FC(T_j)} w(t_l)}. \quad (6)$$

Once we have completed the fusion of T_j with all the immediate transitions of $FC(T_j)$, we can delete from the net both T_j and all the transitions of $FC(T_j)$ together with all their input places.

Repeating this process for all possible free-choice conflict sets, we produce an SPN. In Section C of the appendix, we prove that, if we apply the fusion steps described by (5) and (6) to a closed GSPN with free-choice conflicts among immediate transitions, we obtain a (G)SPN having the same tangible state space and, thus, the same CTMC as the original net. Figs. 2 and 3 show the two possible cases that the fusion process has to manage. Fig. 2a depicts the case where there exists only one timed transition T_j such that $\mathcal{F}c_i = FC(T_j)$, while Fig. 3a illustrates the case where there are several timed transitions, $T_j, \ldots, T_l$, such that $FC(T_j) = FC(T_l) = \cdots = \mathcal{F}c_i$.

Whenever the fusion process involves a pair of (timed and immediate) transitions covered by the same closed T-semiflow in the original GSPN $\mathcal{N}$, then the fused

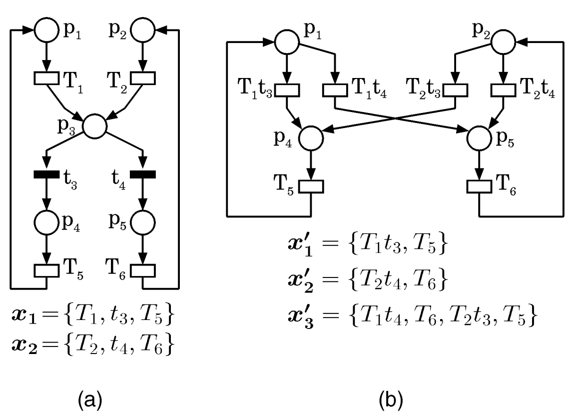


Fig. 3. Effect of the fusion on a "Case b" free choice conflict: (a) GSPN before the fusion, (b) GSPN after the fusion, and (c) closed T-semiflows before and after the fusion.

5. In the remainder of the paper, we will often focus our discussion on the impact that free-choice and nonfree-choice conflicts among immediate transitions have on the analysis of GSPNs. As we already pointed out when we introduced the concept of conflict in GSPNs, to keep the presentation simpler, unless explicitly needed, we will always refer to these conflicts without mentioning that they involve immediate transitions only.

transition of the new GSPN $\mathcal{N}'$ is covered by a T-semiflow that easily derives from the original one. Formally, we can say that

$$||\boldsymbol{x}'|| = ||\boldsymbol{x}|| \cup \{T_j t_{i_h}\} - \{T_j, t_{i_h}\},$$

where $\boldsymbol{x}$ is the closed T-semiflow that in the original GSPN $\mathcal{N}$ covers T_j and t_{i_h} and $\boldsymbol{x}'$ is the corresponding closed T-semiflow that in the new GSPN $\mathcal{N}'$ covers the fused transition $T_j t_{i_h}$.

If instead the transitions involved in the fusion process are covered by different closed T-semiflows, as it happens in the case of the conflict of Fig. 3a, then a new minimal closed T-semiflow is generated that does not correspond to a specific T-semiflow of the original net and that is expressed formally in the following way:

$$||\boldsymbol{x}'|| = ||\boldsymbol{x}_a|| \cup ||\boldsymbol{x}_b|| \cup \{T_j t_{i_h}\} - \{T_j, t_{i_h}\},$$

where $\boldsymbol{x}_a$ and $\boldsymbol{x}_b$ are two minimal closed support T-semiflows of the original GSPN $\mathcal{N}$ such that $\boldsymbol{x}_a \neq \boldsymbol{x}_b$, $T_j \in ||\boldsymbol{x}_a||, t_{i_h} \in ||\boldsymbol{x}_b||, \boldsymbol{x}'$ is the minimal closed support T-semiflow of the new GSPN $\mathcal{N}'$ that covers the fused transition $T_j t_{i_h}$.

The proof of these results can be obtained by considering the transformation of the incidence matrices of the original and of the transformed GSPNs. Section B of the appendix contains such a proof together with some additional lemmas.

Figs. 2b and 3b present the SPNs resulting from the transformation of the GSPNs of Figs. 2a and 3a. In particular, Fig. 3b points out that when we fuse transitions covered by different minimal closed support T-semiflows, we "generate" a new minimal closed support T-semiflow that is not simply a transformation of a minimal closed support T-semiflow of the original GSPN. A T-semiflow corresponding to this new minimal closed support T-semiflow also exists in the original GSPN, but it is nonminimal. In fact, in the GSPN of Fig. 3a, there is the closed (nonminimal) T-semiflow with support set $\{T_1, t_4, T_6, T_2, t_3, T_5\}$ that in the transformed net becomes $\{T_1 t_4, T_6, T_2 t_3, T_5\}$ which is both closed and minimal.

Routing in PFS GSPNs with free-choice conflicts. In Section 3, we pointed out the role played by the minimal closed T-semiflows in the produc-form analysis and, in particular, the relationship between the number of FR^* classes and the number of independent submatrices contained in the routing probability matrix. Given that the fusion process applied to the immediate transitions of free-choice conflicts may increase the number of minimal closed T-semiflows that can be found in the transformed SPN (with respect to that of the original GSPN), such a feature is worth investigation. Due to Definition 3, we can observe that, in the case of free-choice conflicts, the increasing of the number of minimal closed T-semiflows does not change the number of FR^* classes. This means that the routing probability matrix contains the same number of submatrices as the one that can be derived from the original GSPN, i.e., the transformation does not add new routing submatrices. As we will observe in the next section, this will not be the case for GSPNs with nonfree choice conflicts among immediate transitions.

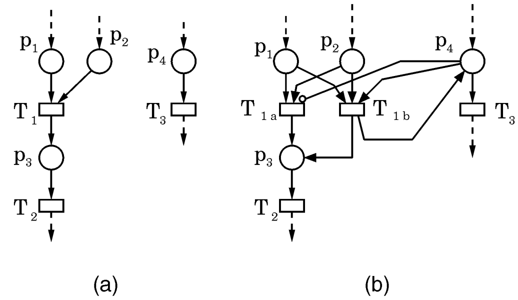


Fig. 4. Example of stratification: (a) original and (b) transformed.

Remark. We can observe that the fusion of timed and immediate transitions in case of free-choice conflicts is a transformation that preserves the closure property. That is, if we start from a closed GSPN with immediate transitions in free-choice conflict among them, by applying the fusion of timed and immediate transitions, as explained in the previous section, we obtain a (G)SPN that satisfies the closure property.

Another observation that comes from the fusion of free-choice conflicts concerns the minimal T-semiflows. As it has been shown in this section there are cases where the fusion increases the number of minimal closed T-semiflows. This is an important aspect that we also find in the treatment of nonfree choice conflicts. However, in case of free-choice conflicts, the fusion does not increase the number of FR^*-classes.

5.2 Closed GSPN with Nonfree Choice Conflicts

We now turn our attention to GSPNs satisfying Definitions 4 and 5 and with nonfree choice conflicts among immediate transitions.

The basic idea is still to show that the original GSPN can be transformed into a PF-SPN of the type discussed in [4], [6], [9], [10] and, thus, that a product-form solution for this type of model exists too. With respect to the free-choice case discussed before, the transformation is more complex, but many similarities exist between the two methods.

As a first step, we introduce a transformation called *stratification of a transition* T_j with respect to a subset of places $\mathcal{P}' \subset \mathcal{P}$. The idea behind this transformation is that of constructing the power set $\wp'$ of $\mathcal{P}'$ and of substituting T_j with a series of transitions $T_{j_1}, T_{j_2}, \ldots$, one for each member of $\wp'$. Before discussing this transformation and its use, it is useful to see an example.

Example 1 [Stratification of a transition]. Fig. 4a presents a portion of a GSPN and the effect of the stratification of transition T_1 with respect to place p_4. In this case, $\mathcal{P}' = \{p_4\}$ and $\wp' = \{\emptyset, \{p_4\}\}$. In the transformed net (Fig. 4b), instead of T_1, there are two transitions, T_{1a} and T_{1b}, that account for all the possible markings of place p_4: transition T_{1a} represents the enabling of T_1 when place p_4 is empty ($\emptyset$), while transition T_{1b} accounts for the enabling of T_1 when place p_4 is marked ($\{p_4\}$). The firing of T_{1a} is conditioned on the marking of p_4 by means of an inhibitor arc, while that of T_{1b} depends on the state of p_4 by means of a test-arc (a pair of input/output arcs). Transitions T_{1a} or T_{1b} are *mutually exclusive* (in fact, HME, according to the definitions in [1]). Given any marking of p_1, p_2, and p_4 that enables T_1 in the original GSPN and either T_{1a} or T_{1b} in the stratified one, it is easy to see that the firing of T_1 and of either T_{1a} or T_{1b} yield the same (new) marking. Obviously, the stratification can be performed with respect to a larger subset of places. For instance, if we stratify one transition with respect to two places, the transformed net will exhibit four transitions that account for all the possible markings of these places.

Given that the GSPNs under investigation are covered by P-semiflows and, hence, have finite state spaces, the stratification can always be performed without the use of inhibitor arcs, as they can be removed with the introduction of complementary places and multiple arcs (see [15] for details).

Definition 6. *Given a GSPN $\mathcal{N}$, a transition T_j, and a subset of places $\mathcal{P}'$, such that $\mathcal{P}' \cap {}^\bullet T_j = \emptyset, \mathcal{P}' \cap T_j^\bullet = \emptyset$, and $\mathcal{P}' \cap {}^\circ T_j = \emptyset$. Let us denote by $\mathbf{s}_i$ (with $i = 1, 2, \ldots, 2^{|\mathcal{P}'|}$) the ith subset of the power set of $\mathcal{P}'$. The stratification of T_j is a transformation that removes this transition from the GSPN and substitutes it with $2^{|\mathcal{P}'|}$ new transitions. The labels of these transitions are $T_j\mathbf{s}_i$ (with $i = 1, 2, \ldots, 2^{|\mathcal{P}'|}$)}, while the input, output, and inhibitor sets are:*

$$
\begin{aligned}
&{}^\bullet T_j\mathbf{s}_i = {}^\bullet T_j \cup \{\mathbf{s}_i\}, T_j\mathbf{s}_i^\bullet = T_j^\bullet \cup \{\mathbf{s}_i\},\\
&{}^\circ T_j\mathbf{s}_i = {}^\circ T_j \cup (\mathcal{P}' - \{\mathbf{s}_i\}).
\end{aligned}
$$

Since the GSPNs we consider are ordinary, the input vector $\boldsymbol{i}(T_j\mathbf{s}_i)$ is a vector having $|\mathcal{P}|$ elements, one for each place of the net. The lth component of $\boldsymbol{i}(T_j\mathbf{s}_i)$ is 1 if $p_l \in {}^\bullet T_j\mathbf{s}_i$ and 0 otherwise. In the same manner, we can obtain the output, and the inhibitor vectors, $\boldsymbol{o}(T_j\mathbf{s}_i)$, and $\boldsymbol{h}(T_j\mathbf{s}_i)$.

The stratification can be applied to timed and to immediate transitions; it has a simple impact on the properties of the GSPN since it does not change its reachability set (a formal proof can be found in Section C of the appendix), but it introduces new minimal T-semiflows that can be easily derived from those of the original net.

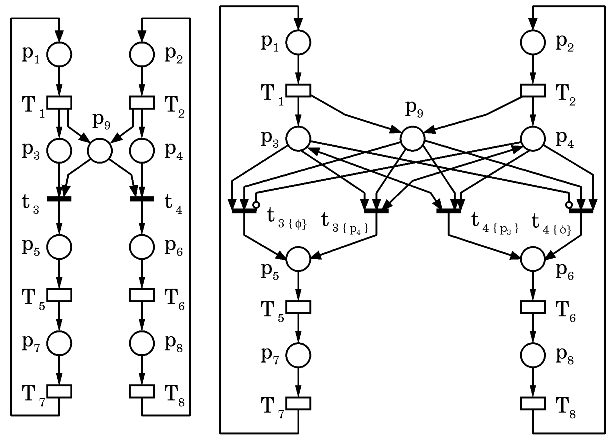
Lemma 2. *Let $\mathcal{N}$ be a GSPN, T_j one of its transitions, and $\mathcal{N}'$ the GSPN obtained by applying the stratification to transition T_j with respect to the subset of places $\mathcal{P}'$ such that $\mathcal{P}' \cap {}^\bullet T_j = \emptyset$, $\mathcal{P}' \cap T_j^\bullet = \emptyset$, and $\mathcal{P}' \cap {}^\circ T_j = \emptyset$. If we denote by $\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots,\}$ the minimal T-semiflows of $\mathcal{N}$, we can derive the set of minimal T-semiflows of $\mathcal{N}'$, $\mathcal{X}'$, as follows:*

- $\mathcal{X}' \leftarrow \emptyset$,
- $\forall\ \boldsymbol{x} \in \mathcal{X}$:
  - *if $T_j \notin ||\boldsymbol{x}||$, then $\mathcal{X}' \leftarrow \mathcal{X}' \cup \boldsymbol{x}$,*
  - *if $T_j \in ||\boldsymbol{x}||$, then for any new transition $T_j\mathbf{s}_i$, we generate a new minimal T-semiflow $\boldsymbol{x}_{\mathbf{s}_i}$ with the following support set*

$$||\boldsymbol{x}_{\mathbf{s}_i}|| = ||\boldsymbol{x}|| - \{T_j\} \cup \{T_j\mathbf{s}_i\}.$$

  - $\mathcal{X}' \leftarrow \mathcal{X}' \cup \boldsymbol{x}_{\mathbf{s}_i}$.

All the details of this proof can be found in Section B of the appendix.



(a) (b)

Fig. 5. Example of a stratification of a nonfree choice conflict among immediate transitions: (a) original and (b) transformed GSPN.

The *stratification of a nonfree choice conflict* of immediate transitions is the stratification of all these conflicting transitions. If we denote by $\mathcal{I}c = \{t_a, t_b, \ldots\}$ this set, any transition $t_l \in \mathcal{I}c$ is stratified with respect to the set of places $\mathcal{P}^{[l]}$ defined as follows:

$$
\begin{aligned}
\mathcal{P}^{[l]} = \{p_i \in \mathcal{P} : \exists t_m \in \mathcal{I}c, \text{ such that } t_m \neq t_l \text{ and}\\
p_i \in {}^\bullet t_m \text{ and } p_i \notin {}^\bullet t_l\}.
\end{aligned}
$$

Before proceeding with the formal definition of this transformation, let us discuss an illustrative example.

Example 2 [Stratification of a subset of nonfree choice conflicting transitions]. Fig. 5a depicts a GSPN with a nonfree choice conflict of the type we are considering. Let us first identify the set of conflicting transitions $\mathcal{I}c = \{t_3, t_4\}$. The first step of the stratification consists of the construction, for any transition belonging to $\mathcal{I}c$, of the set of places with respect to which we perform the stratification. For our example, we have that $\mathcal{P}^{[3]} = \{p_4\}$ and $\mathcal{P}^{[4]} = \{p_3\}$. In the transformed net (Fig. 5b), instead of transition t_3 there is the set of transitions $\{t_{3\{\emptyset\}}, t_{3\{p_4\}}\}$, that account for all the possible markings of place p_4, i.e., $t_{3\{\emptyset\}}$ represents the enabling of t_3 when place p_4 is empty, while transition $t_{3\{p_4\}}$ accounts for the enabling of t_3 when p_4 is marked. In the same manner, the set $\{t_{4\{\emptyset\}}, t_{4\{p_3\}}\}$ accounts for all the possible markings of place p_3.

We can observe that, in the transformed GSPN, transition $t_{3\{\emptyset\}}$ (respectively, $t_{4\{\emptyset\}}$) accounts for the enabling of transition t_3 (respectively, t_4) when this transition is enabled alone. On the other hand, transitions $t_{3\{p_4\}}$ and $t_{4\{p_3\}}$ account for the enabling of t_3 in conflict with t_4.

This example is sufficient to highlight the main point of this transformation: **the nonfree choice conflict has been replaced by a series of mutually exclusive equal conflicts.** In particular, immediate transition $t_{3\{\emptyset\}}$ represents the case when transition t_3 is enabled, but t_4 is not; on the contrary $t_{3\{p_4\}}$, that is in free-choice conflict with $t_{4\{p_3\}}$, represents the case when t_3 and t_4 are both enabled. In a similar manner,

TABLE 1
Transform a Non-Free-Choice Conflict Among Immediate Transitions

Procedure_TransformNFC($\mathcal{N}$, $\mathcal{I}c$)
{Transform a non free-choice conflict}
{$\mathcal{N}$ is the closed GSPN and $\mathcal{I}c = \{t_{i_1}, t_{i_2}, \ldots\}$}
{is set of the immediate transitions in non free-choice conflict}
for all $t_l \in \mathcal{I}c$ **do**
$\mathcal{P}^{[l]} \leftarrow \{p_i \in \mathcal{P} : \exists t_m \in \mathcal{I}c$, such that $t_m \neq t_l$ and $p_i \in {}^\bullet t_m$ and $p_i \notin {}^\bullet t_l\}$
Procedure_Stratify(t_l, $\mathcal{P}^{[l]}$)
{Mark t_h "to be deleted"}
end for
{Delete all the immediate transitions marked "to be deleted"}
End Procedure

Procedure_Stratify(t_l, $\mathcal{P}^{[l]}$)
{Stratify a transition with respect to the set of places $\mathcal{P}^{[l]}$}
Let $\wp$ be the power set of $\mathcal{P}^{[l]}$
for all $\mathsf{s}_i \in \wp$ **do**
{Create a transition $t_l\mathsf{s}_i$}
${}^\bullet t_l\mathsf{s}_i \leftarrow {}^\bullet t_l \cup \{\mathsf{s}_i\}$
$t_l\mathsf{s}_i^\bullet \leftarrow t_l^\bullet \cup \{\mathsf{s}_i\}$
${}^\circ t_l\mathsf{s}_i \leftarrow {}^\circ t_l \cup \{\mathcal{P}^{[l]} - \{\mathsf{s}_i\}\}$
$w(t_l\mathsf{s}_i) \leftarrow w(t_l)$
end for
End Procedure

transition $t_{4\{\emptyset\}}$ accounts for the case when t_4 is enabled, but t_3 is not.

Definition 7. *Given a GSPN $\mathcal{N}$ with all its places covered by some P-semiflow; let $\mathcal{I}c = \{t_a, t_b, \ldots, t_h\}$ be a set of immediate transitions that form a nonfree choice conflict and that satisfy the property of (4). For any transition $t_l \in \mathcal{I}c$, we denote by $\mathcal{P}^{[l]}$ the subset of places defined as:*

$$\mathcal{P}^{[l]} = \{p_i \in \mathcal{P} : \exists t_m \in \mathcal{I}c, \text{ such that } t_m \neq t_l \text{ and } p_i \in {}^\bullet t_m \text{ and } p_i \notin {}^\bullet t_l\}.$$

The stratification of the conflict $\mathcal{I}c$ is a transformation that substitutes any transition t_l belonging to $\mathcal{I}c$ with its stratification with respect to the subset of places $\mathcal{P}^{[l]}$.

The effects of the stratification on the T-semiflows of the nets follow by Lemma 2. Table 1 outlines a procedure that allows us to stratify a nonfree choice conflict among immediate transitions. We consider now the effect of the stratification of a nonfree choice conflict.

Example 3 [Stratification of nonfree choice conflicts in a closed GSPN]. Fig. 5a depicts a GSPN that satisfies Definitions 4 and 5. In this GSPN, there are two minimal closed T-semiflows , x_1 and x_2, whose support sets are $||\boldsymbol{x}_1|| = \{T_1, t_3, T_5, T_7\}$ and $||\boldsymbol{x}_2|| = \{T_2, t_4, T_6, T_8\}$, respectively.

The set of the two immediate transitions in nonfree choice conflict is $\mathcal{I}c = \{t_3, t_4\}$. For transition t_3, the stratification set of places is $\mathcal{P}^{[3]} = \{p_4\}$ (see (7), while, for transition t_4, we have that $\mathcal{P}^{[4]} = \{p_3\}$. The net obtained after the stratification is depicted in Fig. 5b. Lemma 2 tells us how to derive the minimal T-semiflows for the GSPN resulting after the stratification. From the T-semiflow of the original GSPN $||\boldsymbol{x}_1|| = \{T_1, t_3, T_5, T_7\}$, we get two minimal T-semiflows $||\boldsymbol{x}_{1_a}|| = \{T_1, t_{3\{\emptyset\}}, T_5, T_7\}$ and $||\boldsymbol{x}_{1_b}|| = \{T_1, t_{3\{p_4\}}, T_5, T_7\}$. In the same manner from the T-semiflow $||\boldsymbol{x}_2|| = \{T_2, t_4, T_6, T_8\}$, we get two minimal T-semiflows $||\boldsymbol{x}_{2_a}|| = \{T_2, t_{4\{\emptyset\}}, T_6, T_8\}$ and $|\boldsymbol{x}_{2_b}|| = \{T_2, t_{4\{p_3\}}, T_6, T_8\}$.

Looking at the net resulting after this transformation, it is possible to observe that the new GSPN has the same reachability set as the original one (in fact, this can be formally proven as it is shown in Section C of the appendix) and replaces the nonfree choice conflict of the original net with a set of "independent" (actually HME) free-choice conflicts. Unfortunately, this transformation also has the effect of destroying the closure property of the original net since the new GSPN does not satisfy Definition 1 any longer.

The first problem comes from the fact that Definition 1 does not account for inhibitor arcs. This issue can be easily overcome by observing that the GSPNs we are considering are always covered by P-semiflows and, hence, have finite state spaces. This means that the stratification can always be performed without the use of inhibitor arcs, as they can be removed with the introduction of complementary places and multiple arcs. On the other hand, this argument can also be easily accounted for by extending Definition 1 with the additional condition that all the transitions belonging to a closed set must have the same inhibition vector. However, even if Definition 1 is extended, none of the new T-semiflows of the transformed GSPN is closed! For instance, if we consider the T-semiflow $\boldsymbol{x}_{1_b}$, it is easy to note that ${}^\bullet t_{3\{p_4\}} = \{p_3, p_9, p_4\}$, and $\boldsymbol{i}(t_{3\{p_4\}}) = [0, 0, 1, 1, 0, 0, 0, 0, 1]$, but there is no other transition covered by $\boldsymbol{x}_{1_b}$ with output vector equal to this input vector. The same consideration can be made for all the other T-semiflows. In order to recover from this (undesirable) situation, an additional transformation step can be performed following the arguments that will be discussed in the next section.

Remark. Before the presentation of the additional transformations that will recover the closure property we show the result that can be obtained by performing the fusion of timed and immediate transitions (after the stratification of the nonfree choice conflict). Fig. 6 shows the SPN derived from the application of such a fusion to the net of Fig. 5b. The state space of the obtained SPN is equal to the tangible state space of the original GSPN and also the two corresponding CTMCs are identical.

Even if we provide a version of the closure definition that accounts for inhibitor arcs (in the following this will be done by means of Definitions 8 and 9), the SPN of Fig. 6 does not satisfy the (new version) of the closure property. From this, we can conclude that a direct fusion of immediate and timed transitions, in case of nonfree choice conflicts does not preserve the closure property (please note the difference between nonfree choice and free-choice conflicts). We can conclude that the SPN of Fig. 6 does not belong to the class of PF-SPNs (as defined by [4], [6], [9], [8], [10]), although, further transformations on their structure, will finally yield an SPN that is equivalent (in terms of state space and CTMC) to that depicted in Fig. 6 and that has a product-form solution.

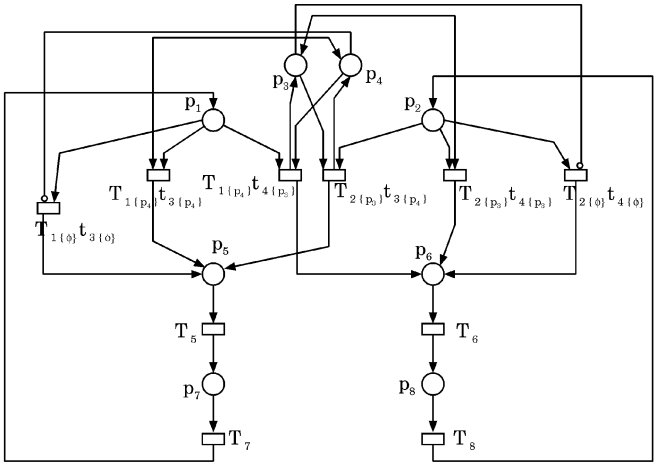


Fig. 6. Example of a nonclosed SPN obtained by a direct fusion of immediate and timed transitions.

5.2.1 Closure

The final observation of the previous section highlights the fact that a new notion of the closure property that takes into account inhibitor arcs needs to be introduced. To recover from this situation, we propose a new transformation.

We start from a closed minimal T-semiflow of the original GSPN; let $\boldsymbol{x}$ be such a T-semiflow. If there is an immediate transition $t_j \in ||\boldsymbol{x}||$ that has been stratified with respect to a set of places $\mathcal{P}^{[j]}$, we can stratify all transitions belonging to $||\boldsymbol{x}|| - \{t_j\}$ with respect to the same set of places $\mathcal{P}^{[j]}$ to obtain a new GSPN in which part of the closure feature has been reconstructed. When we repeat the procedure for all the immediate transitions that have been stratified, we obtain a final GSPN that is closed. Fig. 7 shows the GSPN resulting from the transformation of the net of Fig. 5b.

Note that, we do not state that all the T-semiflows of the GSPN of Fig. 7 are closed; we are only saying that all the transitions of this net are covered by minimal closed T-semiflows. In particular, the closed minimal support T-semiflows are:

$$||\boldsymbol{x}_{1\ \{\emptyset\}}|| = \{T_{1\{\emptyset\}}, t_{3\{\emptyset\}}, T_{5\{\emptyset\}}, T_{7\{\emptyset\}}\},$$
$$||\boldsymbol{x}_{1\ \{p_4\}}|| = \{T_{1\{p_4\}}, t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\},$$
$$||\boldsymbol{x}_{2\ \{\emptyset\}}|| = \{T_{2\{\emptyset\}}, t_{4\{\emptyset\}}, T_{6\{\emptyset\}}, T_{8\{\emptyset\}}\}, \text{ and}$$
$$||\boldsymbol{x}_{2\ \{p_4\}}|| = \{T_{2\{p_3\}}, t_{4\{p_3\}}, T_{6\{p_3\}}, T_{8\{p_3\}}\}.$$

The first step for deriving a formalization of this transformation is to provide a version of Definition 1 and Definition 2 that accounts for inhibitor arcs.

Definition 8 (Closed set of transitions with inhibitor arcs). *For $\mathcal{T}' \subseteq \mathcal{T}$, let $\mathcal{K}(\mathcal{T}')$ be the set of input and output vectors for all the transitions in $\mathcal{T}'$; formally, $\mathcal{K}(\mathcal{T}')$ is represented by the following expression:*

$$\mathcal{K}(\mathcal{T}') = \bigcup_{t \in \mathcal{T}'} \{\boldsymbol{i}(t) \cup \boldsymbol{o}(t)\}.$$

The subset of transitions $\mathcal{T}'$ is said to be h-closed if, for any $l \in \mathcal{K}(\mathcal{T}')$, there exist $t_i, t_j \in \mathcal{T}'$ such that $\boldsymbol{l} = \boldsymbol{i}(t_i)$, $\boldsymbol{l} = \boldsymbol{o}(t_j)$, and $\boldsymbol{h}(t_i) = \boldsymbol{h}(t_j)$; i.e., each output vector is also an input vector for some transition in $\mathcal{T}'$ and, vice versa, each input vector is also an output vector and all the transitions belonging to $\mathcal{T}'$ have the same inhibitor vector.

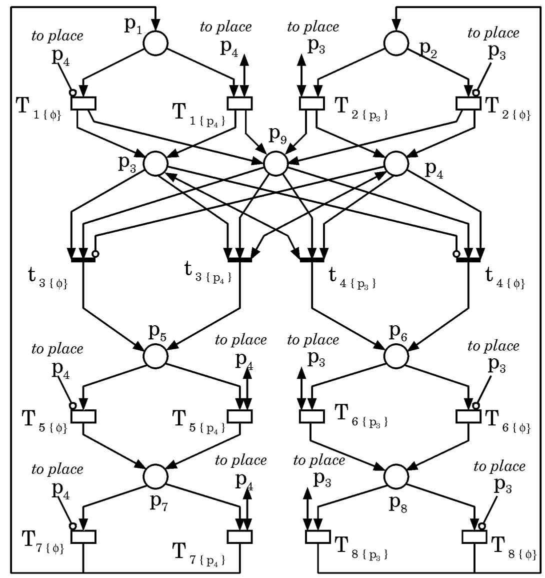


Fig. 7. Closure of the GSPN of Fig. 5b.

Definition 9 (Structural constraints with inhibitor arcs). *A GSPN $(\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), H(.,.), \Pi(.), w(.), \boldsymbol{m}_0)$ is said to be h-closed iff $\forall\, t \in \mathcal{T}$ there exists a minimal T-semiflow $\boldsymbol{x}$ such that $t \in ||\boldsymbol{x}||$, and $||\boldsymbol{x}||$ is an h-closed set.*

To show the effects of this transformation and to point out its consequences, we focus our attention on closed T-semiflows (of the original GSPN) that cover at most one immediate transition belonging to a nonfree choice conflict.[6] The procedure reported in Table 2 allows us to transform a GSPN obtained by using the procedure of Table 1 into a GSPN that satisfies Definition 9.

The application of the procedure of Table 2 to the GSPN of Fig. 5 yields the GSPN depicted in Fig. 7.

Remark. It is interesting to point out the effect of these transformations (stratification of a nonfree choice conflict and stratification for obtaining the closure) on the closed T-semiflows: each minimal closed T-semiflow of the original GSPN has been replaced by a series of minimal h-closed T-semiflows that account for all the possible markings of the places involved in the stratification of the nonfree choice conflict. For instance, if we compare the GSPNs of Figs. 7 and 5a, the original minimal closed T-semiflow $||\boldsymbol{x}_1|| = \{T_1, t_3, T_5, T_7\}$ has been replaced in the GSPN of Fig. 7 with two minimal h-closed T-semiflows $||\boldsymbol{x}_{1\ \{\emptyset\}}|| = \{T_{1\{\emptyset\}}, t_{3\{\emptyset\}}, T_{5\{\emptyset\}}, T_{7\{\emptyset\}}\}$ and $||\boldsymbol{x}_{1\ \{p_4\}}|| = \{T_{1\{p_4\}}, t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\}$. The first accounts for the cases when p_4 is not marked, while the second refers to the cases when p_4 is marked. The same situation happens for the original minimal closed T-semiflow $||\boldsymbol{x}_2|| = \{T_2, t_4, T_6, T_9\}$.

6. In Section 5.2.3, this restriction will be removed and the analysis will be extended to closed T-semiflows that cover more than a single immediate transition in nonfree choice conflict.

TABLE 2
Procedure that Yields an h-Closed GSPN

Procedure_Close($\mathcal{N}$, $\mathcal{N}'$)
  {Let $\mathcal{X}$ be the set of closed minimal T-semiflows of $\mathcal{N}$}
  for all $\boldsymbol{x} \in \mathcal{X}$ **do**
    for all $t_l \in ||\boldsymbol{x}||$ **do**
      if t_l is an immediate transition involved in a non-free choice conflict $\mathcal{I}c$ **then**
        $\mathcal{P}^{[l]} \leftarrow \{p_i \in \mathcal{P} : \exists t_m \in \mathcal{I}c,$ such that $t_m \neq t_l$ and $p_i \in {}^\bullet t_m$ and $p_i \notin {}^\bullet t_l\}$
        for all $T_j \in ||\boldsymbol{x}|| - \{t_l\}$ **do**
          Procedure_Stratify(T_j, $\mathcal{P}^{[l]}$)
        end for
      end if
    end for
  end for
End Procedure

5.2.2 Fusion of Immediate and Timed Transitions

Once we have obtained an equivalent GSPN that is closed, we can perform the final transformation step that reduces the GSPN to a PF-SPN. The fusion of timed and immediate transitions uses a method similar to that of Section 5.1. The only difference is the construction of the sets $\{\mathcal{F}c_1, \mathcal{F}c_2, \ldots,\}$, where we must account for the inhibition vectors too. In this case, for the immediate transitions of a given $\mathcal{F}c_i = \{t_{i_1}, t_{i_2}, \ldots\}$, we have that $\boldsymbol{i}(t_{i_1}) = \boldsymbol{i}(t_{i_2}) = \ldots$ and $\boldsymbol{h}(t_{i_1}) = \boldsymbol{h}(t_{i_2}) = \ldots$. The construction of the new h-closed minimal T-semiflows is exactly the same as that presented in Section 5.1. The derivation of the new h-closed T-semiflows follows the results presented in Section B of the appendix. The final result is an SPN that is h-closed. In Fig. 8, we present the SPN transformation of the GSPN presented in Fig. 5a.

Some interesting considerations can be made on the new h-closed T-semiflows of the SPN of Fig. 8 by comparing them with the original closed T-semiflows of the GSPN of Fig. 5a. Table 3 shows the h-closed T-semiflows of the SPN of Fig. 8 and the T-semiflows of the GSPN of Fig. 5a. In Table 3, we split the new h-closed minimal T-semiflows into three sets.

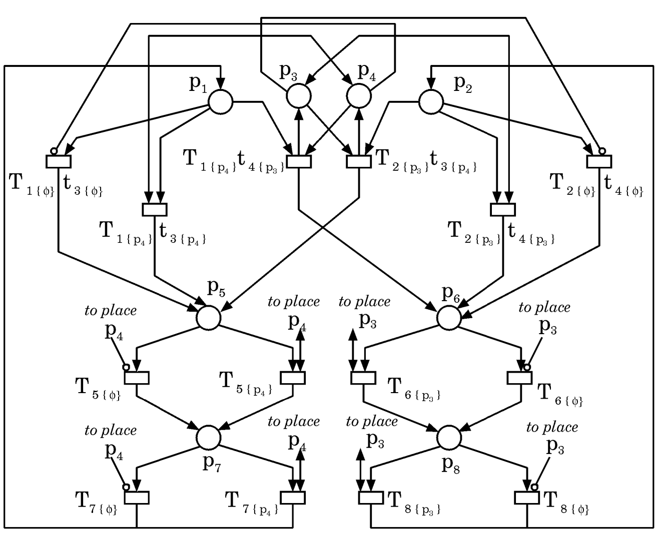


Fig. 8. The h-closed SPN transformation of the GSPN depicted in Fig. 5a.

TABLE 3
Minimal Closed T-semiflows of the GSPN of Fig. 5a (First Column), and Minimal h-Closed T-Semiflows of the Transformed SPN (Second Column)

Closed T-semiflows of the GSPN of Fig. 5(a)	h-closed T-semiflows of the SPN of Figure 8
$\{T_1, t_3, T_5, T_7\}$	$\{T_{1\{\emptyset\}} t_{3\{\emptyset\}}, T_{5\{\emptyset\}}, T_{7\{\emptyset\}}\}$ $\{T_{1\{p_4\}} t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\}$
$\{T_2, t_4, T_6, T_8\}$	$\{T_{2\{\emptyset\}} t_{4\{\emptyset\}}, T_{6\{\emptyset\}}, T_{8\{\emptyset\}}\}$ $\{T_{2\{p_3\}} t_{4\{p_3\}}, T_{6\{p_3\}}, T_{8\{p_3\}}\}$
	$\{T_{1\{p_4\}} t_{4\{p_3\}}, T_{6\{p_3\}}, T_{8\{p_3\}},$ $T_{2\{p_3\}} t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\}$

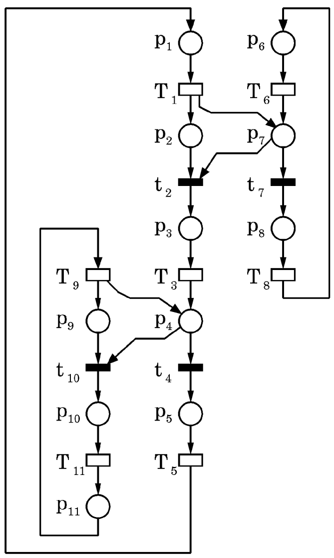
The first set contains the h-closed minimal T-semiflows that represent the behavior of the closed T-semiflow of the GSPN $\{T_1, t_3, T_5, T_7\}$ conditioned with respect to place p_4 (marked versus nonmarked). The second set contains the h-closed minimal T-semiflows that represent the behavior of the closed T-semiflow of the GSPN $\{T_2, t_4, T_6, T_8\}$, conditioned with respect to place p_3 (marked versus nonmarked). The last set contains only one h-closed minimal T-semiflow. This has been originated by the fusion process and represents the closed (nonminimal) T-semiflow of the GSPN having the following support set $\{T_1, t_4, T_6, T_8, T_2, t_3, T_5, T_7\}$. This new h-closed minimal T-semiflow is the important consequence of the presence of nonfree choice conflicts. When the GSPN contains such conflicts, the number of T-semiflows that have to be considered for deriving the routing process (closed or h-closed T-semiflows) is greater than that derived from the incidence matrix of the original GSPN. In particular, some of the minimal new T-semiflows that we must consider are nonminimal T-semiflows of the original GSPN. Said in a different way, this means that when a GSPN of this type contains a nonfree choice conflict among immediate transitions its structural analysis, to be complete, needs to take into consideration certain types of nonminimal T-semiflows that we now know how to characterize on the basis of the discussion performed in this paper.

5.2.3 Closed GSPNs with Nonfree Choice Conflicts in Cascade

Let $\mathcal{N}$ be a closed GSPN satisfying Definition 5, with $\mathcal{I}c = \{t_{i_1}, \ldots, t_{i_h}\}$ and $\mathcal{I}c' = \{t_{j_1}, \ldots, t_{j_k}\}$ being two sets of nonfree choice conflicting immediate transitions such that $\mathcal{I}c \cap \mathcal{I}c' = \emptyset$. If there exists a closed T-semiflow $\boldsymbol{x}$ that covers a transition $t_{i_a} \in \mathcal{I}c$ and a transition $t_{j_b} \in \mathcal{I}c'$, i.e., $t_{i_a} \in ||\boldsymbol{x}||$ and $t_{j_b} \in ||\boldsymbol{x}||$, with $t_{i_a} \in \mathcal{I}c$ and $t_{j_b} \in \mathcal{I}c'$, we need to consider all the possible interleavings of the stratification processes.

For any transition $t_j \in \mathcal{T}$ (timed or immediate) we define the set of places $\mathcal{P}^{[j]}$. For all the immediate transitions t_l, involved in a nonfree choice conflict, we update $\mathcal{P}^{[l]}$ according to (7) (set of control places). We then extend the definition of this set of places at the level of the minimal closed T-semiflows. In particular, for any minimal closed T-semiflow $\boldsymbol{x}$, we define the set of places

$$\mathcal{P}^{[\boldsymbol{x}]} = \bigcup_{t_l \in ||\boldsymbol{x}||} \mathcal{P}^{[l]}. \quad (8)$$



$$||\boldsymbol{x_1}|| = \{T_1, t_2, T_3, t_4, T_5\}$$
$$||\boldsymbol{x_2}|| = \{T_6, t_7, T_8\}$$
$$||\boldsymbol{x_3}|| = \{T_9, t_{10}, T_{11}\}$$
$$\mathcal{I}c_1 = \{t_2, t_7\}$$
$$\mathcal{I}c_2 = \{t_4, t_{10}\}$$
$$\mathcal{P}^{[\boldsymbol{x_1}]} = \{p_9\}$$
$$\mathcal{P}^{[\boldsymbol{x_2}]} = \{p_2\}$$
$$\mathcal{P}^{[\boldsymbol{x_3}]} = \{\emptyset\}$$

Fig. 9. A closed GSPN with T-semiflows that involve two nonfree choice conflicts.

As the next step, we introduce a relation on the set of minimal closed T-semiflows. We define that two minimal closed T-semiflows $\boldsymbol{x}'$ and $\boldsymbol{x}''$ are *adjacent* (denoted $\boldsymbol{x}'A\boldsymbol{x}''$) in the following manner

$$\begin{aligned}&\boldsymbol{x}'\mathbf{A}\,\boldsymbol{x}'' \text{ iff } \exists\, t_a \in ||\boldsymbol{x}'|| \;\wedge \exists\, t_b \in ||\boldsymbol{x}''|| \;\wedge\\ &t_a \text{ and } t_b \text{ imm. trans. in nonfree choice conflict } \wedge\\ &\left(\mathcal{P}^{[\boldsymbol{x}']} \cap {}^\bullet t_b \neq \emptyset\right) \;\wedge \left(\mathcal{P}^{[\boldsymbol{x}'']} \cap {}^\bullet t_a = \emptyset\right).\end{aligned} \tag{9}$$

Then, assume that the adjacent relation is transitive, i.e., there can be three minimal closed T-semiflows $\boldsymbol{x}', \boldsymbol{x}''$, and $\boldsymbol{x}'''$ such that $\boldsymbol{x}'\ \mathbf{A}\ \boldsymbol{x}''$ and $\boldsymbol{x}''\ \mathbf{A}\ \boldsymbol{x}'''$. In this case, we derive that $\boldsymbol{x}'\ \mathbf{A}\ \boldsymbol{x}'''$. We denote the transitive closure of the adjacent relation by means of $\mathbf{A}^*$ By using the relation $\mathbf{A}^*$, we define the *final* set of control places for any minimal closed T-semiflow $\boldsymbol{x}$ (denoted as $\mathcal{P}^{[\boldsymbol{x}]^{\mathbf{A}}}$) as:

$$\mathcal{P}^{[\boldsymbol{x}]^{\mathbf{A}}} = \mathcal{P}^{[\boldsymbol{x}]} \bigcup_{\boldsymbol{x}\ \mathbf{A}^*\ \boldsymbol{x}'} \mathcal{P}^{[\boldsymbol{x}']}. \tag{10}$$

Example 4 [T-semiflows that involve several nonfree choice conflicts]. Fig. 9 shows a closed GSPN. In this net, there are three minimal closed T-semiflows, and two disjoint sets of immediate transitions in nonfree conflict. We can see that $\boldsymbol{x}_2\ \mathbf{A}\ \boldsymbol{x}_1$ because $\exists\, t_7 \in ||\boldsymbol{x}_2||$, $t_2 \in ||\boldsymbol{x}_1||$, with $\mathcal{P}^{[\boldsymbol{x}_2]} \cap {}^\bullet t_2 = \{p_2\}$; and $\boldsymbol{x}_1\ \mathbf{A}\ \boldsymbol{x}_3$ because $\exists\, t_4 \in ||\boldsymbol{x}_1||$, $t_{10} \in ||\boldsymbol{x}_3||$ with $\mathcal{P}^{[\boldsymbol{x}_1]} \cap {}^\bullet t_{10} = \{p_9\}$. We can derive that $\boldsymbol{x}_2\ \mathbf{A}^*\ \boldsymbol{x}_1$ and $\boldsymbol{x}_2\ \mathbf{A}^*\ \boldsymbol{x}_3$ and using (10), we can derive the *final* set of control places for all the minimal closed T-semiflows. In particular, we have that $\mathcal{P}^{[\boldsymbol{x}_1]^{\mathbf{A}}} = \{p_9\}$, $\mathcal{P}^{[\boldsymbol{x}_2]^{\mathbf{A}}} = \{p_2, p_9\}$, and $\mathcal{P}^{[\boldsymbol{x}_3]^{\mathbf{A}}} = \{\emptyset\}$.

To derive the h-closed GSPN in this case we need to set $\mathcal{P}^{[l]} \leftarrow \mathcal{P}^{[\boldsymbol{x}]^{\mathbf{A}}}$, where $\boldsymbol{x}$ is the minimal closed T-semiflow that covers transition t_l.[7] The h-closed GSPN is obtained by applying the **Procedure_Stratify** (t_l, $\mathcal{P}^{[l]}$), defined in Table 1, to all transitions $t_l \in \mathcal{T}$. A procedure that yields an h-closed GSPN based on the previous considerations is presented in Table 4.

TABLE 4
Procedure that Yields an h-Closed GSPN

Procedure_GeneralNFC($\mathcal{N}$)
  {Let $\mathcal{I}c_1, \mathcal{I}c_2, \ldots,$ be the disjoint sets of immediate transitions}
  {in non-free choice conflict and $\mathcal{X}_{\text{cd}}$ the set of minimal closed T-semiflows of $\mathcal{N}$}
  for all $t_l \in \mathcal{T}$ **do**
    $\mathcal{P}^{[l]} \leftarrow \emptyset$
    if t_l is an immediate transition involved in a non-free choice conflict $\mathcal{I}c_h$ **then**
      $\mathcal{P}^{[l]} \leftarrow \left\{p_i \in \mathcal{P} : \exists t_m \in \mathcal{I}c_h, \text{ such that } t_m \neq t_l \text{ and } p_i \in {}^\bullet t_m \text{ and } p_i \notin {}^\bullet t_l\right\}$
    end if
  end for
  for all $\boldsymbol{x} \in \mathcal{X}_{\text{cd}}$ **do**
    $\mathcal{P}^{[\mathbf{x}]} \leftarrow \emptyset$
    for all $t_j \in ||\boldsymbol{x}||$ **do**
      $\mathcal{P}^{[\mathbf{x}]} \leftarrow \mathcal{P}^{[\mathbf{x}]} \cup \mathcal{P}^{[j]}$
    end for
  end for
  for all $\boldsymbol{x} \in \mathcal{X}_{\text{cd}}$ **do**
    {Find all $\boldsymbol{x'}$ such that $\boldsymbol{x}\,\mathsf{A}^*\,\boldsymbol{x'}$}
  end for
  for all $\boldsymbol{x} \in \mathcal{X}_{\text{cd}}$ **do**
    $\mathcal{P}^{[\mathbf{x}]^{\mathsf{A}}} \leftarrow \mathcal{P}^{[\mathbf{x}]} \bigcup_{\mathbf{x}\,\mathsf{A}^*\,\mathbf{x}'} \mathcal{P}^{[\mathbf{x}']}$
  end for
  for all $t_l \in \mathcal{T}$ **do**
    $\mathcal{P}^{[l]} \leftarrow \mathcal{P}^{[\mathbf{x}]^{\mathsf{A}}}\ \{t_l \in ||\boldsymbol{x}||\}$
    Procedure_Stratify(t_l, $\mathcal{P}^{[l]}$)
  end for

This is a general version that allows us to Manage T-semiflows with more than a single immediate transition in nonfree choice conflict.

Before illustrating the structure of the new h-closed T-semiflows generated from the GSPN of Fig. 9, we have to point out that we can avoid all the steps resulting from (8), (9), and (10). We can stratify all the transitions of the GSPN with respect to the union of all control places. The result in this case is still an h-closed GSPN, the only difference is that with a simplified stratification procedure it could be possible to generate redundant h-closed T-semiflows.

From the GSPN of Fig. 9, after the stratification step, we obtain an h-closed GSPN having the following h-closed minimal T-semiflows (we label the new h-closed minimal T-semiflows by using the same idea of the labeling of the transitions):

$$\begin{aligned}
||\boldsymbol{x}_{1,\{\emptyset\}}|| &= \{T_{1\{\emptyset\}}, t_{2\{\emptyset\}}, T_{3\{\emptyset\}}, t_{4\{\emptyset\}}, T_{5\{\emptyset\}}\}\\
||\boldsymbol{x}_{1,\{p_9\}}|| &= \{T_{1\{p_9\}}, t_{2\{p_9\}}, T_{3\{p_9\}}, t_{4\{p_9\}}, T_{5\{p_9\}}\}\\
||\boldsymbol{x}_{2,\{\emptyset\}}|| &= \{T_{6\{\emptyset\}}, t_{7\{\emptyset\}}, T_{8\{\emptyset\}}\}\\
||\boldsymbol{x}_{2,\{p_2\}}|| &= \{T_{6\{p_2\}}, t_{7\{p_2\}}, T_{8\{p_2\}}\}\\
||\boldsymbol{x}_{2,\{p_9\}}|| &= \{T_{6\{p_9\}}, t_{7\{p_9\}}, T_{8\{p_9\}}\}\\
||\boldsymbol{x}_{2,\{p_2,p_9\}}|| &= \{T_{6\{p_2,p_9\}}, t_{7\{p_2,p_9\}}, T_{8\{p_2,p_9\}}\}\\
||\boldsymbol{x}_3|| &= \{T_9, t_{10}, T_{11}\}.
\end{aligned}$$

It is interesting to point out the meaning of the new h-closed T-semiflows. For instance the T-semiflow $\boldsymbol{x}_{2,\{\emptyset\}}$ accounts for

7. If there is more that one minimal closed T-semiflow that covers t_l, we choose the one with the largest set of places $\mathcal{P}^{[\boldsymbol{x}]^{\mathbf{A}}}$.

	$\boldsymbol{i}(T_1)$	$\boldsymbol{i}(T_3)$	$\boldsymbol{i}(T_5)$	$\boldsymbol{i}(T_7)$	$\boldsymbol{i}(T_2)$	$\boldsymbol{i}(T_4)$	$\boldsymbol{i}(T_6)$	$\boldsymbol{i}(T_8)$
$\boldsymbol{i}(T_1)$	0	1	0	0				
$\boldsymbol{i}(T_3)$	0	0	1	0				
$\boldsymbol{i}(T_5)$	0	0	0	1				
$\boldsymbol{i}(T_7)$	1	0	0	0				
$\boldsymbol{i}(T_2)$					0	1	0	0
$\boldsymbol{i}(T_4)$					0	0	1	0
$\boldsymbol{i}(T_6)$					0	0	0	1
$\boldsymbol{i}(T_8)$					1	0	0	0

(a)

	$\boldsymbol{i}(T_{1\{\emptyset\}}t_{3\{\emptyset\}})$	$\boldsymbol{i}(T_{5\{\emptyset\}})$	$\boldsymbol{i}(T_{7\{\emptyset\}})$	$\boldsymbol{i}(T_{2\{\emptyset\}}t_{4\{\emptyset\}})$	$\boldsymbol{i}(T_{6\{\emptyset\}})$	$\boldsymbol{i}(T_{8\{\emptyset\}})$	$\boldsymbol{i}(T_{1\{p_4\}}t_{3\{p_4\}})$	$\boldsymbol{i}(T_{5\{p_4\}})$	$\boldsymbol{i}(T_{7\{p_4\}})$	$\boldsymbol{i}(T_{2\{p_3\}}t_{4\{p_3\}})$	$\boldsymbol{i}(T_{6\{p_3\}})$	$\boldsymbol{i}(T_{8\{p_3\}})$
$\boldsymbol{i}(T_{1\{\emptyset\}}t_{3\{\emptyset\}})$	0	1	0	0	0	0	0	0	0	0	0	0
$\boldsymbol{i}(T_{5\{\emptyset\}})$	0	0	1	0	0	0	0	0	0	0	0	0
$\boldsymbol{i}(T_{7\{\emptyset\}})$	1	0	0	0	0	0	0	0	0	0	0	0
$\boldsymbol{i}(T_{2\{\emptyset\}}t_{4\{\emptyset\}})$	0	0	0	0	1	0	0	0	0	0	0	0
$\boldsymbol{i}(T_{6\{\emptyset\}})$	0	0	0	0	0	1	0	0	0	0	0	0
$\boldsymbol{i}(T_{8\{\emptyset\}})$	0	0	0	1	0	0	0	0	0	0	0	0
$(T_{1\{p_4\}}t_{3\{p_4\}})$	0	0	0	0	0	0	0	α	0	0	β	0
$\boldsymbol{i}(T_{5\{p_4\}})$	0	0	0	0	0	0	0	0	1	0	0	0
$\boldsymbol{i}(T_{7\{p_4\}})$	0	0	0	0	0	0	1	0	0	0	0	0
$\boldsymbol{i}(T_{2\{p_3\}}t_{4\{p_3\}})$	0	0	0	0	0	0	0	γ	0	0	δ	0
$\boldsymbol{i}(T_{6\{p_3\}})$	0	0	0	0	0	0	0	0	0	0	0	1
$\boldsymbol{i}(T_{8\{p_3\}})$	0	0	0	0	0	0	0	0	0	1	0	0

(b)

Fig. 10. (a) Routing matrix of an SPN obtained from the GSPN of Fig. 7a by simply replacing transitions t_3 and t_4 with two timed transitions. (b) Routing matrix underlying the GSPN of Fig. 7a and constructed on the basis of the list of minimal h-closed T-semiflows of Table 3.

the behavior of the original T-semiflow $\boldsymbol{x}_2$ when places p_2 and p_9 are non marked, the T-semiflow $\boldsymbol{x}_{2,\{p_2\}}$ accounts for the behavior of $\boldsymbol{x}_2$ when p_2 is marked but p_9 is not marked, in the same manner $\boldsymbol{x}_{2,\{p_9\}}$ accounts for the behavior of $\boldsymbol{x}_2$ when p_9 is marked but p_2 is not marked, and $\boldsymbol{x}_{2,\{p_2,p_9\}}$ accounts for the behavior of $\boldsymbol{x}_2$ when both p_2 and p_9 are marked. In the same manner the other new h-closed T-semiflows account for the possible marking situations of the sets of places derived by using (8), (9), and (10).

The final step, that is the fusion of timed and immediate transitions, does not require different treatment with respect to the procedure presented in Section 5.2.2.

5.2.4 Routing in PFS GSPNs with Nonfree-Choice Conflicts

The construction of the routing matrix for GSPNs with nonfree choice conflicts is based on the minimal h-closed T-semiflows that have been computed by using all the steps presented in the previous sections. When all the minimal h-closed T-semiflows are obtained, by using Definition 3, we can derive the FR^* classes and from them the routing matrix. It is interesting to compare the structure of the routing matrix of a PF-GSPN with nonfree choice conflicts with that of a similar SPN obtained by replacing the immediate with timed transitions. Fig. 10a shows the routing matrix of the SPN obtained by replacing, in the GSPN of Fig. 7a, transitions t_3 and t_4 with two timed transitions. The FR^* classes of this SPN can be obtained by the closed T-semiflows of the original GSPN (first column of Table 3) by relabeling transition t_3 and t_4.

Fig. 10b shows the routing matrix of the GSPN obtained through the transformation presented in the previous sections (we can derive the FR^* classes by the list of the minimal h-closed T-semiflows listed in Table 3). The FR^* classes to build this routing matrix are obtained by the h-closed minimal T-semiflows of the transformed SPN (second column of Table 3). The third FR^* class is derived by applying Definition 3 and comprises the following h-closed minimal T-semiflows: $\{T_{1\{p_4\}}t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\}$, $\{T_{2\{p_3\}}t_{4\{p_3\}}, T_{6\{p_3\}}, T_{8\{p_3\}}\}$, and

$$\{T_{1\{p_4\}}t_{4\{p_3\}}, T_{6\{p_3\}}, T_{8\{p_3\}}, T_{2\{p_3\}}t_{3\{p_4\}}, T_{5\{p_4\}}, T_{7\{p_4\}}\}.$$

Note that, since $\boldsymbol{i}(T_{1\{p_4\}}t_{3\{p_4\}}) = \boldsymbol{i}(T_{1\{p_4\}}t_{4\{p_3\}})$ and $\boldsymbol{i}(T_{2\{p_3\}}t_{4\{p_3\}}) = \boldsymbol{i}(T_{2\{p_3\}}t_{3\{p_4\}})$ in the routing matrix of Fig. 10b there are only $\boldsymbol{i}(T_{1\{p_4\}}t_{3\{p_4\}})$ and $\boldsymbol{i}(T_{2\{p_3\}}t_{4\{p_3\}})$. In this matrix, the values of α, β, γ, and δ are computed by using (2). It is interesting to point out that, although we can compute the routing matrix of a PF-GSPN on the transformed SPN by computing all the minimal T-semiflows and by selecting those that satisfy the h-closure property, we can avoid the recomputation of all minimal T-semiflows because all the h-closed minimal T-semiflows that are needed for the computation of the routing matrix can be obtained by applying the transformation rules presented in the previous sections.

6 Product-Form Results for GSPNs

The SPNs resulting from the different transformation steps introduced in the previous sections are characterized by what we called the h-closedness property. The name of this property points out the fact that it is a generalization of the closedness property introduced for PFS SPNs (see [4]). We can conclude that the transformations proposed in this paper allow the identification of the routing process embedded in the GSPNs satisfying Definitions 4 and 5 and, thus, of finding the solution for the traffic equations. We can also observe that all the derivations performed in the previous sections can be interpreted as the steps of a lengthy proof of the following:

Theorem 1. *Every GSPN that satisfies Definitions 4 and 5 admits a solution for its traffic equations.*

The existence of a solution for the traffic equations is not a sufficient condition to assert a PFS for the GSPN, but the meaning of this theorem is that the GSPNs that satisfy the criteria expressed by Definitions 4 and 5 are structurally suitable for a product-form solution. To determine whether these GSPNs admit PFS, we need to apply the results derived in [6] and [8] to the equivalent SPNs constructed using the procedure described in this paper. When this is the case, it is possible to obtain performance measures by using the computational algorithms for PF-SPNs proposed in [13], [14].

7 Conclusion and Further Developments

In this paper, we have shown that closed GSPNs admit a Product-Form solution as long as they satisfy some additional constraints. The proof of this claim has been carried out by showing that, given a GSPN of this type, an equivalent PF-SPN can be constructed using a sequence of transformation steps driven by the structural analysis of the GSPN performed disregarding time considerations.

The identification of the routing process embedded in PF-GSPNs has been inspired by the similar analysis conducted for PF-SPNs, but allows us to point out some important differences among the properties of these two classes of stochastic Petri nets:

- The structural constraints proposed for the PF-SPNs are not enough to ensure that the GSPNs that satisfy them have product-form solutions too. Indeed, in some cases, they do not even ensure the liveness of the net under investigation (in the case of SPNs, these constraints ensure the structural liveness [4], [8]).
- The analysis cannot be based on minimal T-semiflows only, but additional nonminimal (in the original GSPN) T-semiflows that cover the immediate transitions of nonfree choice conflicts of the GSPN have to be considered.

These new T-semiflows could be computed from the standard structural analysis of the equivalent SPN that is constructed by the algorithm that implements the transformation steps discussed in the paper. In this paper, we have shown, however, that a direct method exists that builds on the results of the analysis of the structural component of the GSPN model avoiding the complexity of the equivalent SPN. At the present stage of our research, the incidence matrix of the equivalent SPN model that is much larger than that of the original GSPN is only used to test the validity of the Rank theorem [6] that is needed to prove that SPN models whose traffic equations admit a solution are also Product-Form. This way of deriving important invariant properties of GSPNs from the analysis of their structural components augmented with considerations on the priority of immediate transitions shows a direction of research that can be pursued in order to obtain a structural analysis of Petri nets with priorities whose qualitative behaviors are similar to those of GSPNs.

Finally, we may observe that, in the process of constructing a PF-SPN from a closed GSPN, there exist certain SPNs that have a product-form solution even if they do not satisfy the produc-form constraints defined in [4], [6], [9]. This can be seen by considering the SPN depicted in Fig. 6. This SPN does not satisfy the produc-form constraints but, as we showed in the paper, by defining some transformations on the net it is possible to obtain an equivalent model that satisfies the PF-SPN constraints. One of the ongoing research efforts on this topic is the derivation of other transformation rules that allow us to **recognize** producform models that "apparently" do not satisfy the classical constraints defined for PF-SPNs.

Appendix 1
Functional Properties of Closed GSPNs

A GSPN net $\mathcal{N}$ is *live* when every transition can ultimately fire from every reachable marking and it is *structurally live* when *there exists* an initial marking $\boldsymbol{m}_0$ such that $(\mathcal{N}, \boldsymbol{m}_0)$ is live. A marking $\boldsymbol{m}$ is an *home state* iff it is reachable from every reachable marking, and $(\mathcal{N}, \boldsymbol{m}_0)$ is *reversible* iff $\boldsymbol{m}_0$ is a home state.

We assume that the initial marking $\boldsymbol{m}_0$ is a tangible marking. If this is not the case (i.e., if $\boldsymbol{m}_0$ is a vanishing marking) we can repeat the analysis for all the tangible markings that can be reached starting from $\boldsymbol{m}_0$.

Lemma 3. *Given a GSPN system $(\mathcal{N}, \boldsymbol{m}_0)$ that satisfies Definition 4, if $t \in \mathcal{T}$ is enabled in $\boldsymbol{m} \in R(\mathcal{N}, \boldsymbol{m}_0)$, then for any t' belonging to the same closed support T-semiflow of t, there exists a finite firing sequence σ such that $\boldsymbol{m} \xrightarrow{\sigma} \boldsymbol{m}'$ and t' is enabled in $\boldsymbol{m}'$.*

Proof. Let $\boldsymbol{m}$ be a marking that enables transition t. From Definition 4, it follows that there exists a transition t' such that $\boldsymbol{o}(t) = \boldsymbol{i}(t')$; hence, the firing of t yields a marking $\boldsymbol{m}'$ that enables t'. The firing of t' enables t'' and so on. □

Lemma 4. *Given a GSPN system $(\mathcal{N}, \boldsymbol{m}_0)$ that satisfies Definition 4 and 5.*

1. *If $t \in \mathcal{T}$ can be fired in $\boldsymbol{m}_0$, then $(\mathcal{N}, \boldsymbol{m}_0)$ is reversible.*
2. *The net $\mathcal{N}$ is structurally live.*

Proof.

1. For any $\boldsymbol{m} \in R(\mathcal{N}, \boldsymbol{m}_0)$ there is a finite firing sequence $\sigma = t_{\delta_1}, t_{\delta_2}, \ldots, t_{\delta_l}$ such that $\boldsymbol{m}_0 \xrightarrow{t_{\delta_1}} \boldsymbol{m}_1 \ldots \boldsymbol{m}_{l-1} \xrightarrow{t_{\delta_l}} \boldsymbol{m}$. Now, we have to prove that there is a finite firing sequence η such that $\boldsymbol{m} \xrightarrow{\eta} \boldsymbol{m}_0$. We prove this result by induction on the length of the finite firing sequence σ. The basic step of the induction is a finite firing sequence $\sigma = t_{\delta_1}$ (i.e., the firing sequence contains only one transition). Since we have assumed that $\boldsymbol{m}_0$ is a tangible marking, it follows from Lemma 3 that there is a finite firing sequence $\eta = ||\boldsymbol{x}|| - \{t_{\delta_1}\}$, where $||\boldsymbol{x}||$ is the support set of the closed T-semiflow to which t_{δ_1} belongs. From this, we have that $\boldsymbol{m}_0 \xrightarrow{t_{\delta_1}} \boldsymbol{m}_1 \xrightarrow{\eta} \boldsymbol{m}_0$.

   Now, we assume that the lemma holds for all finite firing sequences $\sigma' = t_{\delta_1}, t_{\delta_2}, \ldots, t_{\delta_i}$ (i.e., finite firing sequences containing i transitions).

Let $\sigma = t_{\delta_1}, t_{\delta_2}, \ldots, t_{\delta_i}, t_{\delta_{i+1}}$ be a finite firing sequence such that

$$\boldsymbol{m}_0 \xrightarrow{t_{\delta_1}} \boldsymbol{m}_1 \xrightarrow{t_{\delta_2}} \boldsymbol{m}_2 \cdots \xrightarrow{t_{\delta_i}} \boldsymbol{m}_i \xrightarrow{t_{\delta_{i+1}}} \boldsymbol{m}_{i+1}.$$

If the marking $\boldsymbol{m}_i$ is a tangible marking, then we can apply the basic inductive step. We have that

$$\boldsymbol{m}_i \xrightarrow{t_{\delta_{i+1}}} \boldsymbol{m}_{i+1} \xrightarrow{\eta'} \boldsymbol{m}_i,$$

with $\eta' = ||\boldsymbol{x}'|| - \{t_{\delta_{i+1}}\}$, where $||\boldsymbol{x}'||$ is the support set of the closed T-semiflow to which $t_{\delta_{i+1}}$ belongs. From the inductive hypothesis, we know that the lemma holds for all finite firing sequences containing i transitions. This implies that there exists a finite firing sequence η'' such that $\boldsymbol{m}_i \xrightarrow{\eta''} \boldsymbol{m}_0$. From this, it follows that

$$\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}_{i+1} \xrightarrow{\eta'} \boldsymbol{m}_i \xrightarrow{\eta''} \boldsymbol{m}_0.$$

We now address the case when $\boldsymbol{m}_i$ is a vanishing marking. Let $\mathcal{T}_{ck} = \{t_{i_1}, t_{i_2}, \ldots, t_{i_k}\}$ be the subset of conflicting immediate transitions that are enabled in marking $\boldsymbol{m}_i$ ($t_{\delta_{i+1}} \in \mathcal{T}_{ck}$). Since the GSPN satisfies Definition 5, the firing of $t_{\delta_{i+1}}$ disables all the other transitions belonging to $\mathcal{T}_{ck}$. From Lemma 3, if follows that all the transitions of the closed T-semiflow at which $t_{\delta_{i+1}}$ belongs can fire and, hence,

$$\boldsymbol{m}_i \xrightarrow{t_{\delta_{i+1}}} \boldsymbol{m}_{i+1} \xrightarrow{\eta'} \boldsymbol{m}_i,$$

with $\eta = ||\boldsymbol{x}'|| - \{t_{\delta_{i+1}}\}$, where $||\boldsymbol{x}'||$ is the support set of the closed T-semiflow to which $t_{\delta_{i+1}}$ belongs. From the induction hypothesis, we know that there exists a finite firing sequence η'' such that $\boldsymbol{m}_i \xrightarrow{\eta''} \boldsymbol{m}_0$. From this it follows that

$$\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}_{i+1} \xrightarrow{\eta'} \boldsymbol{m}_i \xrightarrow{\eta''} \boldsymbol{m}_0.$$

Note that, if the GSPN does not satisfy Definition 5 after the firing of $t_{\delta_{i+1}} \in \mathcal{T}_{ck}$, there can be another transition $t' \in \mathcal{T}_{ck}$ that remains enabled in marking $\boldsymbol{m}_{i+1}$. Since t' is an immediate transition, it must fire before any other timed transition. In this manner, we can have situations similar to the GSPN of Fig. 1b, that is, the marking $\boldsymbol{m}_i$ cannot be reproduced.

2. To prove that $\mathcal{N}$ is structurally live, we have to prove that there exists an initial marking $\boldsymbol{m}_0$ such that $(\mathcal{N}, \boldsymbol{m}_0)$ is live. Let $\boldsymbol{m}_0$ be a marking that enables at least one transition in each closed T-semiflow. In this case, from Lemma 3, it follows that $\forall\, t \in \mathcal{T}$ there exists a finite firing sequence σ such that $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}$ and $\boldsymbol{m}$ enables t. We have that $\forall\, t \in \mathcal{T}$ and $\boldsymbol{m} \in R(\mathcal{N}, \boldsymbol{m}_0)$ from Statement 1 of this lemma. It follows that there is a finite firing sequence η such that $\boldsymbol{m} \xrightarrow{\eta} \boldsymbol{m}_0$. Since we know that $\boldsymbol{m}_0$ enables at least a transition in each closed T-semiflow, using Lemma 3, we prove the structural liveness of $\mathcal{N}$. □

APPENDIX 2

EFFECTS OF THE FUSION AND OF THE STRATIFICATION ON THE CLOSED T-Semiflows

Lemma 5. *We have that:*

1. *If $\boldsymbol{x}$ is a closed minimal support T-semiflow, then for any $t \in ||\boldsymbol{x}||$, there exist a unique $t' \in ||\boldsymbol{x}||$ with $\boldsymbol{o}(t') = \boldsymbol{i}(t)$ and a unique $t'' \in ||\boldsymbol{x}||$ with $\boldsymbol{i}(t'') = \boldsymbol{o}(t)$.*
2. *If $\mathcal{T}' \subseteq \mathcal{T}$ and for all $t \in \mathcal{T}'$ there exist a unique $t' \in \mathcal{T}'$ with $\boldsymbol{o}(t') = \boldsymbol{i}(t)$ and a unique $t'' \in \mathcal{T}'$ with $\boldsymbol{i}(t'') = \boldsymbol{o}(t)$ then $\mathcal{T}'$ is the support set of a closed minimal T-semiflow.*

Proof. Assume that there are two transitions $t_a, t_b \in ||\boldsymbol{x}||$ such that $\boldsymbol{o}(t_a) = \boldsymbol{o}(t_b)$. As a consequence of Lemma 1, there must be two transitions $t_c, t_d \in ||\boldsymbol{x}||$ such that $\boldsymbol{i}(t_c) = \boldsymbol{i}(t_d) = \boldsymbol{o}(t_a) = \boldsymbol{o}(t_b)$. Definition 1 implies that we can build two cycles $\{t_{h_0}, t_{h_1}, \ldots, t_{h_{l-1}}\}$ and $\{t_{k_0}, t_{k_1}, \ldots, t_{k_{h-1}}\}$ such that:

- $t_{h_i} \in ||\boldsymbol{x}||$, and $\boldsymbol{o}(t_{h_i}) = \boldsymbol{i}(t_{h_{(i+1) \bmod l}})$ (for $i = 0, \ldots, l-1$);
- $t_{k_i} \in ||\boldsymbol{x}||$, and $\boldsymbol{o}(t_{k_j}) = \boldsymbol{i}(t_{k_{(j+1) \bmod h}})$ (for $j = 0, \ldots, h-1$);
- $t_{h_0} = t_a$, $t_{h_1} = t_c$, $t_{k_0} = t_b$, and $t_{k_1} = t_c$.

Since $t_a \neq t_c$ and $t_b \neq t_d$ we have that these cycles are the support sets of two different closed T-semiflows with $\{t_{h_0}, t_{h_1}, \ldots, t_{h_{l-1}}\} \subset ||\boldsymbol{x}||$, and $\{t_{k_0}, t_{k_1}, \ldots, t_{k_{h-1}}\} \subset ||\boldsymbol{x}||$, but this is a contradiction of the assumption that $\boldsymbol{x}$ is a minimal closed support T-semiflow.

On the other hand, let us assume that $\mathcal{T}'$ is a set of transitions such that for any $t \in \mathcal{T}'$ there exist a unique $t' \in \mathcal{T}'$ with $\boldsymbol{o}(t') = \boldsymbol{i}(t)$ and a unique $t'' \in \mathcal{T}'$ with $\boldsymbol{i}(t'') = \boldsymbol{o}(t)$. We can start from transition t (call it t_{h_0}); and, as a consequence of Definition 1, we can order all the transitions of $\mathcal{T}'$ in a cycle $\mathcal{C} = \{t_{h_0}, t_{h_1}, \ldots, t_{h_{l-1}}\}$ such that $\mathcal{C} = |\mathcal{T}'|$ (i.e., all the transitions of $\mathcal{T}'$ are involved in the cycle) and $\boldsymbol{o}(t_{h_i}) = \boldsymbol{i}(t_{h_{(i+1) \bmod l}})$ for $i = 0, \ldots, l-1$. By assumption, we know that $\mathcal{T}'$ is a closed set. We can also verify that $\mathcal{T}'$ is the support of a T-semiflow because

$$\begin{aligned} \boldsymbol{C}[\mathcal{P}t_{h_0}] + \ldots + \boldsymbol{C}[\mathcal{P}t_{h_{l-1}}] &= \boldsymbol{o}(t_{h_0})^T - \boldsymbol{i}(t_{h_0})^T + \boldsymbol{o}(t_{h_1})^T - \\ &\quad + \boldsymbol{i}(t_{h_1})^T + \ldots + \boldsymbol{o}(t_{h_{l-1}})^T - \boldsymbol{i}(t_{h_{l-1}})^T \\ &\overset{\text{Def.1}}{=} -\boldsymbol{i}(t_{h_0})^T + \boldsymbol{o}(t_{h_{l-1}})^T \overset{\text{Def.1}}{=} \mathbf{0}. \end{aligned}$$

We only have to prove that $\boldsymbol{x}$, i.e., the T-semiflow that has $\mathcal{T}'$ as support set, is minimal. Assume that it is not a minimal T-semiflow. In this case, there must be a T-semiflow $\tilde{\boldsymbol{x}}$ such that $\tilde{\boldsymbol{x}} \leq \boldsymbol{x}$. This means that there exists a subset of transitions $\{t_{k_0}, \ldots, t_{k_{h-1}}\}$ that are a proper subcycle of $\mathcal{C}$. If $\mathcal{C}$ can be decomposed in subcycles, then there must exist two transitions $t', t'' \in \mathcal{C}$ such that $t' \neq t''$ and $\boldsymbol{i}(t') = \boldsymbol{i}(t'')$. In this case, we obtain a contradiction of the assumption that for all transitions $t \in \mathcal{C}$ there exists a unique transition $\tilde{t} \in \mathcal{C}$ with $\boldsymbol{o}(\tilde{t}) = \boldsymbol{i}(t)$. □

Let $\mathcal{N} = (\mathcal{P}, \mathcal{T}, I(.,.), O(.,.), H(.,.), \Pi(.), w(.), \boldsymbol{m}_0)$ be a GSPN satisfying Definition 4, let $\mathcal{F}c = \{t_{i_1}, t_{i_2}, \ldots, t_{i_h}\}$ be the set of immediate transitions in free-choice conflict

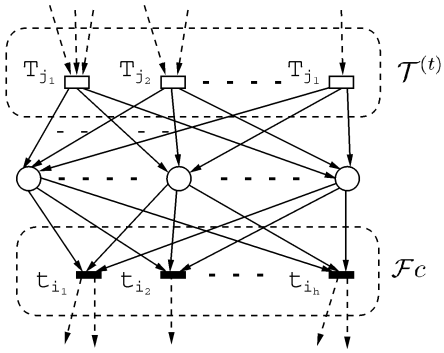


Fig. 11. Timed and immediate transitions involved in the fusion process.

involved in the fusion process, and $\mathcal{T}^{(t)} = \{T_{j_1}, T_{j_2}, \dots, T_{j_l}\}$ be the set of all the transitions such that $FC(T_{j_a}) = \mathcal{F}c$, for $a = 1, \dots, l$ (see Fig. 11).

These subsets are defined such that $\nexists t' \in \mathcal{T}$ such that $\boldsymbol{i}(t') = \boldsymbol{i}(t_{i_1}) = \dots = \boldsymbol{i}(t_{i_h})$ and $t' \notin \mathcal{F}c$, and $\mathcal{T}^{(t)}$ contains all the transitions T' with $\boldsymbol{o}(T') = \boldsymbol{i}(t_{i_1}) = \dots = \boldsymbol{i}(t_{i_h})$. We denote by $\mathcal{X}^{(c)}$ the set of minimal closed support T-semiflows that cover at least one transition belonging to $\mathcal{F}c$ or to $\mathcal{T}^{(t)}$, i.e.,

$$\mathcal{X}^{(c)} = \{\boldsymbol{x} : \boldsymbol{x} \text{ is a m. c. s. } T\text{-semiflow and}$$
$$\Big(\exists\, T_{j_a} \in \mathcal{T}^{(t)} \wedge T_{j_a} \in ||\boldsymbol{x}||\Big) \bigvee \Big(\exists\, t_{i_c} \in \mathcal{F}c \wedge t_{i_c} \in ||\boldsymbol{x}||\Big)\Big\}.$$

Let $T_{j_a} \in \mathcal{T}^{(t)}$ and define $\mathcal{X}^{(c)}(T_{j_a})$ as the subset of $\mathcal{X}^{(c)}$ with the following characteristic:

$$\mathcal{X}^{(c)}(T_{j_a}) = \Big\{\boldsymbol{x} : \boldsymbol{x} \in \mathcal{X}^{(c)} \wedge T_{j_a} \in ||\boldsymbol{x}||\Big\}.$$

Lemma 6. *We have that*

1. $\bigcup_{T_{j_a} \in \mathcal{T}^{(t)}} \mathcal{X}^{(c)}(T_{j_a}) = \mathcal{X}^{(c)}$,
2. $\forall\, T_{j_a}, T_{j_b} \in \mathcal{T}^{(t)}$, $T_{j_a} \neq T_{j_b} \Rightarrow \mathcal{X}^{(c)}(T_{j_a}) \cap \mathcal{X}^{(c)}(T_{j_a}) = \emptyset$.

Proof. For the first statement of the lemma, we have to prove that there is not a T-semiflow $\boldsymbol{x}$ such that $\exists\, t_{i_c} \in \mathcal{F}c$ and $t_{i_c} \in ||\boldsymbol{x}||$, but $\nexists\, T_{j_a} \in \mathcal{T}^{(t)}$ such that $T_{j_a} \in ||\boldsymbol{x}||$. A similar closed T-semiflow cannot exist, i.e., it would be a contradiction of Definition 1 (closure).

To prove the second statement, we can prove the following statements:

- $\nexists \boldsymbol{x} \in \mathcal{X}^{(c)}$ such that $\exists T_{j_a}, T_{j_b} \in \mathcal{T}^{(t)}$ with $T_{j_a} \in ||\boldsymbol{x}||$ and $T_{j_b} \in ||\boldsymbol{x}||$,
- $\nexists \boldsymbol{x} \in \mathcal{X}^{(c)}$ such that $\exists t_{i_c}, t_{i_d} \in \mathcal{F}c$ with $t_{i_c} \in ||\boldsymbol{x}||$ and $t_{i_d} \in ||\boldsymbol{x}||$.

We can observe that both the previous statements follow from Lemma 5 (point 2) and, hence, this concludes the proof of the lemma. □

Lemma 7. *Let $\mathcal{N}$ be a closed GSPN and $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots,$ be its minimal closed support T-semiflows. Any combination of some of these invariants is still a (nonminimal) closed support T-semiflow.*

Proof. Let $\boldsymbol{x} = \alpha_1 \cdot \boldsymbol{x}_1 + \alpha_2 \cdot \boldsymbol{x}_2 + \dots$ (with $\alpha_i \in \mathbb{N}$) be a combination of the minimal closed support T-semiflows $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots,$. It is immediate to see that

$$\begin{aligned}\boldsymbol{C} \cdot \boldsymbol{x} &= \boldsymbol{C} \cdot (\alpha_1 \cdot \boldsymbol{x}_1 + \alpha_2 \cdot \boldsymbol{x}_2 + \dots) \\ &= \alpha_1 \boldsymbol{C}\boldsymbol{x}_1 + \alpha_2 C\boldsymbol{x}_2 + \dots = 0,\end{aligned}$$

hence, $\boldsymbol{x}$ is a nonminimal T-semiflow. The closedness immediately follows from Definition 1. □

The following lemma proves that, in case of free-choice conflict between immediate transitions, all the new fused transitions are covered by a minimal closed support T-semiflow. The lemma also gives a constructive manner for deriving the closed T-semiflows for the new GSPN.

In the following, we denote by C and $\boldsymbol{C}'$ the incidence matrices of the original and of the new GSPN, respectively. With $C[\mathcal{P}, t_i]$ (respectively $\boldsymbol{C}'[\mathcal{P}, t_i]$) we denote the column of C (respectively $\boldsymbol{C}'$) corresponding to transition t_i.

Lemma 8. *Let $\mathcal{T}^{(t)} = \{T_{j_1}, T_{j_2}, \dots, T_{j_l}\}$ be a subset of timed transitions such that*

$$FC(T_{j_1}) = FC(T_{j_2}) = \dots = FC(T_{j_l}) = \mathcal{F}c = \{t_{i_1}, t_{i_2}, \dots, t_{i_h}\},$$

after the fusion in the transformed GSPN $\mathcal{N}'$ all the fused transitions $T_{j_1}t_{i_1}, T_{j_1}t_{i_2}, \dots, T_{j_1}t_{i_h}$, $T_{j_2}t_{i_1}, T_{j_2}t_{i_2}, \dots, T_{j_2}t_{i_h}$, $\dots T_{j_l}t_{i_1}, T_{j_l}t_{i_2}, \dots, T_{j_l}t_{i_h}$ are covered by a minimal closed support T-semiflow.

Proof. The transformed GSPN $\mathcal{N}'$, with $T_{j_a} \in \mathcal{T}^{(t)}$ and $t_{i_c} \in \mathcal{F}c$ (in $\mathcal{N}$). We distinguish two cases:

1. In the original GSPN $\mathcal{N}$, there is a minimal closed support T-semiflow $\boldsymbol{x}$ such that $T_{j_a} \in ||\boldsymbol{x}||$ and $t_{i_c} \in ||\boldsymbol{x}||$.
2. In the original GSPN $\mathcal{N}$, there are two minimal closed support T-semiflows $\boldsymbol{x}'$ and $\boldsymbol{x}''$ such that $T_{j_a} \in ||\boldsymbol{x}'||$, and $t_{i_c} \in ||\boldsymbol{x}''||$, and $\boldsymbol{x}' \neq \boldsymbol{x}''$.

Let $\boldsymbol{x}$ be the minimal closed support T-semiflow that covers t_{i_c} and T_{j_a}. The incidence matrix $\boldsymbol{C}'$ can be obtained in the following manner: let $\boldsymbol{C}[\mathcal{P}, T_{j_a}]$ and $\boldsymbol{C}[\mathcal{P}, t_{i_c}]$ be the columns corresponding to transitions T_{j_a} and t_{i_c}. We have that

$$\begin{aligned}\boldsymbol{C}[\mathcal{P}, T_{j_a}] + \boldsymbol{C}[\mathcal{P}, t_{i_b}] &= \boldsymbol{o}(T_{j_a})^T - \boldsymbol{i}(T_{j_a})^T + \boldsymbol{o}(t_{i_c})^T - \boldsymbol{i}(t_{i_c})^T \\ &\overset{\text{Def.1}}{=} \boldsymbol{o}(t_{i_c})^T - \boldsymbol{i}(T_{j_a})^T \\ &= \boldsymbol{C}'[\mathcal{P}, T_{j_a}t_{i_c}].\end{aligned}$$

By the definition of T-semiflow, we have that $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$, that can be written as

$$\boldsymbol{C}[\mathcal{P}, t_1] \cdot x_1 + \boldsymbol{C}[\mathcal{P}, t_2] \cdot x_2 + \dots + \boldsymbol{C}[\mathcal{P}, t_{|\mathcal{T}|}] \cdot x_{|\mathcal{T}|} = \boldsymbol{0},$$

where x_i (for $i = 1, \dots, |\mathcal{T}|$) is the ith component of $\boldsymbol{x}$. By Lemma 1, we can write that

$$\boldsymbol{C}[\mathcal{P}, t_1] \cdot x_1 + \dots + \boldsymbol{C}[\mathcal{P}, T_{j_a}] \cdot 1 + \boldsymbol{C}[\mathcal{P}, t_{i_c}] \cdot 1 + \dots + \boldsymbol{C}[\mathcal{P}, t_{|\mathcal{T}|}] \cdot x_{|\mathcal{T}|} = \boldsymbol{0}.$$

Let us focus our attention on the terms $\boldsymbol{C}[\mathcal{P}, T_{j_a}] \cdot 1 + \boldsymbol{C}[\mathcal{P}, t_{i_c}] \cdot 1$, we have that

$$\boldsymbol{C}[\mathcal{P}, T_{j_a}] + \boldsymbol{C}[\mathcal{P}, t_{i_c}] = \boldsymbol{C}'[\mathcal{P}, T_{j_a}t_{i_c}].$$

From this, it follows $\boldsymbol{C}' \cdot \boldsymbol{x}' = \boldsymbol{0}$ where $\boldsymbol{x}'$ is a T-semiflow with the following support

$$||\boldsymbol{x}'|| = ||\boldsymbol{x}|| - \{T_{j_a}\} - \{t_{i_c}\} \cup \{T_{j_a}t_{i_c}\}. \quad (11)$$

Since the input and the output vector of $T_{j_a}t_{i_c}$ are equal to the input vector of T_{j_a} and the output vector of t_{i_c}, respectively, it follows that the support set of $\boldsymbol{x}'$ is closed. The "minimality" of $\boldsymbol{x}'$ can be proven by observing that the original closed T-semiflow $\boldsymbol{x}$ satisfies Lemma 5 (point 1). After the fusion, $||\boldsymbol{x}'||$ satisfies point 2 of Lemma 5 and, hence, it is a minimal closed support T-semiflow.

We now address the case when the fused transition comes from transitions covered by different minimal closed support T-semiflows. Assume that $T_{j_a} \in \mathcal{T}^{(t)}$ and $t_{i_c} \in \mathcal{F}c$ and let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be the two minimal T-semiflows that cover such transitions, i.e., $T_{j_a} \in ||\boldsymbol{x}_1||$, $t_{i_c} \in ||\boldsymbol{x}_2||$, with $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Moreover, let us also identify another pair of transitions T_{j_b} and t_{i_d} with the following characteristics: $T_{j_b} \in ||\boldsymbol{x}_1||$ and $t_{i_d} \in ||\boldsymbol{x}_2||$. After the fusion, in the GSPN $\mathcal{N}'$, we can find transitions $T_{j_a}t_{i_c}$ and $T_{j_b}t_{i_d}$. We have that

$$\begin{aligned} \boldsymbol{C}[\mathcal{P}, T_{j_a}] + \boldsymbol{C}[\mathcal{P}, t_{i_c}] &= \boldsymbol{o}(T_{j_a})^T - \boldsymbol{i}(T_{j_a})^T + \\ &\quad \boldsymbol{o}(t_{i_c})^T - \boldsymbol{i}(t_{i_c})^T \\ &\overset{\text{Def.1}}{=} \boldsymbol{o}(t_{i_c})^T - \boldsymbol{i}(T_{j_a})^T \\ &= \boldsymbol{C}'[\mathcal{P}, T_{j_a}t_{i_c}], \end{aligned}$$

and similarly

$$\boldsymbol{C}[\mathcal{P}, T_{j_b}] + \boldsymbol{C}[\mathcal{P}, t_{i_d}] = \boldsymbol{C}'[\mathcal{P}, T_{j_b}t_{i_d}].$$

From Lemma 7, it follows that, in $\mathcal{N}$, $\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2$ is a nonminimal closed support T-semiflow, i.e., $\boldsymbol{C} \cdot \boldsymbol{x} = \boldsymbol{0}$.

Let us focus our attention on the T-semiflow $\boldsymbol{x}$. We have that $\boldsymbol{x}$ covers the transitions T_{j_a}, T_{j_b}, t_{i_c}, and t_{i_d} and no other transitions belonging to $\mathcal{T}^{(t)}$ or to $\mathcal{F}c$ (Lemma 6). In particular, if we denote by x_a, x_b, x_c, x_d the components of $\boldsymbol{x}$ corresponding to these transitions, we can write

$$\begin{aligned} \boldsymbol{C}x &= \ldots + \boldsymbol{C}[\mathcal{P}, T_{j_a}]x_a + \boldsymbol{C}[\mathcal{P}, T_{j_b}]x_b + \boldsymbol{C}[\mathcal{P}, t_{i_c}]x_c + \\ &\quad C[\mathcal{P}, t_{i_d}]x_d + \ldots \\ &= \ldots + \boldsymbol{C}[\mathcal{P}, T_{j_a}]1 + \boldsymbol{C}[\mathcal{P}, T_{j_b}]1 + \boldsymbol{C}[\mathcal{P}, t_{i_c}]1 + \\ &\quad + \boldsymbol{C}[\mathcal{P}, t_{i_d}]1 + \ldots \boldsymbol{C}'[\mathcal{P}, T_{j_a}t_{i_c}]1 + \boldsymbol{C}'[\mathcal{P}, T_{j_b}t_{i_d}]1 + \ldots \\ &= \boldsymbol{C}'\boldsymbol{x}', \end{aligned}$$

where $\boldsymbol{x}'$ is obtained by removing from the support of $\boldsymbol{x}$ the components corresponding to transitions T_{j_a}, T_{j_b}, t_{i_c}, and t_{i_d}, and adding the components corresponding to fused transitions $T_{j_a}t_{i_c}$ and $T_{j_b}t_{i_d}$, i.e.,

$$||\boldsymbol{x}'|| = ||\boldsymbol{x}_1|| \cup ||\boldsymbol{x}_2|| - \{T_{j_a}, T_{j_b}, t_{i_c}, t_{i_d}\} \cup \{T_{j_a}t_{i_c}, T_{j_b}t_{i_d}\}. \quad (12)$$

From this it follows that, in the transformed GSPN, $\boldsymbol{x}'$ is a T-semiflow. The closure property follows from Definition 1. Now, we can prove that $\boldsymbol{x}'$ is a minimal T-semiflow by using Lemma 5 (point 2).

It is easy to verify that $||\boldsymbol{x}'||$ satisfies Lemma 5 (point 2) because in the original (nonminimal) closed T-semiflow $\boldsymbol{x}$ there are four transitions that do not satisfy this assumption: T_{j_a}, T_{j_b}, t_{i_c}, and t_{i_d} (i.e., T_{j_a} and T_{j_b} have the same output vector, and t_{i_c} and t_{i_d} the same input vector). After the fusion, these transitions are substituted by the fused transitions $T_{j_a}t_{i_c}$ and $T_{j_b}t_{i_d}$. Since $\boldsymbol{o}(T_{j_a}t_{i_c}) = \boldsymbol{o}(t_{i_c})$, we have that there is no other transition with this output vector. The same can be observed for the other fused transitions. $\square$

Proof of Lemma 2. The effect of the stratification of a transition on the incidence matrix is simply that the column corresponding to the stratified transition is replaced with a number of columns that is equal to the number of new transitions generated by the stratification. Since the incidence matrix does not account for test (a pair of input and output arcs from and towards the same place) and inhibitor arcs, the new columns are **identical** to the column of the stratified transition. In this manner, if T_j is the stratified transition and $\boldsymbol{x}$ is a T-semiflow with $T_j \in ||\boldsymbol{x}||$, for any transition $T_j\mathbf{s}_i$ generated by the stratification, we have that

$$\begin{aligned} \boldsymbol{C} \cdot x &= \ldots + \boldsymbol{C}[\mathcal{P}, T_j]1 + \ldots = \boldsymbol{0} \\ &= \ldots + \boldsymbol{C}[\mathcal{P}, T_j\mathbf{s}_i]1 + \ldots = \boldsymbol{C}' \cdot \boldsymbol{x}_{\mathbf{s}_i} = \boldsymbol{0}, \end{aligned}$$

where $\boldsymbol{x}_{\mathbf{s}_i}$ is obtained from $\boldsymbol{x}$ by replacing the component corresponding to T_j with the component that corresponds the (new) transition $T_j\mathbf{s}_i$. In other words, for any (new) transition $T_j\mathbf{s}_i$, $\boldsymbol{x}_{\mathbf{s}_i}$ is a copy of the minimal T-semiflow $\boldsymbol{x}$. $\square$

APPENDIX 3

EFFECTS OF THE FUSION AND OF THE STRATIFICATION ON THE STATE SPACE

For a given closed GSPN $\mathcal{N} = (\mathcal{P}, \mathcal{T}, I(\cdot,\cdot), O(\cdot,\cdot), H(\cdot,\cdot), w(\cdot), \boldsymbol{m}_0)$, we group all the immediate transitions that are in free-choice conflict among them. We can denote by $\mathcal{F} = (\mathcal{F}c_1, \mathcal{F}c_2, \ldots)$ the set of these groups. Obviously, $\nexists\ \mathcal{F}c_a, \mathcal{F}c_b \in \mathcal{F}$ such that $\mathcal{F}c_a \neq \mathcal{F}c_b$ and $\mathcal{F}c_a \cap \mathcal{F}c_b \neq \emptyset$. Any $\mathcal{F}c_i = \{t_{i_1}, t_{i_2} \ldots\}$ is "maximal" i.e., there is no transition $t_l \notin \mathcal{F}c_i$ such that t_l is in free-choice conflict with a transition belonging to $\mathcal{F}c_i$.

We assume that the initial marking of $\mathcal{N}$ is tangible. We investigate what happens when we "fuse" the immediate transitions of a maximal free-choice conflict group $\mathcal{F}c_i$. We denote by $\mathcal{N}' = (\mathcal{P}', \mathcal{T}', I'(\cdot,\cdot), O'(\cdot,\cdot), H(\cdot,\cdot), w'(\cdot), \boldsymbol{m}_0)$ the GSPN resulting after the application of the transformation.

Lemma 9. *Let $\mathcal{N}$ be a closed GSPN and $\mathcal{F}c_i = \{t_{i_1}, t_{i_2} \ldots\}$ be a "maximal" subset of immediate transitions in free-choice conflict. If we denote by $\mathcal{N}'$ the GSPN resulting after the application of the fusion process presented in Section 5.1 to $\mathcal{F}c_i$, we can prove that the GSPNs $\mathcal{N}$ and $\mathcal{N}'$ have the same tangible state space and generate the same CTMC.*

Proof. Let $T_j \in \mathcal{T}$ be a timed transition such that $FC(T_j) = \mathcal{F}c_i$. By definition of the transformation, it is immediately seen that any marking $\boldsymbol{m}$ that enables T_j (in

$\mathcal{N}$) enables all the transitions $T_j t_{i_1}, T_j t_{i_2}, \ldots \in \mathcal{T}'$ obtained by fusing T_j with all the immediate transitions belonging to $FC(T_j)$. Starting from $\boldsymbol{m}$, we may observe that, if (in $\mathcal{N}$) T_j followed by t_{i_h} fire, we first obtain a vanishing marking $\boldsymbol{m}'$ and then

$$\boldsymbol{m}''(\boldsymbol{m}\xrightarrow{T_j}\boldsymbol{m}'\xrightarrow{t_{i_h}}\boldsymbol{m}'').$$

In particular, we have that $\boldsymbol{m}'' = \boldsymbol{m} - \boldsymbol{i}(T_j) + \boldsymbol{o}(T_j) - \boldsymbol{i}(t_{i_h}) + \boldsymbol{o}(t_{i_h})$. The closure property ensures that $\boldsymbol{o}(T_j) = \boldsymbol{i}(t_{i_h})$ and, hence, $\boldsymbol{m}'' = \boldsymbol{m} - \boldsymbol{i}(T_j) + \boldsymbol{o}(t_{i_h})$. By definition of "fused" transition, it follows that, in the transformed GSPN, there exists a transition $T_j t_{i_h}$ with $\boldsymbol{i}(T_j t_{i_h}) = \boldsymbol{i}(T_j)$ and $\boldsymbol{o}(T_j t_{i_h}) = \boldsymbol{o}(t_{i_h})$. From this, it follows that, in the transformed GSPN $\mathcal{N}'$, $\boldsymbol{m}\xrightarrow{T_j t_{i_h}}\boldsymbol{m}''$. On the other hand, if in the transformed GSPN $\mathcal{N}'$, a transition $T_j t_{i_h}$ fires, yielding $\boldsymbol{m}\xrightarrow{T_j t_{i_h}}\boldsymbol{m}''$, in the original GSPN $\mathcal{N}$, there must be a firing sequence T_j followed by t_{i_h} such that $\boldsymbol{m} - \boldsymbol{i}(T_j) + \boldsymbol{o}(T_j) - \boldsymbol{i}(t_{i_h}) + \boldsymbol{o}(t_{i_h}) = \boldsymbol{m}''$.

The rates from $\boldsymbol{m}$ to $\boldsymbol{m}''$ in $\mathcal{N}$ and $\mathcal{N}'$ are equal by definition of the fusion process. □

For a given closed GSPN $\mathcal{N} = (\mathcal{P}, \mathcal{T}, I(\cdot,\cdot), O(\cdot,\cdot), H(\cdot,\cdot), w(\cdot), \boldsymbol{m}_0)$, let $T_j \in \mathcal{T}$ be a transition that we stratify[8] with respect to the subset of places $\mathcal{P}'$. According to Definition 6, we obtain a new GSPN $\mathcal{N}' = (\mathcal{P}, \mathcal{T}', I'(\cdot,\cdot), O'(\cdot,\cdot), H'(\cdot,\cdot), w'(\cdot), \boldsymbol{m}_0)$, where $\mathcal{T}' = \mathcal{T} \bigcup_{i=1}^{2^{|\mathcal{P}|}} \{T_j \mathbf{s}_i\} - \{T_j\}$. We can prove that

Lemma 10. *Let $\mathcal{N}$ be a closed GSPN and T_j a transition that we stratify with respect to the subset of places $\mathcal{P}'$ (with $^\bullet T_j \cap \mathcal{P}' = \emptyset$, $T_j^\bullet \cap \mathcal{P}' = \emptyset$, and $^\circ T_j \cap \mathcal{P}' = \emptyset$) yielding a new GSPN $\mathcal{N}'$. The GSPNs $\mathcal{N}$ and $\mathcal{N}'$ have the same tangible state space and generate the same CTMC.*

Proof. By definition of stratification of a transition T_j with respect to the subset of places $\mathcal{P}'$, all the (newly) generated transitions transitions $T_j \mathbf{s}_i$ (with $i = 1, \ldots, 2^{|\mathcal{P}'|}$) are mutually exclusive **HME**. These transitions account for all the possible marking situations (place marked, place not marked) of all the places belonging to the subset $\mathcal{P}'$.

Let $\boldsymbol{m}\xrightarrow{T_j}\boldsymbol{m}'$ be a state transition (in $\mathcal{N}$) due to the firing of T_j. By definition of stratification, there exists a unique $T_j \mathbf{s}_a \in \mathcal{T}'$ enabled in $\boldsymbol{m}$ whose firing yields the marking $\boldsymbol{m}'$, that is, $\boldsymbol{m} \geq \boldsymbol{i}(T_j \mathbf{s}_a)$ and $\boldsymbol{m} < \boldsymbol{h}(T_j \mathbf{s}_a)$ and $\boldsymbol{m}\xrightarrow{T_j \mathbf{s}_a}\boldsymbol{m}'$ (in $\mathcal{N}'$).

On the other hand, if (in $\mathcal{N}'$) there is a state transition $\boldsymbol{m}\xrightarrow{T_j \mathbf{s}_b}\boldsymbol{m}'$, with $T_j \mathbf{s}_b \in \mathcal{T}'$, this means that the marking m (in $\mathcal{N}$) enables transition T_j and, since all the places of the subset $\mathcal{P}'$ are connected with $T_j \mathbf{s}_b$ by means of test (pair of input and output arcs) and inhibitor arcs that do not modify the marking of the places of $\mathcal{P}'$, we have that $\boldsymbol{m}\xrightarrow{T_j}\boldsymbol{m}'$. □

8. Even though we use an uppercase letter for denoting the transition, we do not rely on the assumption that the transition that we stratify is a timed transition. Definition 6 can be applied also to immediate transitions as well.

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