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Product of four Hadamard matrices

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Abstract

We prove that if there exist Hadamard matrices of order 4m, 4n, 4p, and 4q then there exists an Hadamard matrix of order 16mnpq. This improves and extends the known result of Agayan that there exists a Hadamard matrix of order 8mn if there exist Hadamard matrices of order 4m and 4n.

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Note

Product of Four Hadamard Matrices

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We prove that if there exist Hadamard matrices of order $4m$, $4n$, $4p$, and $4q$ then there exists an Hadamard matrix of order $16mnpq$. This improves and extends the known result of Agayan that there exists a Hadamard matrix of order $8mn$ if there exist Hadamard matrices of order $4m$ and $4n$. © 1992 Academic Press, Inc.

A *weighing matrix* [3] of order n with weight k, denoted $W = W(n, k)$, is a $(0, \pm 1)$ matrix satisfying $WW^{\mathrm{T}} = kI_n$. A $W(n, n)$ is an Hadamard matrix.

Two matrices X and Y are said to be *amicable* if $XY^t = YX^t$. They are *disjoint* if $X \cap Y = 0$ (here, $\cap$ denotes the Hadamard, or entry-wise, product of matrices).

Lemma 1. *If there exist Hadamard matrices of order* $4m$ *and* $4n$ *then there exist two* (± 1) *matrices, S and R of order* $4mn$, *satisfying*

(i) $SS^{\mathrm{T}} + RR^{\mathrm{T}} = 8mnI_{4mn}$,

(ii) $SR^{\mathrm{T}} = RS^{\mathrm{T}} = 0$.

Proof. We write

$$H = \begin{pmatrix} \frac{H_1}{H_2} \\ \frac{H_3}{H_4} \end{pmatrix}, \qquad K = \begin{pmatrix} \frac{K_1}{K_2} \\ \frac{K_3}{K_4} \end{pmatrix},$$

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where H and K are the given Hadamard matrices, each H_i being of size $m \times 4m$ and each K_i being of size $n \times 4n$, $i = 1, 2, 3, 4$. Now take

$$R = \tfrac{1}{2}(H_1 + H_2)^T \times K_1 + \tfrac{1}{2}(H_1 - H_2)^T \times K_2,$$
$$S = \tfrac{1}{2}(H_3 + H_4)^T \times K_3 + \tfrac{1}{2}(H_3 - H_4)^T \times K_4.$$

Clearly both R and S are square ± 1 matrices, and $RR^T + SS^T = \frac{1}{2}(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 + H_4^T H_4) \times 4nI$. Since $(H_1^T H_1 + H_2^T H_2 + H_3^T H_3 + H_4^T H_4) = H^T H$, we have $RR^T + SS^T = 8mnI_{4mn}$. Since $K_i K_j^T = 0$ for $i \neq j$, R and S are as claimed.

This is Theorem 3 in [2], where S and R of are called an *orthogonal pair*.

LEMMA 2. *If there exist Hadamard matrices of order* 4*m and* 4*n then there exist two disjoint, amicable* $W(4mn, 2mn)$.

Proof. Let R and S be the matrices constructed in Lemma 1. Let $X = \frac{1}{2}(R + S)$ and $Y = \frac{1}{2}(R - S)$. We calculate

$$XX^T = YY^T = \tfrac{1}{4}(RR^T + SS^T) = 2mn\, I_{4mn}.$$

X and Y are disjoint since R and S are ± 1 matrices. Therefore, X and Y are the desired weighing matrices.

This is Theorem 2 and Lemma 3 of [4], where it was obtained using *M-structures*. It may also be deduced from Theorem 3 and 7 of [2], which follows our method. The fact that the matrices are amicable is not needed for the theorem which follows. The two lemmas are clearly equivalent, for we may also write $S = X + Y$, $R = X - Y$.

THEOREM 1. *If there exist Hadamard matrices of order* 4*m*, 4*n*, 4*p*, 4*q then there exists an Hadamard matrix of order* 16*mnpq*.

Proof. By Lemma 2, there exist two disjoint $W(4mn, 2mn)$, *X and Y*. By Lemma 1, there exist two (± 1) matrices S and R of order $4pq$ satisfying (i) and (ii).

Let $H = X \times S + Y \times R$. Then H is a (± 1) matrix and

$$HH^T = XX^T \times SS^T + YY^T \times RR^T = 2mnI_{4mn} \times (SS^T + RR^T)$$
$$= 2mnI_{4mn} \times 8pqI_{4pq} = 16mnpqI_{16mnpq}.$$

Thus H is the required Hadamard matrix.

Theorem 1 gives an improvement and extension for the result of Agayan [1] that if there exist Hadamard matrices of order $4m$ and $4n$ then there exists an Hadamard matrix of order $8mn$. Using the result of Agayan repeatedly on four Hadamard matrices of order $4m$, $4n$, $4p$, $4q$, gives an Hadamard matrix of order $32mnpq$.

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