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Product Representation of Dyon Partition Function in CHL Models

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Abstract: A formula for the exact partition function of 1/4 BPS dyons in a class of CHL models has been proposed earlier. The formula involves inverse of Siegel modular forms of subgroups of $Sp(2,\mathbb{Z})$. In this paper we propose product formulae for these modular forms. This generalizes the result of Borcherds and Gritsenko and Nikulin for the weight 10 cusp form of the full $Sp(2,\mathbb{Z})$ group.

Contents

1. Introduction and Summary

There exists a proposal for the exact degeneracy of dyons in toroidally compactified heterotic string theory[1, 2, 3, 4, 5] and also in toroidally compactified type II string theory[6]. These formulæ are invariant under the S-duality transformations of the theory, and also reproduce the entropy of a dyonic black hole in the limit of large charges[2]. In [7] this proposal was generalized to a class of CHL models[8, 9, 10, 11, 12, 13], obtained by modding out heterotic string theory on $T^2 \times T^4$ by a $\mathbb{Z}_N$ transformation that involves $1/N$ unit of translation along one of the circles of T^2 and a non-trivial action on the internal conformal field theory (CFT) describing heterotic string compactification on T^4. The values of N considered in [7] were N =2,3,5,7. Using string-string duality[14, 15, 16, 17, 18] one can relate these models to $\mathbb{Z}_N$ orbifolds of type IIA string theory on $T^2 \times K3$, with the $\mathbb{Z}_N$ transformation acting as $1/N$ unit of shift along a circle of T^2 together with an action on the internal CFT describing type IIA string compactification on $K3$.

The proposal of [7] may be summarized as follows. If we denote by Q_e and Q_m the electric and the magnetic charge vectors then the degeneracy $d(Q_e, Q_m)$ of dyons carrying charges (Q_e, Q_m) is of the form

$$d(Q_e, Q_m) = g\left(\frac{1}{2}Q_m^2, \frac{1}{2}Q_e^2, Q_e \cdot Q_m\right), \tag{1.1}$$

where $g(m, n, p)$ is defined through the Fourier expansion

$$\frac{1}{\widetilde{\Phi}_k(U, T, V)} = C_0 \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} e^{2\pi i(mU+nT+pV)} g(m, n, p)\,. \tag{1.2}$$

Here C_0 is a numerical constant and $\widetilde{\Phi}_k(U, T, V)$ is a modular form of weight k under a subgroup $\widetilde{G}$ of $Sp(2, \mathbb{Z}) \equiv SO(2, 3; \mathbb{Z})$ where

$$k = \frac{24}{N+1} - 2\,. \tag{1.3}$$

An explicit algorithm for constructing the Fourier expansion of $\widetilde{\Phi}_k$ in the variables T, U and V was given in [7].

The degeneracy $d(Q_e, Q_m)$ defined through eqs.(1.1), (1.2) is invariant under the T- and S-duality symmetries of the theory. Furthermore it generates integer results for the degeneracies and its behaviour for large charges is consistent with the black hole entropy calculation[7, 19].

In this paper we use the method of [20, 21] to propose an alternative form of $\widetilde{\Phi}_k$ as an infinite product:

$$\begin{aligned}\widetilde{\Phi}_k(U, T, V) = & -(i\sqrt{N})^{-k-2}\ \exp\left(2\pi i\left(\frac{1}{N}T + U + V\right)\right) \\ & \prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}+\frac{r}{N} \\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi i ls/N}\, c^{(r,s)}(4lk'-b^2)} \\ & \prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}-\frac{r}{N} \\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{2\pi i ls/N}\, c^{(r,s)}(4lk'-b^2)}\end{aligned} \tag{1.4}$$

where $(k', l, b) > 0$ means $k' > 0, l \geq 0, b \in \mathbb{Z}$ or $k' = 0, l > 0, b \in \mathbb{Z}$ or $k' = 0, l = 0, b < 0$ and $c^{(r,s)}(n)$ are some calculable coefficients related to the twisted elliptic genus of $K3$. If $\widetilde{g}$ denotes the generator of the $\mathbb{Z}_N$ action on $K3$ that is used in the construction of the CHL model, then we define the twisted elliptic genus of $K3$ as

$$F^{(r,s)}(\tau,z) = \frac{1}{N} Tr^{K3}_{RR;\widetilde{g}^r} \left((-1)^{F_{K3}} (-1)^{\bar{F}_{K3}} \widetilde{g}^s e^{2\pi i z F_{K3}} q^{L_0} \bar{q}^{\bar{L}_0} \right) , \quad 0 \le r, s \le (N-1) , \tag{1.5}$$

where $Tr^{K3}_{RR;\widetilde{g}^r}$ denotes trace in the superconformal field theory associated with target space $K3$ in the $\widetilde{g}^r$ twisted RR sector, $q = e^{2\pi i \tau}$, and F_{K3}, $\bar{F}_{K3}$ denote the left- and right-handed world-sheet fermion numbers in this theory. Here and throughout the rest of the paper L_0 and $\bar{L}_0$ include an additive factor of $-c/24$ so that the RR sector ground state has $L_0 = \bar{L}_0 = 0$. The coefficients $c^{(r,s)}(n)$ are then defined through the Fourier expansion of $F^{(r,s)}(\tau,z)$:

$$F^{(r,s)}(\tau,z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2)\, q^n\, e^{2\pi i z b} . \tag{1.6}$$

Furthermore for the $N = 2$, $k = 6$ case we were able to explicitly compute the functions $F^{(r,s)}(\tau,z)$. They are given by

$$\begin{aligned} F^{(0,0)}(\tau,z) &= 4 \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \right] , \\ F^{(0,1)}(\tau,z) &= 4\, \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} , \quad F^{(1,0)}(\tau,z) = 4 \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} , \quad F^{(1,1)}(\tau,z) = 4\, \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} . \end{aligned} \tag{1.7}$$

For higher values of N we did not evaluate the functions $F^{(r,s)}(\tau,z)$ directly, but were able to guess their forms from general considerations. The results are:

$$\begin{aligned} F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z) , \\ F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) - \frac{2}{N+1} B(\tau,z)\, E_N(\tau) \qquad \text{for } 1 \le s \le (N-1) , \\ F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)} A(\tau,z) + \frac{2}{N(N+1)} E_N\left(\frac{\tau+k}{N} \right) B(\tau,z) , \\ & \qquad\qquad \text{for } 1 \le r \le (N-1),\ 0 \le k \le (N-1) , \end{aligned} \tag{1.8}$$

where

$$A(\tau,z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \right] , \tag{1.9}$$

$$B(\tau, z) = \eta(\tau)^{-6} \vartheta_1(\tau, z)^2 \,, \tag{1.10}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)} \, \partial_\tau \left[\ln \eta(\tau) - \ln \eta(N\tau) \right] = 1 + \frac{24}{N-1} \sum_{\substack{n_1, n_2 \geq 1 \\ n_1 \neq 0 \, mod \, N}} n_1 e^{2\pi i n_1 n_2 \tau} \,. \tag{1.11}$$

Eq.(1.4) gives a generalization of Borcherds and Gritsenko and Nikulin's result[22, 23] of the product representation of $\widetilde{\Phi}_{10}$, – the unique cusp form of weight 10 of the group $Sp(2, \mathbb{Z})$. A systematic procedure for arriving at the product representation for $\widetilde{\Phi}_{10}$ was given in [20]. Our construction of $\widetilde{\Phi}_k$ is essentially based on a generalization of the techniques of [20].

Given the two different constructions of $\widetilde{\Phi}_k$, – one given in [7] and one in the present paper, it is natural to ask if they are the same. For the $N = 2$, $k = 6$ case we have compared 31 different Fourier expansion coefficients of the two proposals and found them to be the same.[1] For other values of N we have compared the expansions up to order $e^{4\pi i T} e^{4\pi i U}$ and all powers of $e^{2\pi i V}$. For general N we also verify that the behaviour of $\widetilde{\Phi}_k$ (and of Φ_k introduced in footnote 1) in the $V \to 0$ limit as well as in the $U \to i\infty$ limit agrees with the results found in [7].

The rest of the paper is organized as follows. In section 2 we outline the strategy that we shall be using for finding $\widetilde{\Phi}_k$. Sections 3 and 4 involve detailed calculations leading to the determination of $\widetilde{\Phi}_6$ associated with the $\mathbb{Z}_2$ orbifold theory. In section 5 we give the final form of $\widetilde{\Phi}_6$ and compare some of its properties with those found in [7]. Section 6 is devoted to the construction of the related quantity Φ_6 described in footnote 1 and its comparison with the corresponding quantity calculated in [7]. In section 7 we describe the construction of $\widetilde{\Phi}_k$ and Φ_k for a general k given in (1.3). The three appendices contain some technical details which were omitted from discussion in the main body of the paper.

2. The Strategy

Our goal is to find a product representation for $\widetilde{\Phi}_k$. In attaining this goal we shall proceed as in the case of ordinary toroidal compactification of heterotic string theory

[1] Actually we compare not the Fourier expansion coefficients of $\widetilde{\Phi}_k$ but those of a closely related object $\Phi_k(U, T, V) = T^{-k} \, \widetilde{\Phi}_k(U - T^{-1}V^2, -T^{-1}, T^{-1}V)$.

or equivalently type II string theory on $T^2 \times K3$. This corresponds to the case $N = 1$, $k = 10$ and the associated modular form $\widetilde{\Phi}_{10}$ is the unique weight 10 cusp form of the Siegel modular group $Sp(2; \mathbb{Z})$. The steps leading to a systematic construction of the product representation of $\widetilde{\Phi}_{10}$ are as follows[20]:

1. We consider a superconformal σ-model with target space $T^2 \times K3$ with y^1, y^2 denoting the T^2 coordinates. We denote by F_{K3} and F_{T^2} the holomorphic parts of the world-sheet fermion number associated with the $K3$ and the T^2 parts and by $\bar{F}_{K3}$ and $\bar{F}_{T^2}$ the anti-holomorphic parts of the world-sheet fermion number associated with the $K3$ and the T^2 parts. We shall be considering an arbitrary T^2 parametrized by the Kähler modulus T and complex structure modulus U, and arbitrary Wilson lines A_1, A_2 corresponding to deforming the world-sheet theory by the marginal operator

$$\sum_{i=1}^{2} A_i \int d^2z \, \bar{\partial} Y^i J_{K3} \, , \tag{2.1}$$

where J_{K3} is the U(1) current corresponding to the charge F_{K3}. We shall denote by V the complex combination $A_2 - iA_1$. V is normalized so that $V \equiv V + 1$.

This theory has an $SO(2,3; \mathbb{Z})$ T-duality group. If we denote by (m_1, m_2) the integers labeling momenta along y^1, y^2, by (n_1, n_2) the integers labeling winding along y^1, y^2, and by b the F_{K3} charge, then the $SO(2,3; \mathbb{Z})$ transformation S acts on these charges and the parameters T, U, V as

$$\begin{pmatrix} m_1' \\ m_2' \\ n_1' \\ n_2' \\ b' \end{pmatrix} = S \begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \\ b \end{pmatrix} , \qquad \begin{pmatrix} T' \\ T'U' - V'^2 \\ -U' \\ 1 \\ 2V' \end{pmatrix} = \lambda \, S \begin{pmatrix} T \\ TU - V^2 \\ -U \\ 1 \\ 2V \end{pmatrix} \tag{2.2}$$

where S is a 5×5 matrix with integer entries, satisfying

$$S^T L S = L, \qquad L = \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} , \tag{2.3}$$

and λ is a number to be adjusted so that the fourth element of the vector on the right hand side of (2.2) is 1. I_n denotes $n \times n$ identity matrix.

Using the equivalence between $SO(2,3)$ and $Sp(2)$ we can represent the T-duality group elements by $Sp(2,\mathbb{Z})$ matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C and D are each 2×2 matrix with integer entries satisfying

$$AB^T = BA^T, \qquad CD^T = DC^T, \qquad AD^T - BC^T = I_2 \,. \tag{2.4}$$

If we define

$$\Omega = \begin{pmatrix} U & V \\ V & T \end{pmatrix}, \tag{2.5}$$

then the duality group acts on Ω as

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \,. \tag{2.6}$$

2. In this theory we define:

$$\mathcal{I}_0(U,T,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} Tr_{RR}\left((-1)^{(F_{K3}+F_{T^2})} (-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})} F_{T^2}\bar{F}_{T^2} q^{L_0}\bar{q}^{\bar{L}_0}\right) \tag{2.7}$$

where $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$ and $q = e^{2\pi i\tau}$. $\mathcal{I}(U,T,V)$ is expected to be invariant under $SO(2,3;\mathbb{Z})$ transformation.

3. Analysis of the integral given in (2.7) shows that it can be expressed in the form

$$\mathcal{I}_0 = -20 \ln\det \mathrm{Im}\,\Omega - 2\ln\widetilde{\Phi}_{10}(\Omega) - 2\ln\widetilde{\Phi}_{10}(\bar{\Omega}) \; + \; \text{constant} \tag{2.8}$$

where $\widetilde{\Phi}_{10}(\Omega)$ is a holomorphic function of T, U and V with a product representation. Since under the duality transformation (2.6)

$$\det \mathrm{Im}\,\Omega \to (\det(C\Omega+D))^{-1}\,(\det(C\bar{\Omega}+D))^{-1}\,\det\mathrm{Im}\,\Omega \,, \tag{2.9}$$

and $\mathcal{I}_0$ is invariant, we must have[2]

$$\widetilde{\Phi}_{10}\left((A\Omega+B)(C\Omega+D)^{-1}\right) = (\det(C\Omega+D))^{10}\,\widetilde{\Phi}_{10}(\Omega) \,. \tag{2.10}$$

Thus $\widetilde{\Phi}_{10}(\Omega)$ must be a Siegel modular form of weight 10. This leads to the construction of the product representation of $\widetilde{\Phi}_{10}$.

[2]In principle there could be Ω independent phases on the right hand side of (2.10), but it is known that they are absent in this case.

Our goal is to construct a modular form $\widetilde{\Phi}_k$ of weight k of an appropriate subgroup $\widetilde{G}$ of $SO(2,3;\mathbb{Z})$ for k given in (1.3). The subgroup $\widetilde{G}$ is the T-duality group of the superconformal field theory with target space $(T^2 \times K3)/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ acts as a $1/N$ unit of shift along a circle on T^2 and as a geometric transformation of order N on $K3$.[3] Thus only those $SO(2,3;\mathbb{Z})$ transformation which commute with the $1/N$ unit of shift along T^2 will be symmetries of the resulting theory.

We shall try to construct $\widetilde{\Phi}_k$ by first defining an analog of the integral $\mathcal{I}_0$ invariant under this subgroup and then splitting it into a sum of an holomorphic piece, an anti-holomorphic piece and a term proportional to $\ln\det \mathrm{Im}\,\Omega$ as in (2.8). A natural candidate integral is

$$\mathcal{I}(U,T,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}\, Tr_{RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar F_{K3}+\bar F_{T^2})} F_{T^2}\bar F_{T^2} q^{L_0}\bar q^{\bar L_0}\right) \qquad (2.11)$$

where the trace is taken over the states in this orbifold superconformal field theory.

For $V=0$ this integral has been calculated for the $\mathbb{Z}_2$ orbifold model in [24]. In the next few sections we shall describe computation of this integral for the $N=2$ case for non-zero V. This will enable us to determine the product form of $\widetilde{\Phi}_6$. Later we shall discuss generalization of this analysis to other values of N.

3. The Integrand for the $\mathbb{Z}_2$ Orbifold Theory

In this section we shall analyze the integrand in eq.(2.11) for the $\mathbb{Z}_2$ orbifold conformal field theory described earlier. We can decompose the contribution to the trace in (2.11) as a sum of the contribution from different sectors characterized by the five charges (m_1, n_1, m_2, n_2, b) introduced earlier.[4] In this case we can factor out the T, U and V dependence of the trace into an overall factor of $q^{p_L^2/2}\bar q^{p_R^2/2}$ where

$$\begin{aligned}\frac{1}{2}p_R^2 &= \frac{1}{4\det Im\Omega}|-m_1U+m_2+n_1T+n_2(TU-V^2)+bV|^2, \\ \frac{1}{2}p_L^2 &= \frac{1}{2}p_R^2+m_1n_1+m_2n_2+\frac{1}{4}b^2\,.\end{aligned} \qquad (3.1)$$

[3]In order to preserve the $\mathcal{N}=4$ target space supersymmetry, the $\mathbb{Z}_N$ action on $K3$ must commute with the (4,4) superconformal symmetry possessed by a supersymmetric σ-model with target space $K3$.

[4]Note that now the twisted sector states carry half integer winding number n_1 along y^1.

Thus $\mathcal{I}(U,T,V)$ has the form

$$\mathcal{I}(U,T,V) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,m_2,n_1,n_2,b} q^{p_L^2/2 - b^2/4} \bar{q}^{p_R^2/2} F_{m_1,m_2,n_1,n_2;b}(\tau) \tag{3.2}$$

where $F_{m_1,m_2,n_1,n_2;b}(\tau)$ is independent of T, U and V and is given by

$$F_{m_1,m_2,n_1,n_2;b}(\tau) = Tr_{m_1,m_2,n_1,n_2;b;RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})} F_{T^2}\bar{F}_{T^2} q^{L_0'}\bar{q}^{\bar{L}_0'}\right) . \tag{3.3}$$

Here

$$L_0' = L_0 - \frac{p_L^2}{2} + \frac{b^2}{4}, \qquad \bar{L}_0' = \bar{L}_0 - \frac{p_R^2}{2} , \tag{3.4}$$

are independent of T, U and V and $Tr_{m_1,m_2,n_1,n_2;b}$ denotes trace over a subspace of the Hilbert space carrying momentum (m_1, m_2) and winding (n_1, n_2) along T^2 and F_{K3} charge b. Note that we have included the $b^2/4$ term in L_0' so that for $V = 0$ when the conformal field theories associated with $K3$ and T^2 parts decouple, L_0' and $\bar{L}_0'$ describe complete contribution from the CFT associated with $K3$ and oscillator contribution from the CFT associated with T^2. Since $F_{m_1,m_2,n_1,n_2;b}(\tau)$ is independent of T, U and V, we can set $V = 0$ while evaluating (3.3).

Let us define

$$F_{m_1,m_2,n_1,n_2}(\tau, z) = \sum_b F_{m_1,m_2,n_1,n_2;b}(\tau)\, e^{2\pi i b z} . \tag{3.5}$$

It then follows from (3.3) that

$$F_{m_1,m_2,n_1,n_2}(\tau, z) = Tr_{m_1,m_2,n_1,n_2;RR}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})} F_{T^2}\bar{F}_{T^2} e^{2\pi i z F_{K3}}\, q^{L_0'}\bar{q}^{\bar{L}_0'}\right) . \tag{3.6}$$

We shall first compute $F_{m_1,m_2,n_1,n_2}(\tau, z)$ and then extract $F_{m_1,m_2,n_1,n_2;b}(\tau)$ using eq.(3.5). Since the contribution to (3.6) from the T^2 part is somewhat trivial, it is useful to separate out this contribution. For this we denote by g' the generator of the $\mathbb{Z}_2$ group with which we take the orbifold of $K3 \times T^2$. Then

$$\begin{aligned} &F_{m_1,m_2,n_1,n_2}(\tau, z) \\ &= \frac{1}{2} \sum_{r,s=0}^{1} Tr^{K3\times T^2}_{m_1,m_2,n_1,n_2;RR;(g')^r}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar{F}_{K3}+\bar{F}_{T^2})} F_{T^2}\bar{F}_{T^2} e^{2\pi i z F_{K3}}\, q^{L_0'}\bar{q}^{\bar{L}_0'}(g')^s\right) , \end{aligned} \tag{3.7}$$

where the superscript $K3 \times T^2$ in Tr indicates that the trace is taken in the superconformal field theory with target space $K3 \times T^2$, and the subscript $(g')^r$ in Tr indicates that the trace is over the sector twisted by $(g')^r$. We now split g' as

$$g' = \hat{g}\,\widetilde{g}\,, \tag{3.8}$$

where $\hat{g}$ and $\widetilde{g}$ represent the action of g' on the T^2 and $K3$ parts respectively. Twisting by $\hat{g}^r$ makes the winding number $n_1 \in \mathbb{Z} + \frac{r}{2}$, and hence the right hand side of (3.7) vanishes unless $n_1 - \frac{r}{2} \in \mathbb{Z}$. The $(\hat{g})^s$ factor inside the trace produces a factor of $(-1)^{m_1 s}$. The non-zero mode bosonic and fermionic oscillator contributions from the T^2 factor always cancel since they are neutral under $\hat{g}$. The fermion zero modes associated with T^2 give a factor of 4 due to 2-fold degeneracy each from the holomorphic and anti-holomorphic sectors, but this cancels with the factor of 1/4 coming from the $F_{T^2}\bar{F}_{T^2}$ factor inside the trace. Thus we can write

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \sum_{s=0}^{1} (-1)^{m_1 s}\, F^{(r,s)}(\tau,z) \qquad \text{for } n_1 \in \mathbb{Z} + \tfrac{r}{2}, \quad r = 0,1 \tag{3.9}$$

where

$$F^{(r,s)}(\tau,z) = \frac{1}{2}\, Tr^{K3}_{RR;\widetilde{g}^r} \left((-1)^{F_{K3}}(-1)^{\bar{F}_{K3}}\, \widetilde{g}^s\, e^{2\pi i z F_{K3}}\, q^{L_0}\bar{q}^{\bar{L}_0}\right) . \tag{3.10}$$

Here $Tr^{K3}_{RR;\widetilde{g}^r}$ denotes trace in the superconformal field theory associated with target space $K3$ in the $\widetilde{g}^r$ twisted RR sector, and L_0, $\bar{L}_0$ inside the trace now includes contribution from $K3$ only. This is twisted elliptic genus of $K3$. These quantities were introduced in [25] in order to calculate the elliptic genus of $\widetilde{g}$ orbifold of $K3$. This would be given by $\sum_{r,s=0}^{1} F^{(r,s)}(\tau,z)$. Here however we need the individual $F^{(r,s)}(\tau,z)$ since we shall be using them for a different purpose.

From the definitions given in (3.10) it follows that[25]

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \exp\left(2\pi i \frac{c\,z^2}{c\tau+d}\right) F^{(cs+ar,ds+br)}(\tau,z)\,, \tag{3.11}$$

for

$$a,b,c,d \in \mathbb{Z}, \qquad ad-bc = 1\,. \tag{3.12}$$

In (3.11) the indices $cs+ar$ and $ds+br$ are to be taken mod 2.

$F_{m_1,m_2,n_1,n_2}(\tau,z)$ has been calculated in appendix A using an orbifold description of $K3$ and the result is given in eq.(A.15). Comparing this with eq.(3.9) we get

$$F^{(0,0)}(\tau,z) = 4\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right],$$
$$F^{(0,1)}(\tau,z) = 4\,\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}, \quad F^{(1,0)}(\tau,z) = 4\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} \quad F^{(1,1)}(\tau,z) = 4\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}. \tag{3.13}$$

Using the known modular transformation laws of $\vartheta_i(\tau,z)$ we can verify that $F^{(r,s)}(\tau,z)$ given in (3.13) satisfy (3.11).

We now use the relations:

$$\begin{aligned}
\vartheta_1^2(\tau,z) &= \vartheta_2(2\tau,0)\vartheta_3(2\tau,2z) - \vartheta_3(2\tau,0)\vartheta_2(2\tau,2z)\\
\vartheta_2^2(\tau,z) &= \vartheta_2(2\tau,0)\vartheta_3(2\tau,2z) + \vartheta_3(2\tau,0)\vartheta_2(2\tau,2z)\\
\vartheta_3^2(\tau,z) &= \vartheta_3(2\tau,0)\vartheta_3(2\tau,2z) + \vartheta_2(2\tau,0)\vartheta_2(2\tau,2z)\\
\vartheta_4^2(\tau,z) &= \vartheta_3(2\tau,0)\vartheta_3(2\tau,2z) - \vartheta_2(2\tau,0)\vartheta_2(2\tau,2z)
\end{aligned} \tag{3.14}$$

to rewrite (3.13) as

$$F^{(r,s)}(\tau,z) = h_0^{(r,s)}(\tau)\,\vartheta_3(2\tau,2z) + h_1^{(r,s)}(\tau)\,\vartheta_2(2\tau,2z) \tag{3.15}$$

where

$$\begin{aligned}
h_0^{(0,0)}(\tau) &= 8\,\frac{\vartheta_3(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_3(2\tau,0)}\\
h_1^{(0,0)}(\tau) &= -8\,\frac{\vartheta_2(2\tau,0)^3}{\vartheta_3(\tau,0)^2\vartheta_4(\tau,0)^2} + 2\,\frac{1}{\vartheta_2(2\tau,0)}\\
h_0^{(0,1)}(\tau) &= 2\,\frac{1}{\vartheta_3(2\tau,0)}, \qquad h_1^{(0,1)}(\tau) = 2\,\frac{1}{\vartheta_2(2\tau,0)},\\
h_0^{(1,0)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_4(\tau,0)^2}, \qquad h_1^{(1,0)}(\tau) = -4\frac{\vartheta_2(2\tau,0)}{\vartheta_4(\tau,0)^2},\\
h_0^{(1,1)}(\tau) &= 4\frac{\vartheta_3(2\tau,0)}{\vartheta_3(\tau,0)^2}, \qquad h_1^{(1,1)}(\tau) = 4\frac{\vartheta_2(2\tau,0)}{\vartheta_3(\tau,0)^2},
\end{aligned} \tag{3.16}$$

Since

$$\vartheta_3(2\tau,2z) = \sum_{b\in 2\mathbb{Z}} e^{2\pi ibz} q^{b^2/4}, \qquad \vartheta_2(2\tau,2z) = \sum_{b\in 2\mathbb{Z}+1} e^{2\pi ibz} q^{b^2/4}, \tag{3.17}$$

we see, by comparing (3.5) and (3.9), (3.15) that

$$F_{m_1,m_2,n_1,n_2;b}(\tau) = q^{b^2/4} \sum_{s=0}^{1} (-1)^{m_1 s} \, h_l^{(r,s)}(\tau) \quad \text{for } n_1 \in \mathbb{Z} + \tfrac{r}{2},\, b \in 2\mathbb{Z} + l$$
$$r, l = 0, 1\,. \tag{3.18}$$

Using (3.18) the original integral $\mathcal{I}(U,T,V)$ given in eq.(3.2) may be written as

$$\mathcal{I}(U,T,V) = \sum_{l,r,s=0}^{1} \mathcal{I}_{r,s,l} \tag{3.19}$$

where

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2 \in \mathbb{Z} \\ n_1 \in \mathbb{Z} + \frac{r}{2}, b \in 2\mathbb{Z}+l}} q^{p_L^2/2} \bar{q}^{p_R^2/2} \, (-1)^{m_1 s} \, h_l^{(r,s)}(\tau)\,. \tag{3.20}$$

From this we see that those $SO(2,3;\mathbb{Z})$ transformations which, acting on a vector (m_1, m_2, n_1, n_2, b) with m_1, m_2, n_2, b integers and n_1 half-integer, preserves m_1 modulo 2, n_1, m_2, n_2 modulo 1 and b modulo 2, will be symmetries of $\mathcal{I}$. This defines the subgroup $\widetilde{G}$.

For later use we define the coefficients $c^{(r,s)}(4n)$ through the expansion

$$h_0^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n\,, \qquad h_1^{(r,s)}(\tau) = \sum_n c^{(r,s)}(4n) q^n\,. \tag{3.21}$$

By examining (3.16) we see that in the expansion of $h_l^{(r,s)}$, $n \in \mathbb{Z} - \frac{l}{4}$ for $r = 0$ and $n \in \frac{1}{2}\mathbb{Z} - \frac{l}{4}$ for $r = 1$. Note that we have used the same symbol $c^{(r,s)}(4n)$ for describing the expansion of $h_0^{(r,s)}(\tau)$ and $h_1^{(r,s)}(\tau)$. This is possible since $c^{(r,s)}(4n)$ has different support for $l = 0$ and $l = 1$.

Using eq.(3.15) and the Fourier expansion (3.17) of ϑ_3 and ϑ_2 we can write the double Fourier expansion of $F^{(r,s)}(\tau, z)$

$$F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n} c^{(r,s)}(4n - b^2)\, q^n \, e^{2\pi i z b}\,, \tag{3.22}$$

where $n \in \mathbb{Z}$ for $r = 0$ and $\frac{1}{2}\mathbb{Z}$ for $r = 1$.

4. The Integral

We shall now proceed to evaluate the integral (3.20). We define

$$Y = \det \mathrm{Im}\Omega = T_2 U_2 - (V_2)^2, \qquad T_2 > 0, \quad U_2 > 0, \quad Y > 0\,. \tag{4.1}$$

where for any complex number u, we denote by u_1 and u_2 its real and imaginary parts respectively. Substituting the values of p_L^2 and p_R^2 from (3.1) into (3.20) we obtain

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,m_2,n_2\in \mathbb{Z}, n_1\in \mathbb{Z}+\frac{r}{2}, b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1 + m_2n_2 + \frac{b^2}{4})\right] \times \\ \exp\left[\frac{-\pi\tau_2}{Y}\left|n_2(TU - V^2) + bV + n_1T - Um_1 + m_2\right|^2\right] (-1)^{m_1 s}\, h_l^{(r,s)}(\tau)\,. \tag{4.2}$$

To evaluate the integral we first perform the Poisson resummation over the momenta m_1, m_2. The basic formula for Poisson resummation we will use is

$$\sum_{m\in\mathbb{Z}} f(m) e^{2\pi i s m/N} = \sum_{k\in\mathbb{Z}+\frac{s}{N}} \int_{-\infty}^{\infty} du f(u)\exp(2\pi i k u) \tag{4.3}$$

for any integer N. Now performing the Poisson resummation over m_1, m_2 and performing the Gaussian integration over the corresponding variables u_1, u_2, we obtain the following

$$\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2} \sum_{n_2,k_2\in\mathbb{Z}, n_1\in\mathbb{Z}+\frac{r}{2}, k_1\in\mathbb{Z}+\frac{s}{2}, b\in 2\mathbb{Z}+l} h_l^{(r,s)}(\tau)\exp\left[\mathcal{G}(\vec{n},\vec{k},b)\right] \tag{4.4}$$

where

$$\begin{aligned}\mathcal{G}(\vec{n},\vec{k},b) = & -\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T\det A \\ & + \frac{\pi b}{U_2}(V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}) - \frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) \\ & + \frac{2\pi i V_2^2}{U_2^2}(n_1 + n_2\bar{U})\mathcal{A} + 2\pi i\tau\frac{b^2}{4}\,,\end{aligned} \tag{4.5}$$

$$A = \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix}, \tag{4.6}$$

$$\mathcal{A} = (1, U) A \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \qquad \tilde{\mathcal{A}} = (1, \bar{U}) A \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \tag{4.7}$$

Using (4.5) we can represent the sum over b in (4.4) as

$$\sum_{b\in 2\mathbb{Z}+l} e^{2\pi i\tau\frac{b^2}{4} + \frac{\pi b}{U_2}(V\tilde{\mathcal{A}} - \bar{V}\mathcal{A})} = \begin{cases} \vartheta_3(2\tau, -i\frac{V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}}{U_2}) & \text{for } l = 0 \\ \vartheta_2(2\tau, -i\frac{V\tilde{\mathcal{A}} - \bar{V}\mathcal{A}}{U_2}) & \text{for } l = 1 \end{cases} \tag{4.8}$$

Substituting this into (4.4) and using (3.15) we get

$$\mathcal{I} \equiv \sum_{l,r,s=0}^{1} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{r,s=0}^{1} \sum_{n_2,k_2\in\mathbb{Z},n_1\in\mathbb{Z}+\frac{r}{2},k_1\in\mathbb{Z}+\frac{s}{2}} \mathcal{J}(A,\tau)\,, \tag{4.9}$$

where

$$\begin{aligned}\mathcal{J}(A,\tau) = \frac{Y}{U_2}\exp\Bigg(&-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A \\ &-\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A}\Bigg) F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{2\,U_2}\right) \\ & r = 2\,n_1 \bmod 2, \quad s = 2\,k_1 \bmod 2\,.\end{aligned} \tag{4.10}$$

In order to interpret the right hand side as a function of the matrix A we need to use eqs.(4.6), (4.7). We may now interpret the sum over r,s and $\vec{n},\vec{k}$ in the right hand side of eq.(4.9) as a sum over all matrices A of the form (4.6) with n_2, k_2 integer, and n_1, k_1 integer or half-integer. (4.9) may then be rewritten as

$$\mathcal{I} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_A \mathcal{J}(A,\tau)\,. \tag{4.11}$$

Now it follows from the modular transformation laws (3.11) and the definition of $\mathcal{J}(A,\tau)$ given in (4.10) that

$$\mathcal{J}\left(A, \frac{a\tau+b}{c\tau+d}\right) = \mathcal{J}\left(A\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tau\right)\,. \tag{4.12}$$

Using this symmetry, we can extend the integration over the fundamental domain to its images under $SL(2,\mathbb{Z})$ and at the same time restrict the summation over A to summation over inequivalent $SL(2,\mathbb{Z})$ orbits. If we denote by $\sum_A'$ the sum over inequivalent $SL(2,\mathbb{Z})$ orbits then we can express $\mathcal{I}$ as

$$\begin{aligned}\mathcal{I} = {\sum_A}' \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\exp\Bigg(&-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T\det A \\ &-\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A}\Bigg) F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{2\,U_2}\right)\end{aligned} \tag{4.13}$$

where now r, s in the label of $F^{(r,s)}$ are to be interpreted as $2\,n_1$ mod 2 and $2\,k_1$ mod 2 respectively. The region of integration $\mathcal{F}_A$ depends on the orbit represented by A.

Following the same procedure as in [28] we now split the integration into the three orbits. These are the zero orbit

$$A = 0\,, \tag{4.14}$$

the non-degenerate orbit

$$A = \begin{pmatrix} k & j \\ 0 & p \end{pmatrix}, \quad 2k-1 \geq 2j \geq 0, \quad p \neq 0, \quad k,j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,, \tag{4.15}$$

and the degenerate orbit

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j,p) \neq (0,0), \quad j \in \frac{1}{2}\mathbb{Z}, \quad p \in \mathbb{Z}\,. \tag{4.16}$$

The contribution from these orbits has been evaluated in appendix B. The final result, as given in (B.39), takes the form

$$\begin{aligned}
\mathcal{I} &= -2\ln\left[\kappa(\det \mathrm{Im}\Omega)^6 \left| \exp\left(2\pi i\left(\frac{1}{2}T + U + V\right)\right)\right.\right. \\
&\qquad \left.\left.\prod_{r,s=0}^{1} \prod_{\substack{(l,b)\in\mathbb{Z},k\in\mathbb{Z}+\frac{r}{2}\\ (k,l,b)>0}} \left\{(1-\exp(2\pi i(kT+lU+bV))^{(-1)^{ls}\,c^{(r,s)}(4kl-b^2)}\right\}\right|^2\right] \\
\kappa &= \left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)^6
\end{aligned} \tag{4.17}$$

and $(k,l,b) > 0$ means $k > 0, l \geq 0, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = 0, l = 0, b < 0$.

5. $\widetilde{\Phi}_6$ and its $V \to 0$ Limit

Eq.(4.17) can be written as

$$\mathcal{I} = -2\left[6\ln\det\mathrm{Im}\Omega + \ln\widetilde{\Phi}_6 + \ln\bar{\widetilde{\Phi}}_6 + \ln\kappa + 8\ln 2\right]\,, \tag{5.1}$$

where

$$\begin{aligned}
\widetilde{\Phi}_6(\Omega) &= \frac{1}{16}\exp\left(2\pi i\left(\frac{1}{2}T + U + V\right)\right) \\
&\qquad \prod_{r,s=0}^{1} \prod_{\substack{l,b\in\mathbb{Z},k\in\mathbb{Z}+\frac{r}{2}\\ k,l,b>0}} \Big[1-\exp\left\{2\pi i(kT+lU+bV)\right\}\Big]^{(-1)^{ls}c^{(r,s)}(4lk-b^2)}\,.
\end{aligned} \tag{5.2}$$

Note that we have normalized $\widetilde{\Phi}_6$ so that the coefficient of $\exp(2\pi i(\frac{1}{2}T + U + V))$ is $1/16$. This agrees with the normalization convention of [7].

Since under a duality transformation by an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\widetilde{G} \subset Sp(2, \mathbb{Z})$

$$\det \mathrm{Im}\Omega \to |\det(C\Omega + D)|^{-2} \det \mathrm{Im}\Omega \,, \tag{5.3}$$

we must have

$$\widetilde{\Phi}_6((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^6 \widetilde{\Phi}_6(\Omega) \,, \tag{5.4}$$

in order that $\mathcal{I}$ given in (5.1) is invariant under this transformation. Thus $\widetilde{\Phi}_6$ transforms as a modular form of weight 6 under $\widetilde{G}$.

We shall now analyze the $V \to 0$ limit of (5.2) and compare this with the corresponding result in [7]. This analysis is facilitated by examining the relation (3.22) at $z = 0$:

$$\sum_n \sum_b c^{(r,s)}(4n - b^2) q^n = F^{(r,s)}(\tau, 0) = \begin{cases} 12 & \text{for} \quad (r,s) = (0,0) \\ 4 & \text{for} \quad (r,s) \neq (0,0) \end{cases} . \tag{5.5}$$

This gives

$$\sum_b c^{(r,s)}(4n - b^2) = \begin{cases} 12\,\delta_{n,0} & \text{for} \quad (r,s) = (0,0) \\ 4\,\delta_{n,0} & \text{for} \quad (r,s) \neq (0,0) \end{cases} . \tag{5.6}$$

Taking $V \to 0$ limit in (5.2) we now get

$$\widetilde{\Phi}_6(U,T,V) \simeq -\frac{4\pi^2 V^2}{16}\, e^{2\pi i\left(\frac{1}{2}T + U\right)} \prod_{\substack{k=1 \\ k\in\mathbb{Z}}}^{\infty} \left\{ \left(1 - e^{2\pi i k T}\right)^8 \left(1 - e^{\pi i k T}\right)^8 \right\}$$
$$\prod_{\substack{l=1 \\ l\in\mathbb{Z}}}^{\infty} \left\{ \left(1 - e^{2\pi i l U}\right)^8 \left(1 - e^{4\pi i l U}\right)^8 \right\} \tag{5.7}$$

where the $-4\pi^2 V^2$ term comes from the $k = l = 0$, $b = -1$ term. This can be rewritten as

$$\widetilde{\Phi}_6(U,T,V) \simeq -\frac{1}{4}\pi^2\, V^2 \eta(T/2)^8 \eta(T)^8 \eta(U)^8 \eta(2U)^8 \,. \tag{5.8}$$

This factorization property, including the overall normalization of $-\frac{1}{4}\pi^2$, agrees with that found in [7].

6. Construction of Φ_6

In the analysis of [7] we introduced another function Φ_6 related to $\widetilde{\Phi}_6$ by:

$$\widetilde{\Phi}_6(U,T,V) = T^{-6} \Phi_6\left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right) , \tag{6.1}$$

or equivalently

$$\Phi_6(U,T,V) = T^{-6}\widetilde{\Phi}_6\left(U - \frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right) . \tag{6.2}$$

From the expressions for $\mathcal{I}_{r,s,l}$ given in (4.2) we see that this transformation may be implemented by

$$m_2 \to n_1, \quad n_1 \to -m_2, \quad m_1 \to -n_2, \quad n_2 \to m_1 . \tag{6.3}$$

Thus in order to find an expression for Φ_6 we can replace $\mathcal{I}_{r,s,l}$ given in (4.2) by $\mathcal{I}'_{r,s,l}$ in which we sum over $m_2 \in \mathbb{Z} + \frac{r}{2}$ instead of $n_1 \in \mathbb{Z} + \frac{r}{2}$, and replace the $(-1)^{m_1 s}$ factor in the summand by $(-1)^{n_2 s}$:

$$\begin{aligned} \mathcal{I}'_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{m_1,n_1,n_2\in\mathbb{Z}, m_2\in\mathbb{Z}+\frac{r}{2}, b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1 n_1 + m_2 n_2 + \frac{b^2}{4})\right] \times \\ \exp\left[\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2) + bV + n_1 T - U m_1 + m_2\right|^2\right] (-1)^{n_2 s}\, h_l^{(r,s)}(\tau) . \end{aligned} \tag{6.4}$$

After Poisson resummation this amounts to summing over only integer values of n_1, n_2, k_1, k_2 and including a factor of

$$(-1)^{k_2 r}(-1)^{n_2 s} , \tag{6.5}$$

in the summand. The integral can now be evaluated following exactly the same procedure as in appendix B, the only difference being that the sum over p in eqs.(B.11), (B.21), (B.26) will contain an additional factor of $(-1)^{pr}$.[5] The net contribution to the full integral comes out to be

$$\begin{aligned} \mathcal{I}' = -2\ln\Bigg[2^8\,\kappa(\det \mathrm{Im}\Omega)^6\Bigg|\exp(2\pi i(T+U+V)) \\ \prod_{r,s=0}^{1} \prod_{\substack{(k,l.b)\in\mathbb{Z} \\ (k,l,b)>0}} \left\{1 - (-1)^r \exp(2\pi i(kT + lU + bV)\right\}^{c^{(r,s)}(4kl-b^2)}\Bigg|^2\Bigg] . \end{aligned} \tag{6.6}$$

[5] An apparent additional complication arises due to the fact that the Fourier expansions of $F^{(1,0)}$ and $F^{(1,1)}$ as given in (3.22) have half integer powers of q. Thus the sum over j in eq.(B.8) will not vanish for non-integer n/k. However since $F^{(1,0)} + F^{(1,1)}$ is invariant under $\tau \to \tau + 1$ due to the modular properties described in (3.11), it has Fourier expansion in integer powers of q. Thus if in analyzing the sum over j in (B.8) we consider the contribution from $F^{(1,0)}$ and $F^{(1,1)}$ together, the sum over j will force n to be a multiple of k.

We can rewrite this as

$$\mathcal{I}' = -2\left[6\ln\det\mathrm{Im}\Omega + \ln\Phi_6 + \ln\bar{\Phi}_6 + \ln\kappa + 8\ln 2\right] , \qquad (6.7)$$

where

$$\Phi_6(\Omega) = -\exp(2\pi i(T+U+V)) \prod_{r,s=0}^{1} \prod_{\substack{(k,l.b)\in \mathbb{Z} \\ (k,l,b)>0}} \left\{1-(-1)^r \exp(2\pi i(kT+lU+bV)\right\}^{c^{(r,s)}(4kl-b^2)} . \qquad (6.8)$$

The normalization of Φ_6 is not arbitrary; it has been chosen so that we have the same additive constant $8\ln 2$ in (6.7) as in (5.1). The phase of Φ_6 can be adjusted. With the choice of phase given in (6.8) the coefficient of the $e^{2\pi i(T+U+V)}$ term matches with that of the corresponding expression in [7]. Following the same argument as in the case of $\widetilde{\Phi}_6$ we can argue that Φ_6 transforms as a modular form of weight 6 under a subgroup G of $Sp(2,\mathbb{Z})$ which is related to the earlier subgroup $\widetilde{G}$ by the conjugation described in (6.3).

Study of the $V\to 0$ limit of this expression is also straightforward. Using the relations (5.6) and the explicit expressions for the coefficients $c^{(r,s)}(0)$ and $c^{(r,s)}(-1)$ given in (B.35), we get

$$\Phi_6(U,T,V) \simeq 4\pi^2\, V^2\, \eta(T)^8\eta(2T)^8\eta(U)^8\eta(2U)^8 . \qquad (6.9)$$

This is the same behaviour as found in [7].

We can also carry out a more detailed comparison between the Φ_6 defined here and those in [7]. The algorithm given in [7] goes as follows:

- We first define a set of coefficients f_n $(n\ge 1)$ through the relation:

$$\sum_{n\ge 1} f_n e^{2\pi i\tau(n-\frac{1}{4})} = \eta(\tau)^2\,\eta(2\tau)^8 , \qquad (6.10)$$

  where $\eta(\tau)$ is the Dedekind function.

- Next we define the coefficients $C(m)$ through

$$C(m) = (-1)^m \sum_{\substack{s,n\in \mathbb{Z} \\ n\ge 1}} f_n \delta_{4n+s^2-1,m} . \qquad (6.11)$$

- Φ_6 is now given by

$$\Phi_6(U,T,V) = \sum_{\substack{n,m,r\in\mathbb{Z}\\ n,m\geq 1,\, r^2<4mn}} a(n,m,r)\, e^{2\pi i(nU+mT+rV)}\,, \tag{6.12}$$

where

$$a(n,m,r) = \sum_{\substack{\alpha\in 2\mathbb{Z}+1\\ \alpha|(n,m,r),\alpha>0}} \alpha^{k-1}\, C\left(\frac{4mn-r^2}{\alpha^2}\right)\,, \tag{6.13}$$

We have compared 31 different coefficients $a(n,m,r)$ defined in (6.13) with the ones obtained from (6.8) and found them to be the same. These results for $a(n,m,r)$ are given in appendix C.

7. Construction of Φ_k and $\widetilde{\Phi}_k$

Generalization of the modular form $\widetilde{\Phi}_6$ to describe the degeneracy of dyons in a $\mathbb{Z}_N$ orbifold of $T^2\times K3$ for $N=2,3,5,7$ was also introduced in [7]. The generator g' of the $\mathbb{Z}_N$ is given by

$$g' = \hat{g}\,\widetilde{g}\,, \tag{7.1}$$

where $\hat{g}$ represents $1/N$ unit of shift along T^2 (which we shall take to be in the y^1 direction) and $\widetilde{g}$ denotes an appropriate $\mathbb{Z}_N$ action on $K3$. $\widetilde{g}$ preserves the harmonic (0,0)-form, (2,2)-form, (0,2)-form and (2,0)-form. Furthermore for each $r\neq 0$, there are $24/(N+1)$ (1,1)-forms with $\widetilde{g}$ eigenvalue $e^{2\pi i r/N}$. The rest of the $20-24(N-1)/(N+1)$ of the (1,1)-forms are invariant under $\widetilde{g}$.

The generating function for the degeneracy is given by $(\widetilde{\Phi}_k)^{-1}$ where

$$k = \frac{24}{N+1} - 2\,, \tag{7.2}$$

and $\widetilde{\Phi}_k$ is a weight k modular form of a subgroup $\widetilde{G}$ of $Sp(2,\mathbb{Z}) = SO(2,3;\mathbb{Z})$ that commutes with $1/N$ unit of shift along a circle of T^2. Associated with $\widetilde{\Phi}_k$ there is a modular form Φ_k of a different subgroup G of $Sp(2,\mathbb{Z})$, related to G by conjugation described in (6.2):

$$\Phi_k(U,T,V) = T^{-k}\widetilde{\Phi}_k\left(U-\frac{V^2}{T}, -\frac{1}{T}, \frac{V}{T}\right)\,. \tag{7.3}$$

Our goal is to find a product representation of Φ_k and $\widetilde{\Phi}_k$. For this we shall start with an analog of eq.(2.11) for the superconformal field theory associated with the $\mathbb{Z}_N$ orbifold of $K3 \times T^2$ and express it as a sum of a holomorphic and an anti-holomorphic term and a term proportional to $\ln \det \mathrm{Im}\Omega$. The holomorphic part can then be identified with Φ_k. Proceeding as in section 2 we arrive at the analog of eq.(3.9), (3.10)

$$F_{m_1,m_2,n_1,n_2}(\tau, z) = \sum_{s=0}^{N-1} e^{2\pi i m_1 s/N} \; F^{(r,s)}(\tau, z) \qquad \text{for } n_1 \in \mathbb{Z} + \tfrac{r}{N}, \quad r = 0, 1, \ldots (N-1)\,, \tag{7.4}$$

where

$$F^{(r,s)}(\tau, z) = \frac{1}{N}\, Tr^{K3}_{RR;\widetilde{g}^r} \left((-1)^{F_{K3}} (-1)^{\bar{F}_{K3}} \, \widetilde{g}^s \, e^{2\pi i z F_{K3}} \, q^{L_0} \bar{q}^{\bar{L}_0} \right) . \tag{7.5}$$

From these definitions it follows that

$$F^{(r,s)} \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \exp\left(2\pi i \frac{cz^2}{c\tau + d} \right) F^{(cs+ar, ds+br)}(\tau, z)\,, \tag{7.6}$$

for

$$a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1\,. \tag{7.7}$$

In (7.6) the indices $cs + ar$ and $ds + br$ are to be taken mod N. Thus for each (r, s), $F^{(r,s)}(\tau, z)$ transforms as a weak Jacobi form[26] of weight zero and index 1 under the group $\Gamma(N)$.

We can now define the coefficients $c^{(r,s)}(n)$ in a manner analogous to (3.22)[6]

$$F^{(r,s)}(\tau, z) = \sum_{b \in \mathbb{Z}, n \in \mathbb{Z}/N} c^{(r,s)}(4n - b^2)\, q^n \, e^{2\pi i z b}\,. \tag{7.8}$$

Contribution to $c^{(0,s)}(l)$ for $l = 0, -1$ comes from geometric data of $K3$ and can be computed easily. In particular untwisted sector states with $n = 0$, $b = 0$ are associated with (1,1)-forms, those with $n = 0$, $b = 1$ are associated with the (2,2) and the (2,0)-forms, and those with $n = 0$, $b = -1$ are associated with the (0,0) and the (0,2)-forms. Thus $Nc^{(0,s)}(0)$ measures the trace of $\widetilde{g}^s$ on the (1,1)-forms of $K3$ and $Nc^{(0,s)}(-1)$ measures the trace of $\widetilde{g}^s$ on the (0,0), (0,2) or (2,0), (2,2)-forms of

[6] In order that $F^{(r,s)}(\tau, z)$ has an expansion of the form given in (7.8) we need to ensure that this can be expressed as a linear combination of $\vartheta_3(2\tau, 2z)$ and $\vartheta_2(2\tau, 2z)$ with z-independent coefficients as in (3.15). This follows from the fact that the z-dependence of $F^{(r,s)}(\tau, z)$ comes from the SU(2) current algebra associated with the superconformal field theory, and this commutes with the $\mathbb{Z}_N$ generator $\widetilde{g}$. $\vartheta_3(2\tau, 2z)$ and $\vartheta_2(2\tau, 2z)$ simply represent the contributions from the even and odd F_{K3} charge sector of this SU(2) sector of the theory.

$K3$. These can be easily computed from the $\widetilde{g}$ action of the cycles described earlier, and we get

$$c^{(0,0)}(0) = \frac{20}{N}\,, \qquad c^{(0,0)}(-1) = \frac{2}{N}\,,$$
$$c^{(0,s)}(0) = \frac{1}{N}\left(20 - \frac{24N}{N+1}\right)\,, \qquad c^{(0,s)}(-1) = \frac{2}{N}\,, \quad \text{for } s = 1, 2, \ldots (N-1)\,. \tag{7.9}$$

Several other useful properties of $c^{(r,s)}$ may be derived without explicitly computing $F^{(r,s)}(\tau, z)$. First note that $F^{(0,0}(\tau, z)$ is $1/N$ times the elliptic genus of $K3$. Hence it is given by

$$F^{(0,0)}(\tau, z) = \frac{8}{N}\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right]\,. \tag{7.10}$$

Next it follows from the definition (7.5) that $F^{(0,s)}(\tau, 0)$ is τ independent since it receives contribution only from the $L_0 = \bar{L}_0 = 0$ states. The modular transformation laws (7.6) together with (7.9) then imply that

$$F^{(r,s)}(\tau, 0) = F^{(0,t)}(\tau, 0)|_{t=g.c.d.(r,s)} = c^{(0,t)}(0) + 2\, c^{(0,t)}(-1) = \frac{24}{N(N+1)}$$
$$\text{for} \quad (r,s) \neq (0,0)\,. \tag{7.11}$$

Substituting (7.10), (7.11) into the expansion (7.8) we get the analog of eq.(5.6)

$$\sum_b c^{(r,s)}(4n - b^2) = \begin{cases} \frac{24}{N}\,\delta_{n,0} & \text{for} \quad (r,s) = (0,0) \\ \frac{24}{N(N+1)}\,\delta_{n,0} & \text{for} \quad (r,s) \neq (0,0) \end{cases}\,. \tag{7.12}$$

Further information about these coefficients comes from the fact that $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ represent the elliptic genus of the super-conformal σ-model with target space $K3/\mathbb{Z}_N$ with the $\mathbb{Z}_N$ generated by $\widetilde{g}$. However for any N this gives us back the superconformal field theory with target space $K3$, and hence $\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z)$ must give us the elliptic genus of $K3$. This in turn is just $NF^{(0,0)}(\tau, z)$. Thus we have

$$\sum_{r,s=0}^{N-1} F^{(r,s)}(\tau, z) = N\, F^{(0,0)}(\tau, z)\,. \tag{7.13}$$

Furthermore the contribution $\sum_{s=0}^{N-1} F^{(r,s)}(\tau, z)$ for a fixed r may be interpreted as the contribution to the elliptic genus from the sector twisted by $\widetilde{g}^r$. For prime values of N, $\widetilde{g}^r$ is an order N transformation for all $r \neq 0$ mod N. Hence we expect the sectors twisted by $\widetilde{g}^r$ to give the same contribution to the elliptic genus for all $r \neq 0$ mod N. This, together with (7.13), gives

$$\sum_{s=0}^{N-1} F^{(r,s)}(\tau,z) = \frac{1}{N-1}\left[N\, F^{(0,0)}(\tau,z) - \sum_{s=0}^{N-1} F^{(0,s)}(\tau,z)\right] \qquad r \neq 0 \text{ mod } N\,. \tag{7.14}$$

Translated to a condition on the coefficients $c^{(r,s)}(m)$, this gives

$$\sum_{s=0}^{N-1} c^{(r,s)}(m) = \frac{1}{N-1}\left[N\, c^{(0,0)}(m) - \sum_{s=0}^{N-1} c^{(0,s)}(m)\right] \qquad \text{for any } m, \quad r \neq 0 \text{ mod } N\,. \tag{7.15}$$

For $m = 0, -1$ we can explicitly evaluate the right hand side of this equation using (7.9). In particular setting $m = -1$ we get

$$\sum_{s=0}^{N-1} c^{(r,s)}(-1) = 0\,, \qquad \text{for } r \neq 0 \text{ mod } N\,. \tag{7.16}$$

Although for $N = 3, 5, 7$ we have not been able to compute $F^{(r,s)}(\tau, z)$ directly, a set of $F^{(r,s)}(\tau, z)$ satisfying the requirements given above are as follows. Let us define

$$A(\tau,z) = \left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right], \tag{7.17}$$

$$B(\tau,z) = \eta(\tau)^{-6}\vartheta_1(\tau,z)^2\,, \tag{7.18}$$

and

$$E_N(\tau) = \frac{12i}{\pi(N-1)}\,\partial_\tau\left[\ln\eta(\tau) - \ln\eta(N\tau)\right] = 1 + \frac{24}{N-1}\sum_{\substack{n_1,n_2\geq 1\\ n_1\neq 0\, mod\, N}} n_1 e^{2\pi i n_1 n_2 \tau}\,. \tag{7.19}$$

Then under an $SL(2,\mathbb{Z})$ transformation $A(\tau, z)$ transforms as a weak Jacobi form of weight 0 and index 1 and $B(\tau, z)$ transforms as a weak Jacobi form of weight -2 and index 1. Furthermore

$$E_N(\tau+1) = E_N(\tau), \qquad E_N(-1/\tau) = -\tau^2\frac{1}{N}E_N(\tau/N)\,. \tag{7.20}$$

From this it follows that $E_N(\tau)$ is a modular form of weight 2 of the group $\Gamma_0(N)$ and hence also of $\Gamma(N)$[27]. Using these properties one can show that the following choice of $F^{r,s}(\tau,z)$ satisfy all the requirements described above:

$$
\begin{aligned}
F^{(0,0)}(\tau,z) &= \frac{8}{N} A(\tau,z)\,, \\
F^{(0,s)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) - \frac{2}{N+1}\, B(\tau,z)\, E_N(\tau) \qquad \text{for } 1 \le s \le (N-1)\,, \\
F^{(r,rk)}(\tau,z) &= \frac{8}{N(N+1)}\, A(\tau,z) + \frac{2}{N(N+1)}\, E_N\left(\frac{\tau+k}{N}\right)\, B(\tau,z)\,, \\
&\qquad\qquad\qquad \text{for } 1 \le r \le (N-1),\ 0 \le k \le (N-1)\,.
\end{aligned}
\tag{7.21}
$$

The rest of the analysis now proceeds exactly as in the $N=2$ case. We arrive at an analog of eq.(4.2) for $\mathcal{I}_{r,s,l}$:

$$
\begin{aligned}
\mathcal{I}_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} & \sum_{m_1,m_2,n_2\in \mathbb{Z},n_1\in \mathbb{Z}+\frac{r}{N},b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right] \times \\
& \exp\left(\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right)\, e^{2\pi i m_1 s/N}\, h_l^{(r,s)}(\tau)\,, \\
& \qquad 0 \le r,s \le (N-1)\,.
\end{aligned}
\tag{7.22}
$$

This can then be Poisson resummed and analyzed using the techniques described in appendix B and be split into holomorphic and anti-holomorphic parts to extract the expression for $\widetilde{\Phi}_k$. On the other hand if we want information about Φ_k we need to use the operation eq.(6.3) to consider a new integral

$$
\begin{aligned}
\mathcal{I}'_{r,s,l} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} & \sum_{m_1,n_1,n_2\in \mathbb{Z},m_2\in \mathbb{Z}-\frac{r}{N},b\in 2\mathbb{Z}+l} \exp\left[2\pi i\tau(m_1n_1+m_2n_2+\frac{b^2}{4})\right] \times \\
& \exp\left(\frac{-\pi\tau_2}{Y}\left|n_2(TU-V^2)+bV+n_1T-Um_1+m_2\right|^2\right)\, e^{-2\pi i n_2 s/N}\, h_l^{(r,s)}(\tau)\,, \\
& \qquad 0 \le r,s \le (N-1)\,.
\end{aligned}
\tag{7.23}
$$

In this case Poisson resummation over m_1, m_2 will give rise to an additional factor of $\exp(2\pi i k_2 r/N)$ and the final sum will be over integer values of n_1, n_2, k_1, k_2. This can again be analyzed using the techniques described in appendix B.

We shall not give the details of the analysis but write down the final expression. The expressions for Φ_k and $\widetilde{\Phi}_k$ obtained this way are:

$$
\Phi_k(U,T,V) = -\exp\left\{2\pi i\,(T+U+V)\right\}
$$

$$\prod_{r,s=0}^{N-1} \prod_{\substack{(k',l,b)\in \mathbb{Z} \\ (k',l,b)>0}} \left\{1 - e^{2\pi i r/N}\ \exp(2\pi i(k'T + lU + bV)\right\}^{\frac{1}{2}c^{(r,s)}(4k'l-b^2)}$$
$$\prod_{r,s=0}^{N-1} \prod_{\substack{(k',l,b)\in \mathbb{Z} \\ (k',l,b)>0}} \left\{1 - e^{-2\pi i r/N}\ \exp(2\pi i(k'T + lU + bV)\right\}^{\frac{1}{2}c^{(r,s)}(4k'l-b^2)}$$
$$(7.24)$$

$$\widetilde{\Phi}_k(U,T,V) = -(i\sqrt{N})^{-k-2}\ \exp\left(2\pi i\left(\frac{1}{N}T + U + V\right)\right)$$
$$\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}+\frac{r}{N} \\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\ c^{(r,s)}(4lk'-b^2)}$$
$$\prod_{r=0}^{N-1} \prod_{\substack{l,b\in \mathbb{Z}, k'\in \mathbb{Z}-\frac{r}{N} \\ k',l,b>0}} \left\{1 - \exp(2\pi i(k'T + lU + bV))\right\}^{\frac{1}{2}\sum_{s=0}^{N-1} e^{2\pi i l s/N}\ c^{(r,s)}(4lk'-b^2)}$$
$$. \qquad (7.25)$$

Φ_k has been normalized so that the coefficient of the $\exp(2\pi i(T+U+V))$ is -1. $\widetilde{\Phi}_k$ is normalized so that the coefficient of the $\exp(2\pi i(\frac{1}{N}T+U+V))$ term is $-(i\sqrt{N})^{-k-2}$. These conventions agree with the one used in [7].

The weight k of the modular form, determined by examining the term proportional to $\ln\det \mathrm{Im}\Omega$ in the final expression for the integral, is given by

$$k = \frac{1}{2}\sum_{s=0}^{N-1} c^{(0,s)}(0) = \frac{24}{N+1} - 2\,, \qquad (7.26)$$

where we have used eq.(7.9). This agrees with (7.2). Furthermore, using eqs.(7.9), (7.12) and (7.16) we can study the $V \to 0$ limits of Φ_k and $\widetilde{\Phi}_k$. We get

$$\Phi_k(U,T,V) \simeq 4\pi^2 V^2 (\eta(T)\eta(NT))^{k+2}(\eta(U)\eta(NU))^{k+2}\,, \qquad (7.27)$$

and

$$\widetilde{\Phi}_k(U,T,V) \simeq (i\sqrt{N})^{-k-2}\, 4\pi^2 V^2\, (\eta(T)\eta(T/N))^{k+2}(\eta(U)\eta(NU))^{k+2}\,, \qquad (7.28)$$

in agreement with [7].

Another important consistency check for eqs.(7.24), (7.25) comes from looking at the coefficient of the terms involving a single power of $e^{2\pi i U}$ and all powers of T

and V. For Φ_k this is given by

$$e^{2\pi iU}\,\eta(T)^{k-4}\,\eta(NT)^{k+2}\,\vartheta_1(T,V)^2\,, \tag{7.29}$$

and for $\widetilde{\Phi}_k$ this is given by

$$(i\sqrt{N})^{-k-2}\,e^{2\pi iU}\,\eta(T)^{k-4}\,\eta(T/N)^{k+2}\,\vartheta_1(T,V)^2\,. \tag{7.30}$$

These agree with the corresponding expressions found in [7].

We have also compared a few terms in the expansion of Φ_k given in (7.24) with the one given in [7]. The results are given in appendix C.

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A. Calculation of the Elliptic Genus

In this appendix we shall calculate

$$\begin{aligned} & F_{m_1,m_2,n_1,n_2}(\tau,z) \\ &= Tr_{RR;m_1,m_2,n_1,n_2}\left((-1)^{(F_{K3}+F_{T^2})}(-1)^{(\bar F_{K3}+\bar F_{T^2})}F_{T^2}\bar F_{T^2}e^{2\pi izF_{K3}}\,q^{L_0'}\bar q^{\bar L_0'}\right)\,, \end{aligned} \tag{A.1}$$

in the superconformal field theory with target space $(K3\times T^2)/\mathbb{Z}_2$. For this we shall use an orbifold description of $K3$. We consider a superconformal σ-model with target space $T^2\times T^4$ with y^1, y^2 denoting the T^2 coordinates and y^3, y^4, y^5, y^6 denoting the T^4 coordinates, and mod out the theory by a $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry generated by elements g and g'. The action of g and g' are given by:

$$\begin{aligned} g:&\quad (y^1,y^2,y^3,y^4,y^5,y^6)\to(y^1,y^2,-y^3,-y^4,-y^5,-y^6) \\ g':&\quad (y^1,y^2,y^3,y^4,y^5,y^6)\to(y^1+\pi,y^2,y^3+\pi,y^4,y^5,y^6)\,. \end{aligned} \tag{A.2}$$

Orbifolding by g produces a $K3\times T^2$ manifold. Further orbifolding by g' produces $(K3\times T^2)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ generator involves a shift along T^2 and a $\mathbb{Z}_2$ involution in $K3$ that preserves the (4,4) superconformal symmetry of the corresponding world-sheet theory. We denote by F_{T^4} and F_{T^2} holomorphic parts of the world-sheet fermion number associated with the T^4 and the T^2 parts and by $\bar{F}_{T^4}$ and $\bar{F}_{T^2}$ the anti-holomorphic parts of the world-sheet fermion number associated with the T^4 and the T^2 parts. We shall be considering an arbitrary T^2 parametrized by the Kähler modulus T and complex structure modulus U, and arbitrary Wilson lines A_1, A_2 corresponding to deforming the world-sheet theory by the marginal operator

$$\sum_{i=1}^{2} A_i \int d^2z\, \bar{\partial} Y^i J_{T^4}\,, \tag{A.3}$$

where J_{T^4} is the U(1) current corresponding to the charge F_{T^4}. We shall denote by V the complex combination $A_2 - iA_1$.

We now define

$$\begin{aligned} &F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z) \\ &= Tr^{T^4\times T^2}_{m_1,m_2,n_1,n_2;RR;g^a,g'^b}\left((-1)^{(F_{T^4}+F_{T^2})}(-1)^{(\bar{F}_{T^4}+\bar{F}_{T^2})}F_{T^2}\bar{F}_{T^2}e^{2\pi i z F_{T^4}}q^{L_0}\bar{q}^{\bar{L}_0}g^c g'^d\right)\,, \end{aligned} \tag{A.4}$$

where L_0', $\bar{L}_0'$ have been defined in eqs.(3.1), (3.4). Here a,b,c,d take values 0 or 1. $Tr^{T^4\times T^2}_{m_1,m_2,n_1,n_2;RR;g^a,g'^b}$ denotes trace in the original CFT associated with the $T^2\times T^4$ target space over RR sector states twisted by $g^a g'^b$ and carrying (m_1,m_2) units of momentum and (n^1,n^2) units of winding along (y^1,y^2). The quantity $F_{m_1,m_2,n_1,n_2}(\tau,z)$ is then given by

$$F_{m_1,m_2,n_1,n_2}(\tau,z) = \frac{1}{4}\sum_{a,b,c,d=0}^{1} F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)\,. \tag{A.5}$$

We shall now calculate $F_{m_1,m_2,n_1,n_2}(a,b;c,d;\tau,z)$. First we note that

$$F_{m_1,m_2,n_1,n_2}(0,0;0,d;\tau,z) = 0 \quad \text{for } d=0,1 \tag{A.6}$$

due to the fermion zero modes associated with the $3,4,5,6$ directions.

Next we have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(0,0;1,d;\tau,z) &= (-1)^{m_1 d}\, 4\,(1+e^{2\pi i z})(1+e^{-2\pi i z}) \\ &\quad \frac{\prod_{n=1}^{\infty}(1+q^n e^{2\pi i z})^2(1+q^n e^{-2\pi i z})^2}{\prod_{n=1}^{\infty}(1+q^n)^4} \\ &= (-1)^{m_1 d}\, 16\, \frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2}\,. \end{aligned} \tag{A.7}$$

In the first line the factor of 4 comes from the anti-holomorphic fermion zero modes associated with the 3,4,5,6 directions and the factor of $(1 + e^{2\pi i z})(1 + e^{-2\pi i z})$ comes from the holomorphic fermion zero-modes. In the second line the numerator comes from the holomorphic non-zero mode fermionic oscillators associated with the 3,4,5,6 directions and the denominator comes from the holomorphic non-zero mode bosonic oscillators associated with the same directions. The contribution from the bosonic and fermionic oscillators associated with the 1 and 2 directions cancel. Also the contributions from all the non-zero mode fermion and bosonic oscillators in the anti-holomorphic sector always cancel. In arriving at (A.7) we have used that the action of g' on the state carrying m_1 units of momentum along y^1 gives a factor of $(-1)^{m_1}$ and the action of g changes the signs of the fermionic and the bosonic oscillators associated with T^4. Also since the action of g reverses the direction of momentum along the 3,4,5,6 directions, only states carrying zero momentum along T^4 contributes to the trace and hence the result is independent of the moduli of T^4. This will be a generic feature of all the terms; either they will vanish due to fermion zero modes or only the zero momentum mode will contribute due to either a g insertion or a twist under g.

Let us now turn to the twisted sector states. First note that there are 16 twisted sector states under g, located as $y^m = 0, \pi$ for $m = 3, 4, 5, 6$. g' (and also gg') exchanges these states pairwise. Thus the action of g' and gg' on these states is off-diagonal and hence the trace of g' and gg' over these states vanish. This gives

$$F_{m_1,m_2,n_1,n_2}(1, 0; c, 1; \tau, z) = 0 \quad \text{for } c = 0, 1\,. \tag{A.8}$$

On the other hand we have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1, 0; c, 0; \tau, z) &= 16\frac{\prod_{n=0}^{\infty}(1 - q^{n+\frac{1}{2}}e^{2\pi i z+i\pi c})^2(1 - q^{n+\frac{1}{2}}e^{-2\pi i z+i\pi c})^2}{\prod_{n=0}^{\infty}(1 - e^{i\pi c}\, q^{n+\frac{1}{2}})^4} \\ &= \begin{cases} 16\,\vartheta_4(\tau, z)^2/\vartheta_4(\tau, 0)^2 & \text{for } c = 0 \\ 16\,\vartheta_3(\tau, z)^2/\vartheta_3(\tau, 0)^2 & \text{for } c = 1 \end{cases}. \end{aligned} \tag{A.9}$$

The factor of 16 is due to the existence of 16 twisted sector states.

Next we consider sectors twisted by g'. In this case the winding number n_1 along y^1 must be half integer and similarly the winding number along y^3 must also be half integer. Since the g' twist just involves a shift and does not affect the world-sheet fermions, the fermion zero modes associated with the 3-6 directions make the contribution vanish unless the g projection is inserted into the trace. This gives:

$$F_{m_1,m_2,n_1,n_2}(0,1;0,d;\tau,z) = 0 \quad \text{for } d = 0,1\,. \tag{A.10}$$

On the other hand the action of g as well as of gg' reverses the sign of the winding number along y^3 and hence these elements are off-diagonal in the sector twisted by g'. This gives

$$F_{m_1,m_2,n_1,n_2}(0,1;1,d;\tau,z) = 0 \quad \text{for } d = 0,1\,. \tag{A.11}$$

Finally let us turn to the sector twisted under gg'. Action of gg' on y^3, y^4, y^5, y^6 gives fixed points at $y^3 = \pi/2, 3\pi/2$, $y^m = 0, \pi$ for $m = 4,5,6$. Although this are not real fixed points due to the shift action $y^2 \to y^2 + \pi$, we can label the 16 twisted sectors by these would be fixed points. Both g and g' exchange these fixed points pairwise and hence are represented by off-diagonal matrices. This gives

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;1,0;\tau,z) &= 0\,, \\ F_{m_1,m_2,n_1,n_2}(1,1;0,1;\tau,z) &= 0\,. \end{aligned} \tag{A.12}$$

On the other hand both the identity element and gg' leave the fixed points invariant and give non-zero answers. We have

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;0,0;\tau,z) &= 16\,\frac{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}e^{2\pi i z})^2(1-q^{n+\frac{1}{2}}e^{-2\pi i z})^2}{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}})^4} \\ &= 16\,\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\,, \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(1,1;1,1;\tau,z) &= 16\,(-1)^{m_1}\,\frac{\prod_{n=0}^{\infty}(1-q^{n+\frac{1}{2}}e^{2\pi i z+i\pi})^2(1-q^{n+\frac{1}{2}}e^{-2\pi i z+i\pi})^2}{\prod_{n=0}^{\infty}(1-e^{i\pi}q^{n+\frac{1}{2}})^4} \\ &= 16\,(-1)^{m_1}\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2}\,. \end{aligned} \tag{A.14}$$

Using eqs.(A.4)-(A.14) we now get

$$\begin{aligned} F_{m_1,m_2,n_1,n_2}(\tau,z) &= 4\left[\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} + \frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} + \frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2}\right] \\ &\quad + 4(-1)^{m_1}\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau,0)^2} \qquad \text{for } n_1 \in \mathbb{Z} \\ &= 4\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau,0)^2} + 4\,(-1)^{m_1}\,\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau,0)^2} \qquad \text{for } n_1 \in \mathbb{Z}+\tfrac{1}{2} \end{aligned} \tag{A.15}$$

B. Evaluation of the Integral

In this appendix we shall evaluate the integral (4.13)

$$\mathcal{I} = \sum_A{}' \int_{\mathcal{F}_A} \frac{d^2\tau}{\tau_2^2}\frac{Y}{U_2}\exp\bigg(-\frac{\pi Y}{U_2^2\tau_2}|\mathcal{A}|^2 - 2\pi i T \det A \\ -\frac{\pi n_2}{U_2}(V^2\tilde{\mathcal{A}} - \bar{V}^2\mathcal{A}) + \frac{2\pi i V_2^2}{U_2^2}(n_1+n_2\bar{U})\mathcal{A}\bigg) F^{(r,s)}\left(\tau, -i\frac{V\tilde{\mathcal{A}}-\bar{V}\mathcal{A}}{2\,U_2}\right). \tag{B.1}$$

The sum over A runs over all integer valued 2×2 matrices of the form (4.6) which are not related to each other by an $SL(2,\mathbb{Z})$ transformation acting from the right. $\mathcal{F}_A$ is the union of images of the fundamental region $\mathcal{F}$ under $SL(2,\mathbb{Z})$ transformations which act non-trivially on A. $\mathcal{A}$, $\tilde{\mathcal{A}}$ are defined in (4.7) and $(r,s) = (2n_1, 2k_1)$ mod 2.

In carrying out the integral we need to introduce some regularization and subtraction scheme. Following [28] we regularize possible divergences in the integral by including a factor of $(1-\exp(-\Lambda/\tau_2))$ in the integrand. For $\tau_2 << \Lambda$ this factor is close to unity, but for $\tau_2 >> \Lambda$ it is close to zero. We also add to the integral a term

$$-\left(c^{(0,0)}(0) + c^{(0,1)}(0)\right)\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\left(1-\exp(-\Lambda/\tau_2)\right). \tag{B.2}$$

As we shall see, this is necessary for getting a finite $\Lambda\to\infty$ limit.

Following the same procedure as in [28] we split the integration into the three orbits.

1. Contribution $\mathcal{I}_1$ from the zero orbit

For $A=0$ we have $(r,s)=(0,0)$ and $\mathcal{F}_A = \mathcal{F}$, – the fundamental region of $SL(2,\mathbb{Z})$. The integral (4.13) reduces to

$$\mathcal{I}_1 = \frac{Y}{U_2}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}F^{(0,0)}(\tau,0) = \frac{Y}{U_2}\frac{\pi}{3}\,12\,, \tag{B.3}$$

using the expression for $F^{(0,0)}(\tau,z)$ given in (3.13).

2. Contribution $\mathcal{I}_2$ from the non-degenerate orbit

Here we consider A to be

$$A = \begin{pmatrix} k & j \\ 0 & p\end{pmatrix}, \quad 2k-1 \geq 2j \geq 0, \quad p\neq 0, \quad k,j\in\frac{1}{2}\mathbb{Z}, \quad p\in\mathbb{Z}\,. \tag{B.4}$$

In this case the region $\mathcal{F}_A$ corresponds to two copies of the upper-half plane (coming from A and $-A$) and the indices (r,s) in (B.1) are given by

$$(r,s) = (2k \bmod 2, 2j \bmod 2)\,. \tag{B.5}$$

Note that for the above form of A,

$$\det A = kp, \quad \mathcal{A} = k\tau + j + pU, \quad \tilde{\mathcal{A}} = k\tau + j + p\bar{U}\,. \tag{B.6}$$

Let us first consider the case $k \in \mathbb{Z}$, $j \in \mathbb{Z}$. In this case j runs from 0 to $k-1$ in steps of 1. The relevant $F^{(r,s)}$ is $F^{(0,0)}$. In order to carry out the integral we replace $F^{(0,0)}(\tau,z)$ in (B.1) by its Fourier expansion (3.22). If we now change the integration variable from τ_1 to

$$\tau_1' = k\tau_1 + j + pU_1\,, \tag{B.7}$$

then $\mathcal{A}$, $\tilde{\mathcal{A}}$ and hence also the exponential factor in (B.1), expressed as a function of τ_1' and τ_2, will have no j dependence. The only j dependence comes from the term

$$\exp(2\pi i n\tau_1) = \exp\left(2\pi i n \frac{1}{k}(\tau_1' - j - pU_1)\right) \tag{B.8}$$

which arises from the factor $c^{(0,0)}(4n-b^2)\exp(2\pi i\tau n)$ in the expansion (3.22) of $F^{(0,0)}(\tau,z)$). Since in this case n is an integer, the summation over j from 0 to $k-1$ in steps of 1 imposes the condition $n = n'k$ where n' is an integer. Furthermore since $n \geq 0$ and $k > 0$, we have $n' \geq 0$. The summation over j also produces a factor of k which cancels the $1/k$ factor arising due to the change of variables from τ_1 to τ_1' in the measure.

Using (B.6)-(B.8) we see that the integration over τ_1' in (B.1) is just a Gaussian integration. The result of carrying out this integral is

$$\begin{aligned}
\mathcal{I}_{2;k,j\in\mathbb{Z}} = & \sum_{\substack{n',k\in\mathbb{Z},\, b,p\in\mathbb{Z} \\ n'\geq 0, k>0, p\neq 0}} \sqrt{Y} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} \exp(\mathcal{F})\, c^{(0,0)}(4n'k - b^2) \\
\mathcal{F} \equiv & -2\pi\tau_2 n'k - \frac{\pi Y}{U_2^2\tau_2}(k\tau_2 + pU_2)^2 - 2\pi i T kp - 2\pi i p n' U_1 \\
& + \frac{\pi b}{U_2}(-2V_2 k\tau_2 - 2ipU_2V_1) \\
& - \frac{2\pi V_2^2}{U_2^2}(k^2\tau_2 + kpU_2) - \frac{\pi B^2 U_2^2 \tau_2}{Y} \\
B \equiv & \; n' + \frac{bV_2}{U_2} + \frac{V_2^2}{U_2^2}k
\end{aligned} \tag{B.9}$$

The τ_2 integral is of the Bessel form and can be performed using

$$\int_0^\infty \frac{du}{u^{3/2}}\, e^{-au-bu^{-1}} = e^{-2\sqrt{ab}}\sqrt{\frac{\pi}{b}}\,. \tag{B.10}$$

This gives

$$\begin{aligned}\mathcal{I}_{2;k,j\in\mathbb{Z}} = & \sum_{\substack{n',k,b\in\mathbb{Z},\, p\in\mathbb{Z}\\ n'\geq 0,k>0,p\neq 0}} \frac{1}{|p|} c^{(0,0)}(4n'k-b^2)\exp\{-2\pi i T k p - 2\pi k|p|T_2 - 2\pi k p T_2 \\ & -2\pi i p n' U_1 - 2\pi|p|U_2 n' \, -2\pi i b p V_1 - 2\pi|p|bV_2\} \\ = & -\ln \prod_{\substack{n',k,b\in\mathbb{Z}\\ n'\geq 0,k>0}} \left\{\left|1-\exp(2\pi i(kT+n'U+bV))\right|^{2c^{(0,0)}(4n'k-b^2)}\right\}\end{aligned} \tag{B.11}$$

Next we consider the contribution from the $k\in\mathbb{Z}$, $j\in\mathbb{Z}+\frac{1}{2}$ terms. In this case j takes values from $\frac{1}{2}$ to $k-\frac{1}{2}$ in steps of 1 and $(r,s)=(0,1)$. The analysis proceeds as in the previous case, the only difference being that the sum over j of (B.8) gives an additional factor of $(-1)^{n'}$ besides forcing the condition $n=n'k$ with $n'\in\mathbb{Z}$. The analog of eq.(B.11) is then

$$\mathcal{I}_{2;k\in\mathbb{Z},j\in\mathbb{Z}+\frac{1}{2}} = -\ln \prod_{\substack{n',k,b\in\mathbb{Z}\\ n'\geq 0,k>0}} \left\{\left|1-\exp(2\pi i(kT+n'U+bV))\right|^{2(-1)^{n'}c^{(0,1)}(4n'k-b^2)}\right\} \tag{B.12}$$

Finally let us consider the case $k\in\mathbb{Z}+\frac{1}{2}$. In this case instead of letting j run from 0 to $k-\frac{1}{2}$ in steps of $\frac{1}{2}$ we can let it run from 0 to $(2k-1)$ in steps of 1 by means of a further SL$(2,\mathbb{Z})$ duality transformation. For each of these terms the relevant (r,s) are $(1,0)$. Proceeding as in the $k,j\in\mathbb{Z}$ case we now see that the sum over j in (B.8) forces the condition $n=4n'k$ with $n'\in\mathbb{Z}$ and when this condition is satisfied we get a factor of $2k$.[7] The rest of the analysis proceeds as in the previous case and we obtain

$$\mathcal{I}_{2;k\in\mathbb{Z}+\frac{1}{2}} = -2\ln \prod_{\substack{n',b\in\mathbb{Z},k\in\mathbb{Z}+\frac{1}{2}\\ n'\geq 0,k>0}} \left\{\left|1-\exp\{2\pi i(kT+n'U+bV)\}\right|^{2c^{(1,0)}(4n'k-b^2)}\right\} \tag{B.13}$$

[7]Note that in this case n is either an integer or a half integer, but the sum over j still forces n to be an integer multiple of k since the sum runs over $2k$ values instead of k values.

Thus the net contribution to the integral from the non-degenerate orbits take the form

$$\mathcal{I}_2 = -\ln\Bigg[\prod_{\substack{n',k,b\in\mathbb{Z}\\ n'\geq 0,k>0}}\left\{\Big|1-\exp(2\pi i(kT+n'U+bV))\Big|^{2c^{(0,0)}(4n'k-b^2)+2(-1)^{n'}c^{(0,1)}(4n'k-b^2)}\right. \\ \prod_{\substack{n',b\in\mathbb{Z},k\in\mathbb{Z}+\frac{1}{2}\\ n'\geq 0,k>0}}\left.\left\{\Big|1-\exp(2\pi i(kT+n'U+bV))\Big|^{4c^{(1,0)}(4n'k-b^2)}\right\}\right]\tag{B.14}$$

3. Contribution $\mathcal{I}_3$ from the degenerate orbit

Here we consider A to be of the form

$$A = \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix}, \quad (j,p)\neq(0,0), \quad j\in\frac{1}{2}\mathbb{Z}, \quad p\in\mathbb{Z}\,. \tag{B.15}$$

In this case the integration region $\mathcal{F}_A$ corresponds to the strip

$$-1/2\leq\tau_1\leq 1/2, \qquad \tau_2\geq 0\,. \tag{B.16}$$

Also we have

$$(r,s)=(0,0) \quad \text{for } j\in\mathbb{Z}, \qquad (r,s)=(0,1) \quad \text{for } j\in\mathbb{Z}+\tfrac{1}{2}\,. \tag{B.17}$$

For A given in (B.15)

$$\mathcal{A}=j+pU, \qquad \tilde{\mathcal{A}}=j+p\bar{U}\,, \qquad \det A=0\,, \tag{B.18}$$

are independent of τ. Thus the exponential factor in (4.13) is independent of τ_1 and the only dependence on τ_1 of the integrand comes from the $\exp(2\pi i\tau n)$ term in the expansion of $F^{(r,s)}(\tau,z)$. The τ_1 integration now forces n to vanish and the coefficients $c^{(r,s)}(4n-b^2)$ multiplying the integrand reduces to $c^{(r,s)}(-b^2)$. It follows from the definition of $c^{(r,s)}(m)$ that these coefficients are non-zero only for $b=0$ and $b=\pm1$.

We first consider the case $j\in\mathbb{Z}$. We begin with the contribution from the $n=0, b=0$ term and proceed as in [28]. We multiply the integrand with the regulating factor $(1-\exp(-\Lambda/\tau_2))$, then integrate over τ_2 and finally perform the sum over j and p. Integrating over τ_2 we obtain

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}} = c^{(0,0)}(0)\Bigg[\frac{U_2}{\pi}\sum_{\substack{(j,p)\neq(0,0)\\ j,p\in\mathbb{Z}}}\left(\frac{1}{|j+Up|^2}-\frac{1}{|j+Up|^2+\Lambda U_2^2/\pi Y}\right) \\ -\int_{\mathcal{F}} d^2\tau\frac{1-\exp(-\frac{\Lambda}{\tau_2})}{\tau_2}\Bigg]\,. \tag{B.19}$$

Note that we have introduced a subtraction term proportional to $\int_{\mathcal{F}} d^2\tau \frac{1-\exp(-\Lambda/\tau_2)}{\tau_2}$ in eq.(B.19), – this is one of the two terms appearing in (B.2). This is necessary in order to get a finite value of the integral in the $\Lambda \to \infty$ limit. The result of the integration in the second terms inside the square brackets is $\ln \Lambda + \gamma_E + 1 + \ln(2/3\sqrt{3})$. To evaluate the summation we use[29]

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \frac{\exp(i\theta j)}{(j+B)^2+C^2} &= \frac{\pi}{C}\exp(-i\theta(B-iC))\frac{1}{1-\exp(-2\pi i(B-iC))} \\
&+ \frac{\pi}{C}\exp(-i\theta(B+iC))\frac{\exp(2\pi i(B+iC))}{1-\exp(2\pi i(B+iC))} \\
&\qquad \text{for} \quad C>0, \quad 0\le\theta\le 2\pi \\
\sum_{\substack{j\in\mathbb{Z}\\ j>0}} \frac{\cos\theta j}{j^2} &= \frac{\theta(\theta-2\pi)}{4} + \frac{\pi^2}{6}\,.
\end{aligned} \tag{B.20}$$

We now regroup the summation in (B.19) as $\sum_{p=0,j\neq 0} + \sum_{j=-\infty,p\neq 0}^{j=+\infty}$ and use (B.20) at $\theta = 0$ to obtain

$$\begin{aligned}
\mathcal{I}_{3,b=0;j\in\mathbb{Z}} = c^{(0,0)}(0)\Bigg[\frac{\pi}{3}U_2 + \sum_{\substack{p>0\\ p\in\mathbb{Z}}}\Bigg\{\frac{2}{p}\frac{e^{-2\pi i p\bar{U}}}{1-e^{-2\pi i p\bar{U}}} + \frac{2}{p}\frac{e^{2\pi i p U}}{1-e^{2\pi i p U}} \\
+ \left(\frac{2}{p} - \frac{2}{\sqrt{p^2+\Lambda/\pi Y}}\right)\Bigg\} - \left(\ln\Lambda + \gamma_E + 1 + \ln(2/3\sqrt{3})\right)\Bigg]\,.
\end{aligned} \tag{B.21}$$

Next we expand

$$\frac{x}{1-x} = \sum_{l=1}^{\infty} x^l\,, \tag{B.22}$$

for $x = e^{-2\pi i p U}$ and $x = e^{2\pi i p\bar{U}}$ in (B.21) and perform the sum over p in the first two terms. Finally we use

$$\sum_{\substack{p>0\\ p\in\mathbb{Z}}}\left(\frac{2}{p} - \frac{2}{\sqrt{p^2+\Lambda/\pi Y}}\right) = -\ln\frac{\pi Y}{\Lambda} + 2\gamma_E - \ln 4 \quad \text{as } \Lambda\to\infty\,, \tag{B.23}$$

to obtain

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}} = c^{(0,0)}(0)\,\left(\frac{\pi}{3}U_2 - \ln Y + \kappa'\right) - \ln\prod_{l\in\mathbb{Z},l>0}\left\{|1-\exp(2\pi i l U)|^{4c^{(0,0)}(0)}\right\} \tag{B.24}$$

where

$$\kappa' = \gamma_E - 1 - \ln(8\pi/3\sqrt{3})\,. \tag{B.25}$$

We now evaluate the contribution of $n = 0$, $b = \pm 1$. The corresponding coefficient is $c^{(0,0)}(-1)$. Integrating over τ_2 we obtain

$$\mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} = c^{(0,0)}(-1)\frac{U_2}{\pi}\sum_{\substack{(j,p)\neq(0,0)\\ j,p\in\mathbb{Z}}}\frac{1}{|j+pU|^2}\exp\left(\frac{2\pi ib}{U_2}(jV_2+p(V_2U_1-V_1U_2))\right) \tag{B.26}$$

We split this summation as before $\sum_{p=0,j\neq0}+\sum_{p\neq0,j}$. We shall assume, for definiteness, that

$$V_2 < 0\,. \tag{B.27}$$

For the $p = 0$ one can apply the second formula in (B.20) to obtain

$$4\pi\, c^{(0,0)}(-1)\left(\frac{V_2^2}{U_2}+V_2+\frac{U_2}{6}\right) \tag{B.28}$$

Let us now turn to the contribution from the $p \neq 0$ terms. Since (B.26) contains the contribution for both $b = 1$ and $b = -1$, care should be taken so that the θ in $(B.20)$ is between $0 \leq \theta \leq 2\pi$. Here $\theta = -2\pi V_2/U_2 \leq 1$. For the $p \neq 0$ case one splits the summation for $p > 0, b = \pm1$ and $p < 0, b = \pm1$, then one changes $j \to -j$ or $p \to -p$ so that one can always apply the formula in (B.20). Carefully taking all these contributions into account one obtains, after using (B.20), the total contribution from the $p \neq 0$ terms to be

$$-\ln\prod_{l\in\mathbb{Z},l>0,b=\pm1}|1-\exp(2\pi i(lU+bV)|^{4c^{(0,0)}(-1)}-\ln|1-\exp(-2\pi iV)|^{4c^{(0,0)}(-1)} \tag{B.29}$$

Thus the net contribution from the $b = \pm1$, $j \in \mathbb{Z}$ terms are

$$\begin{aligned}\mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} &= 4\pi\, c^{(0,0)}(-1)\left(\frac{V_2^2}{U_2}+V_2+\frac{U_2}{6}\right)\\ &\quad-\ln\prod_{\substack{l\in\mathbb{Z},l>0\\ b=\pm1}}\left\{|1-\exp(2\pi i(lU+bV))|^{4c^{(0,0)}(-1)}\right\}\\ &\quad-\ln|1-\exp(-2\pi iV)|^{4c^{(0,0)}(-1)}\end{aligned} \tag{B.30}$$

Note that the last term in the above equation is singular as $V \to 0$.

Next we turn to the contribution from the $j \in \mathbb{Z} + \frac{1}{2}$ terms. In this case $(r,s) = (0,1)$. The analog of (B.20) is obtained by replacing $B \to B + \frac{1}{2}$ in this formula and multiplying the resulting equation by a factor of $e^{i\theta/2}$ on both sides:

$$\sum_{j\in \mathbb{Z}+\frac{1}{2}} \frac{\exp(i\theta j)}{(j+B)^2 + C^2} = \frac{\pi}{C} \exp(-i\theta(B-iC)) \frac{1}{1+\exp(-2\pi i(B-iC))} - \frac{\pi}{C} \exp(-i\theta(B+iC)) \frac{\exp(2\pi i(B+iC))}{1+\exp(2\pi i(B+iC))} \quad \text{for} \quad C>0, \quad 0 \le \theta \le 2\pi \tag{B.31}$$

Using this result we can get the analogs of (B.24) and (B.30):

$$\mathcal{I}_{3,b=0;j\in\mathbb{Z}+\frac{1}{2}} = c^{(0,1)}(0)\ (\pi U_2 - \ln Y + \kappa') - \ln \prod_{l\in\mathbb{Z}, l>0} \left\{ |1-\exp(2\pi i l U)|^{4(-1)^l c^{(0,1)}(0)} \right\} \tag{B.32}$$

$$\begin{aligned} \mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}+\frac{1}{2}} = {} & 4\pi\, c^{(0,1)}(-1) \left(V_2 + \frac{U_2}{2} \right) \\ & - \ln \prod_{\substack{l\in\mathbb{Z}, l>0 \\ b=\pm1}} \left\{ |1-\exp(2\pi i(lU+bV)|^{4(-1)^l c^{(0,1)}(-1)} \right\} \\ & - \ln |1-\exp(-2\pi i V)|^{4c^{(0,1)}(-1)} \end{aligned} \tag{B.33}$$

Adding all the contributions we obtain.

$$\begin{aligned} \mathcal{I}_3 = {} & \mathcal{I}_{3,b=0;j\in\mathbb{Z}} + \mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}} + \mathcal{I}_{3,b=0;j\in\mathbb{Z}+\frac{1}{2}} + \mathcal{I}_{3,b=\pm1;j\in\mathbb{Z}+\frac{1}{2}} \\ = {} & c^{(0,0)}(0) \left(\frac{\pi}{3} U_2 - \ln Y + \kappa' \right) + 4\pi\, c^{(0,0)}(-1) \left(\frac{V_2^2}{U_2} + V_2 + \frac{U_2}{6} \right) \\ & + c^{(0,1)}(0)\ (\pi U_2 - \ln Y + \kappa') + 4\pi\, c^{(0,1)}(-1) \left(V_2 + \frac{U_2}{2} \right) \\ & - \ln \prod_{l\in\mathbb{Z}, l>0} \left\{ |1-\exp(2\pi i l U)|^{4c^{(0,0)}(0)} \right\} - \ln \left\{ |1-\exp(-2\pi i V)|^{4c^{(0,0)}(-1)} \right\} \\ & - \ln \prod_{\substack{l\in\mathbb{Z}, l>0 \\ b=\pm1}} \left\{ |1-\exp(2\pi i(lU+bV)|^{4c^{(0,0)}(-1)} \right\} \\ & - \ln \prod_{l\in\mathbb{Z}, l>0} \left\{ |1-\exp(2\pi i l U)|^{4(-1)^l c^{(0,1)}(0)} \right\} - \ln |1-\exp(-2\pi i V)|^{4c^{(0,1)}(-1)} \\ & - \ln \prod_{\substack{l\in\mathbb{Z}, l>0 \\ b=\pm1}} \left\{ |1-\exp(2\pi i(lU+bV)|^{4(-1)^l c^{(0,1)}(-1)} \right\} \end{aligned} \tag{B.34}$$

Combining the contribution from all the orbits and noting that

$$c^{(0,0)}(0) = 10, \quad c^{(0,0)}(-1) = 1, \quad c^{(0,1)}(0) = 2, \quad c^{(0,1)}(-1) = 1,$$

$$c^{(1,0)}(0) = 4, \quad c^{(1,0)}(-1) = 0, \quad c^{(1,1)}(0) = 4, \quad c^{(1,1)}(-1) = 0\,, \tag{B.35}$$

we can now express the full integral as

$$\begin{aligned} \mathcal{I} &= \mathcal{I}_1 + 2\mathcal{I}_2 + \mathcal{I}_3, \\ &= -2\ln\Bigg[\kappa\,(\det \mathrm{Im}\Omega)^6\Bigg|\exp(2\pi i(\frac{1}{2}T + U + V)) \\ &\quad \prod_{\substack{(k,l.b)\in \mathbb{Z}\\ (k,l,b)>0}} (1-\exp(2\pi i(kT+lU+bV))^{c^{(0,0)}(4kl-b^2)+(-1)^l\,c^{(0,1)}(4kl-b^2)} \\ &\quad \prod_{\substack{l,b\in \mathbb{Z},k\in\mathbb{Z}+\frac{1}{2}\\ l\geq 0,k>0}} \left\{ |1-\exp(2\pi i(kT+lU+bV))|^{2c^{(1,0)}(4lk-b^2)}\right\}\Bigg|^2\Bigg] \end{aligned} \tag{B.36}$$

where

$$\kappa = \left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)^6 \tag{B.37}$$

and $(k,l,b) > 0$ means $k > 0, l \geq 0, b \in \mathbb{Z}$ or $k = 0, l > 0, b \in \mathbb{Z}$ or $k = 0, l = 0, b < 0$. Note that we have $2\mathcal{I}_2$ because of the two copies of the upper half plane.

From the modular transformation laws (3.11) and the series expansion (3.22) it follows that

$$c^{(1,1)}(4lk-b^2) = (-1)^l\,c^{(1,0)}(4lk-b^2) \qquad \text{for } k \in \mathbb{Z}+\tfrac{1}{2},\, l \in \mathbb{Z}\,. \tag{B.38}$$

Using this we can reexpress (B.36) in a more symmetric fashion:

$$\begin{aligned} \mathcal{I} &= -2\ln\Bigg[\kappa\,(\det \mathrm{Im}\Omega)^6\Bigg|\exp\left(2\pi i\left(\frac{1}{2}T+U+V\right)\right) \\ &\quad \prod_{r,s=0}^{1}\ \prod_{\substack{(l,b)\in\mathbb{Z},k\in\mathbb{Z}+\frac{r}{2}\\ (k,l,b)>0}} \left\{(1-\exp(2\pi i(kT+lU+bV))^{(-1)^{ls}\,c^{(r,s)}(4kl-b^2)}\right\}\Bigg|^2\Bigg]\,. \end{aligned} \tag{B.39}$$

C. Explicit Results for $a(n,m,r)$

In this appendix we present the results of explicit computation of the coefficients $a(n,m,r)$ for Φ_k. These were calculated using the expression given in [7] as well as the expression found in the present paper and found to be the same. To write the expansion of Φ_k in a convenient way we define $t = \exp(2\pi iT)$, $u = \exp(2\pi iU)$, $v = \exp(2\pi iV)$. Then for $N = 2$

$$\Phi_6 = \left[(2-\frac{1}{v}-v)u + (-4+\frac{2}{v^2}+2v^2)u^2 + (-16-\frac{1}{v^3}-\frac{4}{v^2}+\frac{13}{v}+13v-4v^2-v^3)u^3\right]t$$

$$
\begin{aligned}
&+\left[(-4+\frac{2}{v^2}+2v^2)u+(32-\frac{16}{v^2}-16v^2)u^2+(-72-\frac{4}{v^4}+\frac{40}{v^2}+40v^2-4v^4)u^3\right]t^2\\
&+\Bigg[(-16-\frac{1}{v^3}-\frac{4}{v^2}+\frac{13}{v}+13v-4v^2-v^3)u\\
&+(-72-\frac{4}{v^4}+\frac{40}{v^2}+40v^2-4v^4)u^2\\
&+\left(336+\frac{13}{v^5}+\frac{40}{v^4}-\frac{87}{v^3}-\frac{64}{v^2}-\frac{70}{v}-70v-64v^2-87v^3+40v^4+13v^5\right)u^3\Bigg]t^3\\
&+\cdots
\end{aligned} \tag{C.1}
$$

For $N=3$

$$
\begin{aligned}
\Phi_4 &= \left((2-\frac{1}{v}-v)u+(\frac{2}{v^2}-\frac{2}{v}-2v+2v^2)u^2\right)t\\
&+\left((\frac{2}{v^2}-\frac{2}{v}-2v+2v^2)u+(4-\frac{2}{v^3}-\frac{6}{v^2}+\frac{6}{v}+6v-6v^2-2v^3)u^2\right)t^2+\cdots
\end{aligned} \tag{C.2}
$$

For $N=5$

$$
\begin{aligned}
\Phi_2 &= \left((2-\frac{1}{v}-v)u+(4+\frac{2}{v^2}-\frac{4}{v}-4v+2v^2)u^2\right)t\\
&+\left((4+\frac{2}{v^2}-\frac{4}{v}-4v+2v^2)u+(28-\frac{4}{v^3}+\frac{10}{v^2}-\frac{20}{v}-20v+10v^2-4v^3)u^2\right)t^2+\cdots
\end{aligned} \tag{C.3}
$$

For $N=7$

$$
\begin{aligned}
\Phi_1 &= \left((2-\frac{1}{v}-v)u+(6+\frac{2}{v^2}-\frac{5}{v}-5v+2v^2)u^2\right)t\\
&+\left((6+\frac{2}{v^2}-\frac{5}{v}-5v+2v^2)u+(52-\frac{5}{v^3}+\frac{19}{v^2}-\frac{40}{v}-40v+19v^2-5v^3)u^2\right)t^2+\cdots
\end{aligned} \tag{C.4}
$$

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