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Product-type system of difference equations with a complex structure of solutions

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Abstract
The solvability of the following system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_{n-2}^c z_{n-2}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$, is studied in detail by using several methods. The system has the most complex structure of solutions of all the related systems studied so far, and some of the forms of solutions appear for the first time.

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1 Introduction

Various types of difference equations and systems have been investigated in the last twenty years [1–31]. Many of the systems studied there, such as the ones in [2, 3, 5–9, 11–13, 17–20, 22–26, 28–31], were essentially obtained as some sorts of symmetrization of scalar ones, and were studied first by Papaschinopoulos and Schinas. One of the basic problems investigated in the field is the solvability (see, e.g., [32–36] for widely known methods). For some recent results, see, e.g., [2, 10, 14, 16–24, 26–31]. Recently several papers presenting formulas for solutions to some difference equations and systems, but without mentioning any theory, have appeared. A theoretical explanation of formulas for such an equation given in our note [14] has attracted some attention. The note shows that the equation is closely related to a known solvable one, and the main idea has been used and developed later in many papers (see, for example, [2, 10, 16, 21, 26, 27] and the references therein).

During the study of some equations and systems (for example, those in [15] and [25]), we have noticed the importance of some classes of solvable ones in their investigation. Eliminating the constant addends in the equations and systems in these two papers, product-type ones were obtained. It is known that product-type equations and systems are solvable if initial values are positive. But the standard method for solving them by using the logarithm is not suitable if some of the values are not positive. The corresponding product-type system in [15] is a special case of the following one:

$$x_n = x_{n-k}^a y_{n-l}^b, \qquad y_n = x_{n-s}^d y_{n-m}^c, \quad n \in \mathbb{N}_0,$$

where $k, l, m, s \in \mathbb{N}$. This observation has motivated us to investigate the solvability of product-type systems with non-positive initial values. However, there are several obstacles in the investigation. An obvious one is that the functions $f_a(z) = z^a$, when a is not an integer, are multi-valued. The obstacle naturally suggests the choice $a \in \mathbb{Z}$ in dealing with the systems. Our first paper on product-type systems [24] investigated the case $k = m = 2$, $l = s = 1$, and our original intention was to study, as usual, the long-term behavior of their solutions. Not so long after that we realized that a related system with three dependent variables was also solvable [19], and that the case $k = m = 1$, $l = s = 2$, was solvable either [28]. Bearing in mind that calculations in these papers were quite technical, we put aside the problem of investigating the long-term behavior of their solutions, and concentrated our efforts on studying the solvability of related systems. In the next phase of the investigation we realized that adding the constant multipliers kept the solvability of such a system [18], which was also verified for an extension of the system in [24], in our paper [29]. Our further investigation showed that the solutions of solvable product-type systems can have several different forms for different values of parameters a, b, c, d, and that the forms can be obtained by a detailed analysis, which we did for the first time in [23] and [31]. Later we investigated another system in [30] where such analysis was unnecessary. A technical difficulty in studying the solvability problem led us to finding another method for getting solutions in [20]. Paper [22] was the first paper which successfully dealt with an associated polynomial to a product-type system of the fourth order in detail. We would also like to say that the max-type system in [17] was also solved by using product-type systems, as well as some equations in [21].

An important fact connected to the investigation is that there are only several solvable product-type systems with two dependent variables due to the finite number of combinations of the sums of two delays which cannot exceed four. Our general task is to present all solvable product-type systems with two dependent variables.

Here we show that the following system

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_{n-2}^c z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C}$, is solvable, complementing our previous results in [18, 20, 22–24, 28–31]. In fact, we will consider the case when $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$, since otherwise trivial or not well-defined solutions to system (1) are obtained. The system has the most complex structure of solutions to all the related systems studied so far, and some of the forms of solutions appear for the first time. The complexity forced us to use and develop almost all methods and tricks that we have used so far. We have to also mention that we regard that $\sum_{i=k}^{k-1} a_i = 0$, $k \in \mathbb{Z}$.

2 Auxiliary results

Here we quote three lemmas which we employ in the section that follows. The first one contains what was proved in [37], formulated in a compact form.

Lemma 1 *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e,$$

$$\Delta_0 = c^2 - 3bd + 12e, \qquad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce,$$

$$\Delta = \frac{1}{27}\left(4\Delta_0^3 - \Delta_1^2\right),$$

$$P = 8c - 3b^2, \qquad Q = b^3 + 8d - 4bc, \qquad D = 64e - 16c^2 + 16b^2c - 16bd - 3b^4.$$

(a) *If* $\Delta < 0$, *then two zeros of* P_4 *are real and different, and two are complex conjugate*;
(b) *If* $\Delta > 0$, *then all the zeros of* P_4 *are real or none is. More precisely,*

1° *if* $P < 0$ *and* $D < 0$, *then all four zeros of* P_4 *are real and different*;
2° *if* $P > 0$ *or* $D > 0$, *then there are two pairs of complex conjugate zeros of* P_4.

(c) *If* $\Delta = 0$, *then and only then* P_4 *has a multiple zero. The following cases can occur*:

1° *if* $P < 0$, $D < 0$ *and* $\Delta_0 \neq 0$, *then two zeros of* P_4 *are real and equal and two are real and simple*;
2° *if* $D > 0$ *or* $(P > 0$ *and* $(D \neq 0$ *or* $Q \neq 0))$, *then two zeros of* P_4 *are real and equal and two are complex conjugate*;
3° *if* $\Delta_0 = 0$ *and* $D \neq 0$, *there is a triple zero of* P_4 *and one simple, all real*;
4° *if* $D = 0$, *then*

4.1° *if* $P < 0$ *there are two double real zeros of* P_4;
4.2° *if* $P > 0$ *and* $Q = 0$, *there are two double complex conjugate zeros of* P_4;
4.3° *if* $\Delta_0 = 0$, *then all four zeros of* P_4 *are real and equal to* $-b/4$.

Two different proofs of the following known lemma can be found, for example, in [28] and [34].

Lemma 2 *If* λ_j, $j = \overline{1,k}$, *are mutually different zeros of the polynomial*

$$P(t) = a_kt^k + a_{k-1}t^{k-1} + \cdots + a_1t + a_0,$$

with $a_ka_0 \neq 0$, *then*

$$\sum_{j=1}^{k} \frac{\lambda_j^l}{P'(\lambda_j)} = 0$$

for $l = \overline{0,k-2}$, *and*

$$\sum_{j=1}^{k} \frac{\lambda_j^{k-1}}{P'(\lambda_j)} = \frac{1}{a_k}.$$

The following lemma contains some known summation formulas which we employ in this paper (see, e.g., [34, 36]). For a general method for calculating this type of sums, see [23].

Lemma 3 *Let* $i \in \mathbb{N}_0$ *and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \cdots + n^i z^{n-1}, \quad n \in \mathbb{N}, \tag{2}$$

where $z \in \mathbb{C}$.

Then

$$s_n^{(0)}(z) = \frac{1-z^n}{1-z}, \tag{3}$$

$$s_n^{(1)}(z) = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}, \tag{4}$$

$$s_n^{(2)}(z) = \frac{1+z-(n+1)^2 z^n + (2n^2+2n-1)z^{n+1} - n^2 z^{n+2}}{(1-z)^3}, \tag{5}$$

$$s_n^{(3)}(z) = \frac{n^3 z^n (z-1)^3 - 3n^2 z^n (z-1)^2 + 3nz^n(z^2-1) - (z^n-1)(z^2+4z+1)}{(1-z)^4} \tag{6}$$

for every $z \in \mathbb{C} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

3 Main results

This section formulates and proves our main results. The first two results deal with the case when $c = 0$ and one of the parameters b and d is also zero.

Theorem 1 *Assume that* $a, d \in \mathbb{Z}$, $b = c = 0$, $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $a \neq 1$, *then the solution to system* (1) *is given by*

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{7}$$

$$w_n = \beta \alpha^{d\frac{1-a^{n-3}}{1-a}} z_0^{da^{n-3}}, \quad n \geq 4. \tag{8}$$

(b) *If* $a = 1$, *then the solution to system* (1) *is given by*

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}, \tag{9}$$

$$w_n = \beta \alpha^{d(n-3)} z_0^d, \quad n \geq 4. \tag{10}$$

Proof Since $b = c = 0$, we have

$$z_{n+1} = \alpha z_n^a, \qquad w_{n+1} = \beta z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{11}$$

from which the following is obtained:

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}. \tag{12}$$

Employing (12) in the second equation in (11), we get

$$w_n = \beta \alpha^{d\sum_{j=0}^{n-4} a^j} z_0^{da^{n-3}}, \quad n \geq 4. \tag{13}$$

From (12) and (13), formulas (7)-(10) easily follow. □

Theorem 2 *Assume that* $a, b \in \mathbb{Z}$, $c = d = 0$, $\alpha, \beta, z_0, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $a \neq 1$, *then the solution to system* (1) *is given by*

$$z_n = \alpha^{\frac{1-a^n}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}} z_0^{a^n} w_0^{ba^{n-1}}, \quad n \geq 2, \tag{14}$$

$$w_n = \beta, \quad n \in \mathbb{N}. \tag{15}$$

(b) *If* $a = 1$, *then the solution to system* (1) *is given by* (15) *and*

$$z_n = \alpha^n \beta^{b(n-1)} z_0 w_0^b, \quad n \geq 2. \tag{16}$$

Proof Since $c = d = 0$, we have

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta, \quad n \in \mathbb{N}_0. \tag{17}$$

Employing the second equation in (17) in the first one, we get

$$z_{n+1} = \alpha \beta^b z_n^a, \quad n \in \mathbb{N},$$

from which, along with the fact $z_1 = \alpha z_0^a w_0^b$, it follows that

$$z_n = \left(\alpha\beta^b\right)^{\sum_{j=0}^{n-2} a^j} z_1^{a^{n-1}} = \alpha^{\sum_{j=0}^{n-1} a^j} \beta^{b\sum_{j=0}^{n-2} a^j} z_0^{a^n} w_0^{ba^{n-1}}, \quad n \geq 2. \tag{18}$$

From (18), formulas (14) and (16) easily follow. □

Theorem 3 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof The assumption $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$ along with (1) implies $z_n w_n \neq 0$, $n \geq -2$. Hence, from (1) we have

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{19}$$

and

$$w_{n+1}^b = \beta^b w_{n-2}^{bc} z_{n-2}^{bd}, \quad n \in \mathbb{N}_0, \tag{20}$$

which together imply

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^a z_{n-1}^c z_{n-2}^{bd-ac}, \quad n \geq 2. \tag{21}$$

We also have

$$\begin{aligned} &z_1 = \alpha z_0^a w_0^b, \qquad z_2 = \alpha^{1+a} \beta^b z_{-2}^{bd} z_0^{a^2} w_{-2}^{bc} w_0^{ab}, \\ &z_3 = \alpha^{1+a+a^2} \beta^{b(1+a)} z_{-2}^{abd} z_{-1}^{bd} z_0^{a^3} w_{-2}^{abc} w_{-1}^{bc} w_0^{a^2 b}. \end{aligned} \tag{22}$$

Now we follow our method presented, for example, in [28], p. 6, and [31], Theorem 3.5. Let $\delta = \alpha^{1-c}\beta^{b}$,

$$a_1 = a, \qquad b_1 = 0, \qquad c_1 = c, \qquad d_1 = bd - ac, \qquad y_1 = 1, \tag{23}$$

then

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \geq 2. \tag{24}$$

Further, from (24) we have

$$\begin{aligned} z_{n+2} &= \delta^{y_1} \left(\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1}\right)^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1} \\ &= \delta^{y_1+a_1} z_n^{a_1 a_1 + b_1} z_{n-1}^{b_1 a_1 + c_1} z_{n-2}^{c_1 a_1 + d_1} z_{n-3}^{d_1 a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} z_{n-3}^{d_2}, \end{aligned} \tag{25}$$

for $n \geq 3$, where

$$\begin{aligned} &a_2 := a_1 a_1 + b_1, \qquad b_2 := b_1 a_1 + c_1, \qquad c_2 := c_1 a_1 + d_1, \\ &d_2 := d_1 a_1, \qquad y_2 := y_1 + a_1. \end{aligned} \tag{26}$$

Suppose

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \tag{27}$$

for $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq k+1$, and

$$\begin{aligned} &a_k = a_1 a_{k-1} + b_{k-1}, \qquad b_k = b_1 a_{k-1} + c_{k-1}, \\ &c_k = c_1 a_{k-1} + d_{k-1}, \qquad d_k = d_1 a_{k-1}, \end{aligned} \tag{28}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{29}$$

Employing (24), where n is replaced by $n-k$ in (27), it follows that

$$\begin{aligned} z_{n+2} &= \delta^{y_k} \left(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1} z_{n-k-2}^{d_1}\right)^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1 a_k + b_k} z_{n-k}^{b_1 a_k + c_k} z_{n-k-1}^{c_1 a_k + d_k} z_{n-k-2}^{d_1 a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} z_{n-k-2}^{d_{k+1}}, \end{aligned} \tag{30}$$

for $n \geq k+2$, where

$$a_{k+1} := a_1 a_k + b_k, \qquad b_{k+1} := b_1 a_k + c_k, \qquad c_{k+1} := c_1 a_k + d_k, \qquad d_{k+1} := d_1 a_k, \tag{31}$$

$$y_{k+1} := y_k + a_k. \tag{32}$$

From (25), (26), (30)-(32) and the induction, we conclude that (27)-(29) hold for $k, n \in \mathbb{N}$ such that $2 \leq k \leq n-1$.

Setting $k = n - 1$ in (27) and employing (22), we get

$$\begin{aligned} z_{n+2} &= \delta^{y_{n-1}} z_3^{a_{n-1}} z_2^{b_{n-1}} z_1^{c_{n-1}} z_0^{d_{n-1}} \\ &= \left(\alpha^{1-c}\beta^b\right)^{y_{n-1}} \left(\alpha^{1+a+a^2}\beta^{b(1+a)} z_{-2}^{abd} z_{-1}^{bd} z_0^{a^3} w_{-2}^{abc} w_{-1}^{bc} w_0^{a^2 b}\right)^{a_{n-1}} \\ &\quad \times \left(\alpha^{1+a}\beta^b z_{-2}^{bd} z_0^{a^2} w_{-2}^{bc} w_0^{ab}\right)^{b_{n-1}} \left(\alpha z_0^a w_0^b\right)^{c_{n-1}} z_0^{d_{n-1}} \\ &= \alpha^{(1-c)y_{n-1}+(1+a+a^2)a_{n-1}+(1+a)b_{n-1}+c_{n-1}} \beta^{by_{n-1}+b(1+a)a_{n-1}+bb_{n-1}} \\ &\quad \times z_{-2}^{abda_{n-1}+bdb_{n-1}} z_{-1}^{bda_{n-1}} z_0^{a^3 a_{n-1}+a^2 b_{n-1}+ac_{n-1}+d_{n-1}} \\ &\quad \times w_{-2}^{abca_{n-1}+bcb_{n-1}} w_{-1}^{bca_{n-1}} w_0^{a^2 ba_{n-1}+abb_{n-1}+bc_{n-1}} \\ &= \alpha^{y_{n+2}-cy_{n-1}} \beta^{by_{n+1}} z_{-2}^{bda_n} z_{-1}^{bda_{n-1}} z_0^{a_{n+2}-ca_{n-1}} w_{-2}^{bca_n} w_{-1}^{bca_{n-1}} w_0^{ba_{n+1}} \end{aligned} \tag{33}$$

for $n \geq 2$.

The equalities in (28) show that for $k \geq 5$

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}. \tag{34}$$

The same equation is also satisfied by sequences b_k, c_k and d_k, $k \in \mathbb{N}$, due to the relations $b_k = a_{k+1} - a_1 a_k$, $c_k = b_{k+1} - b_1 a_k$, $d_k = d_1 a_{k-1}$.

The assumption $d_1 = bd - ac \neq 0$, along with (29) and (34), shows that it must be

$$a_{-3} = a_{-2} = a_{-1} = 0, \quad a_0 = 1, \tag{35}$$

and

$$y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1, \tag{36}$$

(see, for example, the corresponding calculations in [28]).

From (36) and since $y_1 = a_0$, we obtain

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{37}$$

It is clear that closed-form formulas for solutions to problem (34)-(35) can be easily found, from which along with (37) and Lemma 3 closed-form formulas for y_n can also be found, from which along with (33) the solvability of system (21) follows.

Further, we have

$$z_{n-2}^d = \frac{w_{n+1}}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0 \tag{38}$$

and

$$z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0, \tag{39}$$

which together imply

$$w_{n+4} = \alpha^d \beta^{1-a} w_{n+3}^a w_{n+1}^c w_n^{bd-ac}, \qquad n \in \mathbb{N}_0. \tag{40}$$

We also have

$$w_1 = \beta w_{-2}^c z_{-2}^d, \qquad w_2 = \beta w_{-1}^c z_{-1}^d \quad \text{and} \quad w_3 = \beta w_0^c z_0^d. \tag{41}$$

Following the lines of the above method, it is proved that

$$w_{n+4} = \eta^{y_k} w_{n+4-k}^{a_k} w_{n+3-k}^{b_k} w_{n+2-k}^{c_k} w_{n+1-k}^{d_k}, \tag{42}$$

for $k, n \in \mathbb{N}$, $n \geq k-1$, where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ satisfy (23) and (28), and where $(y_k)_{k\in\mathbb{N}}$ satisfies (29) and (36).

From (42) with $k = n+1$ and by using (41), we get

$$\begin{aligned} w_{n+4} &= \eta^{y_{n+1}} w_3^{a_{n+1}} w_2^{b_{n+1}} w_1^{c_{n+1}} w_0^{d_{n+1}} \\ &= \left(\alpha^d \beta^{1-a}\right)^{y_{n+1}} \left(\beta w_0^c z_0^d\right)^{a_{n+1}} \left(\beta w_{-1}^c z_{-1}^d\right)^{b_{n+1}} \left(\beta w_{-2}^c z_{-2}^d\right)^{c_{n+1}} w_0^{d_{n+1}} \\ &= \alpha^{d y_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}+b_{n+1}+c_{n+1}} w_{-2}^{c c_{n+1}} w_{-1}^{c b_{n+1}} w_0^{c a_{n+1}+d_{n+1}} \\ &\quad \times z_{-2}^{d c_{n+1}} z_{-1}^{d b_{n+1}} z_0^{d a_{n+1}} \\ &= \alpha^{d y_{n+1}} \beta^{y_{n+4}-a y_{n+3}} w_{-2}^{c(a_{n+3}-a a_{n+2})} w_{-1}^{c(a_{n+2}-a a_{n+1})} w_0^{a_{n+4}-a a_{n+3}} \\ &\quad \times z_{-2}^{d(a_{n+3}-a a_{n+2})} z_{-1}^{d(a_{n+2}-a a_{n+1})} z_0^{d a_{n+1}} \end{aligned} \tag{43}$$

for $n \in \mathbb{N}_0$.

From this and since the closed-form formulas for a_k and y_k can be found as above, the solvability of (40) follows. It is easily checked that (33) and (43) present a solution to system (1). □

Corollary 1 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$. *Then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_k)_{k\geq -3}$ *satisfies* (34) *and* (35), *while* $(y_k)_{k\geq -3}$ *is given by* (36) *and* (37).

Now we conduct a detailed analysis of the form of sequences a_k and y_k appearing in the proof of Theorem 3. The reason why equation (34) is solvable when $ac \neq bd$ is based on the fact that its characteristic polynomial

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - c\lambda + ac - bd \tag{44}$$

is of the forth degree, so, solvable by radicals. The equation $p_4(\lambda) = 0$ is equivalent to

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\left(\frac{a^2}{4} + s\right)\lambda^2 - \left(\frac{as}{2} - c\right)\lambda + \frac{s^2}{4} + bd - ac\right) = 0. \tag{45}$$

Now choose parameter s so that $(as-2c)^2 = (a^2+4s)(s^2+4bd-4ac)$ (see, for example, [38]), that is,

$$s^3 + ps + q = 0, \tag{46}$$

where $p = 4bd - 3ac$ and $q = (bd-ac)a^2 - c^2$.

Then (45) becomes

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\frac{\sqrt{a^2+4s}}{2}\lambda - \frac{as-2c}{2\sqrt{a^2+4s}}\right)^2 = 0, \tag{47}$$

which implies that

$$\lambda^2 - \left(\frac{a}{2} + \frac{\sqrt{a^2+4s}}{2}\right)\lambda + \frac{s}{2} + \frac{as-2c}{2\sqrt{a^2+4s}} = 0, \tag{48}$$

or

$$\lambda^2 - \left(\frac{a}{2} - \frac{\sqrt{a^2+4s}}{2}\right)\lambda + \frac{s}{2} - \frac{as-2c}{2\sqrt{a^2+4s}} = 0. \tag{49}$$

Recall that solutions to (46) are found in the following form: $s = u + v$, by posing the condition $uv = -p/3$. Since $u^3 + v^3 = -q$ and $u^3v^3 = -p^3/27$, we have that u^3 and v^3 are solutions to the equation $z^2 + qz - p^3/27 = 0$, from which it follows that

$$s = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{50}$$

which, by using the change of variables $p = -\Delta_0/3$ and $q = -\Delta_1/27$, is written in the following form:

$$s = \frac{1}{3\sqrt[3]{2}}\left(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right). \tag{51}$$

For this s quadratic equations (48) and (49) are solved, from which it follows that the zeros of p_4 are

$$\lambda_1 = \frac{a}{4} + \frac{\sqrt{a^2+4s}}{4} + \frac{1}{2}\sqrt{\frac{a^2}{2} - s - \frac{Q}{2\sqrt{a^2+4s}}}, \tag{52}$$

$$\lambda_2 = \frac{a}{4} + \frac{\sqrt{a^2+4s}}{4} - \frac{1}{2}\sqrt{\frac{a^2}{2} - s - \frac{Q}{2\sqrt{a^2+4s}}}, \tag{53}$$

$$\lambda_3 = \frac{a}{4} - \frac{\sqrt{a^2+4s}}{4} + \frac{1}{2}\sqrt{\frac{a^2}{2} - s + \frac{Q}{2\sqrt{a^2+4s}}}, \tag{54}$$

$$\lambda_4 = \frac{a}{4} - \frac{\sqrt{a^2+4s}}{4} - \frac{1}{2}\sqrt{\frac{a^2}{2} - s + \frac{Q}{2\sqrt{a^2+4s}}}, \tag{55}$$

where

$$\Delta_0 = 3(3ac - 4bd), \tag{56}$$

$$\Delta_1 = 27\left(a^3c - a^2bd + c^2\right), \tag{57}$$

$$Q = -a^3 - 8c. \tag{58}$$

According to Lemma 1, we see that the nature of λ_i's, $i = \overline{1,4}$, depends on the signs of

$$\Delta = \frac{1}{27}\left(4\Delta_0^3 - \Delta_1^2\right), \tag{59}$$

$$P = -3a^2 \tag{60}$$

and

$$D = 48ac - 64bd - 3a^4. \tag{61}$$

All zeros of p_4 are different and none of them is equal to 1. By Lemma 1 we see that such a situation appears if $\Delta \neq 0$ and $p_4(1) \neq 0$. From (59) we see that Δ is certainly negative if $\Delta_0 < 0$, that is, if $3ac < 4bd$. For example, if $a = 1$, $c = 2$ and $bd = 3$, then $\Delta < 0$, $p_4(1) = -3 \neq 0$ and

$$p_4(\lambda) = \lambda^4 - \lambda^3 - 2\lambda - 1.$$

Since due to (60), P cannot be positive, p_4 cannot have two pairs of complex-conjugate zeros.

It is well known that in the case the general solution to (34) has the form

$$a_n = \gamma_1\lambda_1^n + \gamma_2\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{62}$$

where γ_i, $i = \overline{1,4}$, are constants.

By Lemma 2 we also have

$$\sum_{j=1}^{4} \frac{\lambda_j^l}{p_4'(\lambda_j)} = 0 \quad \text{for } l = \overline{0,2} \quad \text{and} \quad \sum_{j=1}^{4} \frac{\lambda_j^3}{p_4'(\lambda_j)} = 1. \tag{63}$$

From this and (35), it is obtained

$$\begin{aligned} a_n = {} & \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ & + \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \end{aligned} \tag{64}$$

for $n \geq -3$.

Using (64) into (37) and applying (3) four times, we get

$$y_n = \sum_{j=0}^{n-1}\sum_{i=1}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}, \tag{65}$$

since $\lambda_i \neq 1$, $i = \overline{1,4}$. In fact, (63) shows that (65) holds also for $n = -j$, $j = \overline{0,3}$.

All zeros of p_4 are different and one of them is equal to 1. Polynomial p_4 has 1 as a zero if $p_4(1) = 1 - a - c + ac - bd = 0$, that is, if

$$(a-1)(c-1) = bd. \tag{66}$$

Hence

$$\begin{aligned} p_4(\lambda) &= \lambda^4 - a\lambda^3 - c\lambda + a + c - 1 \\ &= (\lambda - 1)\bigl(\lambda^3 + (1-a)\lambda^2 + (1-a)\lambda + 1 - a - c\bigr). \end{aligned} \tag{67}$$

We may assume that $\lambda_1 = 1$. To calculate the other three zeros of p_4, the following equation should be solved:

$$\lambda^3 + (1-a)\lambda^2 + (1-a)\lambda + 1 - a - c = 0. \tag{68}$$

By using the change of variables $\lambda = s + \frac{a-1}{3}$, equation (68) is transformed in (46) with

$$p = \frac{(1-a)(a+2)}{3} \quad \text{and} \quad q = 1 - a - c - \frac{(a-1)^2}{3} - \frac{2(a-1)^3}{27}. \tag{69}$$

Hence

$$\lambda_j = \frac{a-1}{3} + s_j, \quad j = \overline{2,4}, \tag{70}$$

where

$$s_j = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\,\varepsilon^{j-2} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\,\overline{\varepsilon}^{j-2}, \quad j = \overline{2,4}, \tag{71}$$

ε is a complex zero of the equation $t^3 = 1$, and p and q are given in (69).

For example, if $a = 3$ and $c = 2$, then $bd = 2$ and $\Delta \neq 0$, so by Lemma 1, p_4 has four different zeros such that exactly one of them is equal to 1, and

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 - 2\lambda + 4 = (\lambda - 1)\bigl(\lambda^3 - 2\lambda^2 - 2\lambda - 4\bigr). \tag{72}$$

Formula (64) holds with $\lambda_1 = 1$. Further, we have

$$y_n = \sum_{j=0}^{n-1} \frac{1}{p_4'(1)} + \sum_{j=0}^{n-1}\sum_{i=2}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \frac{n}{4-3a-c} + \sum_{i=2}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)} \tag{73}$$

for $n \in \mathbb{N}$. In fact, using (63) it is shown that (73) also holds for $n = -j$, $j = \overline{0,3}$.

From this and by Corollary 1, we get the following result.

Corollary 2 *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\Delta \neq 0$, and $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If* $(a-1)(c-1)\neq bd$, *then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_n)_{n\ge-3}$ *is given by* (64), $(y_n)_{n\ge-3}$ *is given by* (65), *while* λ_j, $j=\overline{1,4}$, *are given by* (52)-(55).

(b) *If* $(a-1)(c-1)=bd$ *and* $4-3a-c\neq 0$, *then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_n)_{n\ge-3}$ *is given by* (64) *with* $\lambda_1=1$, $(y_n)_{n\ge-3}$ *is given by* (73), $\lambda_1=1$, *while* λ_j, $j=\overline{2,4}$, *are given by* (70) *and* (71).

p_4 has exactly one double zero which is different from 1. If $a=4$, $c=0$ and $bd=-27$, then

$$p_4(\lambda)=\lambda^4-4\lambda^3+27=(\lambda-3)^2\big(\lambda^2+2\lambda+3\big),$$

is a polynomial with exactly one double zero $\lambda_{1,2}=3\neq1$ and two complex conjugate zeros $\lambda_{3,4}=-1\pm i\sqrt{2}$.

In such cases, that is, when $\lambda_1=\lambda_2$, $\lambda_i\neq\lambda_j$, $2\le i,j\le4$, we have

$$a_n=(\gamma_1+\gamma_2 n)\lambda_2^n+\gamma_3\lambda_3^n+\gamma_4\lambda_4^n,\quad n\in\mathbb{N}, \tag{74}$$

where γ_i, $i=\overline{1,4}$, are constants, and the solution satisfying (35) can be obtained, for example, by letting $\lambda_1\to\lambda_2$ in (64), which yields (see [22])

$$\begin{aligned}a_n={}&\frac{\lambda_2^{n+2}((n+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)-\lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2}\\&+\frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)}+\frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}.\end{aligned} \tag{75}$$

Combining (37) and (75) and using Lemma 3, we have

$$\begin{aligned}y_n={}&\sum_{j=0}^{n-1}\bigg(\frac{\lambda_2^{j+2}((j+3)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)-\lambda_2(2\lambda_2-\lambda_3-\lambda_4))}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2}\\&+\frac{\lambda_3^{j+3}}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)}+\frac{\lambda_4^{j+3}}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)}\bigg)\\={}&\frac{\lambda_2^3-n\lambda_2^{n+2}+(n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(1-\lambda_2)^2}\\&+\frac{(\lambda_2^4-2\lambda_2^3\lambda_3-2\lambda_2^3\lambda_4+3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n-1)}{(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2(\lambda_2-1)}\\&+\frac{\lambda_3^3(\lambda_3^n-1)}{(\lambda_3-\lambda_2)^2(\lambda_3-\lambda_4)(\lambda_3-1)}+\frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-\lambda_2)^2(\lambda_4-\lambda_3)(\lambda_4-1)}.\end{aligned} \tag{76}$$

p_4 has exactly one double zero equal to 1. Polynomial p_3 has 1 as a double zero if (66) holds and

$$p_4'(1)=4-3a-c=0, \tag{77}$$

that is, $c=4-3a$, which implies that

$$p_4(\lambda)=\lambda^4-a\lambda^3+(3a-4)\lambda+3-2a=(\lambda-1)^2\big(\lambda^2-(a-2)\lambda+3-2a\big). \tag{78}$$

It will be exactly a double zero if $p_4''(1) = 12 - 6a \neq 0$, that is, $a \neq 2$.

We may assume that $\lambda_{1,2} = 1$. Then from (78) it follows that

$$\lambda_{3,4} = \frac{a - 2 \pm \sqrt{a^2 + 4a - 8}}{2}. \tag{79}$$

In this case we have ([22])

$$\begin{aligned} a_n = {} & \frac{n(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^{n+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)} \end{aligned} \tag{80}$$

and

$$\begin{aligned} y_n = {} & \sum_{j=0}^{n-1} \Bigg(\frac{j(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^{j+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)} \Bigg) \\ = {} & \frac{(n-1)n}{2(1-\lambda_3)(1-\lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - 1)^3(\lambda_3 - \lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - 1)^3(\lambda_4 - \lambda_3)}. \end{aligned} \tag{81}$$

Corollary 3 *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\Delta = 0$ and $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If only one of the zeros of characteristic polynomial* (44) *is double, say $\lambda_1 = \lambda_2$ and $(a-1)(c-1) \neq bd$, then the general solution to system* (1) *is given by* (33) *and* (43), *where $(a_n)_{n\geq -3}$ is given by* (75), *while $(y_n)_{n\geq -3}$ is given by* (76).

(b) *If only a double zero of characteristic polynomial* (44) *is equal to* 1, *say $\lambda_1 = \lambda_2 = 1$, then the general solution to system* (1) *is given by* (33) *and* (43), *where $(a_n)_{n\geq -3}$ is given by* (80), *$(y_n)_{n\geq -3}$ is given by* (81), *while $\lambda_{3,4}$ are given by* (79).

p_4 has two pairs of different double zeros. By Lemma 1 in this case it must be $D = 0$ which is equivalent to $64bd = 48ac - 3a^4$ and $\Delta = 0$ which is equivalent to

$$4\left(3\left(3ac - 4\left(\frac{3}{4}ac - \frac{3}{64}a^4\right)\right)\right)^3 = \left(3^3\left(a^3c - a^2\left(\frac{3}{4}ac - \frac{3a^4}{64}\right) + c^2\right)\right)^2,$$

that is,

$$a^6 = \pm\left(\frac{3}{2}a^6 + 2^3a^3c + 2^5c^2\right).$$

Hence

$$\frac{5}{2}a^6 + 2^3a^3c + 2^5c^2 = 0, \tag{82}$$

or

$$\left(a^3+2^3c\right)^2=0. \tag{83}$$

If it were $a=0$ or $c=0$, then from (82) or (83) we would get $a=c=0$, which would imply $bd=0$, which is impossible. If it were $a,c\in\mathbb{Z}\setminus\{0\}$, from (82) we would get $a^3/c=8(-1\pm 2i)/5$, which is also not possible.

If $ac\neq 0$, then from (83) it follows that $c=-a^3/8$. Let $\lambda=at$, then

$$\begin{aligned}
p_4(\lambda)&=\lambda^4-a\lambda^3+\frac{a^3}{8}\lambda+\frac{a^4}{2^6}\\
&=a^4\left(t^4-t^3+\frac{t}{8}+\frac{1}{64}\right)\\
&=a^4t^2\left(\left(t-\frac{1}{8t}\right)^2-\left(t-\frac{1}{8t}\right)+\frac{1}{4}\right)\\
&=a^4t^2\left(t-\frac{1}{8t}-\frac{1}{2}\right)^2\\
&=a^4\left(t^2-\frac{t}{2}-\frac{1}{8}\right)^2\\
&=\left(\lambda^2-\frac{a\lambda}{2}-\frac{a^2}{8}\right)^2.
\end{aligned}$$

Thus

$$\lambda_{2,4}=\frac{a}{4}(1\pm\sqrt{3}) \tag{84}$$

are two double zeros of p_4 for each $a\neq 0$.

Hence, for every $a\neq 0$, polynomial p_4 has two pairs of equal zeros. It will have integer coefficients only if $a=4\hat{a}$, for some $\hat{a}\in\mathbb{Z}\setminus\{0\}$. Note that since $a(1\pm\sqrt{3})/4\neq 1$, for every $a\in\mathbb{Z}$, 1 cannot be a double zero of p_4.

In such cases the general solution to (34) has the following form:

$$a_n=(\gamma_1+\gamma_2 n)\lambda_2^n+(\gamma_3+\gamma_4 n)\lambda_4^n,\quad n\in\mathbb{N}, \tag{85}$$

where γ_i, $i=\overline{1,4}$, are constants, and the solution satisfying (35) is ([22])

$$\begin{aligned}
a_n=&\frac{\lambda_2^{n+2}(n(\lambda_2-\lambda_4)^2+\lambda_2^2-4\lambda_2\lambda_4+3\lambda_4^2)}{(\lambda_2-\lambda_4)^4}\\
&+\frac{\lambda_4^{n+2}(n(\lambda_4-\lambda_2)^2+\lambda_4^2-4\lambda_2\lambda_4+3\lambda_2^2)}{(\lambda_4-\lambda_2)^4},
\end{aligned} \tag{86}$$

from which along with (37) and by Lemma 3, it follows that

$$\begin{aligned}
y_n=\sum_{j=0}^{n-1}\Bigg(&\frac{\lambda_2^{j+2}(j(\lambda_2-\lambda_4)^2+\lambda_2^2-4\lambda_2\lambda_4+3\lambda_4^2)}{(\lambda_2-\lambda_4)^4}\\
&+\frac{\lambda_4^{j+2}(j(\lambda_4-\lambda_2)^2+\lambda_4^2-4\lambda_2\lambda_4+3\lambda_2^2)}{(\lambda_4-\lambda_2)^4}\Bigg)
\end{aligned}$$

$$= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_4)^2(1-\lambda_2)^2} + \frac{(\lambda_2^4 - 4\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_4^2)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_4)^4(\lambda_2 - 1)}$$
$$+ \frac{\lambda_4^3 - n\lambda_4^{n+2} + (n-1)\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(1-\lambda_4)^2} + \frac{(\lambda_4^4 - 4\lambda_2\lambda_4^3 + 3\lambda_2^2\lambda_4^2)(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^4(\lambda_4 - 1)}. \tag{87}$$

Corollary 4 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$, $\Delta = D = 0$, *and* $\alpha, \beta, z_{-2}, z_{-1}, z_0$, w_{-2}, w_{-1}, $w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $(a-1)(c-1) \neq bd$, *then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_n)_{n\geq -3}$ *is given by* (86), $(y_n)_{n\geq -3}$ *is given by* (87), *while* $\lambda_{2,4}$ *are given by* (84).

(b) *Characteristic polynomial* (44) *cannot have two pairs of double zeros such that one of them is equal to* 1.

p_4 has a triple zero. In this case it must be $\Delta = \Delta_0 = 0$ which is equivalent to $\Delta_0 = \Delta_1 = 0$, that is, if $bd = 3ac/4$ and

$$\Delta_1 = 27c(a^3 + 4c)/4.$$

If $c = 0$, then $bd = 0$ and consequently $ac = bd$, which is impossible. If $c = -a^3/4$, then $bd = -3a^4/16$. Let $\lambda = at$, then

$$\begin{aligned} p_4(\lambda) &= \lambda^4 - a\lambda^3 + \frac{a^3}{4}\lambda - \frac{a^4}{16} \\ &= a^4\left(t^4 - t^3 + \frac{t}{4} - \frac{1}{16}\right) \\ &= a^4\left(t^2 - t + \frac{1}{4}\right)\left(t^2 - \frac{1}{4}\right) \\ &= a^4\left(t - \frac{1}{2}\right)^3\left(t + \frac{1}{2}\right) \\ &= \left(\lambda - \frac{a}{2}\right)^3\left(\lambda + \frac{a}{2}\right). \end{aligned}$$

Hence, for every $a \neq 0$, $a/2$ is a triple zero of p_4, and p_4 cannot have a zero of the fourth order. For $a = 2\hat{a}$, $\hat{a} \in \mathbb{Z} \setminus \{0\}$, polynomial p_4 has integer coefficients, and for $a = 2$ it has a triple zero equal to 1, which could be also obtained by further analyzing the polynomial in (78).

Let λ_j, $j = \overline{1,4}$, be the zeros of polynomial p_4. In the case, we may assume that $\lambda_1 = \lambda_2 = \lambda_3$ and $\lambda_3 \neq \lambda_4$. Then, in the case, the general solution to (34) has the following form:

$$a_n = (c_1 + c_2 n + c_3 n^2)\lambda_1^n + c_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{88}$$

where c_i, $i = \overline{1,4}$, are arbitrary constants.

Since (35) must hold, the following system also holds:

$$\begin{aligned} &c_1 + c_4 = 1, \\ &c_1 - c_2 + c_3 + c_4\lambda^{-1} = 0, \\ &c_1 - 2c_2 + 4c_3 + c_4\lambda^{-2} = 0, \\ &c_1 - 3c_2 + 9c_3 + c_4\lambda^{-3} = 0, \end{aligned} \tag{89}$$

where $\lambda = \lambda_4/\lambda_1$.

By solving system (89), we get

$$c_1 = 1 - \frac{\lambda^3}{(\lambda-1)^3}, \qquad c_2 = \frac{3-5\lambda}{2(\lambda-1)^2}, \qquad c_3 = \frac{1}{2(1-\lambda)}, \qquad c_4 = \frac{\lambda^3}{(\lambda-1)^3},$$

from which along with (88) it follows that

$$a_n = \left(1 - \frac{\lambda^3}{(\lambda-1)^3} + \frac{3-5\lambda}{2(\lambda-1)^2}n + \frac{1}{2(1-\lambda)}n^2\right)\lambda_1^n + \frac{\lambda^3}{(\lambda-1)^3}\lambda_4^n.$$

Hence, if $\lambda_1 = 1$, we have

$$a_n = 1 - \frac{\lambda_4^3}{(\lambda_4-1)^3} + \frac{3-5\lambda_4}{2(\lambda_4-1)^2}n + \frac{1}{2(1-\lambda_4)}n^2 + \frac{\lambda_4^{n+3}}{(\lambda_4-1)^3} \tag{90}$$

for $n \geq -3$, whereas if $\lambda_1 \neq 1$, we have

$$a_n = \left(1 - \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3} + \frac{\lambda_1(3\lambda_1-5\lambda_4)}{2(\lambda_4-\lambda_1)^2}n + \frac{\lambda_1}{2(\lambda_1-\lambda_4)}n^2\right)\lambda_1^n + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)^3} \tag{91}$$

for $n \geq -3$.

Combining (37) and (90), using Lemma 3 and by some calculation, we get

$$\begin{aligned} y_n ={}& \left(1 - \frac{\lambda_4^3}{(\lambda_4-1)^3}\right)n + \frac{(3-5\lambda_4)(n-1)n}{4(\lambda_4-1)^2} + \frac{(n-1)n(2n-1)}{12(1-\lambda_4)} \\ &+ \frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-1)^4} \end{aligned} \tag{92}$$

(recall that λ_4 cannot be equal to 1 in this case).

Combining (37) and (91), using Lemma 3 and by some calculation, we get

$$\begin{aligned} y_n ={}& \left(1 - \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3}\right)\frac{\lambda_1^n-1}{\lambda_1-1} + \frac{\lambda_1^2(3\lambda_1-5\lambda_4)(1-n\lambda_1^{n-1}+(n-1)\lambda_1^n)}{2(\lambda_4-\lambda_1)^2(1-\lambda_1)^2} \\ &+ \frac{\lambda_1^2(1+\lambda_1-n^2\lambda_1^{n-1}+(2n^2-2n-1)\lambda_1^n-(n-1)^2\lambda_1^{n+1})}{2(\lambda_1-\lambda_4)(1-\lambda_1)^3} \\ &+ \frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-\lambda_1)^3(\lambda_4-1)} \end{aligned} \tag{93}$$

for $n \in \mathbb{N}$.

Corollary 5 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac \neq bd$ $\Delta = \Delta_0 = 0$, $D \neq 0$, *and* $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If the triple zero of polynomial* (44) *is different from* 1, *then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_n)_{n\geq -3}$ *is given by* (91), $(y_n)_{n\geq -3}$ *is given by* (93), $\lambda_j = a/2$, $j = \overline{1,3}$ *and* $\lambda_4 = -a/2$.

(b) *If the triple zero of polynomial* (44) *is equal to* 1, *say* $\lambda_1 = \lambda_2 = \lambda_3 = 1$, *which is equivalent to* $a = 2$, $c = -2$ *and* $bd = -3$, *then the general solution to system* (1) *is given by* (33) *and* (43), *where* $(a_n)_{n\geq -3}$ *is given by* (90), $(y_n)_{n\geq -3}$ *is given by* (92), *while* $\lambda_4 = -1$.

Now we deal with the case $ac - bd = 0$, $c \neq 0$.

Theorem 4 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac = bd$, $c \neq 0$ *and* $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof As above, the condition $\alpha, \beta, w_{-2}, w_{-1}, w_0, z_{-2}, z_{-1}, z_0 \in \mathbb{C} \setminus \{0\}$ along with (1) implies $z_n w_n \neq 0$, $n \geq -2$. Hence, (19)-(21) hold, which along with the condition $ac = bd$ yields

$$z_{n+2} = \alpha^{1-c}\beta^b z_{n+1}^a z_{n-1}^c, \quad n \geq 2. \tag{94}$$

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 = a, \qquad b_1 = 0, \qquad c_1 = c, \qquad y_1 = 1. \tag{95}$$

Then equation (94) can be written as

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1}, \quad n \geq 2. \tag{96}$$

Further, from (96), we get

$$\begin{aligned} z_{n+2} &= \delta^{y_1}\left(\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1}\right)^{a_1} z_n^{b_1} z_{n-1}^{c_1} \\ &= \delta^{y_1+a_1} z_n^{a_1 a_1 + b_1} z_{n-1}^{b_1 a_1 + c_1} z_{n-2}^{c_1 a_1} \\ &= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2}, \end{aligned} \tag{97}$$

for $n \geq 3$, where

$$a_2 := a_1 a_1 + b_1, \qquad b_2 := b_1 a_1 + c_1, \qquad c_2 := c_1 a_1, \qquad y_2 := y_1 + a_1. \tag{98}$$

Suppose

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \tag{99}$$

for $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq k+1$, and

$$a_k = a_1 a_{k-1} + b_{k-1}, \qquad b_k = b_1 a_{k-1} + c_{k-1}, \qquad c_k = c_1 a_{k-1}, \tag{100}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{101}$$

Then, by using relation (96) with $n \to n-k$ into (99), we obtain

$$\begin{aligned} z_{n+2} &= \delta^{y_k}\left(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1}\right)^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} \\ &= \delta^{y_k+a_k} z_{n+1-k}^{a_1a_k+b_k} z_{n-k}^{b_1a_k+c_k} z_{n-k-1}^{c_1a_k} \\ &= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}}, \end{aligned} \tag{102}$$

for $n \ge k+2$, where

$$a_{k+1} := a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k, \qquad c_{k+1} := c_1a_k, \tag{103}$$

$$y_{k+1} := y_k + a_k. \tag{104}$$

From (97), (98), (102)-(104) and the induction is obtained that (99)-(101) hold for $k, n \in \mathbb{N}$ such that $2 \le k \le n-1$.

Setting $k = n-1$ in (99) and employing (22), we obtain

$$\begin{aligned} z_{n+2} &= \delta^{y_{n-1}} z_3^{a_{n-1}} z_2^{b_{n-1}} z_1^{c_{n-1}} \\ &= \left(\alpha^{1-c}\beta^b\right)^{y_{n-1}} \left(\alpha^{1+a+a^2}\beta^{b(1+a)} z_{-2}^{abd} z_{-1}^{bd} z_0^{a^3} w_{-2}^{abc} w_{-1}^{bc} w_0^{a^2b}\right)^{a_{n-1}} \\ &\quad \times \left(\alpha^{1+a}\beta^b z_{-2}^{bd} z_0^{a^2} w_{-2}^{bc} w_0^{ab}\right)^{b_{n-1}} \left(\alpha z_0^a w_0^b\right)^{c_{n-1}} \\ &= \alpha^{(1-c)y_{n-1}+(1+a+a^2)a_{n-1}+(1+a)b_{n-1}+c_{n-1}} \beta^{by_{n-1}+b(1+a)a_{n-1}+bb_{n-1}} \\ &\quad \times z_{-2}^{abda_{n-1}+bdb_{n-1}} z_{-1}^{bda_{n-1}} z_0^{a^3a_{n-1}+a^2b_{n-1}+ac_{n-1}} \\ &\quad \times w_{-2}^{abca_{n-1}+bcb_{n-1}} w_{-1}^{bca_{n-1}} w_0^{a^2ba_{n-1}+abb_{n-1}+bc_{n-1}} \\ &= \alpha^{y_{n+2}-cy_{n-1}} \beta^{by_{n+1}} z_{-2}^{bda_n} z_{-1}^{bda_{n-1}} z_0^{aa_{n+1}} w_{-2}^{bca_n} w_{-1}^{bca_{n-1}} w_0^{ba_{n+1}} \end{aligned} \tag{105}$$

for $n \ge 2$.

From (100), we get

$$a_k = a_1a_{k-1} + b_1a_{k-2} + c_1a_{k-3}, \quad k \ge 4, \tag{106}$$

which along with $b_k = a_{k+1} - a_1a_k$ and $c_k = c_1a_{k-1}$ implies that b_k and c_k also satisfy the equation.

Using the condition $c_1 = c \ne 0$ along with (101) and (106), we easily get

$$a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1 \tag{107}$$

and

$$y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1. \tag{108}$$

From (101), (108) and since $y_1 = a_0$, we obtain

$$y_k = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{109}$$

It is clear that closed-form formulas for solutions to (106) can be easily found, from which along with (109) and Lemma 3 closed-form formulas for y_n can be also found, from which along with (105) the solvability of (94) follows.

On the other hand, we also have that (38)-(40) hold, from which along with the condition $ac = bd$ it follows that

$$w_{n+4} = \alpha^d \beta^{1-a} w_{n+3}^a w_{n+1}^c \tag{110}$$

for $n \in \mathbb{N}_0$.

As above, it can be proved that for all $k, n \in \mathbb{N}$ such that $1 \leq k \leq n$

$$w_{n+4} = \eta^{y_k} w_{n+4-k}^{a_k} w_{n+3-k}^{b_k} w_{n+2-k}^{c_k}, \quad n \geq k-1, \tag{111}$$

where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ satisfy (100) and (95), while $(y_k)_{k\in\mathbb{N}}$ satisfies (101) and (108).

From (111) with $k = n+1$ and by using (41), we get

$$\begin{aligned} w_{n+4} &= \eta^{y_{n+1}} w_3^{a_{n+1}} w_2^{b_{n+1}} w_1^{c_{n+1}} \\ &= \left(\alpha^d \beta^{1-a}\right)^{y_{n+1}} \left(\beta w_0^c z_0^d\right)^{a_{n+1}} \left(\beta w_{-1}^c z_{-1}^d\right)^{b_{n+1}} \left(\beta w_{-2}^c z_{-2}^d\right)^{c_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{(1-a)y_{n+1}+a_{n+1}+b_{n+1}+c_{n+1}} w_{-2}^{cc_{n+1}} w_{-1}^{cb_{n+1}} w_0^{ca_{n+1}} \\ &\quad \times z_{-2}^{dc_{n+1}} z_{-1}^{db_{n+1}} z_0^{da_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{y_{n+4}-ay_{n+3}} w_{-2}^{c(a_{n+3}-aa_{n+2})} w_{-1}^{c(a_{n+2}-aa_{n+1})} w_0^{ca_{n+1}} \\ &\quad \times z_{-2}^{d(a_{n+3}-aa_{n+2})} z_{-1}^{d(a_{n+2}-aa_{n+1})} z_0^{da_{n+1}} \end{aligned} \tag{112}$$

for $n \in \mathbb{N}_0$.

From this and since the closed-form formulas for $(a_k)_{k\geq-2}$ and $(y_k)_{k\geq-2}$ can be found as above, the solvability of (110) follows. It is easily checked that (105) and (112) present a solution to system (1) in this case. □

Now we conduct a detailed analysis of the form of sequences a_k and y_k appearing in the proof of Theorem 4. The reason why equation (106) is solvable when $c \neq 0$ is based on the fact that its characteristic polynomial

$$p_3(\lambda) = \lambda^3 - a\lambda^2 - c \tag{113}$$

is of the third degree, so, solvable by radicals.

By using the change of variables $\lambda = s + \frac{a}{3}$, the equation $p_3(\lambda) = 0$ is transformed into the following one:

$$s^3 - \frac{a^2}{3}s - \frac{2a^3 + 27c}{27} = 0. \tag{114}$$

By using formula (50), we get (51) with

$$\Delta_0 = a^2 \quad \text{and} \quad \Delta_1 = 2a^3 + 27c. \tag{115}$$

Hence

$$\lambda_j = \frac{a}{3} + \frac{1}{3\sqrt[3]{2}}\left(\varepsilon^j \sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \overline{\varepsilon}^j \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right), \quad j = \overline{0,2}, \tag{116}$$

where ε is a complex third root of the unity, are the zeros of p_3.

All zeros of p_3 are different and none of them is equal to 1. In this case it must be $\Delta \neq 0$, that is, $\Delta_1^2 \neq 4\Delta_0^3$, from which it follows that $4a^6 \neq (2a^3 + 27c)^2$, which is equivalent to $c(4a^3 + 27c) \neq 0$. Hence, if $0 \neq c \neq -4a^3/27$, then all the zeros of p_3 are different. If, additionally, $p_3(1) \neq 0$, that is, $a + c \neq 1$, then none of them is equal to 1. For example, such a situation appears if $a = c = k \in \mathbb{N}$.

All zeros of p_3 are different and one of them is equal to 1. Polynomial p_3 will have a zero equal to 1 if $a + c = 1$, so that

$$p_3(\lambda) = \lambda^3 - a\lambda^2 + a - 1 = (\lambda - 1)\bigl(\lambda^2 - (a-1)\lambda - (a-1)\bigr).$$

Since $p_3'(1) = 3 - 2a \notin \mathbb{Z}$, for every $a \in \mathbb{Z}$, it follows that 1 cannot be a double zero of p_3. Solving the equation $p_3(\lambda) = 0$, we get $\lambda_3 = 1$,

$$\lambda_{1,2} = \frac{a - 1 \pm \sqrt{a^2 + 2a - 3}}{2}. \tag{117}$$

The general solution to (106) in this case has the following form:

$$a_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \quad n \in \mathbb{N}, \tag{118}$$

where α_i, $i = \overline{1,3}$, are constants, which due to $c_1 = c \neq 0$ can be prolonged for every non-positive index.

From Lemma 2 with $p_3(t) = \prod_{j=1}^3 (t - \lambda_j)$, we have

$$\sum_{j=1}^3 \frac{\lambda_j^l}{p_3'(\lambda_j)} = 0 \quad \text{for } l = 0, 1 \quad \text{and} \quad \sum_{j=1}^3 \frac{\lambda_j^2}{p_3'(\lambda_j)} = 1. \tag{119}$$

This along with (107) implies that

$$a_n = \frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \tag{120}$$

for $n \geq -2$.

From (109) and (120) and the fact $a_0 = 1$, we have

$$y_n = \sum_{i=0}^{n-1}\left(\frac{\lambda_1^{i+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{i+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{i+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right) \tag{121}$$

for $n \in \mathbb{N}$.

If $\lambda_i \neq 1$, $i = \overline{1,3}$, then from formula (121) it follows that

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - 1)} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - 1)} + \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - 1)} \tag{122}$$

for $n \in \mathbb{N}$ (in fact, (122) holds for every $n \geq -2$).

If one of the zeros is equal to one, say λ_3, then $1 \neq \lambda_1 \neq \lambda_2 \neq 1$, and we have

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)^2} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2} + \frac{n}{(\lambda_1 - 1)(\lambda_2 - 1)} \tag{123}$$

for $n \in \mathbb{N}$ (in fact, (123) holds for every $n \geq -2$).

Corollary 6 *Assume that $a, b, c, d \in \mathbb{Z}$, $ac = bd$, $c \neq 0$, $\Delta_1^2 \neq 4\Delta_0^3$ and $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If $a + c \neq 1$, then the general solution to* (1) *is given by* (105) *and* (112), *where $(a_n)_{n\geq-2}$ is given by* (120), *$(y_n)_{n\geq-2}$ is given by* (122), *while λ_j, $j = \overline{1,3}$ are given by* (116).

(b) *If one of the zeros of characteristic polynomial* (113) *is equal to* 1, *say λ_3, i.e., if $a + c = 1$ and $2a \neq 3$, then the general solution to* (1) *is given by formulas* (105) *and* (112), *where $(a_n)_{n\geq-2}$ is given by* (120) *with $\lambda_3 = 1$, $(y_n)_{n\geq-2}$ is given by* (123), *and $\lambda_{1,2}$ are given by* (117).

One of the zeros is double. In this case it must be $\Delta_1^2 = 4\Delta_0^3$, that is, $c(4a^3 + 27c) = 0$. Since the case $c \neq 0$ is excluded, it must be $c = -4a^3/27$, from which it follows that

$$p_3(\lambda) = \lambda^3 - a\lambda^2 + \frac{4}{27}a^3.$$

Since in this case it must be also $p_3'(\lambda) = 0$, it follows that $\lambda_{1,2} = 2a/3$ is a double zero and

$$p_3(\lambda) = \left(\lambda - \frac{2a}{3}\right)^2\left(\lambda + \frac{a}{3}\right).$$

In order that $c \in \mathbb{Z}$, it is clear that it must be $a = 3\hat{a}$ for some $\hat{a} \in \mathbb{Z}$. Since $2a/3 \neq 1$ when $a \in \mathbb{Z}$, the polynomial cannot have 1 as a double zero, and consequently cannot have 1 as a triple zero.

If $\lambda_1 \neq \lambda_2 = \lambda_3$, then the general solution to (106) has the following form:

$$a_n = \hat{\alpha}_1\lambda_1^n + (\hat{\alpha}_2 + \hat{\alpha}_3 n)\lambda_2^n, \quad n \in \mathbb{N}, \tag{124}$$

where $\hat{\alpha}_i \in \mathbb{R}$, $i = \overline{1,3}$. Since, in our case, condition (107) must be satisfied, the solution $(a_n)_{n\geq-2}$ to (106) can be found by letting $\lambda_3 \to \lambda_2$ in (120), so that

$$a_n = \frac{\lambda_1^{n+2} + (\lambda_2 - 2\lambda_1 + n(\lambda_2 - \lambda_1))\lambda_2^{n+1}}{(\lambda_2 - \lambda_1)^2} \tag{125}$$

for $n \geq -2$ (see [31]).

From (109) and (125) and the fact that $a_0 = 1$, we have

$$y_n = \sum_{j=0}^{n-1} a_j = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + (\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1))\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2} \tag{126}$$

for every $n \in \mathbb{N}$.

From (126) and Lemma 3, it follows that

$$\begin{aligned} y_n = {} & \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(\lambda_2 - 2\lambda_1)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} \\ & + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2} \end{aligned} \tag{127}$$

for $n \in \mathbb{N}$ (in fact, (127) holds also for every $n \geq -2$).

If we assume that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 \neq 1$, which is possible if $a = -3$, then from (126) it follows that

$$y_n = \frac{n}{(\lambda_2 - 1)^2} + \frac{(\lambda_2 - 2)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - 1)^3} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - 1)^3} \tag{128}$$

for every $n \in \mathbb{N}$ (in fact, (128) holds also for every $n \geq -2$).

Corollary 7 *Assume that* $a, b, c, d \in \mathbb{Z}$, $ac = bd$, $c \neq 0$, $\Delta_1^2 = 4\Delta_0^3$, *and* $\alpha, \beta, z_{-2}, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $a + c \neq 1$, *then the general solution to* (1) *is given by* (105) *and* (112), *where* $(a_n)_{n\geq -2}$ *is given by* (125), $(y_n)_{n\geq -2}$ *is given by* (127), $\lambda_1 = -a/3$ *and* $\lambda_{2,3} = 2a/3$.

(b) *If only one of the zeros of polynomial* (113) *is equal to one, say* λ_1, *that is, if* $a = -3$ *and* $c = 4$, *then the general solution to system* (1) *is given by* (105) *and* (112), *where* $(a_n)_{n\geq -2}$ *is given by* (125) *with* $\lambda_1 = 1$, $(y_n)_{n\geq -2}$ *is given by* (128), $\lambda_1 = 1$ *and* $\lambda_{2,3} = -2$.

(c) *It is not possible that two zeros of polynomial* (113) *are equal to one.*

Triple zero case. In this case it must be $p_3(\lambda) = p_3'(\lambda) = p_3''(\lambda) = 0$. From $p_3''(\lambda) = 0$ it is obtained that $\lambda = a/3$. Since $p_3'(\lambda) = 3\lambda^2 - 2a\lambda$, we see that $a/3$ is its root only if $a = 0$, which would imply that $p_3(\lambda) = \lambda^3 - c$, but this polynomial has a triple zero if and only if $c = 0$, which contradicts the assumption $c \neq 0$. Hence, polynomial (113) cannot have a triple zero.

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author has contributed solely to the writing of this paper. He read and approved the manuscript.

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