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Production and propagation of Hermite-sinusoidal-Gaussian laser beams

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Hermite-sinusoidal-Gaussian solutions to the wave equation have recently been obtained. In the limit of large Hermite-Gaussian beam size, the sinusoidal factors are dominant and reduce to the conventional modes of a rectangular waveguide. In the opposite limit the beams reduce to the familiar Hermite-Gaussian form. The propagation of these beams is examined in detail, and resonators are designed that will produce them. As an example, a special resonator is designed to produce hyperbolic-sine-Gaussian beams. This ring resonator contains a hyperbolic-cosine-Gaussian apodized aperture. The beam mode has finite energy and is perturbation stable. © 1998 Optical Society of America [S0740-3232(98)02809-9]

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1. INTRODUCTION

It has long been known that the modes of a rectangular metal waveguide are sinusoidal functions. The amplitude of the sinusoids is approximately zero at the metal waveguide walls, which impose well-defined boundary conditions. When ends are put on these waveguides, they become resonators, such as those used for the first maser oscillators. With the advent of the laser, it was found that because of the small divergence of coherent laser light, the rectangular waveguide walls were no longer necessary.¹ The rectangular-symmetry beam modes of these open resonators (with slightly spherical end mirrors) were found to be describable in terms of real-valued Hermite polynomials multiplied by complex-valued Gaussian functions.² These Hermite-Gaussian modes have been highly successful in characterizing the resonant fields of low-diffraction-loss resonators. They also have the attractive property that they retain their beam shape and spherical phase fronts as they propagate through free space.

The real-argument Hermite-Gaussian modes apply to laser resonators that do not contain spatial loss variations that are due to, for example, Gaussian-profiled apodized apertures and laser amplifiers with a radial gain profile. For lasers with these elements, alternative complex-argument Hermite-Gaussian^{3,4} beam modes are found as solutions to the paraxial wave equation. These beam modes are called complex argument on the basis that the arguments of the Hermite polynomials are complex. The original real-argument modes are a special case of the complex-argument modes.⁵ Unlike the real-argument modes, the complex-argument modes do not, in general, retain their beam shape as they propagate

through free space; and their phase fronts may be much more complicated than those of a spherical wave.

Although the sinusoidal beams used previously in the study of rectangular waveguides are well known, these beams have been only recently proposed for use in free-space propagation. Since these beams would require infinite energy, interest has centered on truncated sinusoidal beams for pseudonondiffracting beam applications.⁶ An important alternative to these beams is given by the newly obtained sinusoidal-Gaussian beams.⁷ In particular, the much squarer hyperbolic-cosine-Gaussian (cosh-Gaussian) beam shape may be more useful than the Gaussian beam shape at extracting energy from a laser amplifier.

In an associated study Hermite-sinusoidal-Gaussian beams are obtained as rectangular-symmetried solutions to the paraxial wave equation.⁸ In the limit of large Hermite-Gaussian beam size, the sinusoidal factors are dominant, and the solutions become the modes of the rectangular waveguide. In the opposite limit of large sinusoidal beam size, the beams reduce to the complex-argument beam modes. The basic theory governing the propagation of these beams through misaligned optical systems representable by complex *ABCDGH* matrices is discussed in Section 2. The lowest-order beam mode is the sinusoidal Gaussian.⁷ However, no resonator has been identified previously that produces this mode. In Section 3 a variable-reflectivity mirror resonator is designed that has a hyperbolic sinusoidal Gaussian as its cavity mode. Variable-reflectivity mirrors and graded-phase mirrors are commonly used to increase the mode volume of a laser resonator.⁹⁻¹¹ In this study the resonator is designed to ensure that the sinusoidal-Gaussian

beam profile repeats after a round trip. In addition to satisfying this oscillation condition, however, the cavity mode must also be stable with respect to inevitable perturbations.¹² In Section 4 it is shown that the designed resonator mode is perturbation stable.

2. PROPAGATION OF HERMITE-SINUSOIDAL-GAUSSIAN BEAMS

As may be shown by a rigorous density-matrix derivation, media that have gain (or loss) can be represented with a complex propagation constant k . If this is done, the Heaviside form of Maxwell's equations may be combined to form the following Helmholtz equation:

$$\nabla^2 E'(x, y, z) + k^2(x, y, z)E'(x, y, z) = 0, \quad (1)$$

where the complex scalar amplitude of the electric field is related to the real vector electric field by the relation

$$\mathbf{E}(x, y, z, t) = \text{Re} \left[E'(x, y, z) \exp(i\omega t) \begin{Bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \end{Bmatrix} \right]. \quad (2)$$

The terms in the large curly brackets are meant to suggest a possible superposition of unit vector components of the form $a\mathbf{i}_x + b\mathbf{i}_y$, where a and b are complex constants. If, for example, $a = 1$ and $b = -i$, the light field would be circularly polarized. In deriving Eq. (1) from Maxwell's equations, it has been assumed that the propagation constant has only slow spatial variations, so that the scalar approximation may be used.¹³

If we further assume that $E'(x, y, z)$ propagates paraxially, then for linear media with spatially varying gain (or loss) $\alpha(x, y, z)$ and/or refractive index $n(x, y, z)$ of the form

$$k(x, y, z) \equiv \frac{2\pi}{\lambda} n(x, y, z) + i\alpha(x, y, z) \quad (3)$$

$$= k_0(z) - k_{1x}(z)x/2 - k_{1y}(z)y/2 - k_{2x}(z)x^2/2 - k_{2y}(z)y^2/2 \quad (4)$$

the Hermite-sinusoidal-Gaussian beam solutions to Eq. (1) are⁸

$$E'(x, y, z) = E'_0(z) \exp \left\{ -i \left[\frac{Q_x(z)}{2} x^2 + \frac{Q_y(z)}{2} y^2 + S_x(z)x + S_y(z)y + P(z) \right] \right\} \\ \times H_m \left(\frac{\sqrt{2}}{W_x(z)} [x - \delta_x(z)] \right) \\ \times H_n \left(\frac{\sqrt{2}}{W_y(z)} [y - \delta_y(z)] \right) \\ \times \begin{Bmatrix} \cosh[\Omega_x(z)x + \Phi_x(z)] \\ \sinh[\Omega_x(z)x + \Phi_x(z)] \\ \cos[\Omega_x(z)x + \Phi_x(z)] \\ \sin[\Omega_x(z)x + \Phi_x(z)] \end{Bmatrix} \\ \times \begin{Bmatrix} \cosh[\Omega_y(z)y + \Phi_y(z)] \\ \sinh[\Omega_y(z)y + \Phi_y(z)] \\ \cos[\Omega_y(z)y + \Phi_y(z)] \\ \sin[\Omega_y(z)y + \Phi_y(z)] \end{Bmatrix}, \quad (5)$$

where again the curly brackets designate a linear combination of function solutions. In deriving Eq. (5), it has also been assumed that the transverse spatial variation of the medium is slow, so that the square of the complex propagation constant k in Eq. (4) is at most quadratic in x and y . Equation (5) is identical to one in Ref. 8 if $a \equiv \sqrt{2}/W$, $b \equiv -\sqrt{2}\delta/W$, $a' \equiv \Omega$, $b' \equiv \Phi$, and appropriate changes are made in the phase parameter.

Though it would seem that Eq. (5) is valid only for continuous media, it can easily be shown that it is also valid for "thin" optical elements such as thin lenses, curved mirrors, thin prisms, etc. This follows because each of these elements can be viewed as a special case of Eq. (4). For example, the x variation of an aligned and lossless optical element has $\alpha(x, y, z) = 0$, both y -subscripted variables are zero, and $n_{1x}(z) = 0$. In the limit as $z \rightarrow 0$ and $n_2(z) \rightarrow n_0 f^{-1} z^{-1}$, Eq. (4) is valid for a thin lens. This can be verified by comparing the beam matrix of a lens-like medium in those limits with the beam matrix for a thin lens.¹²

For a system of optical components, it is common to designate the various reference planes numerically. In particular, Eq. (5) may be rewritten as

$$E'_2(x, y) = E'_{02} \exp \left[-i \left(\frac{k_{02}}{2q_{x2}} x^2 + \frac{k_{02}}{2q_{y2}} y^2 + S_{x2}x + S_{y2}y + P_2 \right) \right] \\ \times H_m \left(\frac{\sqrt{2}}{W_{x2}} (x - \delta_{x2}) \right) H_n \left(\frac{\sqrt{2}}{W_{y2}} (y - \delta_{y2}) \right) \\ \times \begin{Bmatrix} \cosh(\Omega_{x2}x + \Phi_{x2}) \\ \sinh(\Omega_{x2}x + \Phi_{x2}) \\ \cos(\Omega_{x2}x + \Phi_{x2}) \\ \sin(\Omega_{x2}x + \Phi_{x2}) \end{Bmatrix} \\ \times \begin{Bmatrix} \cosh(\Omega_{y2}y + \Phi_{y2}) \\ \sinh(\Omega_{y2}y + \Phi_{y2}) \\ \cos(\Omega_{y2}y + \Phi_{y2}) \\ \sin(\Omega_{y2}y + \Phi_{y2}) \end{Bmatrix}, \quad (6)$$

where the 2 subscript indicates the output plane for a given optical element or system. A 1 subscript would indicate the corresponding input plane. The significance of the parameters of the Gaussian portion of the beam is contained in the relations

$$\frac{1}{q_x} = \frac{1}{R_x} - i \frac{\lambda_m}{\pi w_x^2} \quad (7)$$

$$\frac{1}{q_y} = \frac{1}{R_y} - i \frac{\lambda_m}{\pi w_y^2}, \quad (8)$$

$$\frac{S_x}{\beta_0} = -\frac{1}{q_x} d_{xa} + d'_{xa}, \quad (9)$$

$$\frac{S_y}{\beta_0} = -\frac{1}{q_y} d_{ya} + d'_{ya}, \quad (10)$$

where R_x and R_y are the radii of the phase-front curvatures, w_x and w_y are the spot sizes, d_{xa} and d_{ya} are the transverse displacements of the beam from the z axis, and d'_{xa} and d'_{ya} are the slopes of the propagating beam in the x and y directions, respectively. It has also been assumed in writing Eqs. (7)–(10) that the input and output planes of the optical system are in a medium with a low gain per wavelength.

With this formalism an optical element is fully characterized by three matrices: a 3×3 generalized beam matrix for each of the x and y directions and what can be viewed as a 1×1 matrix for the plane-wave portion of the beam:

$$\begin{pmatrix} u_x \\ u_x/q_x \\ S_x u_x \end{pmatrix}_2 = \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ G_x & H_x & 1 \end{bmatrix} \begin{pmatrix} u_x \\ u_x/q_x \\ S_x u_x \end{pmatrix}_1, \quad (11)$$

$$\begin{pmatrix} u_y \\ u_y/q_y \\ S_y u_y \end{pmatrix}_2 = \begin{bmatrix} A_y & B_y & 0 \\ C_y & D_y & 0 \\ G_y & H_y & 1 \end{bmatrix} \begin{pmatrix} u_y \\ u_y/q_y \\ S_y u_y \end{pmatrix}_1, \quad (12)$$

$$(E'_0)_2 = A_J (E'_0)_1. \quad (13)$$

The $ABCD$ portion of the generalized beam matrices are identical to ordinary beam matrices. The complex G and H terms account for element displacement or misalignment.¹⁴ For astigmatic media the matrix in Eq. (11) would be different from the matrix in Eq. (12). For thin optical elements, A_J in Eq. (13) would be unity. Anisotropic media may be treated approximately by replacing the 1×1 matrix in Eq. (13) with a 2×2 Jones matrix. As an example, a length d of free space would be represented by

$$M_x = \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (14)$$

$$M_y = \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (15)$$

$$A_J = \exp(-i\beta_0 d). \quad (16)$$

In Eq. (16) the wave number is related to the wavelength by the relation $\beta_0 = 2\pi/\lambda$. Since free space is stigmatic, $M_x = M_y$. Since free space is also isotropic, it does not require a Jones matrix. Tables with the generalized beam matrix representation of several optical elements are given in Ref. 14. Additionally, the matrix representation for a grating is given in Ref. 15.

The transformations for the beam parameters q_x and q_y and the displacement parameters S_x and S_y may be obtained by dividing rows of Eqs. (11) and (12):

$$\frac{1}{q_{x2}} = \frac{C_x + D_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (17)$$

$$\frac{1}{q_{y2}} = \frac{C_y + D_y/q_{y1}}{A_y + B_y/q_{y1}}, \quad (18)$$

$$S_{x2} = \frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}, \quad (19)$$

$$S_{y2} = \frac{S_{y1} + G_y + H_y/q_{y1}}{A_y + B_y/q_{y1}}. \quad (20)$$

These transformations are valid for continuous media, thin optical elements, and optical systems. The other required transformations are⁸

$$W_{x2}^2 = W_{x1}^2 (A_x + B_x/q_{x1})^2 + 4iB_x(A_x + B_x/q_{x1})/k_{01}, \quad (21)$$

$$W_{y2}^2 = W_{y1}^2 (A_y + B_y/q_{y1})^2 + 4iB_y(A_y + B_y/q_{y1})/k_{01}, \quad (22)$$

$$\delta_{x2} = \delta_{x1}(A_x + B_x/q_{x1}) + S_{x1}B_x/k_{01} + (B_xG_x - A_xH_x)/k_{01}, \quad (23)$$

$$\delta_{y2} = \delta_{y1}(A_y + B_y/q_{y1}) + S_{y1}B_y/k_{01} + (B_yG_y - A_yH_y)/k_{01}, \quad (24)$$

$$\Omega_{x2} = \frac{\Omega_{x1}}{A_x + B_x/q_{x1}}, \quad (25)$$

$$\Omega_{y2} = \frac{\Omega_{y1}}{A_y + B_y/q_{y1}}, \quad (26)$$

$$\Phi_{x2} = \Phi_{x1} - \frac{\Omega_{x1}B_x}{k_{01}} \left(\frac{S_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}} \right) + \frac{\Omega_{x1}H_x}{k_{01}}, \quad (27)$$

$$\Phi_{y2} = \Phi_{y1} - \frac{\Omega_{y1}B_y}{k_{01}} \left(\frac{S_{y1} + G_y + H_y/q_{y1}}{A_y + B_y/q_{y1}} \right) + \frac{\Omega_{y1}H_y}{k_{01}}, \quad (28)$$

$$\begin{aligned} P_2 - P_1 &= \frac{i}{2} \ln(A_x D_x - B_x C_x) \\ &\quad - \frac{i}{2} \ln(A_x + B_x/q_{x1}) - \frac{i}{2} \ln(A_y + B_y/q_{y1}) \\ &\quad + i \frac{m}{2} \ln \left(1 + \frac{4i}{k_{01} W_{x1}^2} \frac{B_x}{A_x + B_x/q_{x1}} \right) \\ &\quad + i \frac{n}{2} \ln \left(1 + \frac{4i}{k_{01} W_{y1}^2} \frac{B_y}{A_y + B_y/q_{y1}} \right) \\ &\quad \mp i \left[\frac{\Omega_{x1}^2}{2k_{01}} \left(\frac{B_x}{A_x + B_x/q_{x1}} \right) \right] \\ &\quad \mp i \left[\frac{\Omega_{y1}^2}{2k_{01}} \left(\frac{B_y}{A_y + B_y/q_{y1}} \right) \right] \\ &\quad - \frac{B_x}{2k_{01}} \frac{(S_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} \\ &\quad - \frac{B_y}{2k_{01}} \frac{(S_{y1} + G_y + H_y/q_{y1})^2}{A_y + B_y/q_{y1}} \\ &\quad + \frac{H_x}{2k_{01}} (2S_{x1} + G_x + H_x/q_{x1}) \\ &\quad + \frac{H_y}{2k_{01}} (2S_{y1} + G_y + H_y/q_{y1}) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2k_{01}} \int_0^z \left(G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right) dz' \\
 & + \frac{1}{2k_{01}} \int_0^z \left(G_y \frac{dH_y}{dz'} - H_y \frac{dG_y}{dz'} \right) dz'. \quad (29)
 \end{aligned}$$

Transformation equations (21)–(24) govern the propagation of the Hermite portion of the beam, and transformation equations (25)–(28) govern the propagation of the sinusoidal portion of the beam. Equation (29) governs the propagation of the axial phase and gain that are due to the Gaussian, Hermite, and sinusoidal portions of the beam. This equation has a \mp sign. The minus sign is to be used with the hyperbolic sinusoids, and the plus sign is used for the conventional sinusoids.

The general procedure for obtaining the detailed propagation characteristics for the beam parameter of the Gaussian portion of the beam through an optical system is as follows:

1. Determine the system matrix by multiplying the individual optical element matrices (as given in Refs. 14 and 15, for example) in the reverse of the order in which they are encountered by the input beam.
2. Determine the input complex beam parameters q_x and q_y given an input spot size, radius of curvature, and wavelength from Eqs. (7) and (8).
3. Determine the output complex beam parameters from the Kogelnik transformation equations (17) and (18).
4. The output spot sizes and radii of curvature may be determined by again using Eqs. (7) and (8).

Hermite–sinusoidal–Gaussian beams contain other parameters and associated beam transformations. However, the basic procedure to analyze these other parameters of the beam is essentially the same as the four steps given above for each parameter.

As an example, consider the propagation of a cosh-Gaussian beam in a Fourier-transforming lens–free-space optical system. If the lens has a focal length f and the free-space segment of the optical systems has a length of f , then the system matrix is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}. \quad (30)$$

Note that since the system is aligned, there are no G and H elements, and thus conventional complex 2×2 beam matrices were used. With the beam matrix in Eq. (30), the Kogelnik transformation equation (17) governing the beam parameter reduces to

$$\frac{1}{q_2} = \frac{-f^{-1} + q_1^{-1}}{f/q_1}. \quad (31)$$

Similarly, Eq. (25) becomes

$$\Omega_2 = \frac{\Omega_1}{f/q_1}. \quad (32)$$

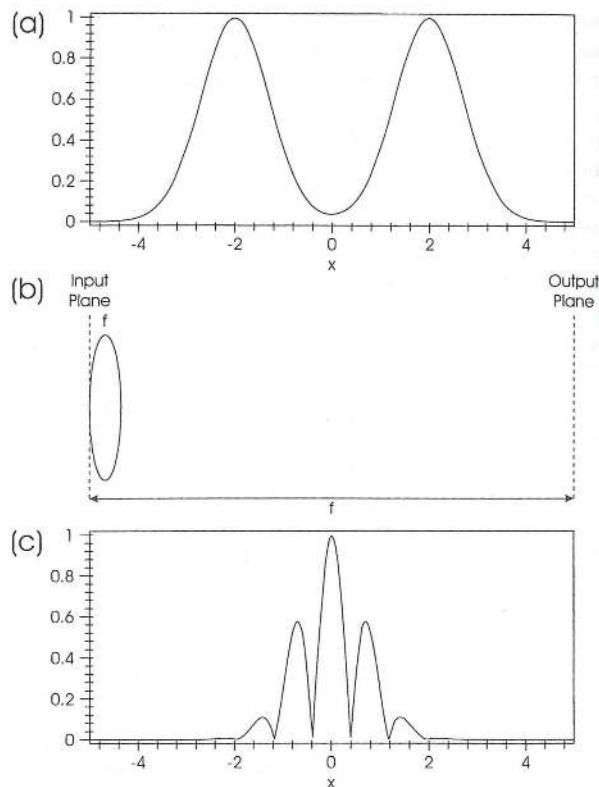


Fig. 1. (a) Field amplitude before the light beam enters the optical system, (b) example optical system, (c) field amplitude after the beam propagates through the optical system.

Here we consider an input field whose complex amplitude is

$$|E'_{\text{in}}| = \exp(-x^2/w_1^2) \cosh(\Omega_1 x), \quad (33)$$

so that the phase fronts are initially flat ($R_1 = \infty$); then

$$\frac{1}{q_1} = -i \frac{\lambda}{\pi w_1^2}. \quad (34)$$

Combining Eqs. (31)–(34), it follows that the complex amplitude of the output field is proportional to

$$|E'_{\text{out}}| = \exp[-x^2/(\lambda f/\pi w_1^2)] \cosh(i\Omega_1 \pi w_1^2 \lambda^{-1} f^{-1} x), \quad (35)$$

where the axial amplitude factor has been ignored, since only the beam shape is of interest here. If we further choose the focal length of the lens, f , to be

$$f = \frac{\pi w_1^2}{\lambda}, \quad (36)$$

then the output field reduces to

$$|E'_{\text{out}}| = \exp(-x^2/w_1^2) \cosh(i\Omega_1 x), \quad (37)$$

which is identical to the input field except for the imaginary unit in the argument of the cosh factor. These input and output fields have been normalized and plotted in Fig. 1.

3. HYPERBOLIC-SINE-GAUSSIAN-MODE RESONATOR DESIGN

Although it is important to understand the propagation of sinusoidal-Gaussian beams in optical systems, it is also of interest to be able to design laser resonators that can produce these beams. An example of such a design is given here. We begin with the observation that in addition to complex sine-Gaussian and cosine-Gaussian beams, there are also hyperbolic-sine-Gaussian (sinh-Gaussian) and cosh-Gaussian beams, which are governed by similar transformations. These alternative modes have certain important properties for practical applications, and our example will involve a mode of this type. The basic criterion for determining whether a given field profile is a mode of a laser resonator is the oscillation condition, which requires that the profile reproduce itself after a round trip through the resonator. Thus we start by assuming that the initial field configuration corresponds to an x -varying on-axis beam of the form

$$E'_1 = E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \sinh(\Omega_{x1}x), \quad (38)$$

where $\beta_0 = 2\pi n_0/\lambda$ represents the dominant real part of the propagation constant k_0 . There are several approaches that one could now take in designing a resonator that would support the mode given in Eq. (38). The oscillation condition mentioned above can be used with Eqs. (17) and (25) to develop constraints on the parameters $1/q_{x1}$ and Ω_{x1} . In this way a resonator can, in principle, be designed to produce the field distribution in Eq. (38) by using only conventional optical elements that are represented by $ABCD$ beam matrices. But, applying the oscillation condition to Eq. (25) results in $\Omega = 0$, which is not a beam of the desired type. The novel sinh function in Eq. (6) also suggests the possibility of designing a resonator that includes optical elements that do not have an $ABCD$ matrix representation. This possibility is illustrated here.

In our resonator design the field given in Eq. (38) is imagined to immediately strike an apodized aperture whose amplitude transmission is

$$t(x) = t_{\text{aper}} \cosh(\Omega_{\text{aper}}x). \quad (39)$$

This aperture does not have an $ABCD$ matrix representation. Now if we choose $\Omega_{x1} = \Omega_{\text{aper}}$, the field distribution after the aperture becomes

$$E'_2 = \frac{1}{2} t_{\text{aper}} E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \sinh(2\Omega_{x1}x), \quad (40)$$

where we have used a standard product identity for the hyperbolic trigonometric functions. It is important to note that with this particular aperture choice the sinh-Gaussian mode remains a sinh-Gaussian mode. Only the effective width of the sinh factor is changed when the beam is transmitted through the aperture.

In this ring laser design, the field distribution encounters the output coupler, the amplifier, and an optical system that is otherwise representable in terms of a beam matrix before returning to plane 1. Thus the field after a round trip is

$$\begin{aligned} E'_3 = & \frac{1}{2} r_{\text{mirror}} t_{\text{aper}} \exp(g_0 l/2) E'_{1,0} \\ & \times \exp\left(-i\left\{\frac{\beta_0}{2}\left(\frac{C_x + D_x/q_{x1}}{A_x + B_x/q_{x1}}\right)x^2\right.\right. \\ & \left.\left.+ \left[P_1 - \frac{i}{2} \ln(A_x + B_x/q_{x1})\right.\right.\right. \\ & \left.\left.+ \frac{2}{\beta_0} \frac{\Omega_{x1}^2 B_x}{A_x + B_x/q_{x1}}\right]\right\}) \\ & \times \sinh\left(\frac{2\Omega_{x1}}{A_x + B_x/q_{x1}} x\right), \end{aligned} \quad (41)$$

where r_{mirror} is the amplitude reflectivity of the output coupler, g_0 is the intensity gain coefficient of the amplifier, and l is the amplifier length. Equations (17), (25), and (29) have been the basis for the transformations of the parameters $1/q_x$, Ω_x , and P , respectively, that have been incorporated in Eq. (41).

Examining the argument of the sinh functions in Eqs. (38) and (41), it can be seen that the sinh portion of the field will repeat if, for example, $A_x = 2$ and $B_x = 0$. Since the beam matrix must be unimodular ($A_x D_x - B_x C_x = 1$), it follows that the most general form of the beam matrix under these conditions is

$$M_x = \begin{bmatrix} 2 & 0 \\ C_x & 1/2 \end{bmatrix}. \quad (42)$$

Our next objective is to find a matrix element C_x that will satisfy the resonator constraints. It will be postulated, without loss of generality, that this element can be written as

$$C_x = -\frac{3}{2L} - i \frac{2\lambda}{\pi w_{\text{aper}}^2}. \quad (43)$$

If these elements are substituted into Eq. (41), the output field is given by

$$\begin{aligned} E'_3 = & 2^{-3/2} r_{\text{mirror}} t_{\text{aper}} \exp(g_0 l/2) E'_{1,0} \exp\left\{-i\left[\frac{\beta_0}{4}\left(\frac{-3}{2L}\right.\right.\right. \\ & \left.\left.\left.- i \frac{2\lambda}{\pi w_{\text{aper}}^2} + \frac{1}{2q_{x1}}\right)x^2 + P_1\right]\right\} \sinh(\Omega_{x1}x). \end{aligned} \quad (44)$$

If we further make the choices

$$\frac{1}{q_{x1}} = -\frac{1}{L} - i \frac{4\lambda}{3\pi w_{\text{aper}}^2}, \quad (45)$$

$$r_{\text{mirror}} t_{\text{aper}} \exp(g_0 l/2) = 2^{-3/2}, \quad (46)$$

we find that the right-hand side of Eq. (44) reduces to Eq. (38). Therefore the assumed field distribution does indeed repeat after a round trip through the resonator.

It remains to determine a sequence of optical elements that can be represented by the matrix in Eq. (42) with the C_x element given in Eq. (43). The procedure for synthesis of Gaussian beam optical systems has been given previously,¹⁶ and only the results for one possible realization will be given here. It can be shown by direct multiplication that Eq. (42) can be factored as

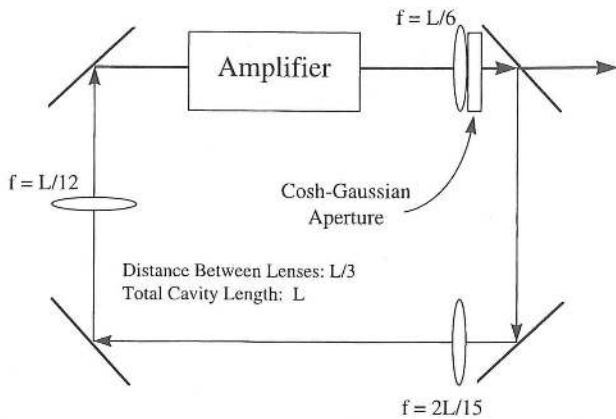


Fig. 2. Resonator design that produces a sinh-Gaussian beam.

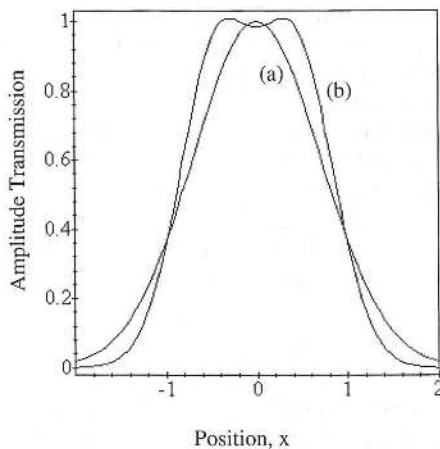


Fig. 3. Curve (a) Gaussian, curve (b) cosh-Gaussian apodized aperture transmission profiles.

$$M_x = \begin{bmatrix} 1 & 0 \\ -i\lambda/(\pi w_{\text{aper}}^2) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -6/L & 1 \end{bmatrix} \begin{bmatrix} 1 & L/3 \\ 0 & 1 \end{bmatrix} \\ \times \begin{bmatrix} 1 & 0 \\ -1/2L & 1 \end{bmatrix} \begin{bmatrix} 1 & L/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -15/(2L) & 1 \end{bmatrix} \begin{bmatrix} 1 & L/3 \\ 0 & 1 \end{bmatrix}. \quad (47)$$

However, each of these matrices can be represented by either a length of free space, a lens, or a Gaussian apodized aperture. This ring resonator configuration is shown in Fig. 2.

It should be noted that the two apertures in this design are adjacent to each other. Therefore, in practice, they could be combined into a single apodized aperture whose variable transmission is

$$t_{\text{total}}(x) = t_{\text{aper}} \cosh(\Omega_{\text{aper}} x) \exp(-x^2/w_{\text{aper}}^2). \quad (48)$$

This transmission profile is plotted in Fig. 3, where it is also compared with the more familiar Gaussian transmission factor.

Equation (39) represents only one of a class of aperture transmission functions that may be used to design sinusoidal-Gaussian beam modes. In particular, Eq. (39) is a special case of

$$t(x) = \frac{t_{\text{aper}} \sinh(c_{\text{aper}} \Omega_{\text{aper}} x)}{c_{\text{aper}} \sinh(\Omega_{\text{aper}} x)}. \quad (49)$$

The denominator represents the input beam, and the numerator represents the output beam. When c_{aper} is chosen to be 2, Eq. (49) reduces to Eq. (39). The fundamental mode of a resonator whose transmission function is given by Eq. (49) would be sinh Gaussian. Similarly, if the sinh functions in Eq. (49) were replaced with cosh functions, the corresponding resonator mode would be cosh Gaussian.

It should also be noted that the cosh-Gaussian beams described here may in some cases have significant advantages over conventional Gaussian beams. In particular, the squarer mode profiles that can be obtained resemble the super-Gaussian field distributions that are known to be more efficient at extracting energy from an amplifying medium.⁷

As a corollary result, if Eq. (49) were replaced with a ratio of Hermite polynomials, one could use the synthesis procedure demonstrated in this paper to design a resonator that would produce a specific Hermite-Gaussian mode. A desired Laguerre-Gaussian beam mode could also be obtained in this way. However, in some situations the transmission function may be difficult to manufacture.

4. RESONATOR MODE STABILITY

The basic design procedure has been to assume that the mode at some plane within a resonator has the desired form and to show that after a round trip the mode reproduces the guessed form. In this way the mode satisfies the oscillation condition. However, physically realizable beam modes must also possess finite energy and be stable with respect to inevitable perturbations.¹² For the sinh-Gaussian mode to be stable, it must be stable with respect to both Gaussian and hyperbolic sinusoidal perturbations. Both of these types of perturbation are considered here.

A. Mode Stability with Respect to Gaussian Perturbations

The Gaussian portion of the beam is unchanged by the cosh aperture. It is stable if¹²

$$F_s = |A + B/q_\infty| > 1. \quad (50)$$

However, from Eq. (42), $A_x = 2$ and $B_x = 0$, and thus $F_s = 2$, which indicates that the Gaussian portion of the beam is stable.

B. Mode Stability with Respect to Sinusoidal Perturbations

The unperturbed field is given by Eq. (38), and the corresponding perturbed field is

$$\tilde{E}'_1 = E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \sinh[(\Omega_{x1} + \delta)x], \quad (51)$$

where δ is the perturbation of Γ_{x1} . After striking the cosh aperture, this field becomes

$$\begin{aligned} \tilde{E}'_2 &= t_{\text{aper}} E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \\ &\quad \times \sinh[(\Omega_{x1} + \delta)x] \cosh(\Omega_{x1}x) \end{aligned} \quad (52)$$

$$\begin{aligned} &= \frac{t_{\text{aper}}}{2} E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \\ &\quad \times \{\sinh[(2\Omega_{x1} + \delta)x] + \sinh(\delta x)\}. \end{aligned} \quad (53)$$

Thus the aperture converts the perturbed field into two sinusoidal-Gaussian beams. Next we must propagate these beams through the *ABCD* portion of the optical system. Since $A_x = 2$ and $B_x = 0$, it follows from Eq. (25) that the arguments of each of the sinh terms are reduced by a factor of 2, so that

$$\begin{aligned} \tilde{E}'_3 &= \frac{r_{\text{mirror}} t_{\text{aper}} \exp(g_0 l/2)}{2} E'_{1,0} \\ &\quad \times \exp\left[-i\left(\frac{\beta_0}{2q_{x3}} x^2 + P_3\right)\right] \\ &\quad \times \{\sinh[(\Omega_{x1} + \delta/2)x] + \sinh(\delta x/2)\}. \end{aligned} \quad (54)$$

However, the perturbed field after a round trip may be written in terms of a new perturbation δ' as

$$\tilde{E}'_3 = E'_{1,0} \exp\left[-i\left(\frac{\beta_0}{2q_{x1}} x^2 + P_1\right)\right] \{\sinh[(\Omega_{x1} + \delta')x]\}. \quad (55)$$

By equating Eqs. (54) and (55), we can compare the initial perturbation δ with its corresponding perturbation after a round trip, δ' . It follows that the curly bracketed terms in Eq. (54) must equal the curly bracketed term in Eq. (55):

$$\sinh[(\Omega_{x1} + \delta')x] = \sinh[(\Omega_{x1} + \delta/2)x] + \sinh(\Omega_{x1}x/2). \quad (56)$$

For small perturbations, $\delta x \ll 1$ and $\delta'x \ll 1$, and Eq. (56) becomes

$$\left|\frac{\delta'}{\delta}\right| = \left|\frac{1 + \cosh(\Omega_{x1}x)}{2 \cosh(\Omega_{x1}x)}\right|. \quad (57)$$

The mode is stable if the magnitude of the perturbation amplitude is decreased after a round trip. Since $\cosh(\cdot)$ is always greater than or equal to unity, it follows that this ratio is always less than or equal to unity. Thus the magnitude of the perturbation is reduced after a round trip through the resonator, and the proposed mode is stable.

One may follow a similar derivation to show that if a sinh-Gaussian aperture were used, the corresponding cosh-Gaussian mode would also be stable. However, if a cosine-Gaussian aperture were used, then Eq. (57) would become

$$\left|\frac{\delta'}{\delta}\right| = \left|\frac{1 + \cos(\Omega_{x1}x)}{2 \cos(\Omega_{x1}x)}\right|, \quad (58)$$

which is always greater than or equal to unity. Thus the cosine-Gaussian and sine-Gaussian beam modes obtained by using the design procedure shown here would be un-

stable. This result is consistent with Fig. 2 of Ref. 7, which shows that the cosine-Gaussian beam profile changes as it propagates through free space. In that figure the cosine-Gaussian beam quickly converts (in less than a Rayleigh length) into a predominantly cosh-Gaussian beam.

5. SUMMARY

A straightforward procedure to design novel laser resonators whose beam mode consists of a sinh function multiplied by a Gaussian function has been shown. The schematic for an example ring sinh-Gaussian-mode resonator design is shown in Fig. 2. A prerequisite for physical realization of the sinh-Gaussian beam mode is that it be stable to perturbations, and it has been shown that the designed resonator produces a perturbation-stable mode. However, it has also been shown that the corresponding resonator design for sine-Gaussian beams does not produce perturbation-stable modes. It is therefore impossible to design a resonator that produces sine-Gaussian modes by using these methods.

As discussed, the procedure outlined herein may be used to design novel laser resonators whose beam mode is cosh Gaussian. Such beams resemble super-Gaussian beams, which are known to be more efficient at extracting energy from a laser amplifier.

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