PRODUCTION CONTROL IN MULTI-STAGE SYSTEMS WITH VARIABLE YIELD LOSSES

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ABSTRACT

Many manufacturing processes involved in the fabrication and assembly of "high tech" components have highly variable yields, complicating the planning and control of production. We develop a procedure which determines optimal input quantities at each stage of a serial production system in which process yields at each stage of production may be stochastic. The procedure is applied to an example in the manufacture of a light-emitting diode (LED) display using actual yield data. We also provide a brief analysis of quantifiable savings obtained by reducing the variability of the yield at one production stage.

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1.0 INTRODUCTION

Many manufacturing processes involved in the fabrication of "high-tech" components have highly variable yields, complicating the planning and control of production. Specific examples include wafer fabrication and microelectronic assembly. Similar problems also arise in chemical and other process industries. In many instances, the processes are intrinsically variable and the short term objective is simply to deal with the variability as well as possible. Over the longer horizon, management may be interested in assessing the economic benefits of improved equipment or process controls to improve the yield rate distribution.

A few early papers addressing the problem of controlling production systems with uncertain yields focus on systems in which the yield is observed only after production of the entire order is completed. In these models (e.g., Giffler (1960) and Levitan (1960)), the entire production process is viewed as a single stage and the objective is to find a reject allowance which optimizes the tradeoff between shortages and overages. Klein (1966) and White (1967) investigate other single-stage situations in which inspection may be done for sub-batches. They demonstrate that optimal sub-batch sizes can be found for single stage production systems using linear programming.

In simulation studies of single-stage systems, Whybark and Williams (1976) found that safety stock is a more effective and efficient buffer against quantity uncertainties (including stochastic yields) than safety time. Other recent work on production control in systems with uncertain yields (e.g., Porteus (1986), Sepehri, Silver, and New (1986), and Lee and Rosenblatt (1986)) is also limited to single-stage systems. Wagner (1980) points out a need for methods to handle stochastic aspects of multi-stage systems, including yields, and suggests that they be developed with a view toward "more realistic approaches, better approximations, and clearer insights about design and decision tradeoffs."

We describe and formulate a single-period production control problem in general serial systems with stochastic yields in the next section. It bears some resemblance to a problem addressed by Beebe, et al. (1968) in the manufacture of transistors. There, the critical decisions were the amounts of impurities to add to pure crystal in order to achieve a desired resistivity. The resistivity actually achieved was random due to a variety of factors.

We then develop a relatively simple procedure to determine optimal input quantities for each process stage. The procedure is applied to a light-emitting diode (LED) fabrication facility using scaled actual data. We then provide results of a brief analysis of the economic impact of changes in the yield variability at one production stage. Finally, we discuss directions for future research.

2.0 PROBLEM DEFINITION AND FORMULATION

The system under study is an N-stage serial production environment in which the yield at each process stage may be stochastic. We assume that the yield rate probability distribution functions at the various stages are mutually independent and that each has a range whose endpoints are greater than zero and less than or equal to 1. Throughout the paper, we use the term yield rate to refer to the fraction of acceptable parts. We assume that the yield rate distributions are invariant with the size of the input batch (i.e., no economies or diseconomies of scale). The p.d.f. of the yield rate at stage n is denoted as $f_n(\cdot)$.

We assume that there is a single product and that the product has a fixed or predetermined routing through the production facility which can be represented conceptually as a serial system. There is a single known requirement or demand for the finished product, D, to be satisfied, and the problem is one of determining the optimal input quantity at each process stage so as to minimize total cost.

We assume that the finished product requirements have been specified so that the capacity limitations of any batch processes are likely to be satisfied by the optimal input quantities. (Our approach can be generalized to capacity constrained situations with little difficulty, however). We further assume that setup costs, if any, are either sunk costs by virtue of the fixed routing, or are considered outside this framework, perhaps to specify the requirement (or lot size), D, if a "make to plan" policy is followed.

The flow of items in the system can be represented diagrammatically as shown in Figure 1. The stages are numbered so that the stage of production to be performed first is denoted as stage N, while the final stage of production is referred to as stage 1. The decision at stage n, n=1 to N, is the input quantity at stage n, which we denote as u_n. The input to stage n is obviously constrained by the output of stage n+1, which, in turn, depends upon the input at stage n+2, etc. The output of stage n is $y_n = p_n u_n$, where p_n denotes the actual (random) yield fraction at stage n. There is a yield loss of $(1-p_n)u_n$ and an overage quantity of $(y_n-u_{n-1})^+$, where $(\cdot)^+$ indicates the positive part. At the final stage of production, the shortage quantity is $(D-y_1)^+$. We assume that initial inventory of all semi-finished items is zero. Later in the paper we explain how the approach can be generalized to accommodate positive initial inventory.

FIGURE 1

Costs to be included are (1) w_n = marginal production cost per unit of input at stage n (material, labor, and inspection), (2) h_n = net cost of disposing a unit produced at stage n but not used at stage n-1 (overages), where stage n = 0 corresponds to finished product demand, and (3) π = shortage cost for each unit of unsatisfied demand.

All of these costs reflect cash flows. Therefore, the net cost of disposing an unused unit is typically the cost of removing each unit of overage from the production facility, less the salvage value if it can be sold. Hence, h_n can be negative, reflecting the cash inflow arising from the net salvage value. Similarly, the shortage cost reflects lost revenue, plus any additional shortage penalties.

We assume that all defective units are disposed at no additional cost. It is also assumed that $\pi \geq |h_1|$. If h_1 is positive, this condition implies that one would prefer to hold a unit of the finished product in inventory rather than to incur a shortage. If h_1 is negative, this condition simply requires that the shortage cost (lost revenue) be at least as large as the net salvage value. All other parameters and costs are assumed to be non-negative, and the integrals of $f_n(\cdot)$ are assumed to exist and to be continuous and twice differentiable.

3.0 SOLUTION APPROACH

Let y_{N+1} denote the available input (raw material) for production stage N. We might formulate the problem as:

$$\underset{u_1,\ldots,u_N}{\text{minimize}} \quad E\{\sum_{n=1}^{N} [w_n u_n + h_{n+1}(y_{n+1}-u_n)^+] + h_1(y_1-D)^+ + \pi(D-y_1)^+\}, \qquad (1)$$

subject to $0 \leq u_n \leq y_{n+1}$, $\quad n = 1,\ldots,N,$

$y_n = p_n u_n$, $\quad n = 1,\ldots,N,$

where p_n has density $f_n(\cdot)$.

This formulation of the problem requires that the decision variables, u_n, be selected before the y_{n+1} values are known. It therefore does not take advantage of the dynamic nature of the problem. In reality, one does not need to specify u_n until y_{n+1} is known. The dynamic programming formulation given below reflects the dynamic nature of the problem.

Let $C_n(y_{n+1})$ = expected cost of operating the system optimally from stage n through stage 1, given that the output from stage n+1 is y_{n+1}, n = 1 to N. Then $C_N(y_{N+1})$ represents the minimum total cost of operating the system.

Clearly at each stage we must have $u_n \leq y_{n+1}$ (i.e., input cannot exceed output of the preceding stage). Therefore, we can write the dynamic programming recursion relationships as:

$$C_n(y_{n+1}) = \min_{0 \leq u_n \leq y_{n+1}} \{w_n u_n + h_{n+1}(y_{n+1} - u_n) + E[C_{n-1}(p_n u_n)]\}, \qquad n = 2 \text{ to } N, \tag{2}$$

and

$$C_1(y_2) = \min_{0 \leq u_1 \leq y_2} \{w_1 u_1 + h_1\ E(y_1 - D)^+ + h_2(y_2 - u_1) + \pi\ E(D - y_1)^+\}. \tag{3}$$

To simplify the presentation, let

$$g_n(u_n | y_{n+1}) = w_n u_n + h_{n+1}(y_{n+1} - u_n) + E\ [C_{n-1}(p_n u_n)], \qquad n=2,\ldots,N, \tag{4}$$

$$g_1(u_1 | y_2) = w_1 u_1 + h_2(y_2 - u_1) + h_1 E(p_1 u_1 - D)^+ + \pi E(D - p_1 u_1)^+, \tag{5}$$

and

u_n^* = optimal value of u_n.

We note that $g_1(u_1|y_2)$ is a function only of u_1. Also $g_n(u_n|y_{n+1})$ is a function of u_n which affects only downstream stages n-1,...1. Therefore, we can determine u_1^* from equation (3), and successively determine u_n^* given y_{n+1} for n = 2,..., N in sequence from equation (2). We next demonstrate that the form of the optimal policy does, in fact, have a "single critical number" at each stage, and that these values exist.

3.1 STRUCTURE OF THE OPTIMAL POLICY

We shall show that there exist $S_1,\ldots,S_N$ such that the optimal production decisions satisfy

$$u_n^* = \begin{cases} y_{n+1} & \text{if } y_{n+1} < S_n \\ S_n & \text{otherwise,} \end{cases} \qquad n=1,\ldots, N.$$

In words, this states that the optimal policy has a single critical number for each stage (S_n), which is independent of the output of the preceding stages.

Two conditions on costs are required for the theorem which follows to hold. For notational simplicity, let $\bar{p}_n$ denote $E(p_n)$. The conditions are:

Condition I: For all n = 1, ..., N,

$$w_n + h_n\bar{p}_n > h_{n+1}$$

Condition II: For all n = 1, ..., N,

$$-h_{n+1} + \sum_{j=0}^{n-1} [w_{n-j} \prod_{i=1}^{j} \bar{p}_{n-i+1}] \leq \pi \prod_{i=1}^{n} \bar{p}_i .$$

The first condition ensures that it is less expensive to dispose an item at one stage than to process it at the next stage and to dispose the expected output of the item at that stage. Stating this in terms of the net salvage values (when the h_n values are negative), we could rewrite condition I as:

$$-h_{n+1} > -h_n p_n - w_n$$

This states that the cash inflow from salvaging at stage n+1 exceeds the expected cash inflow from salvaging at stage n, less the cost of processing at stage n. If this condition is not satisfied, it would be optimal to set u_n =

y_{n+1}, i.e, to input all available units. Condition II essentially ensures that it is profitable to produce the product.

We next present the main result of the paper:

Theorem 1: If conditions I and II hold, then there exists a unique set of finite values S_1, $S_2, \ldots, S_N$ such that $S_N \geq S_{N-1} \geq \ldots \geq S_1$, and such that the optimal production decisions satisfy

$$u_n^* = \begin{cases} y_{n+1} & \text{if } y_{n+1} < S_n \\ S_n & \text{otherwise} \end{cases}$$

Proof: The proof consists of two parts. In the first part we establish (for all cases) the optimality of the critical number policy and the relationship among the S_n values. We also prove uniqueness and existence of S_n assuming that S_{n-1} is positive. In part two, we show that the uniqueness and existence results are valid even when S_{n-1} is zero.

Part 1: First, we establish that the critical number policy given by the S_ns is optimal, and that S_n is unique if $S_{n-1} > 0$. We then show that conditions I and II are sufficient for the existence of a finite value of S_n if $S_{n-1} > 0$. We also demonstrate that the values of S_n, if they exist, have the property that $S_N \geq S_{N-1} \geq \ldots \geq S_1$. Details of several lemmas needed for the proof appear in the Appendix.

We will prove optimality of the critical number policy by induction.

Consider n=1. Then we have

$$C_1(y_2) = \underset{0 \leq u_1 \leq y_2}{\text{Min}} \quad g_1(u_1 | y_2)$$

Consider $g_1(u_1|y_2)$ as defined in (5).

Clearly, the first derivative with respect to u_1 is

$$w_1 - h_2 - \pi \int_{p_1=0}^{D/u_1} p_1 f_1(p_1)dp_1 + h_1 \int_{p_1=D/u_1}^{1} p_1 f_1(p_1)dp_1 \tag{6}$$

Moreover, the second derivative with respect to u_1 is

$$(\pi + h_1)(D^2/u_1^3)f_1(D/u_1) > 0$$

Hence, the u_1^* that minimizes $g_1(u_1|y_2)$ is the value of u_1 which equates the expression in (6) to zero.

Let S_1 be the value of u_1 which satisfies this first order condition. By the convexity of $g_1(u_1|y_2)$, if $y_2 < S_1$, then $u_1^* = y_2$, whereas if $y_2 \geq S_1$, then $u_1^* = S_1$. The expected cost of using $u_1 = y_2$ is simply $g_1(y_2|y_2)$, and the expected cost of using $u_1 = S_1$ is $g_1(S_1|y_2)$.

Now suppose that the theorem is true for some stage n-1. The expected cost of using the optimal decision rules for stages n-1, n-2, ..., 1 is

$$C_{n-1}(y_n) = \begin{cases} g_{n-1}(y_n|y_n) & \text{if } y_n < S_{n-1} \\ g_{n-1}(S_{n-1}|y_n) & \text{if } y_n \geq S_{n-1}. \end{cases}$$

Let us now consider stage n, given output y_{n+1} from stage n+1.

$$C_n(y_{n+1}) = \min_{0 \leq u_n \leq y_{n+1}} \{w_n u_n + h_{n+1}(y_{n+1} - u_n) + E[C_{n-1}(p_n u_n)]\}$$

The expression in braces on the right hand side of the expression above can be written as

$$\gamma_n(u_n) = w_n u_n + h_{n+1}(y_{n+1} - u_n) + \int_{p_n=0}^{S_{n-1}/u_n} g_{n-1}(p_n u_n|p_n u_n)f_n(p_n)dp_n$$

$$+ \int_{p_n=S_{n-1}/u_n}^{1} g_{n-1}(S_{n-1}|p_n u_n)f_n(p_n)dp_n.$$

Let $g_n'(x|x) = \partial g_n(x|x)/\partial x,$

$g_n'(S_n|x) = \partial g_n(S_n|x)/\partial x,$

$g_n''(x|x) = \partial^2 g_n(x|x)/\partial x^2$

$g_n''(S_n|x) = \partial^2 g_n(S_n|x)/\partial x^2$

The first derivative of γ_n with respect to u_n is

$$\gamma_n'(u_n) = w_n - h_{n+1} + \int_{p_n=0}^{S_{n-1}/u_n} g_{n-1}'(p_n u_n|p_n u_n) p_n f_n(p_n) dp_n$$

$$+ \int_{p_n=S_{n-1}/u_n}^{1} g_{n-1}'(S_{n-1}|p_n u_n) p_n f_n(p_n) dp_n \qquad (7)$$

All other terms cancel. The last term can be simplified by using Lemma 1, so $\gamma_n'(u_n)$ becomes

$$w_n - h_{n+1} + \int_{p_n=0}^{S_{n-1}/u_n} g_{n-1}'(p_n u_n|p_n u_n) p_n f_n(p_n) dp_n$$

$$+ \int_{p_n=S_{n-1}/u_n}^{1} h_n p_n f_n(p_n) dp_n \qquad (8)$$

The second partial derivative of γ_n is

$$\gamma_n''(u_n) = \int_{p_n=0}^{S_{n-1}/u_n} g_{n-1}''(p_n u_n|p_n u_n) p_n^2 f_n(p_n) dp_n \qquad (9)$$

since all other terms cancel by Lemma 2. Using the result of Lemma 3, it can be shown that $\gamma_n''(u_n) > 0$ if $S_{n-1} > 0$. Hence, $\gamma_n(u_n)$ is strictly convex in u_n for $n = 1,\ldots,N$. Let S_n be the value of u_n which equates $\gamma_n'(u_n)$ to 0. By the convexity of $\gamma_n(u_n)$, the optimal policy has the critical number form, and by its strict convexity, the values of S_n, $n = 1,\dots,N$ are unique.

We next show that S_n exists and is finite when $S_{n-1} > 0$. Let $\xi_n(S_n) = \gamma_n'(S_n)$. Now

$$\xi_n(0) = \sum_{j=0}^{n-1} [w_{n-j} \prod_{i=1}^{j} \bar{p}_{n-i+1}] - h_{n+1} - \pi \prod_{i=1}^{n} \bar{p}_i, \quad \text{by Lemma 4.}$$

Therefore, if condition II holds, $\xi_n(0) \leq 0$. Also, from equation (8)

$$\lim_{S_n \to \infty} \xi_n(S_n) = w_n - h_{n+1} + h_n \bar{p}_n$$

Therefore, if condition I holds, then $\lim_{S_n \to \infty} \xi_n(S_n) > 0$. By the strictly increasing property of $\xi_n(S_n)$ when $S_{n-1} > 0$, there must exist a finite S_n that satisfies $\xi_n(S_n) = 0$.

To establish that $S_N \geq S_{N-1} \geq \dots \geq S_1$, we make the observation that, for any n, if $S_n < S_{n-1}$, then $\xi_n(S_n)$ has the same value as $\xi_n(S_n = S_{n-1})$. Therefore, if $S_n < S_{n-1}$, we simply set $S_{n-1} = S_n$. This proves the first part of the theorem for all n.

Part 2: We shall show that a finite, unique S_n exists even when $S_{n-1} = 0$. If $S_{n-1} = 0$, then from (9) $\gamma_n''(u_n) = 0$ for all positive values of u_n, so $\gamma_n'(u_n)$ is linear over this domain. By condition I, we have from (8) that $\gamma_n'(u_n) > 0$ for all positive u_n. Now consider what happens at $u_n = 0$. We have from (9) and Lemma 3 that $\gamma_n''(0) > 0$. Thus, γ_n is strictly convex at $u_n = 0$, and is linearly increasing for all positive u_n. It is therefore optimal to set $S_n = 0$. By induction, it is also optimal to set $S_{n+1},\dots,S_N$ to zero. Observe that the optimal values exist and are unique and finite. This proves the second part of the theorem for all n. □

We next examine solution characteristics when either of the two conditions is not satisfied. If condition I is not satisfied at some stage n, then for that stage, $\lim_{S_n \to \infty} \xi_n(S_n) < 0$. This implies that $\xi_n(S_n) < 0$ for all $S_n \geq 0$ since $\xi_n(S_n)$ is increasing for all value of S_n. Hence, for this stage, from equation (8), we have $\Upsilon_n'(u_n) = \xi_n(u_n) < 0$, for all $u_n \geq 0$, so that we can always reduce $\Upsilon_n(u_n)$ by increasing u_n. As a result, the optimal value of u_n is y_{n+1}, which is the largest available quantity. In other words, the entire output of stage n+1 should be input to stage n. This implies that we must redefine $\Upsilon_n(y_{n+1})$ accordingly. We can then proceed as before to solve for S_{n+1} using $\Upsilon_{n+1}' = 0$. Thus, whenever condition I is not satisfied, we set $S_n = \infty$, and proceed as before.

If condition II is not satisfied at some stage n, then for this stage we have $\lim_{S_n \to 0} \xi_n(S_n) > 0$, which implies that $\xi_n(S_n) > 0$ for all $S_n \geq 0$.

Therefore, for this stage, from (8) we have $\Upsilon_n'(u_n) = \xi_n(u_n) > 0$, for all $u_n \geq 0$. As a result, the optimal value of u_n is 0, irrespective of the value of y_{n+1}. In other words, it is not profitable to process any unit from stage n onward. In this case, the solution to the original problem should be not to produce the product at all.

It is now evident that conditions I and II are conditions which ensure that the S_ns are positive and finite. However, the methodology can be applied (with appropriate modifications) even when the conditions do not apply. To find the critical numbers, we simply solve $\partial g_1(S_1|y_2)/\partial S_1 = 0$ for S_1 and $\xi(S_n) = 0$, n = 2,...,N, recursively for $S_2,\ldots,S_N$, by equating the expression in (8) to zero.

3.2 CASE OF POSITIVE INITIAL INVENTORY

If the initial inventory of any semi-finished part or the finished product is positive, the procedure still can be applied with minor modifications. Let

I_n denote the inventory of items which have completed stage n but not stage n-1. Then we can rewrite the dynamic programming recursion relations as:

$$C_n(y_{n+1} + I_{n+1}) = \min_{0 \le u_n \le y_{n+1} + I_{n+1}} \{w_n u_n + h_{n+1}(y_{n+1} + I_{n+1} - u_n) + E[C_{n-1}(p_n u_n + I_n)]\} \qquad n = 2, \ldots N$$

and

$$C_1(y_2 + I_2) = \min_{0 \le u_1 \le y_2 + I_2} \{w_1 u_1 + h_2(y_2 + I_2 - u_1) + h_1 E(y_1 + I_1 - D)^+ + \pi E(D - I_1 - y_1)^+\}$$

Let $y_n' = y_n + I_n$ for all n. By substituting y_n' for y_n throughout the analysis, it can be shown that relationships and results established earlier hold even when $I_n > 0$ for some or all n. To solve for the optimal values of S_n, $n = 1,\ldots,N$, the following equations should be used instead of $g_1'(S_1|y_2) = 0$ and $\xi_n(S_n) = 0$:

$$w_1 - h_2 - \pi \int_{p_1=0}^{(D-I_1)^+/S_1} p_1 f_1(p_1)dp_1 + h_1 \int_{p_1=(D-I_1)/S_1}^{1} p_1 f_1(p_1)dp_1 = 0$$

and

$$w_n - h_{n+1} + \int_{p_n=0}^{(S_{n-1} - I_n)^+/S_n} g'_{n-1}(p_n S_n|p_n S_n)p_n f_n(p_n)dp_n$$

$$+ \int_{p_n=(S_{n-1}-I_n)^+/S_n}^{1} g_{n-1}(S_{n-1}|p_n S_n)p_n f_n(p_n)dp_n = 0$$

Observe that these equations use $D-I_1$ rather than D and $S_{n-1} - I_n$ rather than S_{n-1} to find S_1 and S_{n-1} respectively. Thus, the net "requirement" (for either finished product demand or the target input quantity for the appropriate successor stage) is used instead of the gross requirement. The optimal critical number for stage n, S_n, is defined in the same way as before. However, it may be supplied from two sources: existing inventory at stage n+1 (I_{n+1}) and the output of stage n+1 which is $y_{n+1} = p_{n+1} u_{n+1}$. The sequential procedure described above can still be used.

The only special cases that need to be considered are: (i) $I_1 > D$ and (ii) $I_n > S_{n-1}$. In the former case, nothing needs to be produced, and $S_1 = 0$, which implies $S_n = 0$ for n = 2,...,N by part 2 of Theorem 1. In the latter case, there is sufficient input to stage n-1, so $S_n = 0$ is optimal, and $S_j = 0$ for j = n+1,..,N as well (also by Theorem 1).

Thus, there are minor computational differences between the case with positive initial inventories and the case without initial inventories and only minor modifications to the procedure are required.

4.0 COMPUTATIONAL RESULTS

For most commonly observed yield rate distributions, it is not possible to obtain a closed form solution for any of the S_ns. The solution procedure requires a search in conjunction with numerical integration. However, in many real-world applications historical yield data are available and empirical distributions can be used rather than fitted distributions. Moreover, since the relevant cost functions are convex, the first derivatives are non-decreasing, making bisection or Fibonacci search effective search procedures.

Actual yield rate data were obtained for the three major stages of production for a light-emitting diode (LED) fabrication facility. The fabrication proeess is depicted in Figure 2. The die attach process involves placement of LED chips into appropriate locations within a frame or base. These LED chips are then electrically connected in the wire bond process. Finally, a reflecting cavity is cast around each LED display to enclose it and to provide appropriate diffusion and reflection. There are several other steps in the production process but they have been omitted because their yield rates are essentially 100%. The scaled yield rates are reflected in histograms in Figure 3. Variable production costs (scaled to preserve confidentiality) are \$0.82, \$0.63, and \$1.45, at stages 1, 2, and 3, respectively. The shortage cost is \$5.29, which is the selling price per unit less other variable production expenses. The variable production costs include inspection costs as indicated in Table 1.

The plant produces three shifts per day, seven days a week, so the actual problem involves multiple periods. In order to reflect this as accurately as possible within the context of a one-period problem, we let the w_n values represent the opportunity cost of producing a unit in this period rather than next period (due to discounting effects). We assumed that other inventory holding costs were negligible. Similarly, we let π be the opportunity cost of selling a unit in the next period rather than this period. We used a discount rate of 30% per year. The production requirement for one shift is 7000 units; we therefore let D = 7000.

FIGURES 2 AND 3 AND TABLE 1

To find S_1 we need to solve:

$$w_1 - h_2 - \pi\int_{p_1=0}^{D/S_1} p_1 f_1(p_1)dp_1 + h_1 \int_{p_1=D/S_1}^{1} p_1 f_1(p_1)dp_1 = 0$$

which is the first order necessary condition for the one-stage cost function. Observe that we only need to find the ratio D/S_1, or the appropriate fractile of the yield distribution. For the given data, we obtain $S_1^* = 7857$.

Now, given S_1, we solve for S_2 using the first order condition for the two-stage cost function:

$$w_2 - h_3 + w_1 \int_{p_2=0}^{S_1/S_2} p_2 f_2(p_2)dp_2$$

$$+ h_1 \int_{p_1=0}^{S_1/S_2} \int_{p_1=D/p_2S_2}^{1} p_1f_1(p_1)p_2f_2(p_2)dp_1dp_2$$

$$- \pi \int_{p_2=0}^{S_1/S_2} \int_{p_1=0}^{D/p_2S_2} p_1f_1(p_1)p_2f_2(p_2)dp_1dp_2$$

$$+ h_2 \int_{p_2=S_1/S_2}^{1} p_2f_2(p_2)dp_2 = 0$$

which was obtained from equation (7) by substituting for g_1'. We obtain a solution $S_2^* = 9481$.

Finally, we solve for S_3 in the first order condition for the three-stage cost function:

$$w_3 + w_2 \int_{p_3=0}^{S_2/S_3} p_3f_3(p_3)dp_3 + h_3 \int_{p_3=S_2/S_3}^{1} p_3f_3(p_3)dp_3$$

$$+ h_2 \int_{p_3=0}^{S_2/S_3} \int_{p_2=S_1/p_3S_3}^{1} p_2f_2(p_2)p_3f_3(p_3)dp_2dp_3$$

$$+ \int_{p_3=0}^{S_2/S_3} \int_{p_2=0}^{S_1/p_3S_3} [\ w_1 + h_1 \int_{p_1=D/p_2p_3S_3}^{1} p_1f_1(p_1)dp_1$$

$$\pi \int_{p_1=0}^{D_1/p_2p_3S_3} p_1f_1(p_1)dp_1\ ]\ p_2f_2(p_2)p_3f_3(p_3)dp_2dp_3=0$$

which was derived by recursively substituting for g_2' and then g_1'. The solution for S_3^* is 10,000. Substituting for S* in the (true) total cost function, we obtain a cost of $27,970, given D=7000.

We next examine the effect of "improved" yield rate distributions on the total cost of producing the batch. This may be of concern when there are opportunities to purchase new equipment or to make the investments to improve the yield (e.g., process controls) and the justification for the purchase depends, in part, on a reduction of the variable cost per unit. Suppose the LED fabrication facility under consideration has opportunity to purchase die attach equipment which provides an improved yield distribution as depicted in Figure 4. The average yield is the same as for the current equipment, but the variance is reduced. For the new yield distribution, the optimal policy is $S_1 = 7857$, $S_2 = 9481$, $S_3 = 10054$, with a total cost of $27849. With a three-shift, five day per week operation, the annual savings would be over $90,000. If the yield rate in die attach were perfectly deterministic with the same averages as before, we would obtain $S_1^* = 7857$, $S_2^* = 9481$, $S_3^* = 10116$, with a total cost of $27,712. Of the total cost, $20,300 represents the minimum variable production costs necessary to satisfy demand (i.e., $\sum_n w_n$ = $2.90 multiplied by 7000 units). The remaining $7400 or more is attributable to yield losses and yield variability, i.e., more than $1 per unit on an item whose gross margin before yield-related costs is only $2.39. It is evident that a reduction of the yield variance may make an important contribution to the economic viability of new equipment.

FIGURE 4

Observe that finding S_n involves solving a first order condition with n-tuple integrals (n+1 in the case of uncertain demand). The computational complexity of the solution procedure thus increases exponentially with the number of stages. However, for most systems with empirical yield rate probability mass functions with a moderate number of (less than ten) support points, up to 10 stages can conceivably be handled. The solution procedure is also affected by the magnitude of D, and we found this to be much more critical in our test problems with only three stages. If, however, one is willing to sacrifice a little with regard to optimality by considering a limited number of input batch sizes at each stage (e.g., multiples of a dozen or a hundred) rather than all possible integers, computation time can be reduced dramatically. Nevertheless, the test problems in our study were solved very quickly even on a micro-computer.

5.0 SUMMARY AND CONCLUSIONS

We have developed an approach for production control in serial production systems in which the yield at each stage may be stochastic. A procedure is developed and is shown to provide optimal solutions for any N-stage system. We also illustrate how the procedure can be used by way of an example using scaled actual data from an LED fabrication facility.

The viability of this type of procedure for fairly general yield distributions and the sequential nature of the decision-making procedure indicate that extensions to more complex production environments may be possible. In addition, further research needs to be done to incorporate other realistic factors, such as setup costs, demand uncertainty, possible rework, multiple production batches at a stage, and multiple periods, into the current model.

6.0 ACKNOWLEDGEMENT

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TABLE 1

COST DATA

	Variable Production Cost/Unit	Inspection Cost/ Unit	Total Production and Inspection Cost Per Unit
Die Attach	1.40	0.05	$1.45
Wire Bond	0.55	0.08	$0.63
Cast and Post Cast	0.72	0.10	$0.82

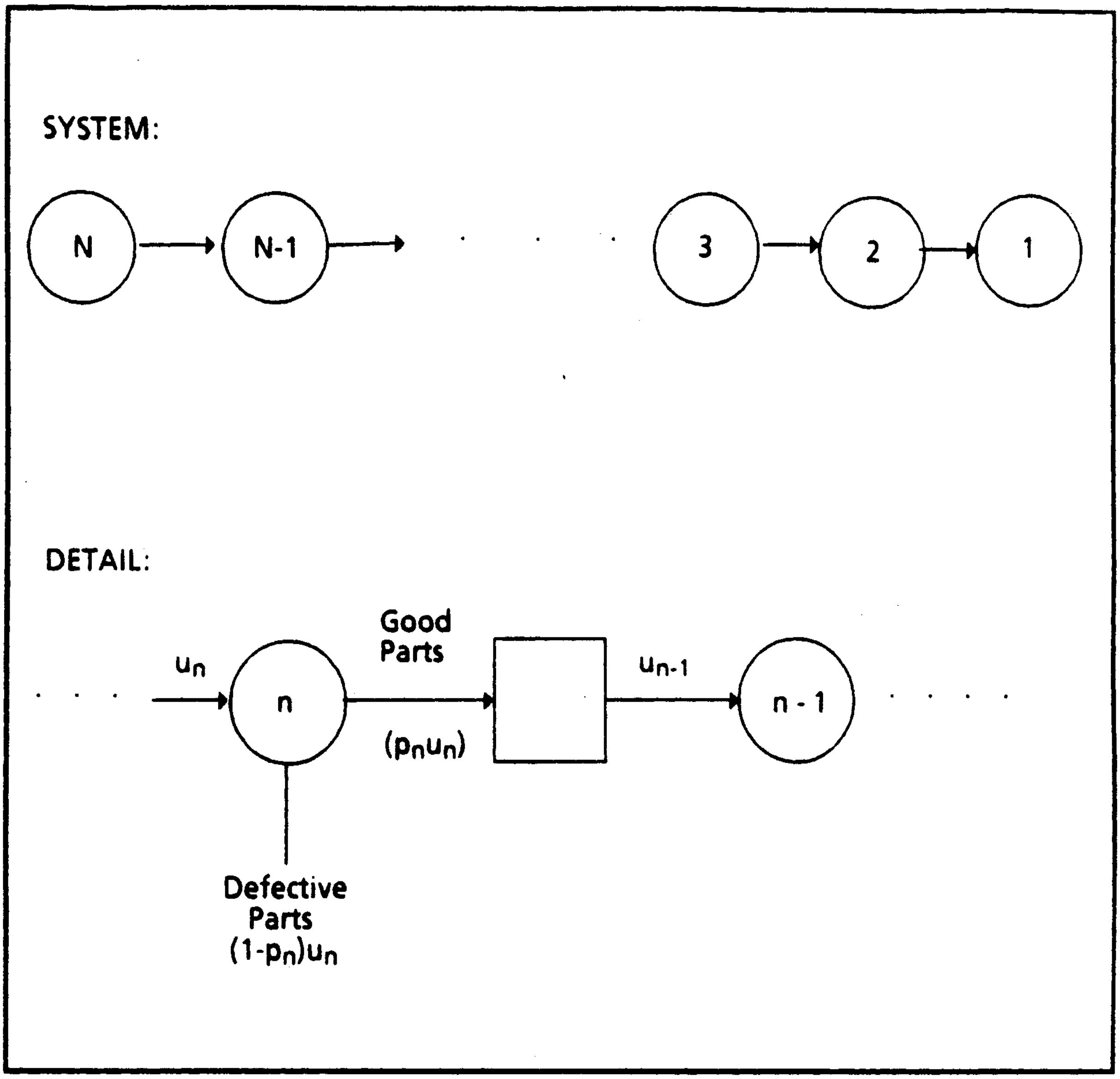


Figure 1

Serial Production System with

Yield Losses

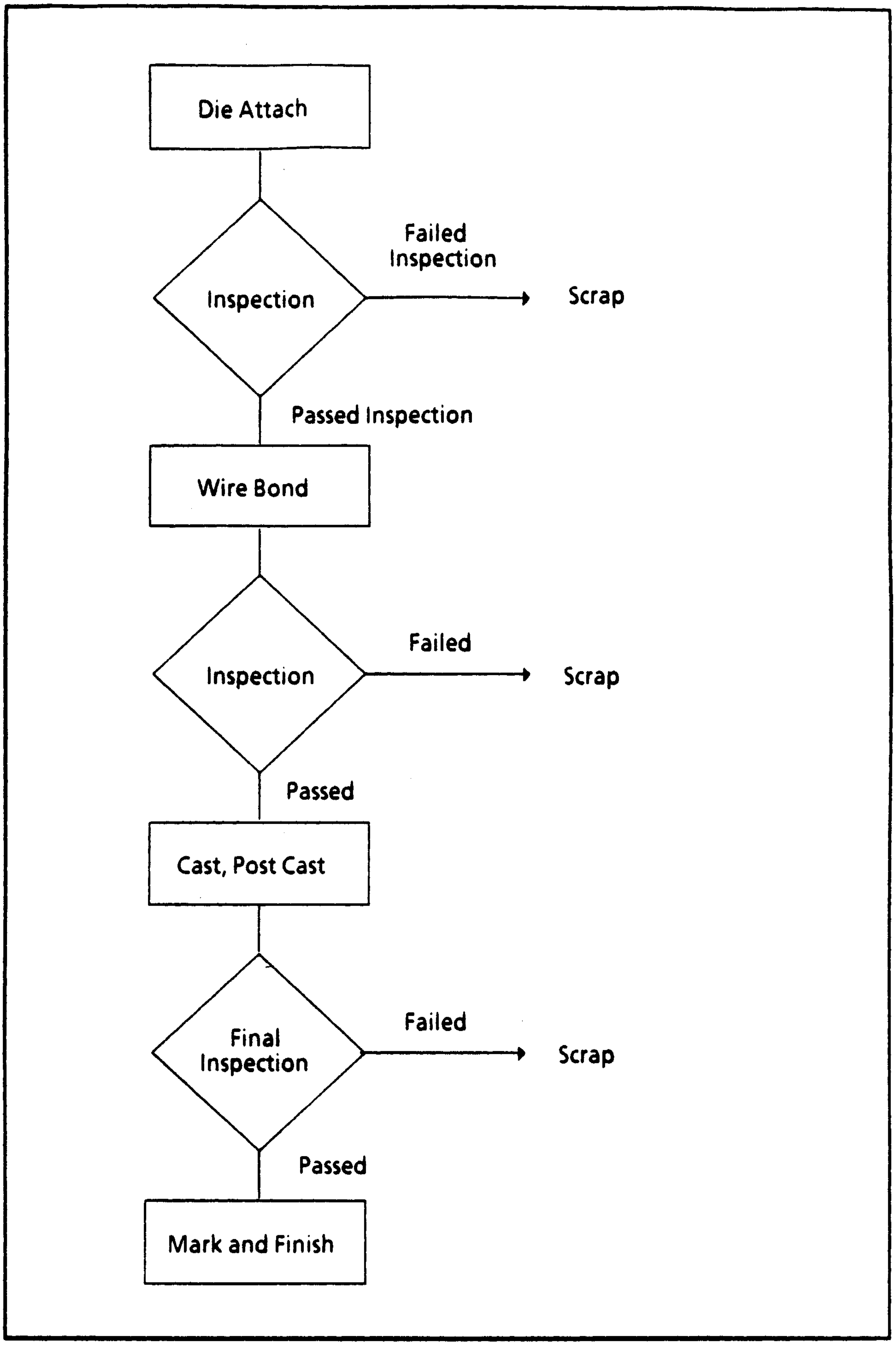
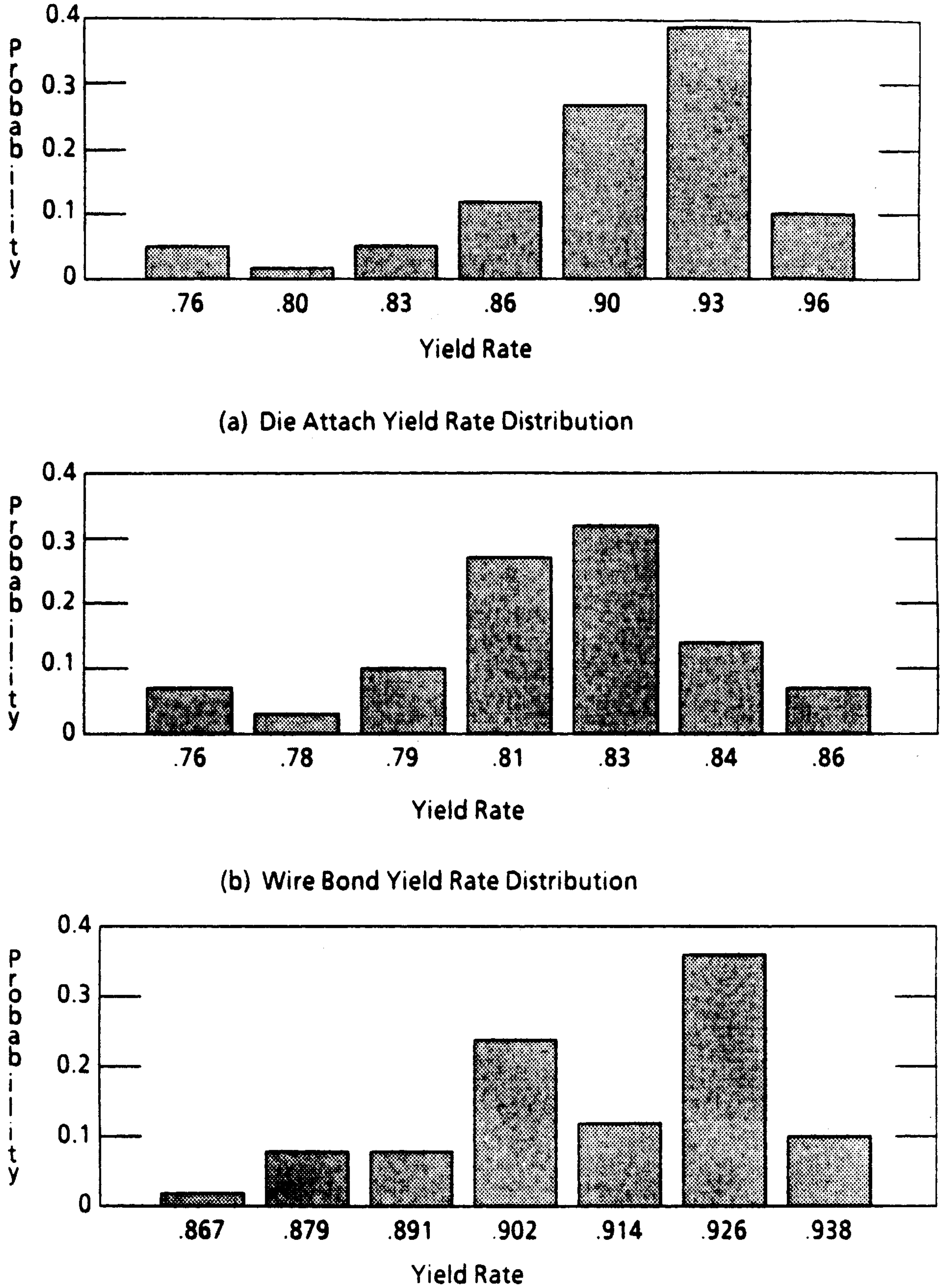


Figure 2: LED Display Production Flowchart



(a) Die Attach Yield Rate Distribution

(b) Wire Bond Yield Rate Distribution

(c) Casting/Final Test Yield Rate Distribution

Figure 3

Yield Rate Distributions

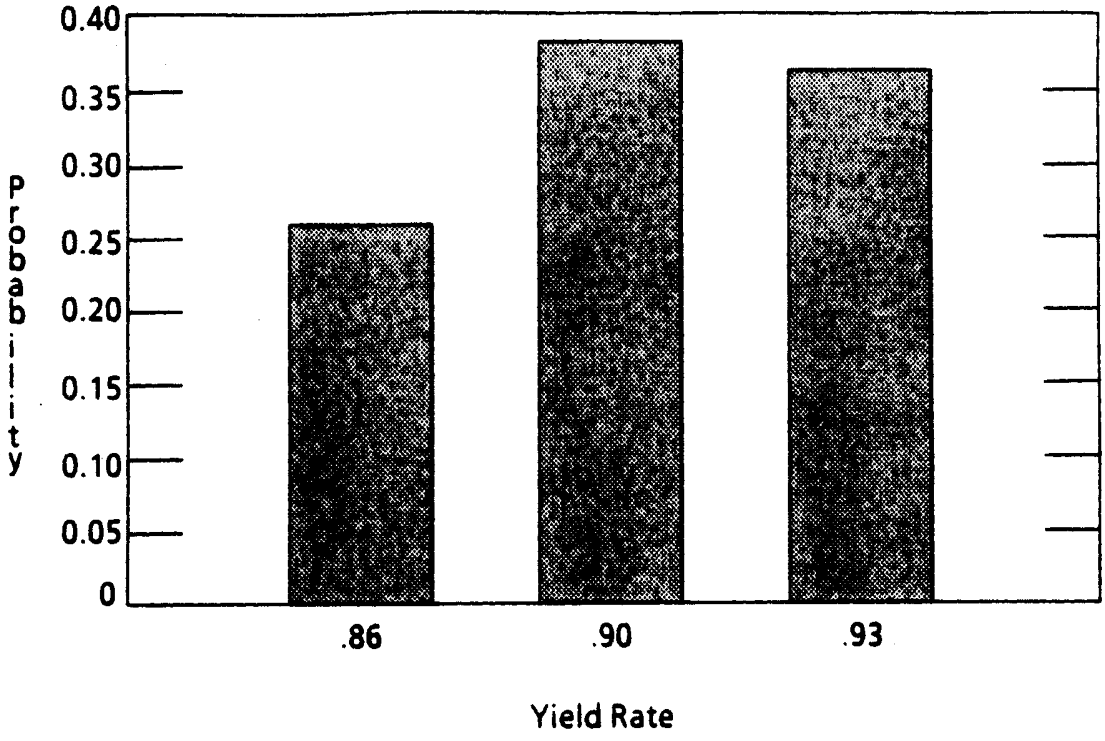


Figure 4

"Improved" Yield Rate Distribution

for Die Attach

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APPENDIX

The proofs in the appendix derive largely from the first order necessary conditions for the one- and n-stage problems:

$$w_1 - h_2 - \pi \int_{p_1=0}^{D/u_1} p_1 f_1(p_1) dp_1, + h_1 \int_{p_1=D/u_1}^{1} p_1 f_1(p_1) dp_1 = 0, \qquad \text{(A-1)}$$

and

$$w_n - h_{n+1} + \int_{p_n=0}^{S_{n-1}/u_n} g'_{n-1}(p_n u_n | p_n u_n) p_n f_n(p_n) dp_n$$

$$+ \int_{p_n=S_{n-1}/u_n}^{1} g'_{n-1}(S_{n-1} | p_n u_n)\, p_n f_n(p_n) dp_n = 0; \quad n=2,\ldots,N. \qquad \text{(A-2)}$$

where

$$g_n(p_n y_{n+1} | p_n y_{n+1}) = w_n\, y_{n+1} + \int_{p_n=0}^{S_{n-1}/y_{n+1}} g_{n-1}(p_n y_{n+1} | p_n y_{n+1})\, f_n(p_n) dp_n$$

$$+ \int_{p_n=S_{n-1}/y_{n+1}}^{1} g_{n-1}\,(S_{n-1} | p_n y_{n+1})\, f_n(p_n) dp_n, \qquad n = 2,\ldots,N,$$

$$g_n(S_n | y_{n+1}) = (w_n - h_{n+1}) S_n + h_{n+1} y_{n+1}$$

$$+ \int_{p_n=0}^{S_{n-1}/S_n} g_{n-1}(p_n S_n | p_n S_n) f_n(p_n) dp_n + \int_{p_n=S_{n-1}/S_n}^{1} g_{n-1}(S_{n-1} | p_n S_n) f_n(p_n) dp_n,$$

$$n = 2,\ldots N.$$

Lemma 1: $g_n'(S_n|y_{n+1}) = h_{n+1}$, $\quad n=1, \ldots, N.$

Proof:

We first note that

$$g_n(S_n|y_{n+1}) = (w_n - h_{n+1})S_n + h_{n+1}y_{n+1}$$

$$+ \int_{p_n=0}^{S_{n-1}/S_n} g_{n-1}(p_nS_n|p_nS_n)f_n(p_n)dp_n$$

$$+ \int_{p_n=S_{n-1}/S_n}^{1} g_{n-1}(S_{n-1}|p_nS_n)f_n(p_n)dp_n$$

It is clear that

$$g_n'(y_{n+1}) = h_{n+1} \quad , \quad n = 2, \ldots, N.$$

Moreover,

$$g_1'(S_1|y_2) = \partial g_1(S_1|y_2)/\partial y_2$$

$$= h_2 \qquad \text{(from equation (5))}.$$

Hence, we have, in general,

$$g_n'(S_n|y_{n+1}) = h_{n+1}, \qquad n=1, \ldots, N. \qquad \square$$

Lemma 2:

For all $n = 1, \ldots, N$,

$$\partial g_n(S_n|S_n)/\partial S_n = h_{n+1}$$

Proof:

We have

$$g_n(y_{n+1}|y_{n+1}) = w_n y_{n+1} + \int_{p_n=0}^{S_{n-1}/y_{n+1}} g_{n-1}(p_n y_{n+1}|p_n y_{n+1}) f_n(p_n)dp_n + \int_{p_n=S_{n-1}/y_{n+1}}^{1} g_{n-1}(S_{n-1}|p_n y_{n+1}) f_n(p_n)dp_n$$

Differentiating this with respect to y_{n+1}, we get

$$g_n'(y_{n+1}|y_{n+1}) = w_n + \int_{p_n=0}^{S_{n-1}/y_{n+1}} g_{n-1}'(p_n y_{n+1}|p_n y_{n+1}) p_n f_n(p_n)dp_n + h_n \int_{p_n=S_{n-1}/y_{n+1}}^{1} p_n f_n(p_n)dp_n \qquad \text{(A-3)}$$

Hence, from equations (7) and (A-3), we have

$g_n'(S_n|S_n) = h_{n+1}$. □

We can now use Lemma 2 to establish:

Lemma 3:

For all $n = 1,\ldots,N$,

$g_n''(y_{n+1}|y_{n+1}) > 0$, for $y_{n+1} \geq 0$.

Proof: Differentiating equation (A-3) with respect to y_{n+1}, we get

$$g_n''(y_{n+1}|y_{n+1}) = \int_{p_n=0}^{S_{n-1}/y_{n+1}} g_{n-1}''(p_n y_{n+1}|p_n y_{n+1}) p_n^2 f_n(p_n)dp_n - g_{n-1}'(S_{n-1}|S_{n-1})(S_{n-1}^2/y_{n+1}^3) f_n(S_{n-1}/y_{n+1})$$

$$+ \quad h_n(S_{n-1}^2/y_{n+1}^3)f_n(S_{n-1}/y_{n+1})$$

$$= \int_{p_n=0}^{S_{n-1}/y_{n+1}} g''_{n-1} \, (p_n y_{n+1} | p_n y_{n+1}) p_n^2 f_n(p_n) dp_n$$

The last equality results from Lemma 2.

Now $p_n > 0$ and $f_n(p_n) \geq 0$, so we need only to establish that $g''_1 \, (y_2|y_2) > 0$ to prove that $g''_n \, (y_{n+1}|y_{n+1}) > 0$ for all n. From equation (5) we have

$$g_1(y_2|y_2) = w_1 y_2 + h_1 \int_{p_1=D/y_2}^{1} (p_1 y_2 - D) f_1(p_1) dp_1 + \pi \int_{p_1=0}^{D/y_2} (D - p_1 y_2) f_1(p_1) dp_1$$

$$g'_1(y_2|y_2) = w_1 + h_1 \int_{p_1=D/y_2}^{1} p_1 f_1(p_1) dp_1 - \pi \int_{p_1=0}^{D/y_2} p_1 f_1(p_1) dp_1 \qquad \text{(A-4)}$$

$g''_1(y_2|y_2) = h_1(D^2/y_2{}^3) f_1(D/y_2) + \pi(D^2/y_2{}^3) f_1(D/y_2) > 0$ since $\pi > |h_1|$ by assumption. □

Finally, we first state and prove a lemma regarding the form of $g'_n(0)$ which is useful in proving the second part of Theorem 1.

Lemma 4:

For all n=1,..., N,

$$g'_n(0|0) = \sum_{j=0}^{n-1} [w_{n-j} \prod_{i=1}^{j} \bar{p}_{n-i+1}] - \pi \prod_{i=1}^{n} \bar{p}_i$$

where, for simplicity of presentation, we define the expectation of p_n as

$$\bar{p}_n = \int_{p_n=0}^{1} p_n f_n(p_n) dp_n$$

and we use the convention $\prod_{i=a}^{b} = 1$ if a > b.

Proof:

We will prove the Lemma by induction. From equation (A-4), for n=1 we have,

$$g_1'(0|0) = w_1 - \pi\bar{p}_1 ,$$

so the Lemma is true for n=1.

Suppose the Lemma is true up to n. From equation (A-3), we have

$$g_{n+1}'(y_{n+2}|y_{n+2}=0) = w_{n+1} + \int_0^1 \sum_{j=0}^{n-1} [w_{n-j} \prod_{i=1}^{j} \bar{p}_{n-i+1}] - \pi \prod_{i=1}^{n} \bar{p}_i \quad p_{n+1} f_{n+1}(p_{n+1}) dp_{n+1}$$

$$= \sum_{j=0}^{n} w_{n+1-j} \prod_{i=1}^{j} \bar{p}_{n+2-i} - \pi \prod_{i=1}^{n+1} \bar{p}_i ,$$

so the Lemma is true for n + 1, and therefore for all n. □