Production of Uniform Electrostatic Fields by a Slotted Conducting Spherical Shell

CARL E. BAUM, FELLOW, IEEE

Abstract—**For some applications one desires a uniform quasi-static electric field. Along the already well-developed lines concerning magnetostatic fields, in this case electric potential is specified on a spherical surface. Dividing the surface on lines of constant latitude (polar angle), the resulting bands are constrained to have particular voltages. The particular case of three conducting surfaces with voltages V_1, 0, and $-V_1$ is considered in detail with optimum angles θ_1 and $\pi - \theta_1 \simeq 63.43°$. This makes for a very uniform electric field near the center of the sphere.**

Key Words—**Uniform electric field source, slotted conducting spherical shell, GLOBUS.**
Index Code—**A3j.**

I. Introduction

Much attention has been paid over the years to the production of uniform magnetic fields. There are the well-known Helmholtz and Maxwell coils involving, respectively, two and three coaxial coils optimally positioned on a spherical surface with optimally specified currents for maximum magnetic field uniformity near the center of the sphere [6]. This case has been extended to an arbitrary number of coils on a spherical surface [5]. The optimum criterion has traditionally been to make the maximum number of derivatives of the field zero at the coordinate origin (center of the sphere). A recent note describes the construction of a Helmholtz coil for scale-model measurements [2].

The dual problem, which has received much less attention, is the production of uniform electric fields. For scale-model measurements of surface charge density on "perfectly conducting" objects, it would be useful to have some comparable kinds of electrostatic uniform-field generators. This paper addresses a certain kind of spherical structure which specifies electric potential on a spherical surface to produce a uniform electric field near the center of the sphere.

II. Slotted Sphere: General Considerations

As in Fig. 1, consider a sphere of radius a. On this spherical surface, let us specify a potential $V(\theta)$. As a practical matter, this surface is constructed of segments of "perfectly conducting" metal with potentials V_n with

$$V(\theta) = V_n, \qquad \text{for} \quad \theta_{n-1} < \theta < \theta_n$$
$$0 \leqq \theta \leqq \pi$$
$$n = 1, 2, \cdots, N$$

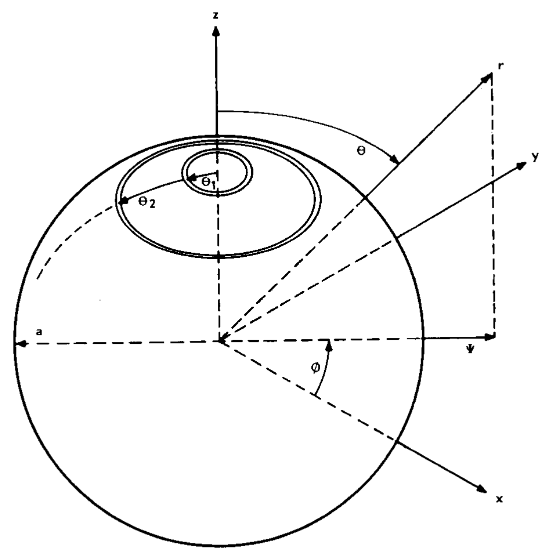

Fig. 1. Perfectly conducting sphere with narrow slots at constant latitudes.

$$\theta_0 = 0$$
$$\theta_N = \pi. \tag{1}$$

The usual Cartesian, cylindrical, and spherical coordinate systems are also indicated in Fig. 1 with

$$x = \Psi \cos(\theta)$$
$$y = \Psi \sin(\theta)$$
$$z = r \cos(\theta)$$
$$\Psi = r \sin(\theta). \tag{2}$$

As a practical matter, (1) is approximate in that each θ_n is actually the center of a narrow slot which insulates one conducting segment from the next. This slot is assumed to be sufficiently narrow that it negligibly affects the electric field near the center of the sphere. A particular advantage of this design concept is that by specifying the potential on $r = a$, the presence of conductors and insulators outside the sphere ($r > a$) does not affect the potential and electric field for $r < a$. Such additional external items might include various electric equipment used to enforce the potential distribution on $r = a$.

In the usual spherical coordinates, the solution for the

Manuscript received September 23, 1987.
The author is with the Air Force Weapons Laboratory, Department of the Air Force, Kirtland Air Force Base, NM 87117. Tel. (505) 844-9816.
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Laplace equation for the electric potential is expressed as

$$\Phi = \sum_{n,m,{}^e_0} a_{n,m,{}^e_0} r^n Y_{n,m,{}^e_0}(\theta, \phi) + a'_{n,m,{}^e_0} r^{-n-1} Y_{n,m,{}^e_0}(\theta, \phi)$$

$$Y_{n,m,{}^e_0}(\theta, \phi) = P_n^{(m)}(\cos(\theta)) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix}$$

$$\equiv \text{scalar spherical harmonics}$$

$$P_n^{(m)}(\xi) \equiv (-1)^m (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi)$$

$$\equiv \text{associated Legendre functions}$$

$$P_n(\xi) \equiv P_n^{(0)}(\xi) \equiv \frac{1}{2^n n!} \frac{d^n}{d\xi^n} [(\xi^2-1)^n]$$

$$\equiv \text{Legendre functions.} \tag{3}$$

Now, restricting our attention to $0 \leqq r \leqq a$, only the nonnegative powers need be considered. Furthermore, noting that the assumed axially symmetric geometry allows only an axially symmetric potential, we can now write

$$\Phi = \sum_{n=0}^{\infty} a_n r^n P_n(\cos(\theta)). \tag{4}$$

Constraining the geometry (and the potential) to be antisymmetric with respect to the (x, y) plane [3], we also have

$$\Phi = \sum_{n=1}^{\infty, 2} a_n r^n P_n(\cos(\theta)) \tag{5}$$

so that only odd n need be considered.

The scalar spherical harmonics are orthogonal on the unit sphere [4, eq. (B19)] as

$$\int_0^{\pi} \int_0^{2\pi} Y_{n,m,\sigma}(\theta, \phi) Y_{n',m',\sigma'}(\theta, \phi) \sin(\theta)\, d\phi\, d\theta$$

$$= [1 + [1_{e,\sigma} - 1_{0,\sigma}] 1_{0,m}] \frac{2\pi}{2n+1} \frac{(n+m)!}{(n-m)!}$$

$$\cdot 1_{n,n'} 1_{m,m'} 1_{\sigma,\sigma'}. \tag{6}$$

For our case of axial symmetry we have

$$\sigma = e, \qquad m = 0$$

$$\int_0^{\pi} \int_0^{2\pi} P_n(\cos(\theta)) P_{n'}(\cos(\theta)) \sin(\theta)\, d\phi\, d\theta = \frac{4\pi}{2n+1} 1_{n,n'}$$

$$\int_0^{\pi} P_n(\cos(\theta)) P_{n'}(\cos(\theta)) \sin(\theta)\, d\theta = \frac{2}{2n+1} 1_{n,n'}$$

$$1_{n,n'} = \begin{cases} 1, & \text{for } n = n' \\ 0, & \text{for } n \neq n' \end{cases}$$

$$\equiv \text{Kronecker delta.} \tag{7}$$

Now solve for the a_n by noting that

$$\Phi = V(\theta), \qquad \text{for } r = a. \tag{8}$$

This gives

$$a_n = a^{-n} \frac{2n+1}{2} \int_0^{\pi} V(\theta) P_n(\cos(\theta)) \sin(\theta)\, d\theta \tag{9}$$

as the general solution for an arbitrary axially symmetric potential distribution on the spherical surface.

The static electric field is given by

$$\vec{E} = -\nabla\Phi = -\vec{1}_r \frac{\partial\Phi}{\partial r} - \vec{1}_\theta \frac{1}{r} \frac{\partial\Phi}{\partial\theta} - \vec{1}_\phi \frac{1}{r \sin(\theta)} \frac{\partial\Phi}{\partial\phi}. \tag{10}$$

For the axially symmetric case there is no ϕ component. Consider the $n = 1$ term as

$$\Phi_1 = a_1 r P_1(\cos(\theta)) = a_1 r \cos(\theta)$$

$$= a_1 z. \tag{11}$$

For this special case we have the corresponding electric field as

$$\vec{E}_1 = -\nabla\Phi_1 = -a_1 \vec{1}_z \tag{12}$$

which is a uniform electric field in the z direction.

This uniform z-directed electric field is the important term in our expansion. If all other terms were zero, this would be the ideal result. From (7) and (9), if we were to choose

$$V_{\text{ideal}}(\theta) = -aE_0 P_1(\cos(\theta)) = -aE_0 \cos(\theta) \tag{13}$$

this would give

$$a_n = \begin{cases} -E_0, & \text{for } n = 1 \\ 0, & \text{for } n \neq 1 \end{cases} \tag{14}$$

giving an electric field

$$\vec{E} = E_0 \vec{1}_z. \tag{15}$$

Our problem is to choose $V(\theta)$ in a way that approximates (13) in the sense of making $a_n = 0$ for as many $n \neq 1$ as possible with emphasis on the values of n near 1 since they are most significant for r near the origin. Viewed another way, the terms for $n = 2, 3, \cdots$ give the next derivatives for the electric field at $r = 0$.

III. Two-Slot Conducting Sphere

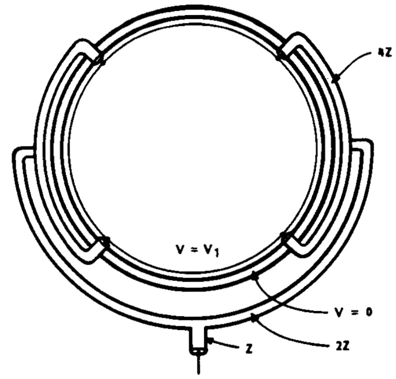
Now consider the special case of $N = 2$ as illustrated in Fig. 2. Evidently constraining

$$\theta_2 = \pi - \theta_1$$

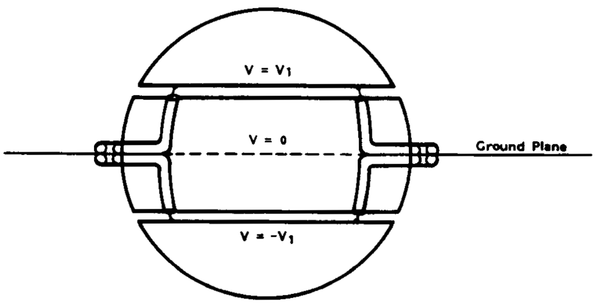
$$V_1 = -V_3$$

$$V_2 = 0 \tag{16}$$

gives symmetry about the (x, y) plane so that only n = odd terms in the potential are included as in (5). The electric field will scale with the single independent potential V_1, leaving only θ_1 to be varied so as to eliminate the $n = 3$ term.



(a)



(b)

Fig. 2. Case of optimally chosen two symmetrical slots. (a) Top view. (b) Side view.

From (9) we have

$$a_3 = a^{-3}\frac{7}{2}\int_0^{\pi} V(\theta)P_3(\cos(\theta))\sin(\theta)\,d\theta$$

$$= a^{-3} 7 V_1 \int_0^{\theta_1} P_3(\cos(\theta))\sin(\theta)\,d\theta$$

$$= a^{-3} 7 V_1 \int_{\cos(\theta_1)}^{1} P_3(\xi)\,d\xi. \quad (17)$$

From (3) we have

$$P_3(\xi) = \frac{5}{2}\xi^3 - \frac{3}{2}\xi \quad (18)$$

so that

$$a_3 = a^{-3} 7 V_1 \left[\frac{5}{8}\xi^4 - \frac{3}{4}\xi^2\right]\Bigg|_{\cos(\theta_1)}^{1}. \quad (19)$$

Setting this to zero gives

$$5\cos^4(\theta_1) - 6\cos^2(\theta_1) + 1 = 0 \quad (20)$$

which is solved by the usual quadratic formula to give

$$\cos^2(\theta_1) = 1, \frac{1}{5}. \quad (21)$$

Neglect the case of 1, which is a degenerate case giving no electric field at the origin. Thus, we have

$$\cos(\theta_1) = \frac{1}{\sqrt{5}} \simeq 0.4472$$

$$\theta_1 \simeq 63.43°$$

$$\cot(\theta_1) = \frac{1}{2}. \quad (22)$$

It may be observed that this solution is the same as that for the Helmholtz coils, the two coils being replaced by slots at the same positions on the sphere containing the two coils.

Having found the special angle, we need

$$a_1 = \frac{3}{2a}\int_0^{\pi} V(\theta)P_1(\cos(\theta))\sin(\theta)\,d\theta$$

$$= \frac{3}{a} V_1 \int_0^{\theta_1} P_1(\cos(\theta))\sin(\theta)\,d\theta$$

$$= \frac{3}{a} V_1 \int_{\cos(\theta_1)}^{1} P_1(\xi)\,d\xi$$

$$P_1(\xi) = \xi \quad (23)$$

$$a_1 = \frac{3}{a} V_1 \frac{\xi^2}{2}\Bigg|_{\cos(\theta_1)}^{1}$$

$$= \frac{3V_1}{2a}[1 - \cos^2(\theta_1)] = \frac{3V_1}{2a}\sin^2(\theta_1)$$

$$= \frac{6}{5}\frac{V_1}{a}.$$

This is a very convenient result which gives, when compared to (14) and (15), the electric field at the origin as

$$\vec{E} = E_0 \vec{1}_z$$

$$E_0 = -\frac{6}{5}\frac{V_1}{a}. \quad (24)$$

With these results, the first error term corresponds to $n = 5$. The fourth derivative is the first nonzero derivative of the field at the origin. This also corresponds to the Helmholtz-coil case.

Fig. 2 shows some features of how one might excite the structure with $\pm V_1$. Having a differential source (with small common mode), one would like to distribute it in some uniform way (equal delays from equal-impedance sources) around the slot. Fig. 2 gives an example with four connections around the slot with a set of transmission lines with net parallel impedance Z for each slot. This is similar to the feed system of the HSD sensor for D-dot. Other forms of feed structure are also possible, such as rotating the top and bottom feed structure with respect to each other to achieve a more uniform illumination with respect to ϕ.

IV. Summary

We now have our special spherical-bowl design to give uniform electrostatic fields. As illustrated in Fig. 2, we have something that looks like some kind of dual of the Helmholtz coils. Note the benefit of having most of the spherical surface conductors at specified potentials, thereby shielding the inside volume from external electrostatic disturbances. This allows us to place our transmission lines and other electrical connections on the outside of the sphere to drive the two spherical bowls to $\pm V_1$ in a differential sense at low frequencies. Note also that since the (x, y) plane is a symmetry plane (antisymmetric) it can be replaced by a perfectly conducting surface so that one can use only half of this spherical system (with single-ended drive) if desired. In this single-ended form, this is a FINES type of simulator for illuminating small penetrations [1].

Let us refer to this type of uniform-electric-field illuminator as the graded latitudinally open bowl uniform simulator (GLOBUS, the Latin word for sphere).

Another approach to the present problem involves charged rings (thin strips or wires of small radius) on a spherical surface. Unfortunately, this case does not well shield the interior from exterior charges (scatterers, including leads to the rings). In an idealized sense, though, one can achieve similar uniform electric fields near the center of the sphere. This case is left as an exercise for the reader.

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