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Production, Price, and Inventory Theory

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By George A. Hay*

This paper is an attempt to derive empirically testable hypotheses regarding the principal determinants of firms' decisions on production, price, and finished goods inventory. The general approach to the problem is that many of the same factors which affect the optimal value for one variable will also influence decisions on the other two, and that a "proper" model must take into account the interdependence of these variables and the simultaneous nature of the decisions involving them. This is in contrast to literature on the theory of inventories (see Paul Darling and Michael Lovell) in which the firm is assumed to determine the optimal level of inventories with the rate of production taken as given and with price held constant. Similarly there are theories of price formation (see Otto Eckstein and Gary Fromm) in which the rate of production is assumed to have been determined previously, and in which the level of inventories is often ignored entirely. The present paper will attempt to present an "integrated" model of firm behavior in which decisions on all relevant variables are assumed to result from a single optimization process.

A principal distinction between this study and previous work in this area (see Gerald Childs and Charles Holt) is the inclusion of price as one of the decision variables.[1] In the past the assumption has been made that price is fixed and therefore that quantity demanded is a datum to the firm. In anything but a purely competitive model, however, the firm does exercise some control over price. The rational firm would use its current pricing policy to select the specific price-quantity combination on the demand curve that best contributes to its overall scheme of profit maximization. Thus the firm whose demand curve is not constant over time but fluctuates from period to period on a random and/or seasonal basis might view price adjustments as one means of achieving production stabilization. If this were so we might expect to find that an increase in demand would be met by continuing to produce at or near the previous rate and raising price to clear the market. More realistically, the entire kit of adjustment tools—inventory, backlog, and price—would be used in some combination to absorb all or part of the increased demand, the specific result depending not only on the particular cost structure assumed, but also on the firm's estimates of what demand will be for several periods hence. The important point is that price must certainly be included as one of these tools.

In the remainder of the paper we construct a model which includes many of the variables which are important at the individual firm level, and which treats decisions regarding those variables as being essentially interdependent. The subsequent section discusses the behavioral assumptions which underlie the model and expresses the model in mathematical form. The first-order conditions for maximizing expected profits generate a set of linear decision rules for production, price, and finished goods inventory. On the assump-

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[1] Edwin Mills developed a model which included price. He was able to derive approximations to the true decision rules which held up reasonably well under empirical investigation. The present study attempts to derive an exact set of decision rules.

tion that firms do attempt to maximize expected profits, these decision rules are suitable for empirical investigation with the appropriate data. In Section II, the model is solved numerically with representative cost parameters. The resulting decision rule coefficients provide predictions regarding the actual regression equations which should be fulfilled if the model accurately reflects the working of the real world. Finally, in Section III, the regressions are performed on two industry groups, and the results compared with the predictions.

I. *The Model*

Behavioral Assumptions

The model is intended to represent a firm which chooses the levels of the variables it controls in order to maximize the expected value of discounted future profits over a time horizon. The variables involved are the rates of production and shipments, the level of finished goods inventories, the backlog of unfilled orders, and price. These variables are not independent, being related by various definitional and market constraints, including the demand equation. Each of the variables serves a particular function, and associated with each of these variables are certain costs. These functions and costs form the basic elements of the model.

A positive level of unfilled orders is, practically speaking, an unavoidable phenomenon for most firms which undertake any production to order. The timing of the arrival of new orders is not in general subject to control by the firm. The development, design, and production of each order takes time, so that there will typically be some work still in process when a new order is received. Even beyond this, however, a *higher* level of unfilled orders may be a useful alternative for the firm because it permits smoothing of production within the period and accumulation of optimal size production batches. This is particularly true when production is a multistage process and several items which are eventually individualized to the specific requirements of their respective purchasers could nonetheless go through several stages of the production process together. On the other hand, there are costs associated with a high level of unfilled orders. As the backlog gets larger and the lead time longer, there is increased danger of cancellation of orders, penalty costs for expediting particular orders, and probable loss of future sales.

This suggests that there is some positive level of unfilled orders which balances the cost savings attributable to an order backlog and the penalties associated with too large a backlog so that the net saving is maximized. We might refer to this as the "desired" level of unfilled orders in the sense that, if there were no other considerations involved, this is the level the firm would try to maintain. We will assume that the desired level of unfilled orders, U^*, is a linear function of the rate of production:

$$U_t^* = c_{13} + c_{14}X_t$$

As stated by Childs:

> Penalty costs for cancellation of orders, for expediting particular orders in response to customer requests and the probability of loss of future sales all increase as the size of the backlog increases and the lead time lengthens. However, more flexibility in production arises as backlog mounts. Therefore, as backlog and lead time increase the costs related to inflexibility of production decline. Average lead time is approximately the ratio of backlog to the rate of production. Then for every rate of production, the cost associated with varying size of backlog is the sum of monotonically rising and monotonically falling components over the relevant range and has a minimum, U_t^*, the optimal level of U_t. [p. 10][2]

[2] It is probably true that U_t^* is a function of only

Furthermore, we will assume that the cost of deviating from the desired level increases quadratically with the size of the deviation and is the same in either direction. Thus the net contribution to total cost of a given level of unfilled orders is given by:

$$c_{11} + c_1(U_t - U_t^*)^2$$

where U_t is the actual level of unfilled orders at the end of t, and c_{11} is a constant (which may be negative).

Thus we have separated all the costs and cost savings specifically associated with an order backlog into two parts; the first being the net contribution to total costs of the desired level, c_{11}, and the remainder reflecting the additional costs of deviating from the desired level, recognizing that when *all costs* which the firm incurs are considered, the optimizing level of unfilled orders may differ from the desired level. The rational firm may, for example, be willing to deviate slightly from its desired level if doing so will make it possible to avoid a substantial increase (or decrease) in the rate of production from one period to the next.

A similar argument can be used to explain the existence of a positive level of inventories. Certain cost savings accrue from the added flexibility made possible in production; on the other hand, an inadequate level of inventories can have serious consequences for future sales through loss of goodwill which may be caused by a "stock out"—a firm's inability to fill an order from a customer who requires immediate delivery. A firm not only loses out on the sales corresponding to that particular order, but may suffer the loss of future sales as well if the disappointed firm switches preferences in favor of another supplier. This effect might most conveniently be handled as a cost whose expected value is associated with an inadequate level of finished goods inventories.

We assume that the desired level of inventories is related to the quantity to be shipped during the period. This can be considered an approximation to the optimal lot size formulas in the operations research literature (see Holt et al. pp. 56–57). Specifically we assume that the relation:

$$H_t^* = c_{23} + c_{24}S_t$$

is valid over the relevant range, where H_t^* is the desired level of finished goods inventories at the end of the period t, S_t is shipments, and c_{23} and c_{24} are appropriate constants. Furthermore, we assume that the cost associated with being away from the desired level of inventories may be approximated by a quadratic such that the overall contribution of inventories to total costs is represented by:

$$c_{21} + c_2(H_t - H_t^*)^2$$

where c_{21} reflects the net effect of the "desired" level of finished goods inventories.

Unit costs of production within a time period are assumed constant. Beyond this, however, are costs associated with changes in the rate of production from period to period which are independent of the actual level of production. These include various setting up costs and costs of hiring and firing where a change in the work force is required. We express these costs simply as:

$$c_3(X_t - X_{t-1})^2$$

This captures the notion that changes are costly in either direction and that large changes are likely to be relatively more

that part of X_t which is production-to-order. However, introducing production-to-order and production-to-stock as separate variables complicates the analysis and furthermore requires an arbitrary aggregation procedure at the end since the only observed variable is total production. In addition, the nature of the results not is affected so long as the cost of changing the rate of production (discussed below) is independent of the mix between order and stock. (Similar remarks are applicable to the inventory decision.)

costly over a certain range than small ones. The general idea is that firms do seem to attach considerable weight to the stability of the rate of production and the size of the work force. This has the effect of making the previous period's output a factor of considerable importance in the determination of the output for the current period.

Demand for the product of the firm (in the sense of the entire demand curve) is not constant over time but is assumed to shift from period to period, in response partly to random factors, and partly to factors which the firm observes but cannot control (e.g., the level of national income). To make the model amenable to analytical treatment we must give a specific form for the demand curve. In this model, therefore, we assume that demand is of the form:

$$O_t = Q_t - bP_t$$

when O_t is new orders and P_t is price in period t. This is a linear demand curve of constant slope b which shifts over time in parallel movements, the extent of the shift determined by the quantity intercept term Q_t. For reasons discussed below, this demand curve is the schedule of the quantity demanded from the firm at various prices when all firms charge the same price.

This leads to the final cost to be considered, viz., that associated with changes in price. It may not be common to think about costs of changing price yet we observe that firms are generally reluctant to do so in terms both of frequency and magnitude. There are, of course, certain out-of-pocket costs which must be met—the necessity of revising price books and possibly some additional advertising expense to announce the new price.

Probably a more important influence on firm behavior, however, are those costs which are never actually observed but may be thought of more as a type of opportunity cost, the amount of money a firm would be willing to pay, *ceteris paribus*, to avoid changing price. This reluctance is due generally to the risk associated with imperfect knowledge about reaction by competitors to a price change. We cite William Fellner for a description of the reasoning involved:

> Each firm knows that others have different appraisals [of the appropriate policy for the maximization of industry profits] and that they are *mutually* ignorant of what precisely the rival's appraisal is. Consequently, no firm can be *sure* whether the move of a rival is toward a profit-maximizing quasi-agreement or towards aggressive competition; and no firm can be sure how its own move will be interpreted. This is where the desire to avoid aggressive competition enters as a qualifying factor. Even where leadership exists, the leader's moves may be misinterpreted as aiming at a change in relative positions rather than as being undertaken in accordance with the quasi-agreement. [p. 179]

There are, of course, other arguments which might be offered for the inclusion in the criterion function of a penalty for price changes. The important consideration is that the firm acts "as if" there were costs attached to changes in price, whether these are explicit, out-of-pocket costs, or more of an opportunity-cost, implicit type which arises from uncertainty about rivals' reactions to its own price decisions.

We might assume initially that the cost of changing price can be expressed as:

$$c_4(P_t - P_{t-1})^2$$

The symmetry assumption regarding the cost of changing price is perhaps bothersome but extremely convenient mathematically. If we regard the demand curve as based on the assumption that other firms will always imitate price changes by our firm, then the quadratic penalty can be taken to represent the fear that for a price increase, the firm will not be followed, and on a price decrease, the firm will be undercut.

There may be certain circumstances, however, under which the reluctance to change price might be considerably diminished. If the incentive for a firm to initiate a price increase is the result of an increase in labor or raw materials cost (which presumably would affect all firms in the industry to a similar, if not identical, degree), it is likely that such a move would be welcomed by rivals as an opportunity for them to restore the margin which existed prior to the increase in direct costs, and hence the price increase would be matched. Indeed, failure to do so by any single firm might well be interpreted as aggressive behavior by rivals in the same sense as a price cut in the absence of any changes in cost. To the extent, then, that firms in the industry follow this type of increase completely, the cost associated with initiating such a change is likely to be insignificant. (A similar argument can be given for price cuts which are occasioned by a drop in direct costs.)

Therefore, we can amend the model to assume that the firm is sensitive to changes in markup rather than changes in the absolute level of price. Cost-induced price changes can be initiated without fear of not being followed and hence the firm does not attribute a penalty to such a move. Changes in price which are not related to direct costs, or failure to change price when direct costs vary will be assigned a cost in the firm's decision-making process. In the terms of the model we are replacing

$$c_4(P_t - P_{t-1})^2$$

with:

$$c_4[(P_t - V_t) - (P_{t-1} - V_{t-1})]^2$$

where V_t represents the direct unit costs of production (labor, capital rental, and raw materials) in period t.[3]

Derivation of Linear Decision Rules

We can now bring the cost and revenue terms together into a single equation so that the conditions for optimization can be derived. It is perhaps best to begin by summarizing the notation. Let:

- $X_t \equiv$ rate of production in period t
- $P_t \equiv$ price in period t
- $U_t \equiv$ level of unfilled orders (backlog) at the end of period t
- $U_t^* \equiv$ desired level of unfilled orders at the end of period t
- $H_t \equiv$ level of finished goods inventories at the end of period t
- $H_t^* \equiv$ desired level of finished goods inventories at the end of period t
- $O_t \equiv$ new orders in period t
- $S_t \equiv$ shipments in period t

These variables are constrained by the following identities:

$$O_t - S_t \equiv U_t - U_{t-1}$$
$$X_t - S_t \equiv H_t - H_{t-1}$$

The demand equation relates the endogenous variable O_t to the decision variable P_t and the exogenous term Q_t:

$$O_t = Q_t - bP_t$$

We will treat the problem initially as one of certainty. Furthermore we assume that the firm discounts future profits according to a discount factor $\lambda (0 < \lambda < 1)$. Therefore, to maximize discounted future profits over an N-period horizon, we maximize the Lagrangian expression in equation **(A)** shown on the following page where δ_t and γ_t are Lagrangian multipliers, and λ a discount factor. After taking derivatives for any period t with respect to the decision variables and allowing t to take on

[3] It may be that a more complex lag structure would be appropriate according to the speed with which increases in direct costs are recognized and transmitted into prices, but in the absence of any detailed empirical knowledge on this process and in the desire to use a single model for many industries, this structure will be retained.

(A)

$$L = \sum_{t=1}^{N} \lambda^{t-1} \Big\{ P_t Q_t - b P_t^2 - c_{11} - c_1(U_t - c_{13} - c_{14} X_t)^2 - c_{21} - c_2(H_t - c_{23} - c_{24} S_t)^2$$

$$- V_t X_t - c_3(X_t - X_{t-1})^2 - c_4[(P_t - V_t) - (P_{t-1} - V_{t-1})]^2 \Big\}$$

$$- \delta_t(Q_t - bP_t - S_t - U_t + U_{t-1}) - \gamma_t(X_t - S_t - H_t + H_{t-1})$$

all values to N(where N is large), the system is solved according to the Z transform procedure outlined in Holt et al.[4] The end result is a set of linear decision rules for production, price, and finished goods inventory of the form:

$$(1) \quad X_t = A_{11}X_{t-1} + A_{12}P_{t-1} + A_{13}H_{t-1} + A_{14}U_{t-1} + A_{15}Q_t + A_{16}Q_{t+1} + \dots + A_{17}V_{t-1} + A_{18}V_t + \dots + k_1$$

$$(2) \quad P_t = A_{21}X_{t-1} + A_{22}P_{t-1} + A_{23}H_{t-1} + A_{24}U_{t-1} + A_{25}Q_t + A_{26}Q_{t+1} + \dots + A_{27}V_{t-1} + A_{28}V_t + \dots + k_2$$

$$(3) \quad H_t = A_{31}X_{t-1} + A_{32}P_{t-1} + A_{33}H_{t-1} + A_{34}U_{t-1} + A_{35}Q_t + A_{36}Q_{t+1} + \dots + A_{37}V_{t-1} + A_{38}V_t + \dots + k_3$$

where the k's are constants. There are also rules for U_t and S_t which can be derived from the first three.

Since $Q_t, Q_{t+1} \dots$ and $V_t, V_{t+1} \dots$ are not known quantities but random variables, the firm can do no better than maximize the expected value of profits. According to the Simon-Theil theorem, the same decision rules apply provided we substitute for Q and V their expected values, $\hat{Q}$ and $\hat{V}$. These equations are used to determine the first period's decision for X, P, and H. Each period the rule is used again, with the current period being treated as the first.

In a normative model, the firm would make a forecast of future Q and V and insert the resulting values into its decision rules. Since we do not have any information about what forecasts the firms used, for purposes of regression analysis we must guess at the specific set of forecasts on which the decisions were based. Following previous studies of this type by Childs and Holt et al., we have used $\hat{Q}_t = Q_t$, i.e., on the assumption that the firm is on average an accurate predictor of future events, the demand that actually comes about is used as a proxy for what the firm had anticipated at the time the decisions were made.[5,6] The same formulation is used for $\hat{V}$.

Aggregation

The model outlined above and the resulting equations for production, price, and finished goods inventory dealt with the decision process of the individual firm. The available data cover the industry aggregates corresponding to these variables. Problems of aggregation have been treated elsewhere (see Theil) and will not be discussed here. The main point is that the coefficients of the industry "decision

[4] The exact solution is not presented here due to lack of space.

[5] Naive and distributed lag expectations were tried in our regressions and did not do as well as the perfect forecasts.

[6] Q_t is not directly observed but can be obtained for purposes of regression analysis by the inverse relation $Q_t = O_t + bP_t$. Although b is not known either, an a priori "reasonable" value (see Section II) can be used. As a check, alternative values of b were used to generate the Q series with very little effect on the regression results. Note that for the firm Q is a true exogenous variable. The use of $(O+bP)$ as a proxy for what the econometrician cannot observe does not change the reduced-form nature of the actual decision rules.

rule" will not in general be a simple function only of the corresponding micro-coefficients. To achieve an unbiased set of restrictions would require knowledge at the individual firm level which we simply do not possess. In the absence of such knowledge the sample calculations discussed below will assume that the industry can in fact be treated as a single large firm and will assume that any restrictions on the micro-coefficients will carry over, with appropriate scaling, to the corresponding macro-coefficients.

II. *Calculation of Decision Rules for Particular Cost Structures*

This section reports on an attempt to establish some of the qualitative properties of the system which has equations (1)–(3) as its reduced form. Specifically, some estimate should be provided as to the range within which the regression coefficients are likely to lie if the observed decisions are made in a manner approximated by the decision model. Furthermore, since the A_{ij} in the reduced form are functions of all the parameters in the original structure (b, λ, and the c's), it is desirable to know how the A's will be affected by different assumptions about the values of these parameters.

> Different firms or industries to which the model might be applied have different cost parameters, and one wants to know how this fact will affect the [A's]. In addition, it is known that within a firm or industry these parameters do not usually remain constant over the period covered by a time-series sample. Thus, if the regression coefficients are very sensitive to changes in the underlying parameters, results will be poor when the regressions are estimated from empirical data even if the basic decision model is correct. [Mills, p. 139]

For a less complex structure, the coefficients of the reduced form equations could be derived analytically as functions of the parameters of that structure. With the present model, however, that approach does not appear to be feasible. It would require, in addition to matrix manipulation and determinant evaluation, the analytic solution of an eighth degree equation.

Given the infeasibility of an analytic solution, an alternative approach is attempted in the present study. The system is solved for specific values of the cost and revenue parameters. As these parameter values are changed in successive trials, the results are examined to determine the sensitivity of the A_{ij} to the specific parameters used. Hopefully, for a wide range of values for the b, λ, and c's, the A coefficients will retain at least the same signs, and to a degree, the same relative magnitudes. The extent to which the coefficients *are* sensitive to the parameters gives some indication of the results which might be expected when parameter values other than the ones used in the samples are appropriate.

Parameter Specifications

The first stage of the experiment consists of the selection of an initial set of cost and revenue parameters to be used to arrive at a specific numeric solution for the derivation of the decision rule coefficients. Including λ, the discount factor, and b, the slope of the demand curve, there are twelve parameters to be specified. The following characteristics of the data will help to set the order of magnitude for these parameters.

Price is in index form and is of the order of 100, with a standard deviation of approximately 3.8. The physical series (shipments, inventories, etc.) are reported in millions of dollars. Shipments, production, and new orders averaged approximately \$1000 million with a standard deviation of approximately \$165 million. Inventories and unfilled orders each average around \$600 million with a standard deviation of

TABLE 1—DECISION RULE COEFFICIENTS FOR A PARTICULAR COST STRUCTURE

Independent Variable	X_{t-1}	P_{t-1}	H_{t-1}	U_{t-1}	Q_t	Q_{t+1}	Q_{t+2}[a]	V_{t-1}	V_t	V_{t+1}	V_{t+2}[a]
Dependent Variable											
X_t	.260	−.028	−.345	.495	.492	.089	.019	.028	−.032	−.004	−.001
P_t	−.169	.903	−.058	.066	.156	.106	.081	−.903	1.028	−.010	−.008
H_t	.195	−.007	.354	−.001	−.001	.024	.012	.007	−.009	−.003	−.001

[a] The two infinite series were truncated at this point. This is justified by the exponentially declining weights in the expansion of $Q(Z)$ and $V(Z)$.

approximately \$100 million and \$80 million, respectively.[7]

With these figures in mind, the following were assumed to be the values of the cost parameters:

i) c_{11}, c_{13}, c_{21}, and c_{23} drop out entirely or affect only the intercept term and were assumed equal to zero.

ii) $b=.05$. Given the magnitude of the price and new orders data, this corresponds to a price elasticity of .5 which seems reasonable given the assumptions about the demand curve (i.e. that it specifies quantity demanded from the firm when all firms charge the same price).

iii) $\lambda=.99$. Since the decision period is one month, this corresponds to an implicit annual discount of approximately 12 percent.

iv) $c_{14}=c_{24}=.6$. Thus it is assumed that the observed long-run ratios between inventories and shipments and unfilled orders and production approximate the desired relationship.

v) $c_1=150$; $c_2=100$; $c_3=35$; $c_4=5$. It was assumed that if the actual level of any of the variables deviated from the desired (or previous period's) level by as much as a standard deviation (approximately), the cost would equal 10 percent of total revenue for that month.

The results of the initial sample calculation are presented in Table 1 above. In Tables 2-a, b, and c we attempt to provide some notion of how the A_{ij} depend on the structural parameters. In each of the eight trials a single parameter, indicated in the left-hand column, was varied from the initial set of values. In the case of λ, the value was lowered to .89. In all other cases, the parameter under consideration was doubled.

Looking ahead to the regression analysis for the production and price equations the results of the sample calculations are generally encouraging. The algebraic signs for all the coefficients remain unchanged in every trial. Furthermore, the magnitudes of many of the coefficients were relatively insensitive to changes in the parameters indicating that time-series analysis may yield meaningful results, even though the cost and revenue structure of the firms involved have undoubtedly undergone some degree of change over the time period studied. For the inventory equation, however, the sample calculations produce ambiguous results, with only the signs of X_{t-1} and H_{t-1} remaining the same in every trial.[8]

It is interesting to note that the production and inventory rules are generally unaffected by different specifications regarding the cost of changing price, c_4, and the

[7] There are some small differences between the two industries studied for all of these figures, but not enough to warrant separate treatment.

[8] This is only partially shown in Table 2. Many other parameter specifications were tried but are not reported here.

slope of the demand curve, b. Similarly, the rule for price is relatively insensitive to c_{14} and c_{24}. As evidence of the interdependence among decisions, however, we note the strong impact of c_3, the cost of changing the production rate, on the price decision rule.

In terms of individual coefficients it is worth noting that the coefficients of U_{t-1} and Q_t are virtually identical in the pro-

TABLE 2-a: PRODUCTION DECISION RULE X_t: CHANGE FROM INITIAL SET UNDER ALTERNATIVE COST STRUCTURES

Value of Parameter which is changed	X_{t-1}	P_{t-1}	H_{t-1}	U_{t-1}	Q_t	Q_{t+1}	Q_{t+2}	V_{t-1}	V_t	V_{t+1}	V_{t+2}
$\lambda=.89$	+.003	−.001	0	+.005	+.005	−.004	−.001	+.001	+.001	0	0
$c_1=300$	−.053	−.001	−.026	+.053	+.053	−.021	−.006	+.001	−.001	0	0
$c_2=200$	−.021	−.001	−.028	+.046	+.046	+.012	+.002	+.001	0	−.001	0
$c_3=70$	+.096	+.004	+.078	−.132	−.132	−.006	+.003	−.004	+.007	−.001	−.001
$c_4=10$	−.003	−.001	−.001	+.001	+.002	+.002	+.002	+.001	+.001	−.001	0
$c_{14}=1.2$	−.111	+.006	+.033	−.021	−.020	−.031	−.005	−.006	+.006	+.002	0
$c_{24}=1.2$	−.011	−.005	+.036	+.090	+.090	+.015	−.002	+.005	−.004	−.001	0
$b=.10$	+.015	−.023	+.007	−.008	−.010	−.005	−.003	+.023	−.025	−.004	0

TABLE 2-b: PRICE DECISION RULE P_t: CHANGE FROM INITIAL SET UNDER ALTERNATIVE COST STRUCTURES

Value of Parameter which is changed	X_{t-1}	P_{t-1}	H_{t-1}	U_{t-1}	Q_t	Q_{t+1}	Q_{t+2}	V_{t-1}	V_t	V_{t+1}	V_{t+2}
$\lambda=.89$	−.013	+.032	+.001	+.002	+.005	−.004	−.008	−.032	−.017	0	+.001
$c_1=300$	−.004	0	−.003	+.006	+.007	+.001	0	0	0	0	0
$c_2=200$	−.009	−.001	0	+.002	+.002	+.001	0	+.001	0	0	0
$c_3=70$	−.135	−.005	−.039	+.040	+.039	+.014	+.004	+.005	−.004	−.002	−.001
$c_4=10$	+.079	+.030	+.031	−.035	−.079	−.052	−.038	−.030	−.005	+.005	+.004
$c_{14}=1.2$	−.050	−.001	+.024	−.024	+.027	−.005	0	+.001	+.002	0	0
$c_{24}=1.2$	+.035	+.001	+.001	+.002	+.002	−.005	−.004	−.001	−.003	0	0
$b=.10$	−.135	−.052	−.051	+.060	+.045	+.011	−.006	+.052	−.007	−.012	−.007

TABLE 2-c: INVENTORY DECISION RULE H_t: CHANGE FROM INITIAL SET UNDER ALTERNATIVE COST STRUCTURES

Value of Parameter which is changed	X_{t-1}	P_{t-1}	H_{t-1}	U_{t-1}	Q_t	Q_{t+1}	Q_{t+2}	V_{t-1}	V_t	V_{t+1}	V_{t+2}
$\lambda=.89$	−.001	0	−.001	+.005*	+.004*	+.001	−.001	0	0	+.001	0
$c_1=300$	+.006	+.001	0	−.002	−.002	+.007	+.001	−.001	+.001	0	0
$c_2=200$	−.058	−.002	−.057	+.098*	+.097*	+.006	−.002	+.002	−.001	+.001	0
$c_3=70$	+.079	+.004	+.070	−.115	−.116	−.019	−.001	−.004	+.006	+.001	−.001
$c_4=10$	−.001	0	0	0	0	0	+.001	0	0	0	0
$c_{14}=1.2$	−.054	+.001	−.065	+.116*	+.116*	+.021	+.008	−.001	0	+.002	0
$c_{24}=1.2$	−.009	−.009	+.072	+.207*	+.205*	+.012	−.007	+.009	−.009	0	0
$b=.10$	+.004	−.004	+.002	−.001	−.002	−.002	−.001	+.004	−.003	−.002	−.001

* Indicates a sign change.

duction equation and the inventory equation. (This was true on all trials.) Thus for purposes of these two decisions, an order on hand at the beginning of the period is treated the same as an order anticipated during the period. This result is a consequence of the particular cost structure assumed since Q_t and U_{t-1} enter the cost function symmetrically through the constraint:

$$Q_t - bP_t - S_t = U_t - U_{t-1}$$

Childs (pp. 36–37) obtained the same result in a similar model (without a price equation) and suggests that it demonstrates the possible over-simplification in the U_t^* equation, whereby the desired level of unfilled orders was assumed to depend solely on current production. He suggests that one alternative would be to make U_t^* a function of O_t as well as X_t which would serve to break the rigid equality between the two coefficients, although we would not expect the signs to change. More importantly, he recommends that in the regressions, the coefficients not be constrained to have the same sign (i.e., estimated as a single variable), since this would almost certainly involve some degree of specification error.

Alternatively we might think of this result as arising from our neglect of the production-to-order, production-to-stock distinction. If, as we mentioned earlier, the desired level of unfilled orders is related not to total production but only to that portion of the total which is production-to-order, then the equality between the coefficients would be broken, with U_{t-1} receiving a smaller weight, especially in the inventory equation. However, U_{t-1} would still appear in the rule for finished goods inventories so long as the cost of changes in the rate of production is a function of total production and independent of the mix between order and stock.

Even assuming the plausibility of the original cost structure, this result serves to demonstrate one of the more questionable implications of the certainty equivalent-quadratic criterion function approach. U_{t-1} is a known quantity, given to the decision maker before he must make his decisions on X_t and H_t. Ignoring the perfect forecast, $\hat{Q}_t$ is not a known quantity but the expected value of the decision maker's subjective probability function over all possible values of Q: yet for purposes of decisions on the rate of production and the level of finished goods inventory, they are treated identically. Of course, this is a straightforward result of the certainty equivalence theorem, and, perhaps more suggestively, a consequence of assuming a quadratic criterion function. The point is that this approach in some ways seems to beg the question of risk and uncertainty, and, in so doing, may lead us to overlook what some regard as the essential characteristic of unfilled orders—its ability to serve as a buffer between the past and an uncertain future.

We also note that in the price equation, the coefficients of H_{t-1} and U_{t-1} are opposite in sign and almost equal in absolute value. Hence, our model suggests that for purposes of the price decision, it may be possible to consider unfilled orders as "negative" inventory. However, as before, this result derives from the simple aggregation of order and stock production with a single price for both. In the regression analyses, we would expect the coefficients of H_{t-1} and U_{t-1} to partly reflect the relative weights of order and stock goods in the aggregate price variable, and therefore not display the same coefficient.

Finally we note that P_{t-1} and V_{t-1} have the same absolute coefficients in all equations and the relevant variable for the model is therefore $(P_{t-1} - V_{t-1})$. Because the P series is in index form, however, we do not constrain the variables to have the same coefficients; nor should we expect them to turn out equal if they are not constrained.

III. *Regression Analysis*

The Data

The data consist of monthly observations for the period March 1953—August 1966 on two Standard Industrial Classification (*SIC*) manufacturing groups—Lumber and Wood Products (*SIC* 24), and Paper and Allied Products (*SIC* 26).

The primary sources for the data are manufacturer's shipments, inventories and unfilled orders series compiled by the Industry Division of the Bureau of Census, *U.S.* Department of Commerce,[9,10] and the Wholesale Price Index and average hourly earnings series compiled by the Bureau of Labor Statistics, *U.S.* Department of Labor.

As is generally known, the Wholesale Price Index is commodity-oriented, i.e., the index is the weighted sum of the prices for a group of similar commodities. On the other hand, the Commerce Department series are industry-oriented, i.e., the shipments series, for example, will include all the shipments by a particular industry despite the fact that the specific commodities which originate in a single industry may be quite diverse. Therefore the coverage of a typical Commerce series is likely to be considerably different from that of the most nearly related price series.

For the industries examined in this study, the coverage of the two types of index is reasonably close. Indeed this was the main reason for selecting these particular industries. Hopefully, therefore, the errors introduced by the coverage problems will be minimized, although one's confidence in the specific numerical results is somewhat weakened. It should be mentioned that generally one can expect a closer correspondence of the two types of data at lower levels of aggregation. Unfortunately, although many of the data are available at the 3-digit level, the series for inventories by stage of fabrication is currently available only for 2-digit aggregates.

A further problem arises in attempting to obtain estimates for the direct costs of production. These are to be used in testing the amended specification of the cost of changing price in which penalties are associated with price changes that are correlated to changes in direct costs. Although labor costs can be represented by the series on average hourly earnings, there is considerable difficulty in arriving at a similar index for raw material costs. The problem arises when the output series at the 2-digit level includes as final product its own basic raw material input. For example, the principle raw material input for Lumber and Wood Products (according to *BLS* weights) is lumber. Yet lumber also gets added in as final product, and is, in fact, the largest item in that aggregate. Hence the largest component in the price index for the final product is identical to the largest component in the index of raw material prices and we can expect a rather substantial spurious correlation between the two indices. Therefore, labor costs were the only element of direct costs used in the regression analysis. Also, due to the high degree of collinearity in the series, only values for periods $t-1$ and t were used.

Discussion of Results

Regressions were performed using simple least squares on both seasonally adjusted

[9] Not all the data are published by Commerce. I am grateful to David Belsley for making available to me the series which he obtained under the stipulation that only the results derived from them and not the series themselves be published. For a detailed description of the Commerce data, see Belsley's doctoral dissertation.

[10] Since the theory is in terms of physical quantities, the shipments, inventories and unfilled orders variables which are reported by Census in value terms were deflated by the price variable for the corresponding month. This procedure introduces a slight degree of error if businesses do not value finished goods inventory at finished goods prices.

TABLE 3—REGRESSION COEFFICIENTS
(t-values in parentheses)

	α	X_{t-1}	P_{t-1}	H_{t-1}	U_{t-1}	Q_t	Q_{t+1}	Q_{t+2}	V_{t-1}[a]	V_t	$\bar{R}^2$*	D.W.[b]
LUMBER												
X_t	−3.620	.213	−.004	−.106	.157	.568	.074	.008	−.006	.015	.904	2.03
		(3.99)	(0.34)	(1.59)	(3.18)	(9.47)	(1.24)	(0.16)	(0.47)	(1.04)		
P_t	5.584	−.261	.914	−.318	.063	.130	.172	.366	−.012	.002	.947	1.42
		(2.73)	(39.26)	(2.65)	(0.72)	(1.21)	(1.62)	(4.13)	(0.47)	(0.07)		
H_t	−0.890	−.008	.012	.914	−.017	−.074	.042	.017	.014	−.011	.902	1.62
		(0.26)	(1.63)	(23.98)	(0.59)	(2.16)	(1.25)	(0.59)	(1.74)	(1.34)		
PAPER												
X_t	−1.490	.023	−.043	−.300	.265	.854	−.060	−.056	.107	−.085	.979	1.98
		(0.75)	(4.34)	(3.16)	(6.40)	(26.53)	(1.96)	(2.03)	(5.56)	(4.10)		
P_t	−0.040	−.019	.970	−.554	.123	.086	−.026	.049	−.031	.046	.981	1.80
		(0.31)	(49.20)	(2.90)	(1.48)	(1.32)	(0.42)	(0.89)	(0.80)	(1.12)		
H_t	−0.250	.024	.004	1.003	−.056	.005	.041	.006	.031	−.035	.982	1.59
		(1.56)	(0.80)	(20.67)	(2.66)	(0.32)	(2.61)	(0.39)	(3.11)	(3.31)		

[a] Average hourly wages in cents.
[b] These are presented without inference as to their significance, given the presence of a lagged dependent variable in the equations. In addition, David Grether has pointed out that the relevant serial correlation may be other than first order for monthly data, e.g., $E(\epsilon_t, \epsilon_{t+12}) \neq 0$.
* 150 degrees of freedom.

and nonadjusted data. The latter is reported below although there was very little difference between the two. The results, which are presented in Table 3, are generally consistent with the expectations generated by the sample calculations. Indeed, none of the 20 coefficients which were predicted unambiguously by the sample calculations showed the wrong sign for both industries. Overall, out of 40 such coefficients, 34 were predicted correctly. Even more revealing, out of 18 coefficients which came out with coefficients more than twice their standard error, only 1 was incorrect. The same regressions without an intercept term were performed on first differences. Although the $\bar{R}^2$'s dropped considerably, the matching of coefficients with expectations was not much changed.

When the results are broken down by equation they become somewhat more meaningful. The production equation performed quite well, showing the predicted sign for 14 of 18 coefficients and for 9 of the 10 coefficients which turned out significant. The coefficient of X_{t-1} is somewhat smaller than had been predicted, especially for the paper industry. This indicates that production may be somewhat more flexible than had been assumed. If c_3 were made smaller, the effect would be to lower the coefficient of X_{t-1} and shift more of the weight to U_{t-1} and Q_t. It is rather surprising that in the regressions on first differences, the coefficient of X_{t-1} comes out significantly negative for both industries. This result could not be reproduced in the simple calculations by any reasonable set of cost parameters and one is tempted to attribute it to some statistical problems (e.g., estimating first differences without an intercept).

The price equation is also strong with 17 of 18 signs predicted correctly, and of the

6 coefficients which turned out significant, all had the correct sign. The price equation is rather heavily dominated by the lagged price term, and this may reflect the tendency of the *BLS* index to pick up list prices which do not necessarily reflect the terms at which transactions actually take place. To the extent to which this is a quantitatively significant phenomenon we would not expect other variables to show up significantly since in the extreme $P_t = P_{t-1}$ for long periods, even though actual transactions prices are changing in response to market conditions.

The coefficient of H_{t-1} in the price equation is considerably larger than predicted, and this result holds up in the first difference regressions. No set of cost parameters could be found which would significantly improve the predictions of the simple calculations for this coefficient without throwing off some of the others; for example, increasing b makes the coefficient of H_{t-1} larger (in absolute value) but moves the coefficient of X_{t-1} in the wrong direction. This suggests that there may be some specification error involved here.

There is little that can be said about the inventory equation since the predictions for most signs were ambiguous. If we judge by the results of the initial sample calculation, 14 of 18 coefficients were predicted correctly, including all 7 which came out significant. In one sense the results can be considered favorable since the model predicted that most coefficients in the inventory equation would have values very close to zero and this was borne out. The extremely high coefficient of H_{t-1} was unexpected, but since this result does not hold up for the first difference equations, there is little value in trying to come up with an economic interpretation. Since the Durbin-Watson statistics suggest serial correlation of the residuals in the inventory equation, the value of H_{t-1} is subject to bias, and this might explain the result.

TABLE 4—THE IMPACT OF A ONE UNIT INCREASE IN DEMAND

	Lumber	Paper	Predicted (TABLE 1)
Increase in production	.568	.854	.492
Rise in price	.006	.004	.008
Drawing down of inventories	.074	−.005	.001
Buildup of unfilled orders	.352	.147	.499

Impact of Changes in Demand

It is interesting to ask how the system reacts to changes in demand; specifically, how the impact is spread over the different variables the firm controls—production, price, inventory, unfilled orders, and shipments. The answer is dependent on the coefficient of Q_t in the various equations together with the identity constraints involving those variables. The information is summarized in Table 4 above. Each number indicates what portion of a sudden and temporary one unit increase in demand, Q, would be borne by the variable indicated in the left-hand column.[16] (The amount absorbed by a price increase is found by multiplying the coefficient of Q_t in the price equation by .05, the assumed slope of the demand curve.)

It is perhaps surprising that price change plays such a small role in absorbing increases in demand although that is precisely what is predicted by the model.[17] To some extent this is attributable to the previously mentioned failure of the *BLS* index to reflect all price changes, but it also serves to point out a possible deficiency in

[16] If the increase in demand is foreseen three months in advance, the adjustment process is only slightly different.

[17] Table 4 understates the total contribution of price change even where the rise in demand is unexpected since price continues above its long-run equilibrium level for several periods. The full effect of price increases is to absorb 11, 4, and 7 percent of the demand increase in lumber, paper, and the sample experiment, respectively.

the model. The assumption that all cost parameters remain constant throughout the cycle is extremely convenient mathematically, but almost certainly misrepresents the facts to some degree. For example, as capacity is approached it must be true that c_3 (the cost of increasing production) rises sharply, and it is at this point that price increases are most likely to play an important role. If the model could be amended to include factors such as capacity utilization, the true role of price changes would probably show up more emphatically.

IV. *Summary and Conclusions*

In this paper we have attempted to develop a model which explains decisions on production, price, and finished goods inventory for manufacturing firms. The keynote of the analysis was that decisions on these variables should be treated as simultaneous and interdependent, emphasizing that price should not be considered as given, but rather as one of the variables to be determined within the model.

To what extent can we consider the model a "success?" It is difficult to attach a score to the overall contribution of the analysis but some observations are possible. First, we have demonstrated that it is technically possible to treat the decision variables as being determined simultaneously, and that given certain assumptions with respect to the demand curve, there is no reason why the price decision cannot be included. Second, the model has the desirable property that it is capable of generating empirically testable (and therefore refutable) hypotheses under most conditions. This may explain why as yet there has been relatively little enthusiasm for many of the nonmaximization theories which, although perhaps appealing to our intuitive feelings about the workings of the business world, have not, for the most part, yielded substantive implications with regard to observable phenomena. Finally, the model seems to hold up reasonably well when applied to selected 2-digit industry groups, despite obvious problems with the data. In particular we noted the imperfect matching of the *BLS* price series to the *SIC*-based production series, and the possibility that even the "correct" price list might not always reflect the actual terms of a transaction.

The final point to be made regards the direction of future research in this area. Certainly there is still much work to be done within the framework of the current model. Some different specifications might be tried, for example making H_t^* depend on new orders as well as shipments and similarly for U_t^*. Alternative lag structures could be tried for the direct cost variables and the treatment of expectations about demand might be improved. We might even want to get behind the production variable by actually including a production function in the model. This might enable us to derive decision rules for the size of the work force, amount of raw materials, and inventories at various stages of fabrication. Finally, we might expand the model to include some of the less easily quantified variables such as advertising and other aspects of nonprice competition.

In many ways, however, the most critical need is for better data. The individual problems have been discussed above and it is not necessary to repeat them here. However the point must be stressed that it is only with data which are accurate, consistent, and appropriate to the level of aggregation of the model that any theory in this area can be given a fair and rigorous test.

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