Progress of Theoretical Physics, Vol. 49, No. 5, May 1973

Productions and Decays of Short-Lived Particles in e^+-e^- Colliding Beam Experiments

Shoichiro KAWASAKI, Tadahiko SHIRAFUJI and S. Y. TSAI*

Department of Physics, Chiba University, Chiba
**Department of Physics, College of Science and Engineering Nihon University, Tokyo*

(Received December 7, 1972)

We investigate productions and decays of short-lived particles (e.g., heavy leptons, charmed hadrons and so on) in e^+-e^- colliding beam experiments. Formulae which are useful for computing the cross section for the production and subsequent decay of a single or a pair of narrow-width particles with spin $\frac{1}{2}$ are derived. Approximate but analytical expressions for the combined angular distribution of the decay products from such particles are derived. The production and decay of a single excited muon is also treated.

§ 1. Introduction

Searches for new types of particles always constitute an interesting part of experimental proposals to be performed with high energy machines. Although there is no evidence, up to date, in favor of the existence of any type of such particles in accelerator experiments, there are actually so many theoretical schemes which postulate or predict their existence, and, furthermore, there have been so many experimental reports which mentioned about various anomalies observed in cosmic ray experiments.[1)]

The term "new types of particles" is here used in contrast to the ordinary leptons (e, ν_e, μ, ν_μ and their antiparticles), ordinary photon and ordinary hadrons (including resonances), and, as examples, one may list the following: W-bosons, heavy leptons, ghost photons, various types of fundamental particles (quarks or urbaryons), various types of charmed hadrons and so on.

Some of these particles (e.g., heavy leptons and charmed hadrons) are probably characterized, not only by their large masses, but also by their narrow-widths; the corresponding mean lifetimes are however too short for them artificially produced to leave any trace in detectors. In order to find them, one should be able to reach their production threshold in energy, and then investigate their decay products carefully.*)

An obvious way to search for new types of particles (hereafter referred to as X-particles) is to use electron-positron colliding beams, and the purpose of the present paper is just to discuss the productions and subsequent decays of

*) We do not discuss indirect tests for detection of such particles.

short-lived X-particles in the colliding beam experiments: $e^+ + e^- \to X + \cdots$ and $e^+ + e^- \to X + \bar{X}$.

The paper is arranged as follows. In § 2, as possible candidates for X, we look in some detail at various types of heavy leptons and charmed hadrons. In § 3, we derive the formulae which are useful for computing the cross section for the production and subsequent decay of a single X. The formulae are then applied in § 4 to $e^+ + e^- \to X + \bar{X}$ and subsequent decays of the produced X and $\bar{X}$. In particular, we are interested in the angular correlation between one particular decay product from X and another one from $\bar{X}$ in the overall center-of-mass system. The individual examples are dealt with in § 5, where the variables with respect to X and $\bar{X}$ are explicitly integrated out. In § 6, we discuss the production and decay of a single excited muon: $e^+ + e^- \to \mu^\pm + \mu^{*\mp} \to \mu^\pm + (\mu^\mp + \gamma)$. The final section is devoted to summary and concluding remarks.

Although §§ 3 and 4, and the associated Appendix, may contain things which are familiarly known, we think it appropriate, in order to be reasonably self-contained, to include them in the present work.

§ 2. Heavy leptons and charmed hadrons

——*Examples of short-lived new particles*——

A. *Heavy Leptons*

Theoretical explanation of the mass spectrum of leptons is still lacking, and, in particular, there is no convincing answer to the question, "Why are there muons?"[2] Thus, recently many people inquire, "Are there leptons other than those already known?" and point out that there is actually no experimental evidence for or against the existence of leptons heavier than the kaon.[3]

Concerning the properties of charged heavy leptons, it is important to distinguish the following two cases:*)

i) Charged heavy leptons ($h^\pm$) appear together with their own neutral counterparts (ν_h and $\bar{\nu}_h$), and the pairs (e, ν_e), (μ, ν_μ) and (h, ν_h) participate symmetrically in the electromagnetic and weak interactions:

$$l_\lambda^{\text{e.m.}} = i(\bar{e}\gamma_\lambda e + \bar{\mu}\gamma_\lambda\mu + \bar{h}\gamma_\lambda h), \tag{2·1}$$

$$l_\lambda^{\text{weak}} = \bar{\nu}_e i\gamma_\lambda(1+\gamma_5)e + \bar{\nu}_\mu i\gamma_\lambda(1+\gamma_5)\mu + \bar{\nu}_h i\gamma_\lambda(1+\gamma_5)h\,, \tag{2·2}$$

$l_\lambda^{\text{e.m.}}$ and l_λ^{weak} being the usual electromagnetic and weak leptonic currents, respectively.

An e^+-e^- colliding beam experiment to search for this type of heavy leptons was carried out at Frascati, with negative results.[5] The lifetime and partial decay rates to various modes were calculated by several authors.[6),7] According

*) There could be, of course, many varieties of schemes which lie between or beyond these two cases. See with this respect Ref. 4).

to Ref. 7), $\tau_h=2.2\times10^{-12}$ sec for $M_h=1.2$ GeV and $\tau_h=1.2\times10^{-15}$ sec for $M_h=6$ GeV.*) The main decay modes are

$$h^{\mp}\to\pi^{\mp}+\nu_h\,, \tag{2·3a}$$

$$\quad e^{\mp}+\nu_e+\nu_h\,, \tag{2·3b}$$

$$\quad \mu^{\mp}+\nu_\mu+\nu_h\,. \tag{2·3c}$$

So, in e^+-e^- experiments, one looks at $e^++e^-\to2$ charged+neutrals. If one denotes the detected (charged) decay product by x, its four-momentum by $q_\mu=(\boldsymbol{q},\omega)$, the angular distribution of x from an arbitrary polarized $X=h^\pm$ may be written in the rest frame of X as

$$\frac{d\Gamma^{(s)}}{\Gamma}(X\to x+\cdots)=\frac{1}{4\pi}(1+\alpha\hat{q}^*\cdot\boldsymbol{s}^*)\,d\Omega_x{}^*\,, \tag{2·4}$$

where α is the asymmetry parameter, $\boldsymbol{s}^*$ is the polarization vector of X, $d\Omega_x{}^*$ is the solid angle element of x and $\hat{q}^*=\boldsymbol{q}^*/|\boldsymbol{q}^*|$. Here and in the following, we use an asterisk to denote a quantity viewed in the rest frame of the parent particle (X). The values of α for each process are readily calculated from Eq. (2·2) and the standard current×current weak interaction Lagrangian, and turn out to be[7),17)]

$$\alpha=\mp\tfrac{1}{3}\quad\text{for } h^\mp\to e^\mp\nu_e\nu_h,\ \mu^\mp\nu_\mu\nu_h\,,$$

$$\alpha=\pm1\quad\text{for } h^\mp\to\pi^\mp\nu_h\,.$$

The decay correlations in the reaction $e^++e^-\to h^++h^-$ have already been discussed by Y. S. Tsai,[7)] and actually the contents of our §§ 4 and 5A overlap partly with his work.

ii) Charged heavy leptons appear as excited electrons ($e^{*\pm}$) and/or excited muons ($\mu^{*\pm}$), and electromagnetic transitions of $e^*(\mu^*)$ into $e+\gamma(\mu+\gamma)$ are allowed:[8)]

$$H_{\rm int}=\frac{fe}{M+m_e}\bar{e}^*\sigma_{\lambda\rho}e\partial_\lambda A_\rho+(e\to\mu)+\text{H.c.} \tag{2·5}$$

the coupling constant f being expected to be of order unity.

The early searches for this type of heavy leptons (excited leptons) all give negative results,**) and are summarized in Ref. 9).

---

*) For $h^\pm$ of a higher mass, the total width may be inferred from

$$\begin{aligned}\Gamma&=\Gamma(e\bar{\nu}_e\nu_h)+\Gamma(\mu\bar{\nu}_\mu\nu_h)+\Gamma(\nu_h+\text{hadrons})\\&=(2+a)\times\Gamma(e\bar{\nu}_e\nu_h)\\&=(2+a)\times G^2M_h{}^5/192\pi^3\end{aligned}$$

with $a\equiv\Gamma(\nu_h+\text{hadrons})/\Gamma(e\bar{\nu}_e\nu_h)$ expected to be of order unity. Thus, for example, $\tau_h\approx1.3\times10^{-17}$ sec for $M_h=15$ GeV. The dominant decay modes are (2·3b) and (2.3c).

**) Recall however the anomalies observed by Ramm in the early CERN high energy neutrino experiments.[10)]

From Eq. (2·5), the decay rate for $l^* \to l+\gamma$, $l=e$ or μ, turns out to be

$$\Gamma = \tfrac{1}{2} f^2 \alpha M . \tag{2·6}$$

The angular distribution of l or γ is again given by Eq. (2·4) with $\alpha=0$. Absence of the term $\hat{q}^* \cdot s^*$ is a consequence of parity conservation of the interaction (2·5). Since, as long as f is of order unity, the decay mode $l^* \to l+\gamma$ will overwhelm the other (weak), if any, decay modes, one finds $\tau_{l^*} \approx 0.6 \times 10^{-22}$ sec for $M=3$ GeV and $\tau_{l^*} \approx 1.2 \times 10^{-23}$ sec for $M=15$ GeV from Eq. (2·6).

B. *Charmed hadrons*

The term "charm" is used in variety in different models. We imagine that, besides charge, baryon number and strangeness, there may be a fourth quantum number which is conserved through strong interactions, and call those hadrons which possess this quantum number "charmed hadrons". Such quantum number finds its place in the quartet model, two-triplet model and three-triplet model, etc. If "charm" is not an absolutely conserved quantity, but is violated, say, in the weak interaction, at least the lightest charmed hadron will be metastable, and decay only weakly into ordinary hadrons.

Recently, Niu and his coworkers[11] discovered in cosmic ray showers an interesting event which contains a charged heavy particle X decaying with lifetime $10^{-13} \sim 10^{-14}$ sec into $x+\pi^0$, where x is a charged secondary not identified with an electron. From the fact that the transverse momentum of the π^0 is larger than 500 MeV/c, Hayashi et al.[12] argued that the event observed cannot be put into strangeness scheme and that the X may be a particle that possesses a hitherto undiscovered quantum number. They then interpreted the event as a charm-violating weak decay process in the framework of a quartet model (the New Nagoya model). Later, one of the present authors (S.Y.T.)[13] investigated charm-violating weak interactions in the framework of the $SU(3)' \times SU(3)''$ scheme of the three-triplet model,[14] and showed that this scheme does provide a reasonable framework for describing such a new type of process as just mentioned.

If the mean lifetime of a charmed particle is indeed $\sim 10^{-13}$ sec or shorter,*) its direct verification in detectors in accelerator energy regions is difficult, and what one has to do is, again, to investigate its decay products carefully.

The angular distribution of a decay product from an arbitrary polarized charmed baryon can be written in the form of Eq. (2·4), and its production and subsequent decay can be treated in a way similar to the case of heavy leptons. One cannot however make an unambiguous prediction, because the asymmetry

---

*) As has been shown,[12] if the mass is sufficiently large, it is not unreasonable that the lifetime of a weakly decaying particle can be as short as $10^{-13} \sim 10^{-14}$ sec. According to the data of Niu et al.,[11]

$$M_X = 1.78\,\text{GeV}, \quad \tau_X = 2.2 \times 10^{-14}\,\text{sec}, \quad \text{if } x=\pi^\pm;$$
$$M_X = 2.95\,\text{GeV}, \quad \tau_X = 3.6 \times 10^{-14}\,\text{sec}, \quad \text{if } x=p.$$

parameter is uncalculable in the present case and, moreover, the form factor effects will complicate the problem. We shall therefore confine ourselves to the simplest case, i.e. the production and decay of spinless mesons. Since dominant decay modes of charmed mesons are expected to be nonleptonic, their pair-productions in e^+-e^- experiments would look like multi-hadron productions. This fact may be of some interest in view of the large cross section for $e^+ + e^- \to$hadrons reported from Frascati.*)[,15)]

§ 3. Production and subsequent decay of a short-lived particle

Let us consider a sequence of processes in which a short-lived particle X is produced, together with a system of particles m, in a collision between two particles 1 and 2, and then decays into a specific decay mode n (see Fig. 1):

$$1+2 \to X+m\,, \tag{3·1a}$$

$$X \to n\,. \tag{3·1b}$$

The differential cross section for the production process (3·1a) and the differential decay rate for the decay process (3·1b) in the rest frame of X are given respectively by[17)]

$$d\sigma_X^{(s)} = \frac{1}{4F} |\langle X(s),\ m|T|1, 2\rangle|^2 (2\pi)^4 \delta^4(p_1+p_2-p-p_m) \times \frac{1}{(2\pi)^3} \frac{d^3\mathbf{p}}{2E} dLips(m)\,, \tag{3·2a}$$

$$d\Gamma_n^{(s)} = \frac{1}{2M} |\langle n|T|X(s)\rangle|^2 (2\pi)^4 \delta^4(p-p_n)\, dLips(n)\,, \tag{3·2b}$$

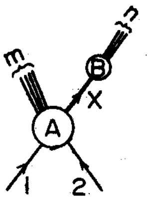


Fig. 1. Production of a short-lived particle X and its subsequent decay into a specific mode in a collision between particles 1 and 2.

---

*) The form factor effects could reduce or enhance the cross section in question. Consider, for example, the $\pi'^+\pi'^-$ production in the framework of the $SU(3)' \times SU(3)''$ scheme of the three-triplet model.[13),14)] In the context of the vector meson dominance model, the form factor at the $\gamma\pi'^+\pi'^-$ vertex is given by $m_{\rho'}^2/(k^2+m_{\rho'}^2)$, k being the four-momentum of the photon. Thus, if $m_{\rho'} > 2m_{\pi'}$, the form factor would enhance the cross section for $e^+ + e^- \to \pi'^+ + \pi'^-$ when $-k^2 = 4E_0^2 \approx m_{\rho'}^2$ (E_0= the beam energy). Here π' and ρ' are members of the $J^P=0^-$ and 1^- mesons belonging to the representation (1, 8) of $SU(3)' \times SU(3)''$.

In passing, we also note that the interpretation of the Niu event as a charm-violating weak decay of a charmed vector meson is indeed an attractive possibility. For details, see Ref. 16).

while the differential cross section for the combined process, (3·1a) followed by (3·1b), is given by

$$d\sigma_{X\to n'}=\frac{1}{4F}|\langle m,\, n|T|1,\,2\rangle|^2(2\pi)^4\delta^4(p_1+p_2-p_m-p_n)$$
$$\times dLips(m)\,dLips(n)\,. \tag{3·3}$$

In the above equations, M, $p=(\boldsymbol{p},\,E)$ and s specify the mass, four-momentum and polarization of X,*) p_1 and p_2 are the four-momenta of the initial particles, F is Møller's invariant flux factor, $F^2=(p_1\cdot p_2)^2-p_1^2p_2^2$, and, for $|m\rangle=|p_a,\,p_b,\cdots p_k\rangle$, $p_a=(\boldsymbol{p}_a,\,E_a)$

$$p_m=p_a+p_b+\cdots+p_k\,,$$
$$dLips(m)=\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_a}{2E_a}\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_b}{2E_b}\cdots\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_k}{2E_k}$$

the latter being the Lorentz invariant phase space element for k particles. It is understood that, if the initial particles are unpolarized and no polarization measurements are undertaken for some or all of particles in m and n, an appropriate "average and sum" rule should be applied to Eqs. (3·2a), (3·2b) and (3·3).

We want to establish an explicit relationship between Eqs. (3·2a) and (3·2b) on the one hand and Eq. (3·3) on the other hand, under the following circumstances: X is a spin 0 or $\frac{1}{2}$ particle with its total width Γ much smaller than its mass M, $\Gamma\ll M$.

First consider the case: X is a spin $\frac{1}{2}$ particle. The matrix elements $\langle X(s),\, m|T|1,\,2\rangle$ and $\langle n|T|X(s)\rangle$ may be written as

$$\langle X(s),\, m|T|1,\,2\rangle=\sqrt{2M}\,\bar{u}_\alpha(p,\,s)\,A_\alpha\,, \tag{3·4a}$$
$$\langle n|T|X(s)\rangle=\sqrt{2M}\,\bar{B}_\alpha u_\alpha(p,\,s)\,, \tag{3·4b}$$

where $u_\alpha(p,\,s)$ is the Dirac spinor describing X, normalized as $\bar{u}u=1\,(\bar{u}=u^\dagger\gamma_4)$. We have then

$$|\langle X(s),\; m|T|1,\,2\rangle|^2=2M\bar{A}_\alpha\left(\Lambda_+(p)\,\frac{1+i\gamma_5\gamma_\mu s_\mu}{2}\right)_{\alpha\beta}A_\beta \tag{3·5a}$$
$$=2M\left(\left(\bar{A}\Lambda_+(p)\frac{1+i\gamma_5\gamma_\mu s_\mu}{2}A\right)\right),$$

$$|\langle n|T|X(s)\rangle|^2=2M\left(\left(\bar{B}\Lambda_+(p)\frac{1+i\gamma_5\gamma_\mu s_\mu}{2}B\right)\right). \tag{3·5b}$$

Here $\Lambda_+(p)$ is the projection operator for the positive energy state

$$\Lambda_+(p)=\frac{-i\gamma_\mu p_\mu+M}{2M}\,,$$

*) For a spinless X, s is, of course, irrelevant.

s_μ is the covariant polarization vector for X satisfying*),[17]

$$p\cdot s=0, \quad s^2=1\,,$$

and (( )) symbolizes a complete contraction over spinor indexes. In terms of the spinors A and B just defined, the matrix element $\langle m, n|T|1, 2\rangle$ is given by

$$\langle m, n|T|1, 2\rangle=\left(\left(\bar{B}\frac{-i\gamma_\mu p_\mu+M}{p^2+M^2-iM\Gamma}A\right)\right). \tag{3·6}$$

In squaring this equation, one may make a narrow width approximation (recall that $\Gamma\ll M$):

$$\left|\frac{1}{p^2+M^2-iM\Gamma}\right|^2\to\frac{\pi}{M\Gamma}\delta(p^2+M^2), \tag{3·7}$$

and, furthermore, make use of the identity**)

$$\begin{aligned}2((\bar{A}\Lambda_+(p)B))((\bar{B}\Lambda_+(p)A))&=((\bar{A}\Lambda_+(p)A))((\bar{B}\Lambda_+(p)B))\\&+\eta_{\mu\nu}((\bar{A}\Lambda_+(p)i\gamma_5\gamma_\mu A))((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\nu B)),\end{aligned} \tag{3·8}$$

which is valid whenever X is on its mass-shell ($p^2+M^2=0$), to obtain

$$\begin{aligned}|\langle m, n|T|1, 2\rangle|^2=&\frac{1}{2}(2M)^2\left(\frac{\pi}{M\Gamma}\right)\delta(p^2+M^2)\\&\times[((\bar{A}\Lambda_+(p)A))((\bar{B}\Lambda_+(p)B))+\eta_{\mu\nu}((\bar{A}\Lambda_+(p)i\gamma_5\gamma_\mu A))\\&\times((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\nu B))]\end{aligned} \tag{3·9}$$

or equivalently

$$\begin{aligned}|\langle m, n|T|1, 2\rangle|^2=&(2M)^2\left(\frac{\pi}{M\Gamma}\right)\delta(p^2+M^2)((\bar{B}\Lambda_+(p)B))\\&\times\left(\left(\bar{A}\Lambda_+(p)\frac{1+i\gamma_5\gamma_\mu n_\mu}{2}A\right)\right).\end{aligned} \tag{3·10}$$

Here we have introduced a polarization-like vector n_μ:

$$n_\mu=\eta_{\mu\nu}((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\nu B))/((\bar{B}\Lambda_+(p)B)), \quad \eta_{\mu\nu}\equiv\delta_{\mu\nu}+p_\mu p_\nu/M^2 \tag{3·11}$$

which satisfies

$$p\cdot n=0, \quad n^2=1\,.$$

*) s_μ reduces to $\mathbf{s}^*$ in the rest frame of X:

$$\mathbf{s}=\mathbf{s}^*+\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{s}^*)\boldsymbol{\beta},$$

$$s_0=\gamma\boldsymbol{\beta}\cdot\mathbf{s}^*,$$

where

$$\gamma=E/M, \quad \boldsymbol{\beta}=\mathbf{p}/E\,.$$

**) We shall give a proof in the Appendix.

Comparing Eq. (3·10) with Eqs. (3·5a) and (3·5b), and then substituting it into Eq. (3·3), one finally obtains

$$d\sigma_{X\to n}=\frac{1}{4F}\left(\frac{\pi}{M\Gamma}\right)\delta(p^2+M^2)|\langle X(n),\ m|T|1,2\rangle|^2\sum_s|\langle n|T|X(s)\rangle|^2$$
$$\times(2\pi)^4\delta^4(p_1+p_2-p_m-p_n)\,dLips(m)\,dLips(n) \tag{3·12}$$
$$=2\times\frac{1}{4F}|\langle X(n),m|T|1,2\rangle|^2(2\pi)^4\delta^4(p_1+p_2-p-p_m)$$
$$\times\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}}{2E}dLips(m)\frac{1}{\Gamma}\times\frac{1}{2M}\times\frac{1}{2}\sum_s|\langle n|T|X(s)\rangle|^2$$
$$\times(2\pi)^4\delta^4(p-p_n)\,dLips(n) \tag{3·13}$$

or symbolically

$$d\sigma_{X\to n}=2d\sigma_X^{(n)}\times\frac{d\Gamma_n}{\Gamma}, \tag{3·14}$$

where $d\Gamma_n\equiv\frac{1}{2}\sum_s d\Gamma_n^{(s)}$ is the differential decay rate for $X\to n$ from an unpolarized X. The formulae derived above provide convenient tools for computing the cross section for the production and subsequent decay of a narrow-width spin $\frac{1}{2}$ particle.

Remarks

1) When summing over all spin states and integrating over all phase space with respect to $|n\rangle$, one would have

$$\sum\int n_\mu d\Gamma_n=0$$

which follows from a simple invariance argument, whence

$$\sum\int d\sigma_{X\to n}=d\sigma_X\times R_n,$$

as it should. Here $d\sigma_X\equiv\sum_s d\sigma_X^{(s)}$ and

$$R_n=\frac{1}{\Gamma}\sum\int d\Gamma_n$$

is the branching ratio for $X\to n$.

2) In some cases, it may happen that, when summing over certain spin states in $|m\rangle$ and $|n\rangle$,

$$\sum((\bar{A}\Lambda_+(p)i\gamma_5\gamma_\mu A))=0$$

or

$$\sum((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\mu B))=0.$$

In such cases, the spin dependence of the intermediary stage becomes irrelevant, and we have

$$d\sigma_{X\to n}=d\sigma_X\times\frac{d\Gamma_n}{\Gamma}. \tag{3·15}$$

3) In the usual spin-density matrix formalism,[18] the square of Eq. (3·6) is written as

$$|\langle m,n|T|1,2\rangle|^2=\left|\frac{1}{p^2+M^2-iM\Gamma}\right|^2(2M)^2((\bar{A}\Lambda_+(p)A))((\bar{B}\rho B)),$$

where $\rho_{\alpha\beta}$ is the spin-density matrix for the production process (3·1a):

$$\rho_{\alpha\beta}=\frac{(\Lambda_+(p)A\bar{A}\Lambda_+(p))_{\alpha\beta}}{((\bar{A}\Lambda_+(p)A))}.$$

The identity (3·8) implies that, when $p^2+M^2=0$,

$$((B\rho B))=\left(\left(\bar{B}\Lambda_+(p)\frac{1+i\gamma_5\gamma_\mu m_\mu}{2}B\right)\right),$$

$$m_\mu\equiv\eta_{\mu\nu}\frac{((\bar{A}\Lambda_+(p)i\gamma_5\gamma_\nu A))}{((\bar{A}\Lambda_+(p)A))},$$

and hence one has

$$d\sigma_{X\to n}=d\sigma_X\times\frac{d\Gamma_n{}^{(m)}}{\Gamma}. \tag{3·16}$$

4) One may write Eqs. (3·2a) and (3·2b) as

$$d\sigma_X{}^{(s)}=A_1+A_{2\mu}s_\mu=A_1+\boldsymbol{A}_2{}^*\cdot\boldsymbol{s}^*, \tag{3·17a}$$

$$\frac{d\Gamma_n{}^{(s)}}{\Gamma_n}=B_1+B_{2\mu}s_\mu=B_1+\boldsymbol{B}_2{}^*\cdot\boldsymbol{s}^*. \tag{3·17b}$$

One has then

$$n_\mu=\eta_{\mu\nu}\frac{B_{2\nu}}{B_1},\quad \frac{d\Gamma_n}{\Gamma}=B_1\times R_n. \tag{3·18}$$

Substitution of Eqs. (3·17a) and (3·18) into Eq. (3·14) yields

$$\begin{aligned}d\sigma_{X\to n}&=2(A_1B_1+\eta_{\mu\nu}A_{2\mu}B_{2\nu})\times R_n\\&=2(A_1B_1+\boldsymbol{A}_2{}^*\cdot\boldsymbol{B}_2{}^*)\times R_n.\end{aligned} \tag{3·19}$$

It is to be remarked that this form of the expression for $d\sigma_{X\to n}$ may be derived from a more intuitive argument,[7] and that this as well as the other two expressions for $d\sigma_{X\to n}$ (Eqs. (3·14) and (3·16)) are valid only when the narrow-width approximation (3·7) is valid. We also note that four-vector $V_\mu{}^*$ (defined in the rest frame of X) is related to the same four-vector V_μ (defined in a moving frame of X) by

$$V^* = V + \left(\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\cdot V - V_0\right)\gamma\boldsymbol{\beta}\,,$$

$$V_0^* = \gamma(V_0 - \boldsymbol{\beta}\cdot V)\,, \tag{3·20}$$

where $\gamma \equiv E/M$ and $\boldsymbol{\beta} = \mathbf{p}/E$ are, respectively, the Lorentz factor and the velocity of X.

When X is a spinless particle, it is trivial to derive

$$d\sigma_{X\to n} = \frac{1}{4F}\frac{\pi}{M\Gamma}\delta(p^2+M^2)\,|\langle X, m|T|1, 2\rangle|^2|\langle n|T|X\rangle|^2$$

$$\times(2\pi)^4\delta^4(p_1+p_2-p_m-p_n)\,dLips(m)\,dLips(n)\,, \tag{3·21}$$

$$= \frac{1}{4F}|\langle X, m|T|1, 2\rangle|^2(2\pi)^4\delta^4(p_1+p_2-p-p_m)\frac{1}{(2\pi)^3}\frac{d^3\mathbf{p}}{2E}dLips(m)$$

$$\times\frac{1}{\Gamma}\frac{1}{2M}|\langle n|T|X\rangle|^2(2\pi)^4\delta^4(p-p_n)\,dLips(n)\,, \tag{3·22}$$

$$= d\sigma_X \times \frac{d\Gamma_n}{\Gamma}\,. \tag{3·23}$$

Remarks

In Eq. (3·13) as well as in Eq. (3·22), one may evaluate a part of integrations over $dLips(m)$ and $dLips(n)$ respectively with the help of $\delta^4(p_1+p_2-p-p_m)$ and $\delta^4(p-p_n)$, and then integrate over $d^3\mathbf{p}$. The last integration can numerically be done in general. If, however, one is interested in the production of X near its threshold, it may be more appropriate to do it approximately but analytically, as will be indicated in § 5.

§ 4. Production and subsequent decays of a pair of short-lived particles in e^--e^+ experiments

A. *General formulae*

Now consider the following processes (Fig. 2):

$$e^- + e^+ \to X + X'\,, \tag{4·1a}$$

$$X \to n\,, \tag{4·1b}$$

$$X' \to m'\,, \tag{4·1c}$$

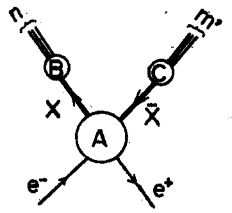


Fig. 2. Production and subsequent decay of a X-$\bar{X}$ pair in an e^--e^+ collision.

$X' \equiv \bar{X}$ being the anti-particle of X. Again, first consider the case: X is a spin 1/2 particle. We denote the four-momenta of e^-, e^+, X and X' by p_1, p_2, p and p', the covariant polarization vectors of X and X' by s and s', and write the matrix elements for (4·1b) and (4·1c) as

$$\langle n|T|X(s)\rangle = \sqrt{2M}\bar{B}_\alpha u_\alpha(p, s), \tag{4·2a}$$

$$\langle m|T|X'(s')\rangle = \sqrt{2M}\bar{v}_\alpha(p', s')C_\alpha. \tag{4·2b}$$

Furthermore, let $d\sigma_{XX'}^{(s,s')}$ ($d\Gamma_n^{(s)}$ and $d\Gamma_{m'}^{(s')}$) be the differential cross section (decay rates) for (4·1a) ((4·1b) and (4·1c)). A repeated application of the formulae derived in § 3 then yields, for the cross section for the combined processes,

$$d\sigma_{XX'\to nm'} = 4d\sigma_{XX'}^{(n,m')} \times \frac{d\Gamma_n}{\Gamma} \times \frac{d\Gamma_{m'}}{\Gamma}, \tag{4·3}$$

where

$$\Gamma_n = \tfrac{1}{2}\sum_s \Gamma_n^{(s)}, \quad \Gamma_{m'} = \tfrac{1}{2}\sum_{s'} \Gamma_{m'}^{(s')}$$

and $d\sigma_{XX'}^{(n,m')}$ is obtained from $d\sigma_{XX'}^{(s,s')}$ by the following substitutions

$$s_\mu \to n_\mu \equiv \eta_{\mu\nu}\frac{((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\nu B))}{((\bar{B}\Lambda_+(p)B))}, \tag{4·4a}$$

$$s_\mu' \to m_\mu' \equiv \eta_{\mu\nu}'\frac{((\bar{C}\Lambda_-(p')i\gamma_5\gamma_\nu C))}{((\bar{C}\Lambda_-(p')C))}. \tag{4·4b}$$

Here

$$\eta_{\mu\nu} = \delta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}, \quad \eta_{\mu\nu}' = \delta_{\mu\nu} + \frac{p_\mu' p_\nu'}{M^2},$$

and $\Lambda_-(p')$ is the projection operator for the negative energy state

$$\Lambda_-(p') = (-i\gamma_\mu p_\mu' - M)/2M.$$

Remark

Our Eq. (4·3) corresponds to Eq. (4·26) of Ref. 7), the latter being derived from a more intuitive argument. In fact, in Ref. 7), the author calculated the spin correlation function for the reaction $e^- + e^+ \to X + X'$ (X being a charged heavy lepton in his case) in the total center-of-mass system, and the energy-angular distributions of the decay products of X and X' in their respective rest frames, and then folded the two results to obtain the correlation of the decay product of X and X' (see with this respect our Remark 4) in § 3).*) However,

*) Incidentally, we found some errors in Ref. 7). Equation (4·11) of Ref. 7) is actually the expression for $4(d\sigma/d\Omega_l)(\mathbf{s}, \mathbf{s}')$, and likewise Eq. (4·25) should read $4(d\sigma/d\Omega_l) = C + D_{ij}s_i s_j$. The factor 4 here, which is lacking in the original equations, just corresponds to the same factor in our Eq. (4·3). The two errors cancel, however, and Eq. (4·34) to Eq. (4·42) of Ref. 7) are all properly normalized.

in the case of interest to us, the angular distributions of the decay products of X and X' have to be given in the overall center-of-mass system and the variables associated with X and X' have to be integrated out. The necessary transformations are implicit in Eq. (3·20) and in Eq. (4·14) below, while the necessary integrations will be carried out in the individual examples to be discussed later.

When X is a spinless particle, we have simply

$$d\sigma_{XX'\to nm'}=d\sigma_{XX'}\times\frac{d\Gamma_n}{\Gamma}\times\frac{d\Gamma_{m'}}{\Gamma}, \tag{4·5}$$

where $d\sigma_{XX'}$($d\Gamma_n$ and $d\Gamma_{m'}$) is the differential cross section (decay rates) for the process (4·1a) ((4·1b) and (4·1c)).

B. *Cross section for $e^-+e^+\to X+X'$*

The explicit expression for $d\sigma_{XX'}^{(s,s')}$ through one-photon intermediate state can be obtained in the standard way:

$$d\sigma_{XX'}^{(s,s')}=\frac{1}{4F}\frac{1}{4}\sum_{\text{spin of } e^\pm}|\langle X(s)X'(s')|T|e^-e^+\rangle|^2$$

$$\times(2\pi)^4\delta^4(p_1+p_2-p-p')\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}}{2E}\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}'}{2E'},$$

$$\sum|\langle X(s)X'(s')|T|e^-e^+\rangle|^2$$

$$=\frac{e^4}{(p_1+p_2)^4}\mathrm{Tr}[(-i\gamma\cdot p_2-m_e)i\gamma_\mu(-i\gamma\cdot p_1+m_e)i\gamma_\nu]$$

$$\times\mathrm{Tr}\Big[(-i\gamma\cdot p+M)\frac{1+i\gamma_5\gamma\cdot s}{2}\Big(i\gamma_\mu F_1-\frac{1}{2M}F_2 i\sigma_{\mu\mu'}k_{\mu'}\Big)$$

$$\times(-i\gamma\cdot p'-M)\frac{1+i\gamma_5\gamma\cdot s'}{2}\Big(i\gamma_\nu F_1^*+\frac{1}{2M}F_2^* i\sigma_{\nu\nu'}k_{\nu'}\Big)\Big], \tag{4·6}$$

where we have written the matrix element of the electromagnetic current between the vacuum and the X-X' state in the form

$$\langle X(s)X'(s')|J_\mu|0\rangle=e(2M)\bar{u}(p,s)\Big[F_1(k^2)i\gamma_\mu$$

$$-\frac{1}{2M}F_2(k^2)i\sigma_{\mu\mu'}k_{\mu'}\Big]v(p',s'), \tag{4·7}$$

with $k=p+p'=p_1+p_2$, and $F_1(k^2)$ and $F_2(k^2)$ are the electric and magnetic form factors of X, normalized in such a way that $F_1(0)=1$ and $F_2(0)=\mu=$ static

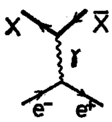


Fig. 3. Production of a X-$\bar{X}$ pair through one-photon intermediate state in an e^--e^+ collision.

anomalous magnetic moment of X, if X is charged; $F_1(0)=0$ and $F_2(0)=\mu$, if it is neutral.

In the center-of-mass system, which is also the laboratory frame in a colliding beam experiment, one finds (the electron mass m_e is neglected)

$$\begin{aligned} d\sigma_{XX'}^{(s,s')} = \frac{\alpha^2}{64E^2}\beta\Big\{ & |F_1+F_2|^2(1+\cos^2\theta_0) + \left|\frac{1}{\gamma}F_1+\gamma F_2\right|^2 \sin^2\theta_0 \\ & + |F_1+F_2|^2[-(s\cdot s')\beta^2\sin^2\theta_0 - 2s_0s_0' \\ & + 2(\hat{p}_1\cdot \boldsymbol{s})(\hat{p}_1\cdot \boldsymbol{s}') - 2\beta\cos\theta_0(s_0\hat{p}_1\cdot \boldsymbol{s}' - s_0'\hat{p}_1\cdot \boldsymbol{s})] \\ & + 2\,\mathrm{Re}[(F_1+F_2)F_2^*][(s\cdot s')\beta^2\sin^2\theta_0 + 2s_0s_0' \\ & + \beta\cos\theta_0(s_0\hat{p}_1\cdot \boldsymbol{s}' - s_0'\hat{p}_1\cdot \boldsymbol{s})] + |F_2|^2\beta^2\gamma^2\sin^2\theta_0[\beta^2(s\cdot s') + 2s_0s_0'] \\ & + \mathrm{Im}[F_1F_2^*]\gamma\beta^2\sin 2\theta_0(\boldsymbol{s}+\boldsymbol{s}')\cdot\frac{\boldsymbol{p}_1\times\boldsymbol{p}}{|\boldsymbol{p}_1\times\boldsymbol{p}|}\Big\} d\Omega_X, \end{aligned} \tag{4·8}$$

where $E(=E'=E_1=E_2)$ is identical to the beam energy, E_0, and

$$\gamma = E/M,\quad \beta = |\boldsymbol{p}|/E,$$
$$\hat{p}_1 = \boldsymbol{p}_1/|\boldsymbol{p}_1| = -\hat{p}_2,$$
$$\cos\theta_0 = \boldsymbol{p}_1\cdot\boldsymbol{p}/|\boldsymbol{p}_1||\boldsymbol{p}|.$$

In Eq. (4·8), the form factors are taken at $k^2=-4E^2$. If X is a point-like charged particle, $F_1(k^2)=1$ and $F_2(k^2)=0$, then Eq. (4·8) reduces to[7]

$$\begin{aligned} d\sigma_{XX'}^{(s,s')} = \frac{\alpha^2}{64E^2}\beta\Big\{ & 1+\cos^2\theta_0 + \left(\frac{1}{\gamma}\right)^2\sin^2\theta_0 \\ & - (s\cdot s')\beta^2\sin^2\theta_0 - 2s_0s_0' + 2(\hat{p}_1\cdot \boldsymbol{s})(\hat{p}_1\cdot \boldsymbol{s}') \\ & - 2\beta\cos\theta_0(s_0\hat{p}_1\cdot \boldsymbol{s}' - s_0'\hat{p}_1\cdot \boldsymbol{s})\Big\} d\Omega_X. \end{aligned} \tag{4·9}$$

Summing over s and s' and integrating over $d\Omega_X$, one obtains the total cross section as[19]

$$\sigma_{XX'} = \frac{\pi\alpha^2}{2E^2}\beta\left(1-\frac{1}{3}\beta^2\right) = \frac{\pi\alpha^2}{2M^2}\beta(1-\beta^2)\left(1-\frac{1}{3}\beta^2\right). \tag{4·10}$$

Similarly, for the production of a pair of spinless charged particles, one gets[19]

$$d\sigma_{XX'} = \frac{\alpha^2}{32E^2}\beta^3|F(k^2)|^2\sin^2\theta_0 d\Omega_X \tag{4·11}$$

and

$$\sigma_{XX'} = \frac{\pi\alpha^2}{12E^2}\beta^3|F(k^2)|^2 = \frac{\pi\alpha^2}{12M^2}|F(k^2)|^2\beta^3(1-\beta^2). \tag{4·12}$$

In Eqs. (4·11) and (4·12), the form factor $F(k^2)$ is taken at $k^2=-4E^2$ and normalized as $F(0)=1$.

C. *Correlation between the decay products of X and X′*

Later we shall be interested in the angular correlation between one particular decay product x from X and another one x' from X' in the overall center-of-mass system.

For X with spin $\frac{1}{2}$, the angular distribution of x from an arbitrary polarized X in the rest frame of X is generally written in the form of Eq. (2·4). The corresponding expression in a moving frame of X is in general rather complicated. If, however, the mass of x should be neglected compared with the mass of X, the desired expression reads*)

$$\frac{d\Gamma_n^{(s)}}{\Gamma_n}=\frac{1}{4\pi}\left(1-\alpha_n\frac{M}{p\cdot q}q\cdot s\right)\left|\frac{d\Omega_x^*}{d\Omega_x}\right|d\Omega_x\,, \tag{4·13a}$$

with $|d\Omega_x^*/d\Omega_x|$ given by[20]

$$\left|\frac{d\Omega_x^*}{d\Omega_x}\right|=\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\,. \tag{4·14a}$$

Here $p=(\boldsymbol{p}, E)$ and $q=(\boldsymbol{q}, \omega)$ are the four-momenta of X and x, $\gamma=E/M$, $\beta=|\boldsymbol{p}|/E$, θ is the angle between the directions of X and x, and $d\Omega_x$ is the solid angle element of x viewed in the moving frame of X. Similarly, for the angular distribution of x' from an arbitrary polarized X' in a moving frame of X', one has

$$\frac{d\Gamma_{m'}^{(s')}}{\Gamma_{m'}}=\frac{1}{4\pi}\left(1-\alpha_{m'}\frac{M}{p'\cdot q'}q'\cdot s'\right)\left|\frac{d\Omega_{x'}^*}{d\Omega_{x'}}\right|d\Omega_{x'} \tag{4·13b}$$

with $|d\Omega_{x'}^*/d\Omega_{x'}|$ given by

$$\left|\frac{d\Omega_{x'}^*}{d\Omega_{x'}}\right|=\frac{1}{\gamma^2(1+\beta\cos\theta')^2}\,. \tag{4·14b}$$

In writing the last equation, we have made use of the fact that, in a colliding beam experiment, the four-momentum of X', p' is given by $p'=(-\boldsymbol{p}, E)$, and the angle between the directions of x' and X (not X') is denoted by θ'.

The procedure developed in the preceding sections now implies that, in the case of interest to us, $d\sigma_{XX'}^{(n,m')}$ in Eq. (4·3) is obtained from Eq. (4·8) or (4·9) by the following substitutions:

$$s_\mu\rightarrow n_\mu\equiv-\alpha_n\frac{M}{p\cdot q}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M^2}\right)q_\nu\,, \tag{4·15a}$$

---

*) If one is interested in the energy-angular correlation between x and x', he may use, instead of Eqs. (4·13a) and Eqs. (4·13b), the following manifestly invariant expression:

$$\frac{d\Gamma_n^{(s)}}{\Gamma_n}=(f_1+f_2q\cdot s)\frac{d^3\boldsymbol{q}}{2\omega}$$

(f_1 and f_2 being some functions of $p\cdot q$) and the corresponding one for $X'\rightarrow m'$.

$$s_\mu' \to m_\mu' = -\alpha_{m'}\frac{M}{p'\cdot q'}\left(\delta_{\mu\nu}+\frac{p_\mu' p_\nu'}{M^2}\right)q_\nu', \tag{4·15b}$$

and $d\Gamma_n/\Gamma$ and $d\Gamma_{m'}/\Gamma$ in Eq. (4·3) are simply

$$\frac{d\Gamma_n}{\Gamma}=\frac{1}{4\pi}\left|\frac{d\Omega_x^*}{d\Omega_x}\right|d\Omega_x\times R_n,\quad \frac{d\Gamma_{m'}}{\Gamma}=\left|\frac{d\Omega_{x'}^*}{d\Omega_{x'}}\right|d\Omega_{x'}\times R_{m'} \tag{4·16}$$

with $|d\Omega_x^*/d\Omega_x|$ and $|d\Omega_{x'}^*/d\Omega_{x'}|$ given by Eqs. (4·14a) and (4·14b).

For spinless $X, d\sigma_{XX'}$ in Eq. (4·5) is given by Eq. (4·11), and $d\Gamma_n/\Gamma$ and $d\Gamma_{m'}/\Gamma$ in Eq. (4·5) take the same forms as Eq. (4·16).

Integration with respect to $d\Omega_X$ remains to be evaluated. This will be carried out explicitly in the individual applications.

§ 5. Applications

A. $e^- + e^+ \to X + X'$, with $X' \to x' +$ neutrino (s) and $X \to x +$ neutrino (s)

$$X=h^-,\qquad X'=h^+,$$
$$x, x'=e \text{ or } \mu \text{ or } \pi.$$

Assuming that the heavy leptons $h^\pm$ are point-like particles, their production cross section is given by Eq. (4·9). We then, obtain, following the procedure described in the previous section, the angular correlation between x and x' in the overall center-of-mass system:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}=&4\times R_n\times R_{m'}\times\left(\frac{1}{4\pi}\right)^2\times\frac{\alpha^2}{64E^2}\beta\int\Big\{1+\cos^2\theta_0+\frac{1}{\gamma^2}\sin^2\theta_0\\&-(n\cdot m')\beta^2\sin^2\theta_0-2n_0m_0'+2(\hat{p}_1\cdot\boldsymbol{n})(\hat{p}_1\cdot\boldsymbol{m}')\\&-2\beta\cos\theta_0(n_0(\hat{p}_1\cdot\boldsymbol{m}')-m_0'(\hat{p}_1\cdot\boldsymbol{n}))\Big\}\\&\times\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\frac{1}{\gamma^2(1+\beta\cos\theta')^2}d\Omega_X\end{aligned} \tag{5·1}$$

with n_μ and m_μ' given by

$$\left.\begin{aligned}\boldsymbol{n}&=\frac{\alpha_n}{\gamma(1-\beta\cos\theta)}[\hat{q}-\gamma^2\beta(1-\beta\cos\theta)\hat{p}],\\ n_0&=\frac{\alpha_n}{\gamma(1-\beta\cos\theta)}[1-\gamma^2(1-\beta\cos\theta)],\\ \boldsymbol{m}'&=\frac{\alpha_{m'}}{\gamma(1+\beta\cos\theta')}[\hat{q}'+\gamma^2\beta(1+\beta\cos\theta')\hat{p}],\\ m_0'&=\frac{\alpha_{m'}}{\gamma(1+\beta\cos\theta')}[1-\gamma^2(1+\beta\cos\theta')].\end{aligned}\right\} \tag{5·2}$$

Here and in the following, the angles are defined as follows:

$$\hat{p}_1\cdot\hat{p}=\cos\theta_0,\quad \hat{p}\cdot\hat{q}=\cos\theta,\quad \hat{p}\cdot\hat{q}'=\cos\theta',$$
$$\hat{p}_1\cdot\hat{q}=\cos\theta_1,\quad \hat{p}_1\cdot\hat{q}'=\cos\theta_1',\quad \hat{q}\cdot\hat{q}'=\cos\theta_2.$$

After substitution of Eq. (5·2) into Eq. (5·1) and rearrangement, one gets

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}=&\left(\frac{1}{4\pi}\right)^2 R_n R_{m'}\frac{\alpha^2}{16E^2}\beta\int\Big\{1+\cos^2\theta_0+\frac{1}{\gamma^2}\sin^2\theta_0\\&+\frac{\alpha_n\alpha_{m'}}{\gamma^2(1-\beta\cos\theta)(1+\beta\cos\theta')}\Big[2\cos\theta_1\cos\theta_1'-\beta^2\cos\theta_2-\gamma^2\beta^4\\&+\gamma^2\beta^3(\cos\theta-\cos\theta')+\gamma^2\beta^2(2-\beta^2)\cos\theta\cos\theta'\\&+2\beta\cos\theta_0(\cos\theta_1-\cos\theta_1')+\beta^2\cos^2\theta_0[\cos\theta_2-2-\gamma^2\beta^2\\&+\gamma^2\beta(\cos\theta-\cos\theta')+\gamma^2\beta^2\cos\theta\cos\theta']\Big]\Big\}\\&\times\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\frac{1}{\gamma^2(1+\beta\cos\theta')^2}d\Omega_X.\end{aligned}\tag{5·3}$$

For the collinear processes, i.e., for $\theta_2=0(\theta'=\theta,\ \theta_1'=\theta_1)$ or $\theta_2=\pi(\theta'=\pi-\theta,\ \theta_1'=\pi-\theta_1)$, integration with respect to $d\Omega_X$ can be carried out exactly and yields

$$\begin{aligned}\left.\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}\right)_{\theta_2=0}=&\frac{\alpha^2}{32\pi E^2}R_n R_{m'}\beta\frac{1}{4\gamma^4}\Big\{\Big[\frac{3-\beta^2}{2(1-\beta^2)}+\frac{5-\beta^2}{4\beta}\log\left|\frac{1+\beta}{1-\beta}\right|\Big]\\&+\cos^2\theta_1\Big[\frac{3-\beta^2}{2(1-\beta^2)}-\frac{3+\beta^2}{4\beta}\log\left|\frac{1+\beta}{1-\beta}\right|\Big]\\&+\alpha_n\alpha_{m'}\Big[1-\frac{1+\beta^2}{2\beta}\log\left|\frac{1+\beta}{1-\beta}\right|+\cos^2\theta_1\Big(1+\frac{3-\beta^2}{2\beta}\log\left|\frac{1+\beta}{1-\beta}\right|\Big)\Big]\Big\},\end{aligned}\tag{5·4}$$

$$\begin{aligned}\left.\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}\right)_{\theta_2=\pi}=&\frac{\alpha^2}{32\pi E^2}R_n R_{m'}\beta\gamma^2\Big\{1-\frac{1}{3}\beta^4+\frac{2}{3}\beta^4\cos^2\theta_1\\&+\alpha_n\alpha_{m'}\Big[-\frac{1}{3}\beta^2+\frac{1}{15}\beta^4-\Big(1-\beta^2+\frac{4}{15}\beta^4\Big)\cos^2\theta_1\Big]\Big\}.\end{aligned}\tag{5·5}$$

For the general process in which x and x' occur in arbitrary directions, we carry out the integration after expanding $(1-\beta\cos\theta)^{-1}$ and $(1+\beta\cos\theta')^{-1}$ with respect to β. One finds

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}=&\frac{\alpha^2}{32\pi E^2}R_n R_{m'}\beta\Big\{1-\frac{1}{3}\beta^2(1+4\cos\theta_2)\\&+\alpha_n\alpha_{m'}\Big[\cos\theta_1\cos\theta_1'-\frac{1}{3}\beta^2[1-3\cos\theta_1\cos\theta_1'(1-3\cos\theta_2)\end{aligned}$$

$$-3(\cos\theta_1-\cos\theta_1')^2]\Big]+O(\beta^4)\Big\}\,. \qquad (5\cdot6)$$

It is easily verified that Eqs. (5·4), (5·5) and (5·6) are mutually compatible; this serves as a check of our results. The value of the asymmetry parameter for each process is already given in § 2A. If one looks for ee or $\mu\mu$ or μe coincidence, $\alpha_n\alpha_{m'}=-1/9$; for $e\pi$ or $\mu\pi$ coincidence, $\alpha_n\alpha_{m'}=1/3$; and for $\pi\pi$ coincidence, $\alpha_n\alpha_{m'}=-1$.

B. $e^-+e^+\to\mu^{*-}+\mu^{*+}$, with $\mu^{*+}\to\mu^++\gamma$ and $\mu^{*-}\to\mu^-+\gamma$

The angular correlation between μ^- and μ^+ is given by Eqs. (5·4), (5·5) and (5·6) with $R_n=R_{m'}\approx1$ and $\alpha_n=\alpha_{m'}=0$. Note that, as for the production of an e^{*-}-e^{*+} pair, another graph shown in Fig. 4 contributes, and we shall treat this process in a separate paper.

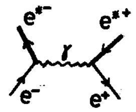


Fig. 4. Another graph contributing to $e^-+e^+\to e^{*-}+e^{*+}$.

C. $e^-+e^+\to X+X'$, with $X'\to x'+\cdots$ and $X\to x+\cdots$

$X, X'=$ spinless particle,
$x, x'=\mu$ or π or$\cdots$.

The procedure described in the preceding sections leads to

$$\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}=\left(\frac{1}{4\pi}\right)^2R_nR_{m'}\times\frac{\alpha^2}{32E^2}\beta^3|F(-4E^2)|^2$$
$$\times\int\sin^2\theta_0\frac{1}{\gamma^2(1-\beta\cos\theta)^2}\frac{1}{\gamma^2(1+\beta\cos\theta')^2}d\Omega_X\,. \qquad (5\cdot7)$$

Integrating over $d\Omega_X$, one finds

$$\left.\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}\right)_{\theta_2=0}=\frac{\alpha^2}{192\pi E^2}R_nR_{m'}|F(-4E^2)|^2\beta^3\times\frac{3}{8\gamma^4\beta^2}$$
$$\times\left\{\frac{1+\beta^2}{1-\beta^2}-\frac{1-\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}-\cos^2\theta_1\left(\frac{3-\beta^2}{1-\beta^2}-\frac{3+\beta^2}{2\beta}\log\frac{1+\beta}{1-\beta}\right)\right\}, \qquad (5\cdot8)$$

$$\left.\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}}\right)_{\theta_2=\pi}=\frac{\alpha^2}{192\pi E^2}R_nR_{m'}|F(-4E^2)|^2\beta^3(1+2\gamma^2\beta^2\sin^2\theta_1) \qquad (5\cdot9)$$

and

$$\frac{d^2\sigma}{d\Omega_x d\Omega_{x'}} = \frac{\alpha^2}{192\pi E^2} R_n R_{m'} |F(-4E^2)|^2 \beta^3$$
$$\times \left\{1 + \frac{1}{5}\beta^2 [2 - 8\cos\theta_2 - \cos^2\theta_1 - \cos^2\theta_1' - 2(\cos\theta_1 - \cos\theta_1')^2] + O(\beta^4)\right\}. \tag{5·10}$$

§ 6. Reaction $e^- + e^+ \to l^{*\mp} + l^\pm$, $l^{*\mp} \to l^\mp + \gamma$

Before going into details, we recall that the condition for this type of reactions to take place is (E_0 = beam energy)

$$E_0 \geqq \frac{M + m_l}{2} \gtrsim \frac{M}{2},$$

to be compared with $E_0 \geqq M$ for those reactions considered in § 5.

If the coupling at the $l^* l \gamma$ vertex takes the form as in Eq. (2·5), which is parity-conserving, and if one is not interested in the spin-orientations of the final particles (l^+, l^- and γ), the spin-orientation of $l^{*\pm}$ is completely irrelevant to the angular distributions of the final $l^\pm$ (see Remark 2 of § 3). From Eqs. (3·15), (4·14) and (4·16), one obtains the angular distributions of l^+ and l^- in the overall center-of-mass system as ($m_l = 0$)

$$\frac{d^2\sigma}{d\Omega_{l^+} d\Omega_{l^-}} = \frac{d\sigma_{l^*}}{d\Omega_l} \times \frac{1}{4\pi} \frac{R}{\gamma^2 (1 + \beta\cos\theta_2)^2}, \tag{6·1}$$

where $d\sigma_{l^*}/d\Omega_l$ is the differential cross section for the production process $e^- + e^+ \to l^{*\pm} + l^\mp$, $R (\approx 1)$ is the branching ratio for $l^{*\pm} \to l^\pm + \gamma$, θ_2 is the angle between the directions of l^+ and l^-, and γ and β are the Lorentz factor and the velocity of l^* given by

$$\gamma = \frac{4E_0^2 + M^2}{4ME_0}, \quad \beta = \frac{4E_0^2 - M^2}{4E_0^2 + M^2}.$$

We now calculate $d\sigma_{\mu^{*+}}/d\Omega_{\mu^-}$, the differential cross section for $e^- + e^+ \to \mu^- + \mu^{*+}$, for which the Feynman graph in Fig. 5 contributes. Denoting the four-momenta of e^-, e^+, μ^{*+} and μ^- by $p_1, p_2, p = (\boldsymbol{p}, E)$ and $q = (\boldsymbol{q}, \omega)$, respectively, one has

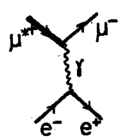


Fig. 5. Production of a single excited muon through one-photon intermediate state in an e^--e^+ collision.

$$d\sigma_{\mu^{*+}}=\frac{1}{4F}\frac{1}{4}\sum_{\text{spins}}|\langle\mu^{*+}\mu^-|T|e^-e^+\rangle|^2(2\pi)^4\delta^4(p_1+p_2-p-q)\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{p}}{2E}\frac{1}{(2\pi)^3}\frac{d^3\boldsymbol{q}}{2\omega},$$

$$\sum|\langle\mu^{*+}\mu^-|T|e^-e^+\rangle|^2=\left(\frac{fe^2}{M+m_\mu}\right)^2\frac{1}{(p_1+p_2)^4}\mathrm{Tr}[(-i\gamma\cdot p_2-m_e)i\gamma_\lambda(-i\gamma\cdot p_1+m_e)\gamma_\nu]$$
$$\times\mathrm{Tr}[(-i\gamma\cdot q+m_\mu)\sigma_{\lambda\lambda'}(p+q)_{\lambda'}(-i\gamma\cdot p-M)\sigma_{\nu\nu'}(p+q)_{\nu'}]. \qquad (6\cdot2)$$

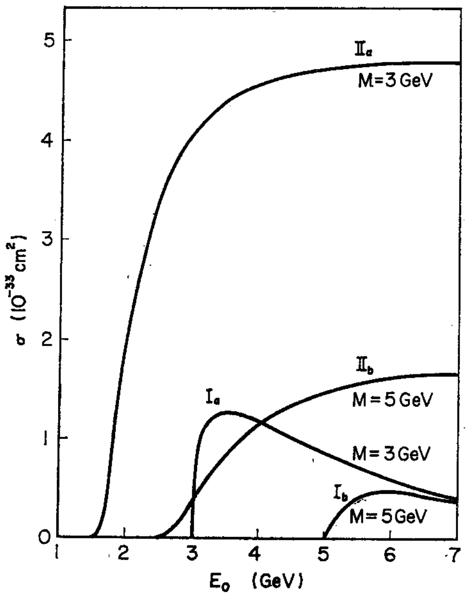
In the center-of-mass system, in which $p_{10}=p_{20}=E_0$ (beam energy), one finds ($m_e=m_\mu=0$)*)

$$\frac{d\sigma_{\mu^{*+}}}{d\Omega_{\mu^-}}=\frac{f^2\alpha^2}{16E_0^2}\left(1-\frac{M^2}{4E_0^2}\right)^2\left(1+\cos^2\theta_1+\frac{4E_0^2}{M^2}\sin^2\theta_1\right), \qquad (6\cdot3)$$

where θ_1 is the angle between the directions of the incident e^- and the outgoing μ^-. Integrating over $d\Omega_{\mu^-}$, we obtain the total cross section as

$$\sigma_{\mu^*}=\frac{\pi f^2\alpha^2}{3E_0^2}\left(1-\frac{M^2}{4E_0^2}\right)^2\left(1+\frac{2E_0^2}{M^2}\right). \qquad (6\cdot4)$$

For comparison, we show in Fig. 6 the total cross sections for $e^-+e^+\to h^-+h^+(\mu^{*-}+\mu^{*+})$, Eq. (4·10), and for $e^-+e^+\to\mu^\pm+\mu^{*\mp}$, Eq. (6·4), as functions



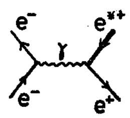


Fig. 7. Another graph contributing to $e^-+e^+\to e^{*\mp}+e^\pm$.

Fig. 6. Total cross sections for $e^-+e^+\to h^-+h^+(\mu^{*-}+\mu^{*+})$ (Curves Ia and Ib) and for $e^-+e^+\to\mu^{*\mp}+\mu^\pm$ (Curves IIa and IIb) as functions of the beam energy E_0 with $f=1$ and $M=3$ or 5 GeV.

---

*) If one retains the muon mass, then one finds

$$\frac{d\sigma_{\mu^{*+}}}{d\Omega_{\mu^-}}=\frac{1}{4}\left(\frac{f\alpha}{M+m_\mu}\right)^2\frac{|\boldsymbol{q}|}{E_0}\left[1+\frac{Mm_\mu}{E_0^2}-\frac{(M^2-m_\mu^2)^2}{16E_0^4}-\frac{|\boldsymbol{q}|^2}{E_0^2}\cos^2\theta_1\right]$$

with $|\boldsymbol{q}|$ given by

$$|\boldsymbol{q}|=\frac{1}{4E_0}[(2E_0+M+m_\mu)(2E_0+M-m_\mu)(2E_0-M+m_\mu)(2E_0-M-m_\mu)]^{1/2}.$$

of the beam energy with $f=1$ and $M=3$ and 5 GeV.

We have also computed the cross section for $e^-+e^+\to e^{*\pm}+e^{\mp}$, for which another graph shown in Fig. 7 contributes. The results will be reported in a separate paper.

§ 7. Summary and concluding remarks

In §§ 3 and 4A, we have derived the general formulae which may be conveniently used for computing the cross section for the production and subsequent decay of a single X or a X-$\bar{X}$ pair, X being a narrow-width particle with spin 0 or $\frac{1}{2}$. In §§ 4B and 4C, we have described the general procedure for derivation of the combined angular distribution of the decay products of X and $\bar{X}$ in the overall center-of-mass system, the X-$\bar{X}$ pair being produced in the reaction $e^-+e^+\to X+\bar{X}$. The procedure has been applied to some processes of interest in § 5, where the variables associated with X and $\bar{X}$ have been explicitly integrated out. Our treatments also include the production and decay of an excited muon through the sequence $e^-+e^+\to\mu^{*\pm}+\mu^{\mp}\to(\mu^{\pm}+\gamma)+\mu^{\mp}$ (§ 6).

Past and future searches for heavy leptons (including excited leptons) have recently been described in detail by Perl.[21] As noted by him and also by others, the important search method for heavy leptons, $h^{\pm}$, in colliding beam experiments includes looking for $e^+\mu^-$ or $e^-\mu^+$ pairs, since the fractional decay rates are relatively large (especially for high-mass heavy leptons) and background problems are less serious for these modes. The expected combined angular distribution of e^+ and μ^- (or e^- and μ^+) is given by Eqs. (5·4), (5·5) and (5·6) with $\alpha_n\alpha_{m'}=-1/9$. As numerical examples, we show in Fig. 8 the differential cross section for $e^-+e^+\to h^-+h^+\to(e^-+\bar{\nu}_e+\nu_h)+(\mu^++\nu_\mu+\bar{\nu}_h)$ or $(\mu^-+\bar{\nu}_\mu+\nu_h)+(e^++\nu_e+\bar{\nu}_h)$ for various special cases, with $M_h=3$ GeV and $R(h\to e\nu_e\nu_h)=R(h\to\mu\nu_\mu\nu_h)=0.24$.*)

The search for excited leptons, $e^{*\pm}$ and $\mu^{*\pm}$, must involve looking for anomalous e^+e^- or $\mu^+\mu^-$ pairs and hence background problems are very serious. The expected angular distribution of a $\mu^+\mu^-$ pair is given by Eqs. (5·4), (5·5) and (5·6) with $\alpha_n=\alpha_{m'}=0$ for $e^-+e^+\to\mu^{*-}+\mu^{*+}\to(\mu^-+\gamma)+(\mu^++\gamma)$, and by Eq. (6·1) supplemented by Eq. (6·3) for $e^-+e^+\to\mu^{*\pm}+\mu^{\mp}\to(\mu^{\pm}+\gamma)+\mu^{\mp}$. Numerical examples for the latter process are illustrated in Fig. 9.

Equations (5·8), (5·9) and (5·10) give the differential cross section for the production and decay of narrow-width spinless mesons (e.g., charmed mesons). The corresponding equations for baryons could have been obtained, if one had started from Eq. (4·8) instead of Eq. (4·9) in § 5A.

We note here that, since we consider the production of heavy leptons near its threshold, the production of a heavy lepton pair by a one-photon exchange mechanism completely dominates the one by a two-photon exchange mechanism.[22]

*) This value of the branching ratio for $h\to l\nu_l\nu_h$, $l=e$ or μ, is taken from Ref. 7).

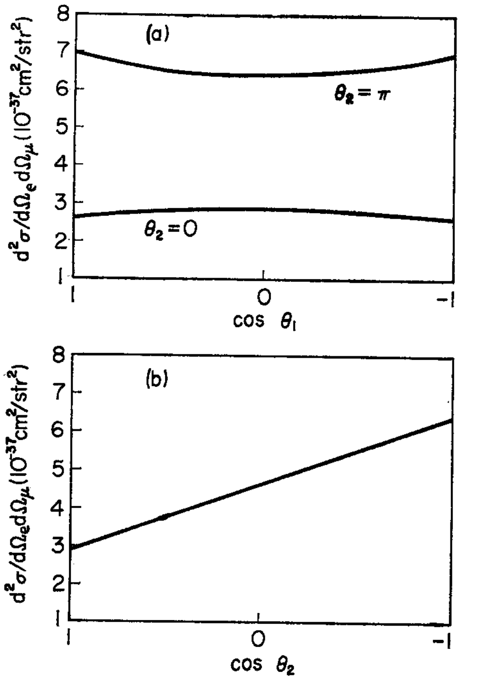


Fig. 8.

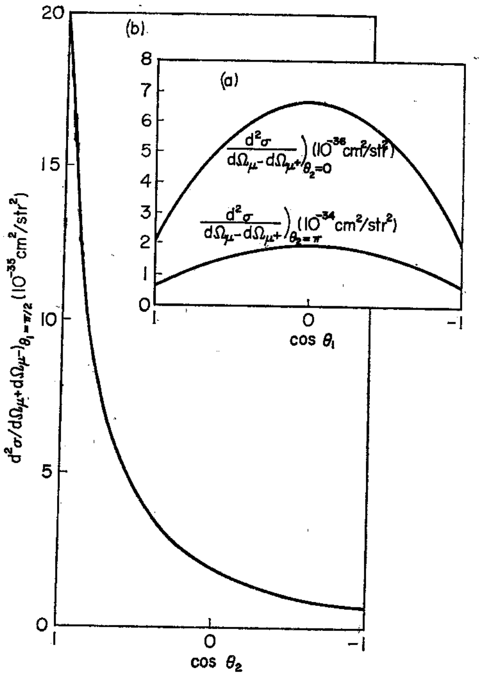


Fig. 9.

Fig. 8. The combined angular distribution of the final e and μ in the reaction $e^-+e^+\rightarrow h^-+h^+\rightarrow(e^\pm\nu_e\nu_h)+(\mu^\mp\nu_\mu\nu_h)$ with $M=3$ GeV, $E_0=3.5$ GeV and $R(h\rightarrow l\nu_e\nu_h)=0.24$, as a function of θ_1 with $\theta_2=0$ or π (Fig. 8(a)) and as a function of θ_2 at $\theta_1=\theta_1'=\pi/2$ (Fig. 8(b)).

Fig. 9. The combined angular distribution of the final μ^+ and μ^- in the reaction $e^-+e^+\rightarrow\mu^\mp+\mu^{*\pm}\rightarrow\mu^\mp+(\mu^\pm\gamma)$ with $f=1$, $M=3$ GeV, $E_0=3.5$ GeV and $R(\mu^*\rightarrow\mu\gamma)=1$, as a function of θ_1 with $\theta_2=0$ or π (Fig. 9(a)) and as a function of θ_2 at $\theta_1=\pi/2$ (Fig. 9(b)).

To conclude, we remark that various considerations made in this paper can be applied to many other similar problems which involve creation of narrow-width particles. For example, application to the reaction, $e^-+e^+\rightarrow\Sigma^0+\bar{\Sigma}^0\rightarrow(\Lambda+\gamma)+(\bar{\Lambda}+\gamma)$ or $e^-+e^+\rightarrow\Sigma^0+\bar{\Lambda}\rightarrow(\Lambda+\gamma)+\bar{\Lambda}$, may yield information on the life-time of Σ^0 or the form factor relevant to the Σ^0-Λ transition.

Acknowledgements

One of the present authors (S.Y.T.) would like to express his sincere thanks to Professor O. Hara, Professor T. Goto, Professor K. Hiida, Dr. J. Otokozawa and Dr. S. Ishida for their encouragement and discussions.

The authors are indebted to Dr. Y. Miyazaki of Faculty of Technology, Chiba University, for his help in numerical works.

Appendix

We shall give a proof of the identity (3·8),

$$2((\bar{A}\Lambda_+(p)B))((\bar{B}\Lambda_+(p)A))=((\bar{A}\Lambda_+(p)A))((\bar{B}\Lambda_+(p)B))$$
$$+\eta_{\mu\nu}((\bar{A}\Lambda_+(p)i\gamma_5\gamma_\mu A))((\bar{B}\Lambda_+(p)i\gamma_5\gamma_\nu B)). \qquad (3\cdot8)$$

We use the following representations for γ matrices:

$$\gamma_i=\begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix},\quad \gamma_4=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \gamma_5=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

In the rest frame of $X(\boldsymbol{p}=0)$

$$\Lambda_+(p)\equiv\frac{-i\gamma\cdot p+M}{2M}\Rightarrow\frac{1+\gamma_4}{2},$$

$$\eta_{\mu\nu}\equiv\delta_{\mu\nu}+\frac{p_\mu p_\nu}{M}\Rightarrow\begin{cases}\delta_{ij}, & (\mu,\nu=i,j)\\ 0, & (\mu,\nu=4)\end{cases}$$

$$i\gamma_5\gamma_i\Rightarrow\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}.$$

Thus, in this frame, the right-hand side of Eq. (3·8) reduces to

$$\mathrm{Tr}\left(\bar{A}\frac{1+\gamma_4}{2}A\right)\mathrm{Tr}\left(\bar{B}\frac{1+\gamma_4}{2}B\right)+\mathrm{Tr}\left(\bar{A}\frac{1+\gamma_4}{2}\boldsymbol{\sigma}A\right)\left(\bar{B}\frac{1+\gamma_4}{2}\boldsymbol{\sigma}B\right)$$
$$=(A_1^\dagger A_1+A_2^\dagger A_2)(B_1^\dagger B_1+B_2^\dagger B_2)+(A_1^\dagger A_2+A_2^\dagger A_1)(B_1^\dagger B_2+B_2^\dagger B_1)$$
$$-(A_1^\dagger A_2-A_2^\dagger A_1)(B_1^\dagger B_2-B_2^\dagger B_1)+(A_1^\dagger A_1-A_2^\dagger A_2)(B_1^\dagger B_1-B_2^\dagger B_2)$$
$$=2(A_1^\dagger A_1B_1^\dagger B_1+A_2^\dagger A_2B_2^\dagger B_2+A_1^\dagger A_2B_2^\dagger B_1+A_2^\dagger A_1B_1^\dagger B_2)$$
$$=2\,\mathrm{Tr}\left(\bar{A}\frac{1+\gamma_4}{2}B\right)\mathrm{Tr}\left(\bar{B}\frac{1+\gamma_4}{2}A\right)$$

which is precisely the left-hand side of Eq. (3·8) viewed in the same frame. This confirms the proof of the identity (3·8).

References

1) Such reports are conceivably too numerous to mention all. We cite some examples:
H. Bergeson et al., Phys. Rev. Letters **19** (1967), 1487.
G. L. Cassiday, Phys. Rev. **D3** (1971), 1109.
B. Baschiera et al., Lett Nuovo Cim. **1** (1971), 961.
M. Nagano et al., INS-Report No. 147 (1970).
S. Mikamo, private communication.

2) The following paper contains a new approach, as well as a brief summary and references of early attempts, to understand the muon puzzle:
K. Tennakone and S. Pakvasa, preprint (UH-511-117-72).

3) See, for example,
S. S. Gerstein and V. N. Folomeshkin, Soviet J. of Nucl. Phys. 8 (1969), 447.
K. W. Rothe and A. M. Wolsky, Nucl. Phys. **B10** (1969), 241.

4) M. L. Perl, preprint (SLAC-PUB-1062 (1972)).
S. Y. Tsai, Lett. Nuovo Cim. **2** (1971), 949.

A. Vancura, preprint.
5) V. Alles-Borelli et al., Lett. Nuovo Cim. **4** (1970), 1156.
6) H. B. Thacker and J. J. Sakurai, Phys. Letters **36**B (1971), 103.
7) S. Y. Tsai, Phys. Rev. **D4** (1971), 2821.
8) F. Low, Phys. Rev. Letters **14** (1965), 238.
A. O. Barut et al., Phys. Rev. **182** (1969), 1844.
9) C. D. Boly et al., Phys. Rev. **167** (1968), 1275.
10) C. A. Ramm, Nature Phys. Sci. **230** (1971), 145.
11) K. Niu et al., Prog. Theor. Phys. **46** (1971), 1644.
12) T. Hayashi et al., Prog. Theor. Phys. **47** (1972), 280.
13) S. Y. Tsai, Prog. Theor. Phys. **47** (1972), 1331.
14) M. Y. Han and Y. Nambu, Phys. Rev. **139** (1965), B1006.
15) C. Bernardini, *1971 International Symposium on Electron and Photon Interactions at High Energies—Cornell, Ithaca.*
16) S. Y. Tsai, Soryushiron Kenkyu (mimeographed circular in Japanese) **45** (1972), 184.
17) See, for example, L. B. Okun', *Weak Interaction of Elementary Particles* (Pergamon Press, 1965).
18) See for example B. E. Y. Svensson, "Multi-Pion System" in *Kinematics and Multiparticle Systems,* edited by M. Nikolic (Gordon and Breach, 1968).
19) N. Cabibbo and R. Gatto, Phys. Rev. **124** (1961), 1577.
20) See for example A. Werbrouck, "Relativistic Kinematics" in *Kinematics and Multiparticle Systems,* edited by M. Nikolic (Gordon and Breach, 1968).
21) M. L. Perl, Ref. 4).
22) S. J. Brodsky et al., Phys. Rev. **D4** (1971), 1532.