Products and Ratios of Characteristic Polynomials of Random Hermitian Matrices

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Abstract

We present new and streamlined proofs of various formulae for products and ratios of characteristic polynomials of random Hermitian matrices that have appeared recently in the literature.

1 Introduction

In random matrix theory, unitary ensembles of $N \times N$ matrices $\{H\}$ play a central role [16]. Such ensembles are described by a measure $d\alpha$ with finite moments $\int_{\mathbb{R}} |x|^k d\alpha(x) < \infty$, $k = 0, 1, 2, \cdots$, and the distribution function for the eigenvalues $\{x_i = x_i(H)\}$ of matrices H in the ensembles has the form

$$d\,\mathrm{P}_{\alpha,N}(x) = \frac{1}{Z_N} \Delta(x)^2 d\alpha(x) \tag{1.1}$$

where $d\alpha(x) = \prod_{i=1}^{N} d\alpha(x_i)$, $\Delta(x) = \prod_{N \geq i > j \geq 1} (x_i - x_j)$ is the Vandermonde determinant for the x_i's, and $Z_N = \int \cdots \int \Delta(x)^2 d\alpha(x)$ is the normalization constant. The special case $d\alpha(x) = e^{-x^2}dx$ is known as the Gaussian Unitary Ensemble (GUE). For symmetric functions $f(x) = f(x_1, \cdots, x_N)$ of the x_i's,

$$\langle f \rangle_\alpha \equiv \frac{1}{Z_N} \int \cdots \int f(x) \Delta(x)^2 d\alpha(x) \tag{1.2}$$

denotes the average of f with respect to $d\,\mathrm{P}_{\alpha,N}$.

Recently there has been considerable interest in the averages of products and ratios of the characteristic polynomials $D_N[\mu, H] = \prod_{i=1}^{N} (\mu - x_i(H))$ of random matrices with respect to various ensembles. Such averages are used, in particular, in making predictions about the moments of the Riemann-zeta function, see [15, 14, 13] (circular ensembles) and [3] (unitary ensembles). Many other uses are described, for example, in [1], [12] and [11].

By (1.2), for unitary ensembles, such averages have the form

$$\left\langle \frac{\prod_{j=1}^{K} D_N[\mu_j, H]}{\prod_{j=1}^{M} D_N[\epsilon_j, H]} \right\rangle_\alpha = \frac{1}{Z_N} \int \cdots \int \frac{\prod_{j=1}^{K} \prod_{i=1}^{N} (\mu_j - x_i)}{\prod_{j=1}^{M} \prod_{i=1}^{N} (\epsilon_j - x_i)} \Delta(x)^2 d\alpha(x). \tag{1.3}$$

In this paper we consider certain explicit determinantal formulae for (1.3) – see (2.6), (2.24), (2.36), (3.3), (3.12) below. Formula (2.6) is due to Brezin and Hikami [3] (see also [17], and when all the μ_j's are equal, see [10]), whereas (2.24), (2.36), (3.3) and (3.12) are due to Fyodorov and Strahov [12, 11]. The papers [12, 11] also contain a discussion of the history of these formulae. The formulae (3.3) and (3.12) are particularly useful in proving universality results for the ratios (1.3) in the Dyson limit as $N \to \infty$ (see [11]). For a discussion of other universality results, particularly the work of Brezin-Hikami and Fyodorov in special cases, we again refer the reader to [11]. The asymptotic analysis in [11] is based on the reformulation of the orthogonal polynomial problem as a Riemann-Hilbert problem by Fokas, Its and Kitaev [9]. The Riemann-Hilbert problem is then analyzed asymptotically using the non-commutative steepest-descent method introduced by Deift and Zhou [8], and further developed with Venakides in [7] to allow for fully non-linear oscillations, and in [6], [5].

Our goal in this paper is to give new, streamlined proofs of (2.6)-(3.12), using only the properties of orthogonal polynomials and a minimum of combinatorics. Along the way we will also need an integral version of the classical Binet-Cauchy formula due to C.Andréief dating back to 1883 (see Lemma 2.1 below).

Let $\pi_j(z) = x^j + \cdots$ denote the jth monic orthogonal polynomial with respect to the measure $d\alpha$,

$$\int_{\mathbb{R}} \pi_j(x)\pi_k(x)d\alpha(x) = c_j c_k \delta_{jk}, \qquad j,k \geq 0, \tag{1.4}$$

where the norming constants c_j's are positive. The key observation in our approach is that for $K = 1$ and $M = 0$ in (1.3)

$$\big\langle D_N[\mu, H] \big\rangle_\alpha = \pi_N(\mu) \tag{1.5}$$

(see [18]). In our words, the orthogonal polynomial $\pi_N(\mu)$ with respect to $d\alpha$ is also precisely the average polynomial $\prod_{i=1}^{N}(\mu - x_i)$ with respect to $d\,\mathrm{P}_{\alpha,N}$. Formula (1.5) appears already in the work of Heine in the 1880's (see [18]). Set

$$d\alpha^{[\ell,m]}(t) \equiv \frac{\prod_{j=1}^{\ell}(\mu_j - t)}{\prod_{j=1}^{m}(\epsilon_j - t)} d\alpha(t), \qquad \ell, m \geq 0, \tag{1.6}$$

($d\alpha^{[0,0]}(t) \equiv d\alpha(t)$), and let $\pi_j^{[\ell,m]}(t)$ denote the jth monic orthogonal polynomial with respect to $d\alpha^{[\ell,m]}$. With this notation we see immediately from (1.3), (1.5) that $\big\langle \frac{\prod_{j=1}^{K} D_N[\mu_j,H]}{\prod_{j=1}^{M} D_N[\epsilon_j,H]} \big\rangle_\alpha$ is proportional to $\pi_N^{[K-1,M]}(\mu_K)$ Using a classical determinantal formula of Christoffel (see [18]) for $\pi_N^{[\ell,0]}(\mu)$ and a more recent formula of Uvarov [19] for $\pi_N^{[0,m]}(\mu)$, we are then led (see Section 2. Formulae of Christoffel-Uvarov type) to (2.6), (2.24) and (2.36) in a rather straightforward way. Formula (3.3) appears to have a different character from (2.6), (2.24), (2.36), and relies on Lemma 2.1 mentioned above, which computes the integral of the product of two determinants: formula (3.12) follows (see Section 3. Formulae of two-point function type) by combining (3.3) with (2.6) and (2.36). In [11] the authors present a variety of additional formulae for $\big\langle \frac{\prod_{j=1}^{K} D_N[\mu_j,H]}{\prod_{j=1}^{M} D_N[\epsilon_j,H]} \big\rangle_\alpha$ for cases of K and M not covered by (2.6)-(3.12): we leave it to the interested reader to verify that the method of this paper can also be used to derive these formulae in a straightforward manner.

Remark 1.1. As is well-known (see e.g., [18]), each measure $d\alpha$ gives rise to a tridiagonal operator

$$J = J(d\alpha) = \begin{pmatrix} a_1 & b_1 & 0 & \\ b_1 & a_2 & b_2 & \\ 0 & b_2 & a_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad b_i > 0 \tag{1.7}$$

with generalized eigenfunctions given by the orthonormal polynomials

$$p_j(x) = c_j^{-1}\pi_j(x), \qquad j = 0, 1, \cdots, \tag{1.8}$$

i.e.

$$b_{j-1}p_{j-1}(x) + a_j p_j(x) + b_j p_{j+1}(x) = x p_j(x), \qquad j \geq 1 \tag{1.9}$$

where $b_0 \equiv 0$. Conversely, modulo certain essential self-adjointness issues, $d\alpha$ is the spectral measure for J in the cyclic subspace generated by J and the vector $e_1 = (1, 0, 0, \cdots)^T$ (see, e.g., [4]). It follows that the transformation of measures

$$d\alpha \to d\alpha^{[\ell,m]} \tag{1.10}$$

leads to the transformation of operators

$$J(d\alpha) \to J(d\alpha^{[\ell,m]}). \tag{1.11}$$

For appropriate choices of $\mu_1, \cdots, \mu_m$ and $\epsilon_1, \cdots, \epsilon_\ell$, such transformations corresponding to removing m points from the spectrum of $J(d\alpha)$ and inserting ℓ points: in the spectral theory literature, such transformations are known as Darboux transformations. The formulae in this paper clearly provide formulae for the generalized eigenfunctions $p_j^{[\ell,m]}(x)$ of the Darboux-transformed operator $J(d\alpha^{[\ell,m]})$, as well as the matrix entries, $a_j^{[\ell,m]}$ and $b_j^{[\ell,m]}$, in terms of the corresponding objects for $J(d\alpha)$. Again we leave the details to the reader. Here the elementary formulae

$$b_n^2(d\alpha) = \frac{n+1}{n+2}\frac{Z_n(d\alpha)Z_{n+2}(d\alpha)}{\big(Z_{n+1}(d\alpha)\big)^2}, \qquad a_n(d\alpha) = \frac{d}{dt}\Big|_{t=0} \log \frac{Z_n(d\alpha_t)}{Z_{n+1}(d\alpha_t)} \tag{1.12}$$

where $d\alpha_t(x) = e^{tx}d\alpha(x)$, are useful.

Technical Remark 1.2. Formulae (2.6)-(3.12) clearly do not make sense for all values of the parameters. In ***all the calculations that follow, we will assume that*** $d\alpha$ ***has compact support, support(***$d\alpha$***)=*** $[-Q, Q]$***, say, and that the*** μ_i***'s and*** ϵ_j***'s are distinct real numbers greater than*** Q: under these assumptions, $d\alpha^{[\ell,m]}(t)$ becomes, in particular, a bona-fide measure, etc. By analytic continuation one sees that the formulae remain true for complex values of $\{\mu_i\}$ and $\{\epsilon_j\}$, as long as they remain distinct. Furthermore, if the μ_i's and ϵ_j's are distinct, and $Im(\epsilon_j) \neq 0$ for all j, then we can let $Q \to \infty$ and so the formulae are true for measures $d\alpha$ with unbounded support. Finally we can, for example, let $\mu_j \to \mu_k$ for some $j \neq k$, which leads to formulae involving derivatives of the π_j's, etc.

2 Formulae of Christoffel-Uvarov type

We use the notations $d\alpha$, π_j, $d\alpha^{[\ell,m]}$, $\pi_j^{[\ell,m]}$, ... of Section 1. In addition, in all the calculations that follow we assume that $d\alpha$, $\{\mu_j\}$, $\{\epsilon_k\}$ satisfy the conditions described in Technical Remark 1.2 above: the natural analytical continuation of the formulae obtained to complex values of the parameters, and the limit $Q \to \infty$, is left to the reader.

The following result of Christoffel (see [18]) plays a basic role in what follows.

Lemma 2.1. *Consider the measure* $d\alpha^{[\ell,0]}(t) = \prod_{j=1}^{\ell}(\mu_j - t)\, d\alpha(t)$, *where* $\ell = 1, 2, \ldots$. *Then the* n^{th} *monic orthogonal polynomial* $\pi_n^{[\ell,0]}(t)$ *associated with the new measure* $d\alpha^{[\ell,0]}(t)$ *can be expressed as follows:*

$$\pi_n^{[\ell,0]}(t) = \frac{1}{(t-\mu_1)\ldots(t-\mu_\ell)} \frac{\begin{vmatrix} \pi_n(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_\ell) & \ldots & \pi_{n+\ell}(\mu_\ell) \\ \pi_n(t) & \ldots & \pi_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} \pi_n(\mu_1) & \ldots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_\ell) & \ldots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix}}. \tag{2.1}$$

Proof. Set

$$q_n^{[\ell,0]}(t) = \begin{vmatrix} \pi_n(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_\ell) & \ldots & \pi_{n+\ell}(\mu_\ell) \\ \pi_n(t) & \ldots & \pi_{n+\ell}(t) \end{vmatrix}. \tag{2.2}$$

We note that $q_n^{[\ell,0]}(t)$ satisfies the condition $\int t^j q_n^{[\ell,0]}(t) d\alpha(t) = 0$ for all $j \in \{0, \ldots, n-1\}$. Also $q_n^{[\ell,0]}(\mu_j) = 0$, $j = 1, \cdots, \ell$, and so $\frac{q_n^{[\ell,0]}(t)}{(\mu_1-t)\cdots(\mu_\ell-t)}$ is a polynomial of degree at most n. Now observe that

$$\int t^j \left[\frac{q_n^{[\ell,0]}(t)}{(\mu_1-t)\ldots(\mu_\ell-t)}\right] d\alpha^{[\ell,0]}(t) = 0, \quad 0 \le j < n \tag{2.3}$$

which means that $q_n^{[\ell,0]}(t)$ divided by the product $(\mu_1 - t)\ldots(\mu_\ell - t)$ is proportional to the n^{th} monic orthogonal polynomial $\pi_n^{[\ell,0]}(t)$ associated with the new measure $d\alpha^{[\ell,0]}(t)$. Now $q_n^{[\ell,0]}(t)$ cannot vanish for any $t = \mu_{\ell+1} > Q$, $\mu_{\ell+1} \notin \{\mu_1, \cdots, \mu_\ell\}$. Indeed, if $q_n^{[\ell,0]}(\mu_{\ell+1}) = 0$, then there exist $\{\alpha_i\}_{i=0}^{\ell}$, not all zero, such that $p(t) \equiv \sum_{i=0}^{\ell} \alpha_i \pi_{n+i}(t)$ vanishes at $\{\mu_i\}_{i=1}^{\ell+1}$. Thus $\tilde{p}(t) \equiv p(t)/\prod_{i=1}^{\ell+1}(\mu_i - t)$ is a polynomial of order $< n$, and as above, $\tilde{p}(t)$ is orthogonal to t^j, $0 \le j < n$, with respect to the measure $d\alpha^{[\ell+1,0]}(t)$. Thus $\tilde{p}(t) \equiv 0$ and hence $\alpha_0 = \cdots = \alpha_\ell = 0$, which is a contradiction. Replacing ℓ by $\ell - 1$, we conclude that

$$\begin{vmatrix} \pi_n(\mu_1) & \ldots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_\ell) & \ldots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix} \neq 0. \tag{2.4}$$

Taking the limit $t \to \infty$ and noting that the coefficient of the highest degree of $\pi_n^{[\ell,0]}(t)$ should be equal to 1, we find the coefficient of proportionality and establish formula (2.1). □

Representation (2.1) for the monic orthogonal polynomials associated with the measure $d\alpha^{[\ell,0]}(t)$ immediately leads to the following result:

Corollary 2.2. *The product of monic orthogonal polynomials* $\prod_{j=0}^{\ell} \pi_n^{[j,0]}(\mu_{j+1})$ *defined with respect to the different measures* $d\alpha^{[j,0]}(t) \equiv (\mu_j - t) \cdots (\mu_1 - t) d\alpha(t)$ *is given by the formula*

$$\prod_{j=0}^{\ell} \pi_n^{[j,0]}(\mu_{j+1}) = \frac{1}{\triangle(\mu)} \begin{vmatrix} \pi_n(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_{\ell+1}) & \cdots & \pi_{n+\ell}(\mu_{\ell+1}) \end{vmatrix} \tag{2.5}$$

where $\triangle(\mu) = \prod_{\ell+1 \geq i > j \geq 1} (\mu_i - \mu_j)$.

We observe that Corollary (2.2) gives the identity for the average of products of random characteristic polynomials obtained first by Brezin and Hikami [3].

Theorem 2.3. *Let* $D_N[\mu, H]$ *be the characteristic polynomial of the Hermitian matrix* H*. The following identity is valid:*

$$\left\langle \prod_{j=1}^{L} D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{1}{\triangle(\mu)} \begin{vmatrix} \pi_N(\mu_1) & \ldots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_N(\mu_L) & \ldots & \pi_{N+L-1}(\mu_L) \end{vmatrix} \tag{2.6}$$

where the average is defined by (1.2).

Proof. To prove formula (2.6) we use the representation for the monic orthogonal polynomials in the case $L = 1$ given in (1.5),

$$\pi_N(\mu) = \frac{1}{Z_N} \int \ldots \int \prod_{i=1}^{N} (\mu - x_i) \triangle^2(x) d\alpha(x). \tag{2.7}$$

Let $Z_N^{[\ell,0]}$ be defined by

$$Z_N^{[\ell,0]} = \int \ldots \int \triangle^2(x) d\alpha^{[\ell,0]}(x), \quad \ell = 1, 2, \cdots . \tag{2.8}$$

where $d\alpha^{[\ell,0]}(x) = \prod_{i=1}^{N} d\alpha^{[\ell,0]}(x_i)$. With this notation, we have

$$\left\langle \prod_{j=1}^{L} D_N[\mu_j, H] \right\rangle_{\alpha} = \frac{Z_N^{[L,0]}}{Z_N} = \frac{Z_N^{[L,0]}}{Z_N^{[L-1,0]}} \frac{Z_N^{[L-1,0]}}{Z_N^{[L-2,0]}} \cdots \frac{Z_N^{[1,0]}}{Z_N}. \tag{2.9}$$

Equation (2.7) implies that $\pi_n^{[\ell-1,0]}(\mu_\ell)$ can be represented as the ratio $Z_N^{[\ell,0]}/Z_N^{[\ell-1,0]}$, where $\pi_N^{[0,0]}(\mu) \equiv \pi_N(\mu)$, and $Z_N^{[0,0]} \equiv Z_N$. Thus we obtain

$$\left\langle \prod_{j=1}^{L} D_N[\mu_j, H] \right\rangle_{\alpha} = \prod_{j=0}^{L-1} \pi_N^{[j,0]}(\mu_{j+1}) \tag{2.10}$$

The above equation together with Corollary (2.2) proves formula (2.6). □

Remark 2.4. Notice (see equations (2.7) and (2.10)) that the average of products of characteristic polynomials can be rewritten as a product of averages. Namely,

$$\left\langle \prod_{j=1}^{L} D_N[\mu_j, H] \right\rangle_{\alpha} = \prod_{j=1}^{L} \langle D_N[\mu_j, H]\rangle_{\alpha^{[j-1,0]}} \tag{2.11}$$

where $\langle \ldots \rangle_{\alpha^{[j-1,0]}}$ means the average defined by equation (1.2) but with respect to the new measure $d\alpha^{[j-1,0]}(x)$, and $d\alpha(x) \equiv d\alpha^{[0,0]}(x)$.

The formula of Christoffel (equation (2.1)) enables us to construct the orthogonal polynomials associated with the measure $d\alpha^{[\ell,0]}(t) = \prod_{j=1}^{\ell}(\mu_j - t)d\alpha(t)$ in terms of the orthogonal polynomials associated with the measure $d\alpha(t)$. Now we derive a formula due to Uvarov [19] expressing the monic orthogonal polynomials $\pi_n^{[0,m]}(t)$ associated with the measure $d\alpha^{[0,m]}(t) = \prod_{j=1}^{m}(\epsilon_j - t)^{-1}d\alpha(t)$, again in terms of the monic orthogonal polynomials $\pi_n(t)$ associated with the measure $d\alpha(t)$.

Lemma 2.5. *Suppose* $0 \le m \le n$. *The monic orthogonal polynomials* $\pi_n^{[0,m]}(t)$ *associated with the measure* $d\alpha^{[0,m]}(t)$ *can be expressed as ratios of determinants,*

$$\pi_n^{[0,m]}(t) = \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \ldots & h_n(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \ldots & h_n(\epsilon_m) \\ \pi_{n-m}(t) & \ldots & \pi_n(t) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \ldots & h_{n-1}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \ldots & h_{n-1}(\epsilon_m) \end{vmatrix}}. \tag{2.12}$$

Here the $h_k(\epsilon_j)$*'s are the Cauchy transformations of the monic orthogonal polynomials* $\pi_k(t)$,

$$h_k(\epsilon_j) = \frac{1}{2\pi i}\int \frac{\pi_k(t)d\alpha(t)}{t - \epsilon_j}. \tag{2.13}$$

Proof. Set

$$q_n^{[0,m]}(t) = \begin{vmatrix} h_{n-m}(\epsilon_1) & \ldots & h_n(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \ldots & h_n(\epsilon_m) \\ \pi_{n-m}(t) & \ldots & \pi_n(t) \end{vmatrix}. \tag{2.14}$$

Now $q_n^{[0,m]}(t)$ is proportional to the n^{th} monic orthogonal polynomial $\pi_n^{[0,m]}(t)$ with respect to the measure $d\alpha^{[0,m]}(t)$. Indeed, first observe that

$$\int \frac{q_n^{[0,m]}(t)}{t - \epsilon_j} d\alpha(t) = 0, \qquad j = 1, \cdots, m. \tag{2.15}$$

Also, for $0 \le k < n$,

$$\frac{t^k}{\prod_{\ell=1}^m (\epsilon_\ell - t)} = \sum_{\ell=1}^m \frac{\beta_\ell}{\epsilon_\ell - t} + p(t) \tag{2.16}$$

for suitable constants $\{\beta_\ell\}$ and for some polynomial of degree $< n - m$. But for $0 \le k < n$,

$$\int t^k q_n^{[0,m]}(t) d\alpha^{[0,m]}(t) = -\sum_{\ell=1}^m \beta_\ell \int \frac{q_n^{[0,m]}(t)}{t - \epsilon_\ell} d\alpha(t) + \int p(t) q_n^{[0,m]}(t) d\alpha(t). \tag{2.17}$$

The terms in the sum are zero by (2.15) and the final integral is zero by the construction (2.14) of $q_n^{[0,m]}(t)$ and the fact that $\deg p(t) < n-m$. Thus $q_n^{[0,m]}(t)$ is proportional to $\pi_n^{[0,m]}(t)$. An argument similar to the proof in Lemma 2.1 that

$$\begin{vmatrix} \pi_n(\mu_1) & \dots & \pi_{n+\ell-1}(\mu_1) \\ \vdots & & \\ \pi_n(\mu_\ell) & \dots & \pi_{n+\ell-1}(\mu_\ell) \end{vmatrix} \neq 0, \tag{2.18}$$

shows that the denominator in (2.12) does not vanish. Letting $t \to \infty$ in (2.14), and matching leading terms, we prove Lemma 2.5. □

Remark 2.6. In [19], Uvarov obtains formulae for $\pi_n^{[0,m]}(t)$ of type (2.12) also in the case $m > n$. These formulae can be used to obtain analogues of (2.24) and (2.36) below in the case $M > N$.

Remark 2.7. As noted in [12, 11], the Cauchy transformations $h_k(\epsilon)$ of the π_k's occur explicitly, together with the π_k's, in the solution of the Fokas-Its-Kitaev Riemann-Hilbert problem for orthogonal polynomials [9].

Lemma (2.5) implies the following analogue of the Christoffel formula for the Cauchy transforms of monic orthogonal polynomials.

Corollary 2.8. *Let $h_k^{[0,m]}(\epsilon)$ be the Cauchy transform of the monic polynomial $\pi_k^{[0,m]}(t)$ with respect to the measure $d\alpha^{[0,m]}(t)$,*

$$h_k^{[0,m]}(\epsilon) = \frac{1}{2\pi i} \int \frac{\pi_k^{[0,m]}(t)}{t - \epsilon} \, d\alpha^{[0,m]}(t). \tag{2.19}$$

Let also $0 \le m \le n$. Then $h_n^{[0,m]}(\epsilon)$ has a representation similar to that for the monic orthogonal polynomials $\pi_n^{[l,0]}(t)$ (equation (2.1)),

$$h_n^{[0,m]}(\epsilon) = \frac{(-1)^m}{(\epsilon - \epsilon_m)\dots(\epsilon - \epsilon_1)} \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \dots & h_n(\epsilon_m) \\ h_{n-m}(\epsilon) & \dots & h_n(\epsilon) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_{n-1}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \dots & h_{n-1}(\epsilon_m) \end{vmatrix}}. \tag{2.20}$$

Proof. The above representation follows from formula (2.12) and from the fact that

$$\frac{1}{(t-\epsilon_{m+1})\dots(t-\epsilon_1)} = \sum_{j=1}^{m+1} \frac{1}{t-\epsilon_j} \prod_{k\neq j} \frac{1}{\epsilon_j-\epsilon_k}. \tag{2.21}$$

Indeed we find from formula (2.12) that $h_n^{[0,m]}(\epsilon)$ is the ratio of the determinants. The elements of the last row of the determinant in the numerator are the integrals

$$\frac{1}{2\pi i} \int \frac{\pi_{n-k}(t)d\alpha(t)}{(t-\epsilon)(t-\epsilon_m)\dots(t-\epsilon_1)}, \qquad 0 \le k \le m.$$

Using identity (2.21) and noting that the only term

$$\frac{1}{t-\epsilon} \frac{1}{(\epsilon-\epsilon_m)\dots(\epsilon-\epsilon_1)}. \tag{2.22}$$

of the sum (2.21) contributes to the determinant, (2.20) follows. □

Equation (2.20) immediately implies the following analogy of (2.5) for the $h_k^{[0,m]}$'s.

Corollary 2.9. *Let $0 \le m \le n$. Then the product of the Cauchy transforms of monic orthogonal polynomials with respect to the measures $d\alpha^{[0,j]}(t)$, $0 \le j \le m$ can be written as a determinant,*

$$\prod_{j=0}^{m} h_{n-m+j}^{[0,j]}(\epsilon_{j+1}) = \frac{(-1)^{\frac{m(m+1)}{2}}}{\triangle(\epsilon)} \begin{vmatrix} h_{n-m}(\epsilon_1) & \dots & h_n(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_{m+1}) & \dots & h_n(\epsilon_{m+1}) \end{vmatrix}. \tag{2.23}$$

Now we derive the identity for the average of the product of inverse random characteristic polynomials.

Theorem 2.10. *Suppose $1 \le M \le N$ and let $\gamma_n = -\frac{2\pi i}{c_n^2}$, where c_n is the normality constant defined by equation* (1.4)*. Then we have the following formula*

$$\left\langle \prod_{j=1}^{M} D_N^{-1}\left[\epsilon_j, H\right] \right\rangle_\alpha = (-1)^{\frac{M(M-1)}{2}} \frac{\prod_{j=N-M}^{N-1}\gamma_j}{\triangle(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \dots & h_{N-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \dots & h_{N-1}(\epsilon_M) \end{vmatrix}. \tag{2.24}$$

Proof. When $M = 1$, we use the identity (2.21) together with (2.7) and the relation (see, e.g., [18])

$$\gamma_{n-1} = -2\pi i n \frac{Z_{n-1}}{Z_n} \tag{2.25}$$

to obtain

$$\left\langle D_N^{-1}[\epsilon, H] \right\rangle_\alpha = \gamma_{N-1} h_{N-1}(\epsilon). \tag{2.26}$$

We rewrite the average in equation (2.24) as follows:

$$\left\langle \prod_{j=1}^{M} D_N^{-1}\left[\epsilon_j, H\right] \right\rangle_\alpha = \frac{Z_N^{[0,M]}}{Z_{N-1}^{[0,M-1]}} \frac{Z_{N-1}^{[0,M-1]}}{Z_{N-2}^{[0,M-2]}} \cdots \frac{Z_{N-M}^{[0,0]}}{Z_N^{[0,0]}} \tag{2.27}$$

where

$$Z_N^{[0,M]} = \int \cdots \int \triangle^2(x) d\alpha^{[0,M]}(x), \tag{2.28}$$

$Z_N^{[0,0]} \equiv Z_N$ and $d\alpha^{[0,0]}(x) = d\alpha(x)$. The following relation can be observed from equations (2.26) and (2.25):

$$\frac{Z_{N-K}^{[0,m]}}{Z_{N-K-1}^{[0,m-1]}} = -2\pi i(N-K)\, h_{N-K-1}^{[0,m-1]}(\epsilon_m). \tag{2.29}$$

Inserting this relation in (2.27) we find

$$\left\langle \prod_{j=1}^{M} D_N^{-1}\left[\epsilon_j, H\right] \right\rangle_\alpha = \prod_{j=1}^{M} \gamma_{N-j} h_{N-j}^{[0,M-j]}(\epsilon_{M-j+1}). \tag{2.30}$$

Our result (2.24) immediately follows from the above equation and formula (2.23). □

We now repeat the above considerations for the case

$$d\alpha^{[\ell,m]}(t) = \frac{(\mu_1 - t)\cdots(\mu_\ell - t)}{(\epsilon_1 - t)\cdots(\epsilon_m - t)} d\alpha(t). \tag{2.31}$$

The first result is a Christoffel type formula for the measure (2.31), which is due to Uvarov [19]:

Lemma 2.11. *Suppose $0 \le m \le n$. Then the monic orthogonal polynomials $\pi_n^{[\ell,m]}(t)$'s with respect to the measure $d\alpha^{\ell,m]}(t)$ have the following representation:*

$$\pi_n^{[\ell,m]}(t) = \frac{1}{(t-\mu_\ell)\ldots(t-\mu_1)} \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \ldots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \ldots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \ldots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \ldots & \pi_{n+\ell}(\mu_\ell) \\ \pi_{n-m}(t) & \ldots & \pi_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \ldots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \ldots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \ldots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \ldots & \pi_{n+\ell}(\mu_\ell) \end{vmatrix}}. \tag{2.32}$$

Proof. As in the previous cases we define $q_n^{[\ell,m]}(t)$ to be the determinant in the numerator of (2.32). Observe that

$$q_n^{[\ell,m]}(\mu_1) = \ldots = q_n^{[\ell,m]}(\mu_\ell) = 0 \tag{2.33}$$

and that

$$\int \frac{q_n^{[\ell,m]}(t)d\alpha(t)}{\epsilon_1 - t} = \ldots = \int \frac{q_n^{[\ell,m]}(t)d\alpha(t)}{\epsilon_m - t} = 0. \tag{2.34}$$

The next steps are the same as in the proofs of Lemma (2.1) and Lemma (2.5). □

Corollary 2.12.

$$\left\langle \prod_{j=1}^{K} D_N[\mu_j, H] \right\rangle_{\alpha^{[0,M]}} = \frac{1}{\triangle(\mu)} \frac{\begin{vmatrix} h_{N-M}(\epsilon_1) & \ldots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \ldots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \ldots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_K) & \ldots & \pi_{N+K-1}(\mu_K) \end{vmatrix}}{\begin{vmatrix} h_{N-M}(\epsilon_1) & \ldots & h_N(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \ldots & h_N(\epsilon_M) \end{vmatrix}}. \tag{2.35}$$

Proof. Identity (2.35) follows from equations (2.10) and (2.32) once we note that equation (2.32) can be rewritten in a similar manner as equation (2.5). □

Finally we generalize Theorem (2.3) and Theorem (2.10) and obtain a formula for the average of ratios of characteristic polynomials.

Theorem 2.13. *Suppose $0 \le M \le N$. Then the average of ratios of characteristic polynomials of $N \times N$ Hermitian matrices H is given by the following formula:*

$$\left\langle \frac{\prod_{j=1}^{K} D_N[\mu_j, H]}{\prod_{j=1}^{M} D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{(-1)^{\frac{M(M-1)}{2}} \prod_{j=N-M}^{N-1} \gamma_j}{\triangle(\mu)\triangle(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \ldots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \ldots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \ldots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_K) & \ldots & \pi_{N+K-1}(\mu_K) \end{vmatrix}. \tag{2.36}$$

Proof. Let $\alpha^{[0,0]} \equiv \alpha$, $\mathcal{Z}_n^{[0,0]} \equiv Z_n$. Then we have

$$\left\langle \frac{\prod_{j=1}^{K} D_N[\mu_j, H]}{\prod_{j=1}^{M} D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \frac{Z_N^{[K,M]}}{Z_N^{[0,0]}} = \frac{Z_N^{[K,M]}}{Z_N^{[0,M]}} \frac{Z_N^{[0,M]}}{Z_N^{[0,0]}} \tag{2.37}$$

i.e.

$$\left\langle \frac{\prod_{j=1}^{K} D_N[\mu_j, H]}{\prod_{j=1}^{M} D_N[\epsilon_j, H]} \right\rangle_{\alpha} = \left\langle \prod_{j=1}^{K} D_N[\mu_j, H] \right\rangle_{\alpha^{[0,M]}} \left\langle \prod_{j=1}^{M} D_N^{-1}[\epsilon_j, H] \right\rangle_{\alpha} . \tag{2.38}$$

We use Corollary (2.12) and Theorem (2.10) to obtain formula (2.36). □

Remark 2.14. Observe that formulae (2.6), (2.24) do not follow immediately as special cases of (2.36): some further algebraic manipulation is required. Similarly, the process of adding and removing zeros is clearly reciprocal. More precisely, given $\epsilon_1, \cdots, \epsilon_\ell$, we can construct the polynomials $\pi_n^{[0,\ell]}(t; d\alpha^{[0,\ell]})$ associated with the measure $d\alpha^{[0,\ell]}(t) = \left(\prod_{i=1}^{\ell}(\epsilon_i - t)^{-1}\right)dt$ by (2.12): We can then construct $\pi_n^{[\ell,0]}\big(t; d(\alpha^{[0,\ell]})^{[\ell,0]}\big)$ with $\mu_i = \epsilon_i$, inserting $\pi_n^{[0,\ell]}(t; d\alpha^{[0,\ell]})$ for $\pi_n(t)$ on the right-hand-side of (2.1). We should find that $\pi_n^{[\ell,0]}\big(t; d(\alpha^{[0,\ell]})^{[\ell,0]}\big) = \pi_n(t; d\alpha)$. However, again, this relation is not immediately clear, and requires further algebraic manipulation.

3 Formulae of two-point function type

The following integral version of the Binet-Cauchy formula is due to Andréief [2], and plays a basic role in our calculations.

Lemma 3.1. *Let $(X, d\mu)$ be a measure space and suppose $f_i, g_j \in L^2(X, d\mu)$ for $1 \le i, j \le k$. Then*

$$\begin{aligned} &\int_X \cdots \int_X \det(f_i(x_j))_{1\le i,j\le k} \det(g_i(x_j))_{1\le i,j\le k} d\mu(x_1)\cdots d\mu(x_k) \\ &\quad = k! \det\left(\int_X f_i(x) g_j(x) d\mu(x)\right)_{1\le i,j\le k} . \end{aligned} \tag{3.1}$$

Proof. Set $c_{ij} = \int_X f_i(x) g_j(x) d\mu(x)$. Then

$$\begin{aligned} &\int_X \cdots \int_X \det(f_i(x_j))_{1\le i,j\le k} \det(g_i(x_j))_{1\le i,j\le k} d\mu(x_1)\cdots d\mu(x_k) \\ &= \sum_{\sigma,\tau\in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) c_{\sigma(1)\tau(1)}\cdots c_{\sigma(k)\tau(k)} \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma) \sum_{\tau} \operatorname{sgn}(\tau\circ\sigma) c_{\sigma(1)\tau\circ\sigma(1)}\cdots c_{\sigma(k)\tau\circ\sigma(k)} \\ &= \sum_{\sigma} (\operatorname{sgn}(\sigma))^2 \sum_{\tau} \operatorname{sgn}(\tau) c_{1\tau(1)}\cdots c_{k\tau(k)} \\ &= k! \det(c_{ij})_{1\le i,j\le k} \end{aligned} \tag{3.2}$$

as desired. In (3.2) we used $\operatorname{sgn}(\tau\circ\sigma) = (\operatorname{sgn}\tau)(\operatorname{sgn}\sigma)$ and the fact that $c_{\sigma(1)\tau\circ\sigma(1)}\cdots c_{\sigma(k)\tau\circ\sigma(k)} = c_{1\tau(1)}\cdots c_{k\tau(k)}$ for all σ. □

Theorem 3.2. *Let $K \geq 1$. Then the following identity is valid:*

$$\left\langle \prod_{j=1}^{K} D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_\alpha = \frac{C_{N,K}}{\Delta(\lambda)\Delta(\mu)} \det\big(W_{I,N+K}(\lambda_i, \mu_j)\big)_{1\leq i,j\leq K} \tag{3.3}$$

where

$$W_{I,N+K}(x,y) = \frac{\pi_{N+K}(x)\pi_{N+K-1}(y) - \pi_{N+K}(y)\pi_{N+K-1}(y)}{x-y} \tag{3.4}$$

and

$$C_{N,K} = \frac{\prod_{\ell=N}^{N+K-1} c_\ell^2}{(c_{N+K-1})^{2K}} \tag{3.5}$$

where c_ℓ is again the norming constant for π_ℓ given in (1.4).

Proof. Let $p_j(x) = c_j^{-1}\pi_j(x)$, $j \geq 0$, denote the orthonormal polynomials with respect to $d\alpha$. From (1.2) we obtain

$$\left\langle \prod_{j=1}^{K} D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_\alpha = \frac{1}{Z_N \Delta(\lambda)\Delta(\mu)} \int \cdots \int \Delta(x,\lambda)\Delta(x,\mu) d\alpha(x). \tag{3.6}$$

Adding columns, we see that the Vandermonde determinant $\Delta(x,\lambda)$ has the form

$$\begin{vmatrix} \pi_0(x_1) & \pi_1(x_1) & \cdots & \pi_{N+K-1}(x_1) \\ \vdots & & & \\ \pi_0(x_N) & \pi_1(x_N) & \cdots & \pi_{N+K-1}(x_N) \\ \pi_0(\lambda_1) & \pi_1(\lambda_1) & \cdots & \pi_{N+K-1}(\lambda_1) \\ \vdots & & & \\ \pi_0(\lambda_K) & \pi_1(\lambda_K) & \cdots & \pi_{N+K-1}(\lambda_K) \end{vmatrix} \tag{3.7}$$

and similarly for $\Delta(x,\mu)$. Here $\pi_j(t) = \pi_j^{[0,0]}(t)$. The determinant $\Delta(x,\lambda)$ can be evaluated by a Lagrange expansion of the form

$$\sum_{0\leq i_1<i_2<\cdots<i_k\leq N+K-1} \sigma_{i_1,\cdots,i_K} \begin{vmatrix} \pi_{i_1}(\lambda_1) & \cdots & \pi_{i_K}(\lambda_1) \\ \vdots & & \\ \pi_{i_1}(\lambda_K) & \cdots & \pi_{i_K}(\lambda_K) \end{vmatrix} \begin{vmatrix} \pi_{j_1}(x_1) & \cdots & \pi_{j_N}(x_1) \\ \vdots & & \\ \pi_{j_1}(x_N) & \cdots & \pi_{j_N}(x_N) \end{vmatrix} \tag{3.8}$$

where $\sigma_{i_1,\cdots,i_K} = \pm 1$ is an appropriate signature and $\{(j_1,\cdots,j_N) : 0 \leq j_1 < j_2 < \cdots < j_N \leq N+K-1\}$ is the complement of $\{i_1,\cdots,i_K\}$ in $\{0,1,\cdots,N+K-1\}$. Multiplying (3.8) by a similar expansion for $\Delta(x,\mu)$, and inserting in (3.6), we obtain a sum of terms of the form

$$\int \cdots \int \begin{vmatrix} \pi_{j_1}(x_1) & \cdots & \pi_{j_K}(x_1) \\ \vdots & & \\ \pi_{j_1}(x_N) & \cdots & \pi_{j_K}(x_N) \end{vmatrix} \begin{vmatrix} \pi_{j_1'}(x_1) & \cdots & \pi_{j_N'}(x_1) \\ \vdots & & \\ \pi_{j_1'}(x_N) & \cdots & \pi_{j_N'}(x_N) \end{vmatrix} d\alpha(x) \tag{3.9}$$

which is equal by Lemma 3.1 to $N!\det\big(\int \pi_{j_i'}(x)\pi_{j_k}(x)d\alpha(x)\big)_{1\le i,k\le N} = N!\det(\delta_{j_i' j_k} c_{j_k}^2)_{1\le i,k\le N}$. From this we see that

$$
\begin{aligned}
&\left\langle \prod_{j=1}^{K} D_N[\lambda_j, H] D_N[\mu_j, H] \right\rangle_\alpha \\
&= \frac{N!}{Z_N \Delta(\lambda)\Delta(\mu)} \sum_{0\le i_1<\cdots<i_k\le N+K-1} \sigma^2_{i_1,\cdots,i_K} \begin{vmatrix} \pi_{i_1}(\lambda_1) & \cdots & \pi_{i_K}(\lambda_1) \\ \vdots & & \\ \pi_{i_1}(\lambda_K) & \cdots & \pi_{i_K}(\lambda_K) \end{vmatrix} \\
&\quad \times \prod_{k=1}^{N} c_{j_k}^2 \begin{vmatrix} \pi_{i_1}(\mu_1) & \cdots & \pi_{i_K}(\mu_1) \\ \vdots & & \\ \pi_{i_1}(\mu_K) & \cdots & \pi_{i_K}(\mu_K) \end{vmatrix} \\
&= \frac{N! \prod_{q=N}^{N+K-1} c_q^2}{Z_N \Delta(x,\lambda)\Delta(x,\mu)} \sum_{0\le i_1<\cdots<i_k\le N+K-1} \det\big(p_{i_j}(\lambda_k)\big)_{1\le j,k\le K} \det\big(p_{i_j}(\mu_k)\big)_{1\le j,k\le K} \\
&= \frac{N! \prod_{q=N}^{N+K-1} c_q^2}{Z_N \Delta(x,\lambda)\Delta(x,\mu)} \det\Bigg(\sum_{0\le i\le N+K-1} p_i(\lambda_j)p_i(\mu_k)\Bigg)_{1\le j,k\le K}
\end{aligned}
\tag{3.10}
$$

where the last line follows by applying Lemma 3.1 to the discrete measure $d\mu = \sum_{i=0}^{N+K-1} \delta_i$. But by the Christoffel-Darboux formula

$$
\sum_{0\le i\le N+K-1} p_i(\lambda_j)p_i(\mu_k) = \frac{\pi_{N+K}(\lambda_j)\pi_{N+K-1}(\mu_k) - \pi_{N+K}(\mu_k)\pi_{N+K-1}(\lambda_j)}{\lambda_j - \mu_k}
\tag{3.11}
$$

which then implies (3.3) as $Z_N = N!\prod_{\ell=0}^{N-1} c_\ell^2$ (see, e.g. [18]). □

Theorem 3.3. *Suppose $1 \le K \le N$. Then the following identity is valid:*

$$
\left\langle \prod_{j=1}^{K} \frac{D_N[\mu_i, H]}{D_N[\epsilon_j, H]} \right\rangle_\alpha = (-1)^{K(K-1)/2} \gamma_{N-1}^K \frac{\Delta(\epsilon,\mu)}{\Delta^2(\epsilon)\Delta^2(\mu)} \det\big(W_{II,N}(\epsilon_i,\mu_j)\big)_{1\le i,j\le K}
\tag{3.12}
$$

where

$$
W_{II,N}(x,y) = \frac{h_N(\epsilon)\pi_{N-1}(\mu) - h_{N-1}(\epsilon)\pi_N(\mu)}{\epsilon - \mu}
\tag{3.13}
$$

and again $h_k(\epsilon) = \frac{1}{2\pi i}\int \frac{\pi_k(t)d\alpha(t)}{t-\epsilon}$ is the Cauchy transform of $\pi_k(t)$ and $\gamma_{N-1} = -2\pi i/C_{N-1}^2$.

Observe first that by linearity

$$\begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_{N+L-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_{N+L-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_L) \end{vmatrix} = \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} \times \begin{vmatrix} \pi_{N-M}(\lambda_1) & \cdots & \pi_{N+L-1}(\lambda_1) \\ \vdots & & \\ \pi_{N-M}(\lambda_M) & \cdots & \pi_{N+L-1}(\lambda_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+L-1}(\mu_L) \end{vmatrix}. \tag{3.14}$$

Inserting (2.36) on the left-hand-side, and using (2.5) to re-express the integrand on the right-hand-side, we obtain the following result, which is of independent interest. The result expresses averages of ratios of characteristic polynomials in terms of averages of products of such polynomials.

Proposition 3.4. *Let* $1 \leq M \leq N$*. Then*

$$\begin{aligned} \left\langle \frac{\prod_{j=1}^L D_N[\mu_i, H]}{\prod_{j=1}^M D_N[\epsilon_j, H]} \right\rangle_\alpha &= \frac{(-1)^{M(M-1)/2} \prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\mu)\Delta(\epsilon)} \\ &\times \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} \Delta(\lambda, \mu) \left\langle \prod_{j=1}^M D_{N-M}[\lambda_j, H] \prod_{j=1}^L D_{N-M}[\mu_j, H] \right\rangle_\alpha . \end{aligned} \tag{3.15}$$

Proof of Theorem 3.2. For $M = L = K \leq N$, by (3.15) and (3.3),

$$\begin{aligned} &\frac{\Delta(\mu)\Delta(\epsilon)}{(-1)^{K(K-1)/2} \prod_{j=N-K}^{N-1} \gamma_j} \left\langle \frac{\prod_{j=1}^K D_N[\mu_i, H]}{\prod_{j=1}^K D_N[\epsilon_j, H]} \right\rangle_\alpha \\ &= \int \cdots \int \frac{d\alpha(\lambda)}{(2\pi i)^M \prod_{j=1}^M (\lambda_j - \epsilon_j)} C_{N-K,K} \prod_{i=1}^K \prod_{j=1}^K (\mu_i - \lambda_j) \det\big(W_{I,N}(\lambda_i, \mu_j)\big)_{1 \leq i,j \leq K}. \end{aligned} \tag{3.16}$$

But

$$\frac{1}{2\pi i}\int \frac{d\alpha(\lambda_j)}{\lambda_j-\epsilon_j}\prod_{i=1}^{K}(\mu_i-\lambda_j)\frac{\pi_N(\lambda_j)\pi_{N-1}(\mu_k)-\pi_{N-1}(\lambda_j)\pi_N(\mu_k)}{\lambda_j-\mu_k}$$
$$=\frac{1}{2\pi i}\int d\alpha(\lambda_j)\left(1-\frac{\mu_1-\epsilon_j}{\lambda_j-\epsilon_j}\right)\left(\prod_{\substack{i=2\\ i\neq k}}^{K}(\mu_i-\lambda_j)\right)\big(\pi_N(\lambda_j)\pi_{N-1}(\mu_k)-\pi_{N-1}(\lambda_j)\pi_N(\mu_k)\big)$$
$$=-\frac{1}{2\pi i}\int d\alpha(\lambda_j)\frac{\mu_1-\epsilon_j}{\lambda_j-\epsilon_j}\left(\prod_{\substack{i=2\\ i\neq k}}^{K}(\mu_i-\lambda_j)\right)\big(\pi_N(\lambda_j)\pi_{N-1}(\mu_k)-\pi_{N-1}(\lambda_j)\pi_N(\mu_k)\big) \tag{3.17}$$

as $\int d\alpha(\lambda_j)\lambda_j^\ell\pi_{N-1}(\lambda_j)=\int d\alpha(\lambda_j)\lambda_j^\ell\pi_N(\lambda_j)=0$ for $0\le\ell\le K-2<N-1$. Continuing in this way, the integral reduces to $\prod_{i=1}^K(\mu_i-\epsilon_j)W_{II,N}(\epsilon_i,\mu_k)$. Thus we find

$$\frac{\Delta(\mu)\Delta(\epsilon)}{(-1)^{K(K-1)/2}\prod_{j=N-K}^{N-1}\gamma_j}\left\langle\frac{\prod_{j=1}^K D_N[\mu_i,H]}{\prod_{j=1}^K D_N[\epsilon_j,H]}\right\rangle_\alpha=\frac{\Delta(\epsilon,\mu)}{\Delta(\epsilon)\Delta(\mu)}\det\big(W_{I,N+K}(\lambda_i,\mu_k)\big)_{1\le i,k\le K} \tag{3.18}$$

and (3.12) follows. □

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