Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

ELA

PRODUCTS OF COMMUTING NILPOTENT OPERATORS*

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Abstract. Matrices that are products of two (or more) commuting square-zero matrices and matrices that are products of two commuting nilpotent matrices are characterized. Also given are characterizations of operators on an infinite dimensional Hilbert space that are products of two (or more) commuting square-zero operators, as well as operators on an infinite-dimensional vector space that are products of two commuting nilpotent operators.

Key words. Commuting matrices and operators, Nilpotent matrices and operators, Factorization, Products.

AMS subject classifications. 15A23, 15A27, 47A68, 47B99.

1. Introduction. Is every complex singular square matrix a product of two nilpotent matrices? Laffey [5] and Sourour [8] proved that the answer is positive: any complex singular square matrix A (which is not 2×2 nilpotent with rank 1) is a product of two nilpotent matrices with ranks both equal to the rank of A. Earlier Wu [9] studied the problem. (Note that [9, Lem. 3] holds but the decomposition given in its proof on [9, p. 229] is not correct since the latter matrix given for the odd case is not always nilpotent.) Novak [6] characterized all singular matrices in $\mathcal{M}_n(\mathbb{F})$, where $\mathbb{F}$ is a field, which are a product of two square-zero matrices. Related problem of existence of k-th root of a nilpotent matrix was studied by Psarrakos in [7].

Similar results were proved for the set $\mathcal{B}(\mathcal{H})$ of all bounded (linear) operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. Fong and Sourour [3] proved that every compact operator is a product of two quasinilpotent operators and that a normal operator is a product of two quasinilpotent operators if and only if 0 is in its essential spectrum. Drnovšek, Müller, and Novak [2] proved that an operator is a product of two quasinilpotent operators if and only if it is not semi-Fredholm. Novak [6] characterized operators that are products of two and of three square-zero operators.

Here we consider similar questions for products of commuting square-zero or commuting nilpotent operators on a finite dimensional vector space or on a infinite-dimensional Hilbert or vector space. The commutativity condition considerably restricts the set of operators that are such products. Namely, if $A = BC$ and B, C are commuting nilpotent operators then A is nilpotent as well and it commutes with both B and C. If in addition B and C are square-zero then so is A.

* Received by the editors 5 February 2007. Accepted for publication 10 August 2007. Handling Editor: Harm Bart. This research was supported in part by the Research Agency of the Republic of Slovenia.

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In the paper we characterize the following sets of matrices and operators:

- Matrices that are products of k commuting square-zero matrices for each $k \geq 2$.
- Matrices that are products of two commuting nilpotent matrices.
- Operators on a Hilbert space that are products of k commuting square-zero operators for each $k \geq 2$.
- Operators on an infinite-dimensional vector space that are products of two commuting nilpotent operators.

2. When is a matrix a product of commuting square-zero matrices?

First we consider the following question:

QUESTION 1. *Which matrices $A \in \mathcal{M}_n(\mathbb{F})$ can be written as a product $A = BC$, where $B^2 = C^2 = 0$ and $BC = CB$?*

Observe that if A, B and C are as above then B and C commute with A.

EXAMPLE 2.1. It can be easily seen that

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

but E_{13} cannot be written as a product of two commuting square-zero matrices. Therefore the set of matrices that can be written as a product of two commuting square-zero matrices is not the same as the set of matrices that are products of two square-zero matrices.

Next, we have that

$$E_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = 0.$$

Thus E_{14} is a product of two commuting square-zero matrices. □

We denote by $J_{\underline{\mu}} = J_{(\mu_1, \mu_2, \ldots, \mu_t)} = J_{\mu_1} \oplus J_{\mu_2} \oplus \ldots \oplus J_{\mu_t}$ the upper triangular nilpotent matrix in its Jordan canonical form with blocks of order $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t > 0$. If A is similar to $J_{\underline{\mu}}$ then we call $\underline{\mu}$ the *partition* corresponding to A. We also

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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say that $\underline{\mu}$ is the Jordan canonical form of A. For a finite sequence of natural numbers $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ we denote by $\mathrm{ord}(\underline{\lambda}) = \underline{\mu}$ the ordered sequence $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_t$.

Let $\iota(A)$ denote the index of nilpotency of matrix A. For a nilpotent matrix A, define a sequence

$$\mathcal{J}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_{\iota(A)}) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where α_i is the number of Jordan blocks of the size i and $\alpha_j = 0$ for $j > \iota(A)$. Note that $\sum_{j=1}^{n} j\alpha_j = n$.

If C commutes with $J_{\underline{\mu}}$ it is of the form $C = [C_{ij}]$, where $C_{ij} \in \mathcal{M}_{\mu_i \times \mu_j}$ and C_{ij} are all upper triangular Toeplitz matrices (see e. g. [4, p. 297]), i.e. for $1 \leq i \leq j \leq t$ we have

$$C_{ij} = \begin{bmatrix} 0 & \ldots & 0 & c_{ij}^0 & c_{ij}^1 & \ldots & c_{ij}^{\mu_i-1} \\ \vdots & & \ddots & 0 & c_{ij}^0 & \ddots & \vdots \\ \vdots & & & \ddots & 0 & \ddots & c_{ij}^1 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & c_{ij}^0 \end{bmatrix} \text{ and } C_{ji} = \begin{bmatrix} c_{ji}^0 & c_{ji}^1 & \ldots & c_{ji}^{\mu_i-1} \\ 0 & c_{ji}^0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{ji}^1 \\ \vdots & & 0 & c_{ji}^0 \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix}. \tag{2.1}$$

If $\mu_i = \mu_j$ then we omit the rows or columns of zeros in C_{ji} or C_{ij} above.

Proposition 2.2. *A matrix A is a product of two commuting square-zero matrices if and only if it has a Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \leq \frac{n}{4}$, i.e. if and only if $\mathcal{J}(A) = (n-2x, x)$ for some $x \leq \frac{n}{4}$.*

Proof. Since $A^2 = B^2C^2 = 0$, it follows that also A is a square-zero matrix. Since $B^2 = 0$, the Jordan canonical form of matrix B is equal to $\underline{\mu} = (2^a, 1^{n-2a})$ for some $0 \leq a \leq \frac{n}{2}$. Suppose that $B = J_{\underline{\mu}}$ is in its Jordan canonical form. Since C commutes with B it is of the form $C = [C_{ij}]$, where C_{ij} are given in (2.1). Following Basili [1, p. 60, Lemma 2.3], the matrix C is similar to

$$\begin{bmatrix} U_1 & X & Y \\ 0 & U_1 & 0 \\ 0 & W & U_2 \end{bmatrix},$$

where $U_1, X \in \mathcal{M}_{a\times a}$, $Y \in \mathcal{M}_{a\times(n-2a)}$, $W \in \mathcal{M}_{(n-2a)\times a}$, $U_2 \in \mathcal{M}_{(n-2a)\times(n-2a)}$ and U_1 and U_2 are strictly upper triangular matrices. Note that B is transformed by the same similarity to

$$\tilde{B} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

Take an invertible matrix P_1 such that $P_1 U_1 P_1^{-1} = J_{\underline{\lambda}}$ and denote

$$\tilde{C} = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} U_1 & X & Y \\ 0 & U_1 & 0 \\ 0 & W & U_2 \end{bmatrix} \begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & P_1^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} J_{\underline{\lambda}} & X' & Y' \\ 0 & J_{\underline{\lambda}} & 0 \\ 0 & W' & U_2 \end{bmatrix}.$$

Note that $\tilde{B}$ does not change under the above similarity. Since $C^2 = 0$, also $\tilde{C}^2 = 0$ and thus $J_{\underline{\lambda}}^2 = 0$. Therefore, $\underline{\lambda} = (2^x, 1^{a-2x})$, where $0 \le x \le \frac{a}{2} \le \frac{n}{4}$. We see that

$$\tilde{B}\tilde{C} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_{\underline{\lambda}} & X' & Y' \\ 0 & J_{\underline{\lambda}} & 0 \\ 0 & W' & U_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & J_{\underline{\lambda}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, it easily follows that $\mathrm{rk}(A) = \mathrm{rk}(\tilde{B}\tilde{C}) = x$. Since $A^2 = 0$, we see that A must have Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{4}$.

Now, take a nilpotent matrix A with its Jordan canonical form $(2^x, 1^{n-2x})$, where $x \le \frac{n}{4}$. Then there exists an invertible matrix Q such that

$$QAQ^{-1} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{x} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{n-4x}.$$

In Example 2.1 we observed that the matrix E_{14} is a product of two commuting square-zero matrices. Then it follows that QAQ^{-1} and A are also products of two commuting square-zero matrices. We have proved the proposition. □

THEOREM 2.3. *A matrix A is a product of k pairwise commuting square-zero matrices if and only if it has a Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{2^k}$, i.e. if and only if $\mathcal{J}(A) = (n-2x, x)$ for some $x \le \frac{n}{2^k}$.*

Proof. Let A be a matrix with Jordan canonical form $(2^x, 1^{n-2x})$ for some $x \le \frac{n}{2^k}$. Then it is similar to a matrix

$$A' = \underbrace{E_{1\,2^k} \oplus E_{1\,2^k} \oplus \ldots \oplus E_{1\,2^k}}_{x} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{n-2^k x},$$

where $E_{1\,2^k} \in \mathcal{M}_{2^k}(\mathbb{C})$ is a matrix with only nonzero element (equal to 1) in the upper-right corner. To prove that A is a product of k pairwise commuting square-zero matrices it is sufficient to show, that $E_{1\,2^k}$ is a product of k pairwise commuting square-zero matrices.

We define matrices

$$C_i = \begin{bmatrix} 0_{2^{i-1}} & I_{2^{i-1}} \\ 0_{2^{i-1}} & 0_{2^{i-1}} \end{bmatrix} \in \mathcal{M}_{2^i}(\mathbb{C})$$

for every $i = 1, 2, ..., k$ and let

$$B_i = \underbrace{C_i \oplus C_i \oplus \ldots \oplus C_i}_{2^{k-i}} \in \mathcal{M}_{2^k}(\mathbb{C}).$$

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A publication of the International Linear Algebra Society
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It is easy to check that $B_i^2 = 0$ and $B_iB_j = B_jB_i$ for every i, j and that

$$E_{1\,2^k} = B_1B_2 \ldots B_k.$$

To prove the converse we have to show that every product of k pairwise commuting square-zero matrices has rank at most $\frac{n}{2^k}$. We will show this by induction. The assertion is true for $k = 2$ by the previous proposition. Suppose that every product of k pairwise commuting square-zero matrices has rank at most $\frac{n}{2^k}$ and let

$$A = B_1B_2 \ldots B_{k+1}$$

where $B_1, B_2, \ldots B_{k+1}$ are pairwise commuting square-zero matrices. Denote by m the rank of B_1. Since $B_1^2 = 0$ we have that $m \leq \frac{n}{2}$. Now the matrix B_1 is similar to a matrix

$$B_1' = \begin{bmatrix} 0_m & I_m & 0 \\ 0_m & 0_m & 0 \\ 0 & 0 & 0_{2n-m} \end{bmatrix}$$

Again following Basili [1, p. 60, Lemma 2.3], we transform the matrices B_i simultaneously by similarity to the matrices

$$B_i' = \begin{bmatrix} X_i & Y_i & Z_i \\ 0 & X_i & 0 \\ 0 & U_i & V_i \end{bmatrix}.$$

Here matrices X_i are square-zero and they pairwise commute. Now

$$A' = B_1'B_2' \ldots B_{k+1}' =$$

$$= \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_2 & Y_2 & Z_2 \\ 0 & X_2 & 0 \\ 0 & U_2 & V_2 \end{bmatrix} \ldots \begin{bmatrix} X_{k+1} & Y_{k+1} & Z_{k+1} \\ 0 & X_{k+1} & 0 \\ 0 & U_{k+1} & V_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & X_2 \ldots X_{k+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the matrix $X_2 \ldots X_{k+1}$ is a product of k pairwise commuting square-zero matrices, so it has the rank at most $\frac{m}{2^k}$ and thus the rank of A is at most $\frac{n}{2^{k+1}}$. □

3. When is a matrix a product of two commuting nilpotent matrices?

In this section we study the following question:

QUESTION 2. *Which matrices $A \in \mathcal{M}_n(\mathbb{F})$ can be written as $A = BC = CB$, where B and C are nilpotent matrices?*

Clearly, A must be nilpotent. Thus, not every singular matrix is a product of two commuting nilpotent matrices.

Moreover, suppose that $\mathrm{rk}(A) = n-1$ and $A = BC = CB$ with B and C nilpotent. Then also $\mathrm{rk}(B) = \mathrm{rk}(C) = n-1$ and thus $B = PJ_nP^{-1}$ and $P^{-1}CP = p(J_n)$, where p is a polynomial such that $p(0) = 0$. Then $A = BC = PJ_np(J_n)P^{-1}$ and thus $\mathrm{rk}(A) < n - 1$, which is a contradiction. Hence not every nilpotent matrix is a product of two commuting nilpotent matrices (for example J_n is not).

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A publication of the International Linear Algebra Society
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Example 3.4. Suppose $A = \begin{bmatrix} J_m & 0 \\ 0 & 0 \end{bmatrix}$, where $m \geq 3$. Can A be written as a product of two commuting nilpotent matrices? Assume that $A = BC$ is such a product. Since B and C commute with A it follows that $B = \begin{bmatrix} T_B & W_B \\ V_B & U_B \end{bmatrix}$ and $C = \begin{bmatrix} T_C & W_C \\ V_C & U_C \end{bmatrix}$, where $T_B, T_C \in \mathcal{M}_m(\mathbb{F})$ are (strictly) upper triangular Toeplitz matrices, $U_B, U_C \in \mathcal{M}_k(\mathbb{F})$ are nilpotent matrices (see Basili [1]), $W_B, W_C \in \mathcal{M}_{m\times k}(\mathbb{F})$ have the only nonzero entries in the first row and $V_B, V_C \in \mathcal{M}_{k\times m}(\mathbb{F})$ have the only nonzero entries in the last column.

Since $A = BC$ it follows that $J_m = T_B T_C + W_B V_C$. The product $W_B V_C$ has the only nonzero entry in the first row and the last column, and T_B and T_C are strictly upper-triangular. The assumption that $m \geq 3$ is needed to conclude that $T_B T_C + W_B V_C$ is upper triangular Toeplitz matrix with zero superdiagonal. This contradicts the fact that J_m has nonzero superdiagonal and implies that A is not a product of two commuting nilpotent matrices. □

What is the Jordan canonical form of J_n^t for $t \geq 2$? It is an easy observation that the partition of n corresponding to J_n^t is equal to $(\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $\lambda_1 - \lambda_t \leq 1$. We denote this partition by $r(n,t)$. If $n = kt + r$, where $0 \leq r < t$, then $r(n,t) = ((k+1)^r, k^{t-r})$. Note that $k = \left\lfloor \frac{n}{t} \right\rfloor$. It follows that $\mathcal{J}(J_n^t) = (\underbrace{0, \ldots, 0}_{\lfloor \frac{n}{t} \rfloor - 1}, t-r, r)$.

Proposition 3.5. *If a nilpotent matrix A has a Jordan canonical form*

$$\operatorname{ord}(r(n_1,t_1), r(n_2,t_2), \ldots, r(n_m,t_m), 1^k),$$

where $n = k + \sum_{i=1}^m n_i$ and $t_i \geq 2$ for all i, then A can be written as a product of two commuting nilpotent matrices.

Proof. Since the Jordan canonical form of $J_{n_i}^{t_i}$ is $r(n_i, t_i)$, matrix A is similar to $J_{n_1}^{t_1} \oplus J_{n_2}^{t_2} \oplus \ldots \oplus J_{n_m}^{t_m} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k}$, which is obviously equal to the product of two commuting nilpotent matrices

$$\left(J_{n_1} \oplus J_{n_2} \oplus \ldots \oplus J_{n_m} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k} \right)$$

and

$$\left(J_{n_1}^{t_1-1} \oplus J_{n_2}^{t_2-1} \oplus \ldots \oplus J_{n_m}^{t_m-1} \oplus \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{k} \right).$$

Thus also A can be written as a product of two commuting nilpotent matrices. □

In the following we show that the converse is true as well.

Theorem 3.6. *For a nilpotent matrix A the following are equivalent:*

(a) A can be written as a product of two commuting nilpotent matrices,

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A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

(b) A has a Jordan canonical form

$$\mathrm{ord}(r(n_1,t_1), r(n_2,t_2), \ldots, r(n_m,t_m), 1^k),$$

where $n = k + \sum_{i=1}^m n_i$ and $t_i \geq 2$ for all i,

(c) $\mathcal{J}(A)$ does **not** *include a subsequence $(0, \underbrace{1, 1, \ldots, 1}_{2l-1}, 0)$ for any $l \geq 1$.*

We first prove the following lemma and propositions.

Lemma 3.7. *If $\mathcal{J}(A) = (\underbrace{0, \ldots, 0}_{m-2l+1}, \underbrace{1, \ldots, 1}_{2l-1})$, where $l \geq 1$ and $m \geq 2l$, then A is* **not** *a product of two commuting nilpotent matrices.*

Proof. Suppose that $A = BC = CB$ is nilpotent matrix with $\mathcal{J}(A)$ as in the statement of the lemma. Let us denote $s = 2l - 1$ and let us assume that $A = J_{(m,m-1,\ldots,m-s+1)}$. Then

$$B = \begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1s} \\ B_{21} & B_{22} & \ldots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \ldots & B_{ss} \end{bmatrix} \text{ and } C = \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1s} \\ C_{21} & C_{22} & \ldots & C_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ C_{s1} & C_{s2} & \ldots & C_{ss} \end{bmatrix},$$

where all B_{ij} and C_{ij} are upper triangular Toeplitz and we use the notation introduced in (2.1).

Since $J_m = B_{11}C_{11} + B_{12}C_{21} + \ldots + B_{1s}C_{s1} = C_{11}B_{11} + C_{12}B_{21} + \ldots + C_{1s}B_{s1}$ and the only possible summands with nonzero superdiagonal are $B_{12}C_{21}$ and $C_{12}B_{21}$, it follows that $b_{12}^0 c_{21}^0 = c_{12}^0 b_{21}^0 = 1$. Since $J_{m-1} = B_{21}C_{12} + B_{22}C_{22} + \ldots + B_{2s}C_{s2} = C_{21}B_{12} + C_{22}B_{22} + \ldots + C_{2s}B_{s2}$ and the only possible summands with nonzero superdiagonals are $B_{21}C_{12} + B_{23}C_{31}$ and $C_{21}B_{12} + C_{23}B_{31}$, it follows that $b_{21}^0 c_{12}^0 + b_{23}^0 c_{32}^0 = c_{21}^0 b_{12}^0 + c_{23}^0 b_{32}^0 = 1$ and therefore $b_{23}^0 c_{32}^0 = c_{23}^0 b_{32}^0 = 0$.

Similarly, we show by induction, that $b_{i,i+1}^0 c_{i+1,i}^0 = c_{i,i+1}^0 b_{i+1,i}^0 = 0$ for all even i and $b_{i,i+1}^0 c_{i+1,i}^0 = c_{i,i+1}^0 b_{i+1,i}^0 = 1$ for all odd i. In particular, it follows that $b_{s-1,s}^0 c_{s,s-1}^0 = c_{s-1,s}^0 b_{s,s-1}^0 = 0$.

Furthermore, $J_{m-s+1} = B_{s1}C_{1s} + B_{s2}C_{2s} + \ldots + B_{ss}C_{ss} = C_{s1}B_{1s} + C_{s2}B_{2s} + \ldots + C_{ss}B_{ss}$ and the only possible summands with nonzero superdiagonals are $B_{s,s-1}C_{s-1,s}$ and $C_{s,s-1}B_{s-1,s}$. It follows that the superdiagonal of J_{m-s+1} is equal to $1 = b_{s,s-1}^0 c_{s-1,s}^0 = c_{s,s-1}^0 b_{s-1,s}^0 = 0$, which is a contradiction. ◻

Proposition 3.8. *If $\mathcal{J}(A) = (\alpha_1, \ldots, \alpha_{m-2l}, 0, \underbrace{1, \ldots, 1}_{2l-1}, 0, \alpha_{m+2}, \ldots, \alpha_n)$, where $l \geq 1$, then A is* **not** *a product of two commuting nilpotent matrices.*

Proof. Denote $\underline{\mu} = (n^{\alpha_n}, (n-1)^{\alpha_{n-1}}, \ldots, (m+2)^{\alpha_{m+2}})$, $\underline{\lambda} = (m, m-1, \ldots, m-2l+2)$ and $\underline{\mu}' = ((m-2l)^{\alpha_{m-2l}}, (m-2l-1)^{\alpha_{m-2l-1}}, \ldots, 1^{\alpha_1})$.

Suppose that $A = BC = CB$ with $\mathcal{J}(A)$ as in the statement. Then we can assume that $A = J_{\underline{\mu}} \oplus J_{\underline{\lambda}} \oplus J_{\underline{\mu}'}$, $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$ and $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$ in the same block partition, where all B_{ij}, C_{ij} are block upper triangular Toeplitz.

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A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

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We compute that $J_{\underline{\lambda}} = B_{21}C_{12} + B_{22}C_{22} + B_{23}C_{32} = C_{21}B_{12} + C_{22}B_{22} + C_{23}B_{32}$. Since $m - 2l + 2 > m - 2l + 1$, it follows that superdiagonals of all blocks of $J_{\underline{\lambda}}$ must be equal to superdiagonals of $B_{22}C_{22}$ and by symmetry to superdiagonals of $C_{22}B_{22}$. We have already seen in the proof of Lemma 3.7 that this is not possible. □

Proposition 3.9. *If $\mathcal{J}(A)$ does* **not** *include a subsequence $(0, \underbrace{1, 1, \ldots, 1}_{2l-1}, 0)$ for any $l \geq 1$, then the Jordan canonical form of a matrix A is equal to*

$$\operatorname{ord}(r(n_1, t_1), r(n_2, t_2), \ldots, r(n_m, t_m), 1^k),$$

where $n = k + \sum_{i=1}^{m} n_i$ and $t_i \geq 2$ for all i.

Proof. If $\mathcal{J}(A)$ does not include a subsequence of the form $(0, \underbrace{1, 1, \ldots, 1}_{2l-1}, 0)$, then for any subsequence of $\mathcal{J}(A)$ of the form

$$(0, \alpha_t, \alpha_{t+1}, \ldots, \alpha_s, 0), \tag{3.2}$$

for some $2 \leq t \leq s \leq n$, where $\alpha_i \neq 0$ for $i = t, t+1, \ldots, s$ holds either

(a) $s - t + 1$ is even or

(b) $s - t + 1$ is odd and there exists j, $t \leq j \leq s$, such that $\alpha_j \geq 2$.

So, the matrix A can be written as a direct sum $A_1 \oplus A_2 \oplus \ldots \oplus A_r$, where each A_i has one of the following forms:

(i) $\mathcal{J}(A_i) = (\alpha_1)$ and the Jordan canonical form of A_i is equal to (1^{α_1}).

(ii) $\mathcal{J}(A_i) = (\underbrace{0, \ldots, 0}_{q_i-1}, \alpha_{q_i})$, where $\alpha_{q_i} \geq 2$ and the Jordan canonical form of A_i is equal to $r(q_i\alpha_{q_i}, \alpha_{q_i})$.

(iii) $\mathcal{J}(A_i) = (\underbrace{0, \ldots, 0}_{q_i-1}, \alpha_{q_i}, \alpha_{q_i+1})$ and the Jordan canonical form of A_i is equal to $r(q_i\alpha_{q_i} + (q_i+1)\alpha_{q_i+1}, \alpha_{q_i} + \alpha_{q_i+1})$.

(iv) $\mathcal{J}(A_i) = (\underbrace{0, \ldots, 0}_{q_i-2}, \alpha_{q_i-1}, \alpha_{q_i}, \alpha_{q_i+1})$, where $\alpha_{q_i} \geq 2$ and the Jordan canonical form of A_i is equal to

$$\operatorname{ord}(r((q_i-1)\alpha_{q_i-1} + q_i(\alpha_{q_i}-1), \alpha_{q_i-1} + \alpha_{q_i} - 1), r(q_i + (q_i+1)\alpha_{q_i+1}, 1 + \alpha_{q_i+1})).$$

In case (a) we can write each block corresponding to subsequences of the form (3.2) as a direct sum of blocks of type (iii). In the case (b) we use types (ii) and (iii) if there is an odd $i \geq 1$ such that $\alpha_{t-1+i} \geq 2$ and types (iii) and (iv) otherwise. If $\alpha_1 \geq 1$ then the block corresponding to the subsequence $(\alpha_1, \alpha_2, \ldots, \alpha_s, 0)$, $\alpha_i \geq 1$, is decomposed as a direct sum of blocks of type (iii) if s is even and a combination of types (i) and (iii) if s is odd. This proves the proposition. □

Proof. (of Theorem 3.6) Since Proposition 3.8 holds also for $m = n$, the implication (a) $\Rightarrow$ (c) follows from Proposition 3.8. The implication (c) $\Rightarrow$ (b) is the statement of Proposition 3.9 and the implication (b) $\Rightarrow$ (a) is the statement of Proposition 3.5. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

4. When is an operator a product of commuting square-zero operators?

In this section we assume that $\mathcal{H}$ is an infinite-dimensional, separable, real or complex Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all operators (i.e., bounded linear transformations) on $\mathcal{H}$.

Question 3. *Which operators $A \in \mathcal{B}(\mathcal{H})$ can be written as a product of two commuting square-zero operators?*

Similarly as in the finite-dimensional case we notice that also A is square-zero. Therefore $\overline{\operatorname{im} A} \subseteq \ker A$. So the space $\overline{\operatorname{im} A} + \ker A$ is closed.

Theorem 4.10. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $A = BC = CB$, where $B^2 = C^2 = 0$, if and only if* $\dim(\ker A \cap \ker A^*) = \infty$ *and* $A^2 = 0$.

Proof. If A is a product of two square-zero operators it follows by [6] that $\dim(\ker A \ominus \overline{\operatorname{im} A}) = \infty$. Since $\ker A \ominus \overline{\operatorname{im} A} = \ker A \cap \left(\overline{\operatorname{im} A}\right)^{\perp} = \ker A \cap \ker A^*$ we have that $\dim(\ker A \cap \ker A^*) = \infty$ and $A^2 = 0$.

It remains to prove the converse. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\mathcal{H}_2 \subseteq \ker A \cap \ker A^*$. The matrix of A relative to this decomposition is of the form $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. Since $A^2 = 0$ also $D^2 = 0$. Therefore we can find a decomposition of space $\mathcal{H}_1 = \mathcal{H}_{11} \oplus \mathcal{H}_{12}$, where both subspaces are infinite-dimensional and $\overline{\operatorname{im} D} \subseteq \mathcal{H}_{11}$. The matrix of D relative to this decomposition is $\begin{bmatrix} 0 & D_1 \\ 0 & 0 \end{bmatrix}$. Since $\mathcal{H}_2$ is infinite-dimensional space, we can write it as a direct sum of two infinite-dimensional subspaces $\mathcal{H}_{21}$ and $\mathcal{H}_{22}$. The form of A relative to the decomposition $\mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22}$ is

$$\begin{bmatrix} 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Define operators B and C on $\mathcal{H}$ by

$$B = \begin{bmatrix} 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & D_1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is evident that $A = BC = CB$ and $B^2 = C^2 = 0$. □

The factorization in the proof above is based on the factorization in the finite-dimensional case. Since $\mathcal{H}_2$ is an infinite-dimensional space, we can write it as a direct sum of k infinite-dimensional subspaces. Using the factorization in the proof of Theorem 2.3 we get the following result.

Corollary 4.11. *An operator A is a product of two commuting square-zero operators if and only if A is a product of k square-zero operators.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

5. When is an operator a product of two commuting nilpotent operators?

Let V be an infinite-dimensional vector space and $A : V \to V$ a nilpotent operator with index of nilpotency n. We proceed to define the sequence

$$\mathcal{J}(A) = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

where now $\alpha_i \in \mathbb{N} \cup \{0, \infty\}$.

For $k = 0, 1, \dots, n-1$ we choose subspaces V_{n-k} such that:

1. For $k = 0$ we have $V = \ker A^n = \ker A^{n-1} \oplus V_n$.
2. For $k > 0$ we have $\ker A^{n-k} = \ker A^{n-k-1} \oplus AW_{n-k+1} \oplus V_{n-k}$, where $W_{n-k+1} = AW_{n-k+2} \oplus V_{n-k+1}$ and $W_{n+1} = 0$.

Then we define $\alpha_i = \dim V_i$, $i = 1, 2, \dots, n$. Observe that if $\dim V < \infty$ then this definition of $\mathcal{J}(A)$ coincides with the one given in §2.

Observe that if an operator A is a product of two commuting nilpotent operators, then A is also a nilpotent operator.

Theorem 5.12. *A nilpotent operator A with $\mathcal{J}(A) = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a product of two commuting nilpotent operators if and only if a matrix B with $\mathcal{J}(B) = (\beta_1, \beta_2, \dots, \beta_n)$, where $\beta_i = \min\{\alpha_i, 2\}$, is a product of two commuting nilpotent matrices.*

Before we prove the theorem let us show the following two lemmas.

Lemma 5.13. *If $\mathcal{J}(A) = (0, \dots, 0, \infty)$, then A is a product of two commuting nilpotent operators.*

Proof. Since $\alpha_i = 0$ for $i \neq n$, it follows that all the indecomposable blocks are of size n. Then A is similar to $\oplus_{k=1}^{\infty} J_{k,n} = \oplus_{k=1}^{\infty}(J_{2k-1,n} \oplus J_{2k,n})$, where $J_{k,n}$, $k \in \mathbb{N}$, are indecomposable blocks of A each of them similar to J_n. Since $J_{2k-1,n} \oplus J_{2k,n}$ is a product of two commuting nilpotent matrices by Theorem 3.6, the assertion follows. □

Lemma 5.14. *Let A be a nilpotent matrix with $\mathcal{J}(A) = (\alpha_1, \alpha_2, \dots, \alpha_n)$, where there exists an index j such that $\alpha_j \geq 2$. Suppose that B is a matrix with $\mathcal{J}(B) = (\beta_1, \beta_2, \dots, \beta_n)$, where $\beta_i = \alpha_i$ for $i \neq j$ and $\beta_j = \alpha_j + 1$. Then A is a product of two commuting nilpotent matrices if and only if B is a product of two commuting nilpotent matrices.*

Proof. By Theorem 3.6 a matrix A is a product of two commuting nilpotent matrices if and only if $\mathcal{J}(A)$ is not of the form

$$(\alpha_1, \dots, \alpha_i, 0, \underbrace{1, \dots, 1}_{2l-1}, 0, \alpha_k, \dots, \alpha_n). \tag{5.3}$$

It follows easily that matrices A and B are simultaneously the products of two commuting nilpotent matrices. □

Proof. (of Theorem 5.12) Suppose that A is a product of two commuting nilpotent operators. Similarly as in the finite-dimensional case we can show that $\mathcal{J}(A)$ can not be of the form (5.3). Hence also $\mathcal{J}(B)$ is not of that form and therefore B is a product of two commuting nilpotent matrices.

To prove the converse write A as a direct sum of A_1 and A_2 with

$$\mathcal{J}(A_j) = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{nj})$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 16, pp. 237-247, September 2007

so that $\alpha_{i1} = \alpha_i$ and $\alpha_{i2} = 0$ if α_i is finite, and $\alpha_{i1} = 2$ and $\alpha_{i2} = \infty$ otherwise. It suffices to show that A_1 and A_2 are the products of commuting nilpotent operators, which follows from the lemmas above. $\square$

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