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PRODUCTS OF COMPOSITION AND DIFFERENTIATION BETWEEN HARDY SPACES

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We shall discuss boundedness and compactness of the products of composition and differentiation between Hardy spaces.

1. INTRODUCTION

Throughout this article, we denote by $\mathbb{U}$ the open unit disk in the complex plane and by H^p $(1 \leqslant p \leqslant \infty)$ the classical Hardy space on $\mathbb{U}$. That is, for $1 \leqslant p < \infty$, H^p is the Banach space of all analytic functions f on $\mathbb{U}$ satisfying

$$\begin{aligned}\|f\|_{H^p}^p &= \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \\ &= \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta < \infty\end{aligned}$$

and H^∞ is the Banach algebra of bounded analytic functions f on $\mathbb{U}$ with the norm

$$\|f\|_\infty = \sup\Big\{\left|f(z)\right| : z \in \mathbb{U}\Big\}.$$

See [**4**] for more information on the Hardy spaces.

Let D be the differentiation operator and C_φ the operator of composition with an analytic self-map φ of $\mathbb{U}$. Then we define the products of these operators by

$$C_\varphi D f(z) = \big(C_\varphi f'\big)(z) = f'\big(\varphi(z)\big)$$

and

$$DC_\varphi f(z) = (C_\varphi f)'(z) = f'\big(\varphi(z)\big)\varphi'(z)$$

for $z \in \mathbb{U}$ and analytic function f on $\mathbb{U}$.

On a general space of analytic functions, D is typically unbounded. On the other hand, it has been showed that C_φ is bounded on various spaces of analytic functions on

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$\mathbb{U}$ ([**3, 7, 9**]), though the products $C_\varphi D$ and DC_φ are possibly still unbounded there. Hibschweiler and Portnoy [**5**] defined the products DC_φ and $C_\varphi D$ and investigated the boundedness and the compactness of DC_φ and $C_\varphi D$ between weighted Bergman spaces using the Carleson-type measures. But such weighted Bergman spaces would not include the Hardy space case in the characterisation of $C_\varphi D$ and so the investigation of boundedness and compactness of $C_\varphi D$ between Hardy spaces would remain open. In this article we shall study this problem. That is, in the next section we give the necessary and sufficient conditions for $C_\varphi D$ to be bounded and compact between Hardy spaces using the Carleson-type conditions. Moreover, in Section 3, we focus the Hilbert Hardy space H^2 and present explicit conditions and examples of $C_\varphi D$ that is bounded and compact on H^2. And we also consider when $C_\varphi D$ is a Hilbert–Schmidt operator on H^2.

2. Between Hardy spaces: Carleson-type criteria

In this section, we give the necessary and sufficient conditions for $C_\varphi D$ to be bounded and compact between Hardy spaces using the Carleson-type conditions. Let φ be an analytic self-map of $\mathbb{U}$. We put

$$\varphi^*(\zeta) = \lim_{r \to 1} \varphi(r\zeta)$$

for $\zeta \in \partial\mathbb{U}$ whenever this limit exists and associate a measure μ to φ^* by setting

$$\mu(E) = \int_{(\varphi^*)^{-1}(E)\cap\partial\mathbb{U}} d\theta/2\pi$$

for $E \subset \overline{\mathbb{U}}$. In other words μ is the measure on $\overline{\mathbb{U}}$ that satisfies

$$\int_{\overline{\mathbb{U}}} h\,d\mu = \int_{\partial\mathbb{U}} (h \circ \varphi^*)\,d\theta/2\pi$$

for measurable function h on $\overline{\mathbb{U}}$.

Then, for $1 \leqslant p < \infty$ we have

$$\begin{aligned}\|C_\varphi D f\|_{H^p}^p &= \int_0^{2\pi} \left|f' \circ \varphi^*(e^{i\theta})\right|^p d\theta/2\pi \\ &= \int_{\overline{\mathbb{U}}} |f'|^p\,d\mu.\end{aligned}$$

So we can obtain the following equivalences: for $1 \leqslant p,\ q < \infty$, $C_\varphi D : H^p \to H^q$ is bounded (compact, respectively) if and only if the differentiation $D : H^p \to L^q(\overline{\mathbb{U}}, d\mu)$ is bounded (compact, respectively).

Here we recall that Luecking [**6**] and Choe, Koo and Smith [**1**] characterise the necessary and sufficient conditions for the differentiation $D : H^p \to L^q(\mathbb{U}, d\mu)$ to be bounded and compact, for a positive finite Borel measure μ on $\mathbb{U}$. We extend their results to the measures on $\overline{\mathbb{U}}$.

For any arc I in $\partial\mathbb{U}$, define the *Carleson square* over I to be

$$S_I = \{re^{i\theta} \in \overline{\mathbb{U}} : 1 - |I| \leqslant r \leqslant 1, e^{i\theta} \in I\}$$

where $|I|$ is $1/2\pi$ times the Euclidean length of I.

Then we have the following Carleson-type criteria, which are the so-called "big-oh" and "little-oh" conditions.

THEOREM 2.1. *Let φ be an analytic self-map of $\mathbb{U}$ and μ be defined as above. Suppose that $1 \leqslant p < q < \infty$ or $2 \leqslant p = q < \infty$. Then the following hold.*

(i) *$C_\varphi D : H^p \to H^q$ is bounded if and only if*

$$\mu(S_I) = O\big(|I|^{q(1+p)/p}\big), \quad I \subset \partial\mathbb{U}.$$

(ii) *$C_\varphi D : H^p \to H^q$ is compact if and only if*

$$\mu(S_I) = o\big(|I|^{q(1+p)/p}\big), \quad |I| \to 0.$$

PROOF: At first suppose that it is satisfied that

$$\mu(S_I) = O\big(|I|^{q(1+p)/p}\big)$$

for all $I \subset \partial\mathbb{U}$. Then we can show by the similar method as in the proof of [**2**, Theorem 2.8] that $\mu|_{\partial\mathbb{U}} = 0$. So we apply [**6**, Theorem 3.1] and obtain the equivalence of (i).

Futhermore we can prove (ii) using the following fact whose proof is an easy modification of that of [**3**, Proposition 3.11]: for $1 \leqslant p, q < \infty$, $C_\varphi D : H^p \to H^q$ is compact if and only if $\|C_\varphi D f_n\|_{H^q} \to 0$ for every bounded sequence $\{f_n\}_n$ in H^p such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{U}$. (Refer to [**1**, Lemma 2.5].) ∎

Here we add a result of the case $q = \infty$.

THEOREM 2.2. *Let $1 \leqslant p \leqslant \infty$ and φ be an analytic self-map of $\mathbb{U}$. Then the following are equivalent:*

(i) *$C_\varphi D : H^p \to H^\infty$ is bounded;*

(ii) *$C_\varphi D : H^p \to H^\infty$ is compact;*

(iii) $\|\varphi\|_\infty < 1$.

PROOF: We prove only that the condition (i) implies (iii). The other implications (ii)⇒(i) and (iii)⇒(ii) are clear.

Suppose that $\|\varphi\|_\infty = 1$ and $|\varphi(\lambda)| = 1$ for some $\lambda \in \partial\mathbb{U}$. Then, for $0 < \alpha < 1$, let

$$f(z) = \frac{p}{\alpha\overline{\varphi(\lambda)}(1 - \overline{\varphi(\lambda)}z)^{\alpha/p}}.$$

Then $f \in H^p$.

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On the other hand,

$$\left|CD_\varphi f(z)\right| = (1 - \overline{\varphi(\lambda)}z)^{-(1+\alpha/p)}$$

and so $CD_\varphi f \notin H^\infty$. ∎

Finally in this section we pose a question: Characterise the boundedness and compactness of $C_\varphi D : H^p \to H^q$ in the case that $1 \leqslant q < p < \infty$.

3. The Hilbert Hardy space case

We would like to obtain the function-theoretic characterisation. For the purpose we focus the Hilbert Hardy space H^2.

Before starting our results, we briefly collect some materials for the Nevanlinna counting function that shall be needed in the sequel (refer to [7]).

The *Nevanlinna counting function* N_φ of φ is defined by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log\frac{1}{|z|}, \quad w \in \mathbb{U} \setminus \{\varphi(0)\}. \tag{3.1}$$

At an extreme case, $N_\varphi(\varphi(0)) = \infty$.

Firstly we shall require the change of variable formula for integral means of analytic functions using the Nevanlinna counting function: for f analytic on $\mathbb{U}$,

$$\|f \circ \varphi\|_{H^2}^2 = \left|f(\varphi(0))\right|^2 + 2\int_{\mathbb{U}} \left|f'(w)\right|^2 N_\varphi(w)\,dA(w) \tag{3.2}$$

where dA is the normalised area measure on $\mathbb{U}$.

The Nevanlinna counting function has the sub-averaging property as follows: suppose that φ is an analytic self-map of $\mathbb{U}$ with $\varphi(0) \neq 0$. If

$$0 < R < \left|\varphi(0)\right|,$$

then

$$N_\varphi(0) \leqslant \frac{1}{R^2}\int_{R\mathbb{U}} N_\varphi\,dA \tag{3.3}$$

where $R\mathbb{U} = \{|z| < R\}$.

So we obtain the explicit conditions for $C_\varphi D$ to be bounded and compact on the Hardy space H^2, which also are the so-called "big-oh" and "little-oh" conditions.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{U}$. Then the following hold.*

(i) *$C_\varphi D$ is bounded on H^2 if and only if*

$$N_\varphi(w) = O\left(\left[\log(1/|w|)\right]^3\right) \quad (|w| \to 1).$$

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(ii) *$C_\varphi D$ is compact on H^2 if and only if*

$$N_\varphi(w) = o\Big(\big[\log(1/|w|)\big]^3\Big) \quad (|w| \to 1).$$

Proof: At first we shall show the case (i).

Suppose that $C_\varphi D$ is bounded on H^2. For $\lambda \in \mathbb{U}$, we take the function

$$f(z) = \frac{\sqrt{1-|\lambda|^2}}{1-\overline{\lambda}z}.$$

Then $f \in H^2$ and $\|f\|_{H^2} = 1$. So, using (3.2), we have

$$\begin{aligned} \|C_\varphi D\|^2 &\geqslant \|C_\varphi D f\|_{H^2}^2 = \|f' \circ \varphi\|_{H^2}^2 \\ &= \big|f'(\varphi(0))\big|^2 + 2\int_{\mathbb{U}} |f''(w)|^2 N_\varphi(w)\,dA(w) \\ &\geqslant \int_{\mathbb{U}} \Big|\frac{2\overline{\lambda}^2(1-|\lambda|^2)^{1/2}}{(1-\overline{\lambda}w)^3}\Big|^2 N_\varphi(w)\,dA(w) \\ &= \int_{\mathbb{U}} \frac{4(1-|\lambda|^2)|\lambda|^4}{|1-\overline{\lambda}w|^6} N_\varphi(w)\,dA(w). \end{aligned}$$

Now substituting $w = \alpha_\lambda(u) = (\lambda - u)/(1-\overline{\lambda}u)$,

$$\begin{aligned} \|C_\varphi D\|^2 &\geqslant \int_{\mathbb{U}} \frac{4|\lambda|^4|1-\overline{\lambda}u|^2}{(1-|\lambda|^2)^3} N_\varphi\big(\alpha_\lambda(u)\big)\,dA(u) \\ &\geqslant \int_{\mathbb{U}/2} \frac{4|\lambda|^4|1-\overline{\lambda}u|^2}{(1-|\lambda|^2)^3} N_\varphi\big(\alpha_\lambda(u)\big)\,dA(u). \end{aligned}$$

Note that $|1-\overline{\lambda}u| \geqslant 1/2$ for $u \in \mathbb{U}/2$. Using the sub-averaging property (3.3) of the Nevanlinna counting function, we obtain

$$\|C_\varphi D\|^2 \geqslant \frac{|\lambda|^4 N_\varphi(\alpha_\lambda(0))}{(1-|\lambda|^2)^3} = \frac{|\lambda|^4 N_\varphi(\lambda)}{(1-|\lambda|^2)^3}$$

for $\lambda \in \mathbb{U} \setminus \{\varphi(0)\}$.

Since $\log(1/|\lambda|)$ is comparable to $1-|\lambda|$ as $|\lambda| \to 1^-$, we obatin

$$N_\varphi(\lambda) = O\Big(\big[\log(1/|\lambda|)\big]^3\Big) \quad (|\lambda| \to 1).$$

We shall see the converse. Suppose that for some $R, 0 < R < 1$, there is a constant M satisfying

$$\sup_{R<|w|<1} N_\varphi(w)/\big[\log(1/|w|)\big]^3 \leqslant M.$$

For f analytic on $\mathbb{U}$, we use (3.2) and have

$$\|C_\varphi Df\|_{H^2}^2 = \left|f'(\varphi(0))\right|^2 + 2\int_{\mathbb{U}} |f''(w)|^2 N_\varphi(w)\,dA(w)$$
$$= \left|f'(\varphi(0))\right|^2 + 2\left(\int_{R\mathbb{U}} + \int_{\mathbb{U}\setminus R\mathbb{U}}\right).$$

The first and the second terms in the right-hand of the equality above are:

$$\left|f'(\varphi(0))\right|^2 \leqslant \left(1 - |\varphi(0)|\right)^{-4} \|f\|_{H^2}^2$$

and

$$\int_{R\mathbb{U}} |f''(w)|^2 N_\varphi(w)\,dA(w) \leqslant \frac{4}{(1-|\varphi(0)|)^6}\|f\|_{H^2}^2.$$

Next we estimate the third one.

$$\int_{\mathbb{U}\setminus R\mathbb{U}} |f''(w)|^2 N_\varphi(w)\,dA(w)$$
$$\leqslant \sup_{R<|w|<1} \frac{N_{\varphi(w)}}{[\log(1/|w|)]^3} \int_{\mathbb{U}\setminus R\mathbb{U}} |f''(w)|^2 \left[\log(1/|w|)\right]^3 dA(w).$$

Here let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$ with $\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2$. Then

$$\int_{\mathbb{U}\setminus R\mathbb{U}} |f''(w)|^2 \left[\log(1/|w|)\right]^3 dA(w)$$
$$\leqslant \sum_{n=2}^{\infty} |a_n|^2 n^2 (n-1)^2 \int_0^1 r^{2(n-2)} \left(\log\frac{1}{r}\right)^3 2r\,dr$$
$$= \sum_{n=2}^{\infty} |a_n|^2 n^2 (n-1)^2 \int_0^1 t^{n-2} \left(\frac{1}{2}\log\frac{1}{t}\right)^3 dt,$$

substituting $t = r^2$. And substitute $u = \log(1/t)$. Then

$$\int_0^1 t^{n-2}\left(\log\frac{1}{t}\right)^3 dt = \int_0^\infty e^{-(n-1)u} u^3\,du.$$

Further substituting $x = (n-1)u$, we have

$$\int_0^\infty e^{-(n-1)u} u^3\,du = \frac{1}{(n-1)^4}\int_0^\infty e^{-x} x^3\,dx = \frac{\Gamma(4)}{(n-1)^4}.$$

So

$$\int_{\mathbb{U}\setminus R\mathbb{U}} |f''(w)|^2 N_\varphi(w)\,dA(w) \leqslant \sum_{n=2}^{\infty} |a_n|^2 n^2 (n-1)^2 \frac{\Gamma(4)}{8(n-1)^4}$$
$$\leqslant 3\sum_{n=2}^{\infty} |a_n|^2 \leqslant 3\sum_{n=0}^{\infty} |a_n|^2.$$

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Consequently we obtain

$$\|C_\varphi Df\|_{H^2}^2 \leqslant \Big(\frac{1}{(1-|\varphi(0)|)^4} + \frac{8}{(1-|\varphi(0)|)^6} + 6M\Big)\|f\|_{H^2}^2.$$

That is, $C_\varphi D$ is bounded on H^2.

To show the case (ii), we take test functions

$$f_n(z) = \frac{\sqrt{1-|\lambda_n|^2}}{1-\overline{\lambda_n}z}$$

for a sequence $\{\lambda_n\}$ in $\mathbb{U}$ such that $|\lambda_n| \to 1$ as $n \to \infty$. Then f_n converges weakly to 0 and so we obtain the desired condition. The converse is routine. ∎

REMARK. Recall that the *essential norm* $\|C_\varphi D\|_e$ of the operator $C_\varphi D$ is defined to its distance from the space of all compact operators on H^2. We obtain the upper and lower estimates of $\|C_\varphi D\|_e$:

$$\limsup_{|w|\to 1} \frac{N_\varphi(w)}{[\log(1/|w|)]^3} \leqslant \|C_\varphi D\|_e^2 \leqslant K \limsup_{|w|\to 1} \frac{N_\varphi(w)}{[\log(1/|w|)]^3},$$

where $K > 0$ is a constant.

In the case that φ is univalent on $\mathbb{U}$, we can easily deduce the following corollary.

COROLLARY 3.2. *Let φ be a univalent analytic self-map of $\mathbb{U}$. Then the following hold.*

(i) *$C_\varphi D$ is bounded on H^2 if and only if*

$$\sup_{w\in\mathbb{U}} \frac{1-|w|}{(1-|\varphi(w)|)^3} < \infty.$$

(ii) *$C_\varphi D$ is compact H^2 if and only if*

$$\lim_{|w|\to 1} \frac{1-|w|}{(1-|\varphi(w)|)^3} = 0.$$

EXAMPLES. We can give explicit examples of $C_\varphi D$ that is bounded or compact. For $0 < \alpha \leqslant 1/3$, let $\varphi_\alpha(z) = 1-(1-z)^\alpha$ or

$$\varphi_\alpha(z) = \frac{\sigma(z)^\alpha - 1}{\sigma(z)^\alpha + 1}$$

where $\sigma(z) = (1+z)/(1-z)$. The latter φ_α is called the lens map. Then both φ_α satisfy

$$1 - \left|\varphi_\alpha(z)\right|^2 \approx |1-z|^\alpha \quad \text{for } z \text{ near } 1.$$

So using Corollary 3.2, we obtain that $C_{\varphi_\alpha}D$ is bounded on H^2 when $0 < \alpha \leqslant 1/3$ and furthermore compact on H^2 when $0 < \alpha < 1/3$.

We also can find other example in Smith's paper [8]. Let $P \subset \overline{\mathbb{U}}$ be a polygon with $P \cap \partial\mathbb{U} = \{1\}$ and with angular aperture $\pi/3$ at $w = 1$. Let φ be a Riemann map of $\mathbb{U}$ onto the interior of P. He showed that for such a polygonal map φ,

$$N_\varphi(w) = O\left(\left[\log(1/|w|)\right]^3\right) \quad \text{as} \quad |w| \to 1.$$

Then $C_\varphi D$ is bounded on H^2.

Furthermore we consider when a product $C_\varphi D$ is a Hilbert-Schmidt operator on H^2.

THEOREM 3.3. *Let φ be an analytic self-map of $\mathbb{U}$. Then $C_\varphi D$ is a Hilbert-Schmidt operator on H^2 if and only if*

$$\sup_{0 \leqslant r < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1 - |\varphi(re^{i\theta})|^2)^3} d\theta < \infty.$$

PROOF: At first we show the "if"-part. For the orthonormal basis $\{z^n\}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \|C_\varphi D z^n\|_{H^2}^2 &= \sum_{n=1}^{\infty} \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| n\varphi(re^{i\theta})^{n-1} \right|^2 d\theta \right\} \\ &\leqslant \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} (n+1)^2 \left|\varphi(re^{i\theta})\right|^{2n} d\theta \right\} \\ &\leqslant \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{2}{(1 - |\varphi(re^{i\theta})|^2)^3} d\theta \right\}. \end{aligned}$$

Thus $\sum_{n=0}^{\infty} \|C_\varphi D z^n\|_{H^2}^2 < \infty$ and so $C_\varphi D$ is a Hilbert–Schmidt operator on H^2.

Conversely suppose that $C_\varphi D$ is a Hilbert–Schmidt operator on H^2. For the orthonormal basis $\{z^n\}$, we have

$$\begin{aligned} \infty &> \sum_{n=0}^{\infty} \|C_\varphi D z^n\|_{H^2}^2 \\ &= \sum_{n=1}^{\infty} \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| n\varphi(re^{i\theta})^{n-1} \right|^2 d\theta \right\} \\ &\geqslant \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0}^{\infty} (n+1)^2 \left|\varphi(re^{i\theta})\right|^{2n} d\theta \right\} \\ &\geqslant \sup_{0 \leqslant r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1 - |\varphi(re^{i\theta})|^2)^3} d\theta \right\}. \end{aligned}$$

So we obtain the desired condition. ∎

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