Products of composition and differentiation operators on the weighted Bergman space

Stevo Stević

Abstract

Motivated by a recent paper by S. Ohno we calculate Hilbert-Schmidt norms of products of composition and differentiation operators on the Bergman space A^2_α, $\alpha > -1$ and the Hardy space H^2 on the unit disk. When the convergence of sequences (φ_n) of symbols to a given symbol φ implies the convergence of product operators $C_{\varphi_n}D^k$ is also studied. Finally, the boundedness and compactness of the operator $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ are characterized in terms of the generalized Nevanlinna counting function.

1 Introduction and an auxiliary result

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, $dm(z) = \frac{1}{\pi} r dr d\theta$ the normalized Lebesgue area measure on $\mathbb{D}$, $dm_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dm(z)$, $\alpha > -1$, (note that $m_\alpha(\mathbb{D}) = 1$), and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$.

The Bergman space $A^p_\alpha = A^p_\alpha(\mathbb{D})$, $p > 0$, $\alpha > -1$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{p,\alpha} = \int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) < \infty.$$

With the norm $\|\cdot\|_{p,\alpha}$ the weighted Bergman space becomes a Banach space, when $p \geq 1$. If $p \in (0,1)$, it is a Frechet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p_{p,\alpha}.$$

The Hardy space $H^p = H^p(\mathbb{D})$, $p > 0$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Received by the editors June 2008.
Communicated by F. Bastin.

Since for every $f \in H^p$

$$\lim_{\alpha \to -1+0} \|f\|_{p,\alpha} = \|f\|_{H^p},$$

H^p can be viewed as A^p_{-1}.

Let D be the differentiation operator, i.e., $Df = f'$, $D^k f = f^{(k)}$, $k \in \mathbb{N}$ and C_φ the composition operator induced by a nonconstant analytic self-map φ of $\mathbb{D}$. For some classical results on composition operators, see, e.g., [2]. For some recent results, see, e.g., [1, 3, 8, 9, 10, 15, 17, 18, 20, 25, 28, 32, 33] and the references therein. Products of integral, differentiation and composition operators on spaces of analytic functions are studied in [4, 5, 11, 12, 13, 14, 16, 19, 29, 31, 34, 35].

The paper is organized as follows. Motivated by [19, Theorem 3.3] which studies products of composition and differentiation on H^2, in Section 2 we calculate the exact value of Hilbert-Schmidt norms of products of composition and differentiation operators $C_\varphi D^k$, $k \in \mathbb{N}$ and DC_φ on the Bergman space A^2_α, $\alpha > -1$ and the Hardy space H^2.

In Section 3 we investigate when convergence of sequences (φ_n) of symbols to a given symbol φ implies the convergence of product operators $C_{\varphi_n} D^k$.

In the last section we characterize the boundedness and compactness of the operator $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ in terms of the generalized Nevanlinna counting function.

We need the following lemma:

Lemma 1. *Let* $c_{n,\alpha} = \left(\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}\right)^{1/2}$, $\alpha \geq -1$. *Then for* $|x| < 1$, *the following formula hold true*

$$\sum_{n=k}^{\infty} (n(n-1)\cdots(n-k+1))^2 c^2_{n,\alpha} x^{n-k} = \prod_{j=0}^{k-1} (\alpha+2+j) \left(\frac{x^k}{(1-x)^{\alpha+2+k}} \right)^{(k)}. \quad (1)$$

Proof. Let

$$f(x) = \sum_{n=k}^{\infty} \Big(\prod_{j=0}^{k-1} (n-j) \Big)^2 c^2_{n,\alpha} x^{n-k}. \quad (2)$$

It is easy to see that $f^{(n)}(x)$, $n \in \mathbb{Z}$ exists for $|x| < 1$ (when for $n < 0$ it denotes the nth antiderivative of f). Integrating (2) k times, we have that for $|x| < 1$

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} f(x_k) dx_k \cdots dx_1 = \sum_{n=k}^{\infty} \Big(\prod_{j=0}^{k-1} (n-j) \Big) c^2_{n,\alpha} x^n = x^k g(x), \quad (3)$$

and

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} g(x_k) dx_k \cdots dx_1 = \sum_{n=k}^{\infty} c^2_{n,\alpha} x^n = \frac{1}{(1-x)^{\alpha+2}} - \sum_{l=0}^{k-1} c^2_{l,\alpha} x^l. \quad (4)$$

Differentiating (4) k times we obtain

$$g(x) = \frac{\prod_{j=0}^{k-1} (\alpha+2+j)}{(1-x)^{\alpha+2+k}}.$$

Replacing this in (3) and differentiating such obtained formula k times, (1) follows. ■

Remark 1. By using Leibnitz's formula and some simple calculations, (1) can be written in the following form

$$\sum_{n=k}^{\infty}\Big(\prod_{j=0}^{k-1}(n-j)\Big)^2 c_{n,\alpha}^2 x^{n-k} = \frac{\sum_{l=0}^{k} C_l^k \prod_{s=l+1}^{k} s\, x^l (1-x)^{k-l}\Gamma(\alpha+k+l+2)}{\Gamma(\alpha+2)(1-x)^{\alpha+2+2k}}. \quad (5)$$

Note also that the polynomial in the numerator is positive on the interval $[0,1)$.

2 Hilbert-Schmidt norm of $C_\varphi D^k$ (or DC_φ) : $A_\alpha^2 \to A_\alpha^2$

If $\mathcal{H}$ is a separable Hilbert space, then the Hilbert-Schmidt norm $\|T\|_{HS}$ of an operator $T:\mathcal{H}\to\mathcal{H}$ is defined by:

$$\|T\|_{HS} = \Big(\sum_{n=1}^{\infty}\|Te_n\|^2\Big)^{1/2}, \quad (6)$$

where $\{e_n\}$ is an orthonormal basis on $\mathcal{H}$. The right-hand side in (6) does not depend on the choice of basis. Hence, it is larger than the operator norm $\|T\|_{op}$ of T.

Let

$$\langle f,g\rangle_\alpha = \int_{\mathbb{D}} f(z)\overline{g(z)}\,dm_\alpha(z),$$

$\alpha \geq -1$, be the scalar product on A_α^2 and $e_n(z) = c_{n,\alpha}z^n$, $n\in\mathbb{N}_0$.

Since for $\alpha > -1$

$$\begin{aligned}\|e_n\|_{2,\alpha}^2 &= (\alpha+1)c_{n,\alpha}^2\int_{\mathbb{D}}|z|^{2n}(1-|z|^2)^\alpha dm(z)\\ &= (\alpha+1)c_{n,\alpha}^2\int_0^1\rho^n(1-\rho)^\alpha d\rho = 1\end{aligned}$$

and $\langle e_n,e_m\rangle_\alpha = 0$ when $m\neq n$, it follows that $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal basis for A_α^2. The proof of this fact, for the case $\alpha=-1$, is simpler.

Theorem 1. *The Hilbert-Schmidt norm of the operator $C_\varphi D^k$ on A_α^2, $\alpha > -1$ is*

$$\|C_\varphi D^k\|_{HS} = \left((\alpha+1)\int_{\mathbb{D}}\frac{P_\alpha(|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^{\alpha+2+2k}}(1-|z|^2)^\alpha dm(z)\right)^{1/2}, \quad (7)$$

where

$$P_\alpha(x) = \frac{1}{\Gamma(\alpha+2)}\sum_{l=0}^{k} C_l^k\Big(\prod_{s=l+1}^{k} s\Big)x^l(1-x)^{k-l}\Gamma(\alpha+k+l+2). \quad (8)$$

Proof. By using the definition of the Hilbert-Schmidt norm, the monotone convergence theorem, Lemma 1 and formula (5), we obtain

$$\begin{aligned}
\|C_\varphi D^k\|_{HS}^2 &= \sum_{n=0}^{\infty} \|C_\varphi D^k(e_n)\|_{2,\alpha}^2 = \sum_{n=0}^{\infty} c_{n,\alpha}^2 \Big(\prod_{j=0}^{k-1}(n-j)\Big)^2 \|\varphi^{n-k}\|_{2,\alpha}^2 \\
&= (\alpha+1)\sum_{n=k}^{\infty} c_{n,\alpha}^2 \Big(\prod_{j=0}^{k-1}(n-j)\Big)^2 \int_{\mathbb{D}} |\varphi(z)|^{2n-2k}(1-|z|^2)^\alpha dm(z) \\
&= (\alpha+1)\int_{\mathbb{D}} \frac{P_\alpha(|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^{\alpha+2+2k}}(1-|z|^2)^\alpha dm(z),
\end{aligned}$$

from which the result follows. ∎

Corollary 1. *The Hilbert-Schmidt norm of the operator $C_\varphi D$ on A^2_α, $\alpha > -1$ is*

$$\|C_\varphi D\|_{HS} = \left((\alpha+1)(\alpha+2)\int_{\mathbb{D}} \frac{1+(\alpha+2)|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+4}}(1-|z|^2)^\alpha dm(z)\right)^{1/2}. \tag{9}$$

Moreover, the operator $C_\varphi D : A^2_\alpha \to A^2_\alpha$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+4}} dm(z) < \infty. \tag{10}$$

Proof. Theorem 1 and Lemma 1 with $k = 1$, imply formula (9). Since $1 \le 1+(\alpha+2)|\varphi(z)|^2 \le \alpha+3$ we get that the integrals in (9) and (10) are comparable, from which the second statement follows. ∎

The next corollary can be proved in the same way. Hence we omit its proof.

Corollary 2. *The Hilbert-Schmidt norm of the operator $C_\varphi D^2$ on A^2_α, $\alpha > -1$ is*

$$\|C_\varphi D^2\|_{HS} = \left(C_\alpha \int_{\mathbb{D}} \frac{2+(4\alpha+12)|\varphi(z)|+(\alpha^2+5\alpha+6)|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+6}} dm_\alpha(z)\right)^{1/2},$$

where $C_\alpha = (\alpha+1)(\alpha+2)(\alpha+3)$.

Moreover, the operator $C_\varphi D^2 : A^2_\alpha \to A^2_\alpha$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\alpha+6}} dm(z) < \infty. \tag{11}$$

Similarly, by Lemma 1 with the case $\alpha = -1$, we can prove the following theorem (for the case $k = 1$ the proof is close to the proof of Theorem 3.3 in [19]):

Theorem 2. *The Hilbert-Schmidt norm of the operator $C_\varphi D^k$ on H^2 is*

$$\|C_\varphi D^k\|_{HS} = \left(\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{P_{-1}(|\varphi(re^{i\theta})|^2)}{(1-|\varphi(re^{i\theta})|^2)^{2k+1}} d\theta\right)^{1/2},$$

where $P_{-1}(x)$ is polynomial (8) with $\alpha = -1$.

Corollary 3. *The Hilbert-Schmidt norm of the operator $C_\varphi D$ on H^2 is*

$$\|C_\varphi D\|_{HS} = \left(\sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1+|\varphi(re^{i\theta})|^2}{(1-|\varphi(re^{i\theta})|^2)^3} d\theta \right)^{1/2}.$$

Moreover, the operator $C_\varphi D : H^2 \to H^2$ is Hilbert-Schmidt if and only if

$$\sup_{0\le r<1} \int_0^{2\pi} \frac{d\theta}{(1-|\varphi(re^{i\theta})|^2)^3} < \infty.$$

3 Hilbert-Schmidt norm of $DC_\varphi : A^2_\alpha \to A^2_\alpha$

Here we consider the operator DC_φ. We will not consider the general case $D^k C_\varphi$, $k \in \mathbb{N}$ since the corresponding formulae are not written in a simple way.

Theorem 3. *The Hilbert-Schmidt norm of the operator DC_φ on A^2_α, $\alpha > -1$ is*

$$\|DC_\varphi\|_{HS} = \left((\alpha+1)(\alpha+2) \int_{\mathbb{D}} \frac{1+(\alpha+2)|\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+4}} |\varphi'(z)|^2 (1-|z|^2)^\alpha dm(z) \right)^{1/2}.$$

Proof. We have

$$\begin{aligned} \|DC_\varphi\|^2_{HS} &= \sum_{n=0}^{\infty} \|DC_\varphi(e_n)\|^2_{2,\alpha} = \sum_{n=0}^{\infty} c^2_{n,\alpha} n^2 \|\varphi^{n-1}\varphi'\|^2_{2,\alpha} \\ &= (\alpha+1) \sum_{n=0}^{\infty} c^2_{n,\alpha} n^2 \int_{\mathbb{D}} |\varphi(z)|^{2n-2} |\varphi'(z)|^2 (1-|z|^2)^\alpha dm(z). \end{aligned}$$

From this, by using the monotone convergence theorem and Lemma 1 with $k = 1$ the result follows. ∎

For the case $\alpha = -1$, the following result holds true.

Theorem 4. *The Hilbert-Schmidt norm of the operator DC_φ on H^2 is*

$$\|DC_\varphi\|_{HS} = \left(\sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1+|\varphi(re^{i\theta})|^2}{(1-|\varphi(re^{i\theta})|^2)^3} |\varphi'(re^{i\theta})|^2 d\theta \right)^{1/2}.$$

Proof. We have

$$\|DC_\varphi\|^2_{HS} = \sum_{n=0}^{\infty} \|DC_\varphi(e_n)\|^2_{H^2} = \sum_{n=0}^{\infty} \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} n^2 |\varphi(re^{i\theta})^{2n-2} (\varphi'(re^{i\theta}))^2| d\theta.$$

From this, since the supremum and sum can interchange their positions (in view of the positivity and continuity of the integral means appearing there), using the monotone convergence theorem and Lemma 1 (case $\alpha = -1$) the formula follows.

∎

4 Continuity with respect to symbols of composition operators

By Theorem 1, similar to the proof of Theorem 1 in [27], the following theorem can be proved. We omit the proof.

Theorem 5. *Let $\alpha \geq -1$, $Q_{\alpha,k}(x) = P_\alpha(x)/(1-x)^{\alpha+2+2k}$, φ be an analytic self-map of $\mathbb{D}$ and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of $\mathbb{D}$ such that $\varphi_n \to \varphi$ a.e. on $\mathbb{D}$,*

$$\int_{\mathbb{D}} Q(|\varphi(z)|^2)dm_\alpha(z) < \infty, \tag{12}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{D}} Q(|\varphi_n(z)|^2)dm_\alpha(z) = \int_{\mathbb{D}} Q(|\varphi(z)|^2)dm_\alpha(z). \tag{13}$$

Then the sequence $(C_{\varphi_n}D^k)_{n\in\mathbb{N}}$ of Hilbert-Schmidt composition operators converges in Hilbert-Schmidt norm to the composition operator $C_\varphi D^k$.

Remark 2. Note that in view of equation (7) Theorem 1 can be written in the following form: Let φ be an analytic self-map of $\mathbb{D}$ and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of $\mathbb{D}$ such that $C_\varphi D^k$ and $(C_{\varphi_n}D^k)_{n\in\mathbb{N}}$ are Hilbert-Schmidt operators, $\varphi_n \to \varphi$ a.e. on $\mathbb{D}$, and $\lim_{n\to\infty} \|C_{\varphi_n}D^k\|_{HS} = \|C_\varphi D^k\|_{HS}$. Then $\lim_{n\to\infty} \|C_{\varphi_n}D^k - C_\varphi D^k\|_{HS} = 0$.

Theorem 6. *If $C_\varphi D^k$ is a Hilbert-Schmidt operator, then*

$$\lim_{n\to\infty} \|C_{\varphi_n}D^k - C_\varphi D^k\|_{HS} = 0 \tag{14}$$

if and only if $\|C_{\varphi_n}D^k\|_{HS} \to \|C_\varphi D^k\|_{HS}$ and $\|\varphi_n - \varphi\|_{2,\alpha} \to 0$ as $n \to \infty$.

Proof. If (14) holds, then clearly $\|C_{\varphi_n}D^k\|_{HS} \to \|C_\varphi D^k\|_{HS}$ as $n \to \infty$. On the other hand,

$$\begin{aligned}\|\varphi_n - \varphi\|_{2,\alpha} &= C\|(C_{\varphi_n}D^k - C_\varphi D^k)(z^{k+1})\|_{2,\alpha} \\ &\leq \|C_{\varphi_n}D^k - C_\varphi D^k\| \, \|z^{k+1}\|_{2,\alpha} \\ &\leq C\|C_{\varphi_n}D^k - C_\varphi D^k\|_{HS},\end{aligned}$$

from which it follows that $\|\varphi_n - \varphi\|_{2,\alpha} \to 0$.

Now assume $\|C_{\varphi_n}D^k\|_{HS} \to \|C_\varphi D^k\|_{HS}$ and $\|\varphi_n - \varphi\|_{2,\alpha} \to 0$ as $n \to \infty$, but

$$\|C_{\varphi_n}D^k - C_\varphi D^k\|_{HS} \not\to 0.$$

Then there are a subsequence $(C_{\varphi_{n_j}}D^k)_{j\in\mathbb{N}}$ and an $\varepsilon_0 > 0$ such that

$$\|C_{\varphi_{n_j}}D^k - C_\varphi D^k\|_{HS} \geq \varepsilon_0 > 0, \qquad j \in \mathbb{N}.$$

Since $\|\varphi_{n_j} - \varphi\|_{2,\alpha} \to 0$, as $j \to \infty$, there is a subsequence $(\varphi_{n_{j_l}})_{l\in\mathbb{N}}$ converging to φ a.e. By Theorem 5 we have

$$\lim_{l\to\infty} \|C_{\varphi_{n_{j_l}}}D^k - C_\varphi D^k\|_{HS} = 0,$$

which is a contradiction. ∎

Theorem 7. *Let φ be an analytic self-map of $\mathbb{D}$ and suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of analytic self-maps of $\mathbb{D}$ such that $\|\varphi_n - \varphi\|_{2,\alpha} \to 0$ as $n \to \infty$. If there is a measurable function $\chi : \mathbb{D} \to [0,1]$ such that for every $n \in \mathbb{N}$, $|\varphi_n| \le |\chi|$ a.e. on $\mathbb{D}$ and*

$$\int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2)dm_\alpha(z) < \infty,$$

then the sequence $(C_{\varphi_n}D^k)_{n\in\mathbb{N}}$ converges in Hilbert-Schmidt norm to $C_\varphi D^k$.

The proof of Theorem 7 is similar to the proof of Theorem 3 in [27] and is omitted.

The next theorem is proved similar to Theorem 4 in [27].

Theorem 8. *For every pair of distinct symbols φ, ψ, let $\chi = \max\{|\varphi|, |\psi|\}$. Then the following upper estimates hold*

$$\|C_\varphi D^k - C_\psi D^k\|_{HS} \le M\left(\int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2)dm_\alpha(z)\right)^{1/2} \|\varphi - \psi\|_\infty,$$

where the constant M is independent of φ and ψ. Therefore, for each $R > 0$, the map $\varphi \to C_\varphi D^k$ is Lipschitz continuous on the set

$$\mathcal{S}_R = \left\{\varphi \;\middle|\; \int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2)dm_\alpha(z) \le R\right\}.$$

5 A characterization of the boundedness and compactness of $C_\varphi D^k$

Here we characterize the boundedness and compactness of $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ in terms of the generalized Nevanlinna counting function. Before we formulate the main result in this section, we need some notation and an auxiliary result.

In view of the following asymptotic relationship

$$1 - |z| \asymp \ln(1/|z|), \quad |z| \to 1, \tag{15}$$

it is easy to see that the Bergman space A^p_α is equivalent with the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{p,\alpha,1} = \int_{\mathbb{D}} |f(z)|^p \left(\ln\frac{1}{|z|}\right)^\alpha dm(z) < \infty.$$

The generalized Nevanlinna counting function $N_{\varphi,\gamma}$, $\gamma > 0$ for φ, is defined as follows

$$N_{\varphi,\gamma}(w) = \sum_{z\in\varphi^{-1}(w)} \left[\ln(1/|z|)\right]^\gamma, \qquad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

It is introduced by Shapiro in [21] to study composition operators from a weighted Bergman space to itself.

By a well known characterization of the Bergman space (see, for example, [22, 23, 30]) and the asymptotics in (15) it follows that

$$\|f\|^2_{A^2_\alpha} \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \Big(\ln \frac{1}{|z|}\Big)^{\alpha+2} dm(z).$$

From this and by applying the Stanton's formula

$$\|f \circ \varphi\|^p_{H^p} = |f(\varphi(0))|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(w)|^{p-2} |f'(w)|^2 N_{\varphi,1}(w) dm(w),$$

with $p = 2$, on the dilations $f_r(z) = f(rz)$, $r \in (0,1)$, it follows that

$$\|f \circ \varphi\|^2_{A^2_\alpha} \asymp |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi,\alpha+2}(w) dm(w). \tag{16}$$

It is also known ([21]) that if φ is an analytic self-map of $\mathbb{D}$ with $\varphi(0) \neq 0$, $\alpha \geq 1$ and if $0 < r < |\varphi(0)|$, then

$$N_{\varphi,\alpha}(0) \leq \frac{1}{r^2} \int_{|z| \leq r} N_{\varphi,\alpha}(z) dm(z). \tag{17}$$

The following result is proved in a standard way (see, for example, the proofs of the corresponding results in [2, 6, 24, 25, 26]).

Lemma 2. *The operator $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ is compact if and only if $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ is bounded and for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in A^2_α converging to zero uniformly on compacts of $\mathbb{D}$ as $n \to \infty$, we have $\lim_{n\to\infty} \|C_\varphi D^k f_n\|_{A^2_\alpha} = 0$.*

Theorem 9. *Let φ be an analytic self-map of $\mathbb{D}$, $k \in \mathbb{N}$ and $\alpha > -1$. Then*

(a) $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ is bounded if and only if

$$N_{\varphi,\alpha+2}(w) = O\Big(\big[\ln(1/|w|)\big]^{\alpha+2+2k}\Big). \tag{18}$$

(b) $C_\varphi D^k : A^2_\alpha \to A^2_\alpha$ is compact if and only if

$$N_{\varphi,\alpha+2}(w) = o\Big(\big[\ln(1/|w|)\big]^{\alpha+2+2k}\Big), \quad \text{as } |w| \to 1. \tag{19}$$

Proof. (a) For $u \in \mathbb{D}$ set

$$f_u(z) = \frac{(1-|u|^2)^{\frac{\alpha+2}{2}}}{(1-\bar{u}z)^{\alpha+2}}, \quad z \in \mathbb{D}. \tag{20}$$

It is known that $\|f_u\|_{A^2_\alpha} = 1$, for each $u \in \mathbb{D}$. By using (16) we have

$$\begin{aligned}\|C_\varphi D^k\|^2 &\geq \|C_\varphi D^k f_u\|^2_{A^2_\alpha} = \|f_u^{(k)} \circ \varphi\|^2_{A^2_\alpha} \\ &\asymp |f_u^{(k)}(\varphi(0))|^2 + \int_{\mathbb{D}} |f_u^{(k+1)}(w)|^2 N_{\varphi,\alpha+2}(w)dm(w) \\ &\geq C\int_{\mathbb{D}} \frac{(1-|u|^2)^{\alpha+2}}{|1-\bar{u}w|^{2\alpha+6+2k}} |u|^{2(k+1)} N_{\varphi,\alpha+2}(w)dm(w).\end{aligned}$$

By the change

$$w = \varphi_u(z) = \frac{u-z}{1-\bar{u}z}$$

and (17) we obtain

$$\begin{aligned}\|C_\varphi D^k\|^2 &\geq C\int_{\mathbb{D}} \frac{|1-\bar{u}z|^{2\alpha+2+2k}}{(1-|u|^2)^{\alpha+2+2k}} |u|^{2(k+1)} N_{\varphi,\alpha+2}(\varphi_u(z))dm(z) \\ &\geq C\int_{|z|\leq 1/2} \frac{|1-\bar{u}z|^{2\alpha+2+2k}}{(1-|u|^2)^{\alpha+2+2k}} |u|^{2(k+1)} N_{\varphi,\alpha+2}(\varphi_u(z))dm(z) \\ &\geq C\frac{|u|^{2(k+1)}}{(1-|u|^2)^{\alpha+2+2k}} \int_{|z|\leq 1/2} N_{\varphi,\alpha+2}(\varphi_u(z))dm(z) \\ &\geq C\frac{|u|^{2(k+1)}}{(1-|u|^2)^{\alpha+2+2k}} N_{\varphi,\alpha+2}(u), \qquad (21)\end{aligned}$$

when when $|\varphi_u(\varphi(0))| > 1/2$, which holds for $|u|$ sufficiently close to 1.

From (21) and (15) we obtain (18).

Now assume that (18) holds. Then for each $r \in (0,1)$, there is $M > 0$ such that

$$\sup_{r<|w|<1} N_{\varphi,\alpha+2}(w)/\big[\ln(1/|w|)\big]^{\alpha+2+2k} \leq M. \qquad (22)$$

We have

$$\begin{aligned}\|C_\varphi D^k f\|^2 &\asymp |f^{(k)}(\varphi(0))|^2 + \int_{\mathbb{D}} |f^{(k+1)}(w)|^2 N_{\varphi,\alpha+2}(w)dm(w) \\ &= |f^{(k)}(\varphi(0))|^2 + \left(\int_{\mathbb{D}\setminus r\mathbb{D}} + \int_{r\mathbb{D}}\right)|f^{(k+1)}(w)|^2 N_{\varphi,\alpha+2}(w)dm(w).\end{aligned} \qquad (23)$$

By a well known estimate (see, e.g., [7]), we have that

$$|f^{(k)}(\varphi(0))|^2 \leq \frac{C}{(1-|\varphi(0)|)^{\alpha+2+2k}} \|f\|^2_{A^2_\alpha}. \qquad (24)$$

Since $N_{\varphi,\alpha+2}$ is bounded on $r\mathbb{D}$, similarly we obtain

$$\begin{aligned}\int_{r\mathbb{D}} |f^{(k+1)}(w)|^2 N_{\varphi,\alpha+2}(w)dm(w) &\leq C\sup_{|w|\leq r} |f^{(k+1)}(w)|^2 \qquad (25) \\ &\leq \frac{C}{(1-|\varphi(0)|)^{\alpha+4+2k}} \|f\|^2_{A^2_\alpha}. \qquad (26)\end{aligned}$$

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$, then we have

$$\begin{aligned}
&\int_{\mathbb{D}\setminus r\mathbb{D}}|f^{(k+1)}(w)|^2N_{\varphi,\alpha+2}(w)dm(w)\\
\le\ &\sup_{r<|w|<1}\frac{N_{\varphi,\alpha+2}(w)}{\big[\ln(1/|w|)\big]^{\alpha+2+2k}}\int_{\mathbb{D}\setminus r\mathbb{D}}|f^{(k+1)}(w)|^2\Big(\ln\frac{1}{|w|}\Big)^{\alpha+2+2k}dm(w)\\
\le\ &M\sum_{n=k+1}^{\infty}\prod_{j=0}^{k}(n-j)^2|a_n|^2\int_0^1 r^{2n-2k-1}\Big(\ln\frac{1}{r}\Big)^{\alpha+2+2k}dr\\
\le\ &M\sum_{n=k+1}^{\infty}n^{2+2k}|a_n|^2\int_0^1 t^{n-k-1}\Big(\ln\frac{1}{t}\Big)^{\alpha+2+2k}dt\\
\le\ &M\sum_{n=k+1}^{\infty}n^{2+2k}|a_n|^2\int_0^{\infty}e^{-(n-k)s}s^{\alpha+2+2k}ds\\
=\ &M\sum_{n=k+1}^{\infty}n^{2+2k}|a_n|^2\frac{\Gamma(\alpha+3+2k)}{(n-k)^{\alpha+3+2k}}\le CM\sum_{n=0}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}},
\end{aligned}\tag{27}$$

where we have used the changes $t=r^2$ and $e^{-s}=t$.

On the other hand, by a direct calculation and Stirling's formula, we have

$$\|f\|^2_{A^2_\alpha}=\sum_{n=0}^{\infty}\frac{|a_n|^2}{c^2_{n,\alpha}}\asymp\sum_{n=0}^{\infty}\frac{|a_n|^2}{n^{\alpha+1}}.\tag{28}$$

From (24), (26), (27) and (28) the boundedness of $C_\varphi D^k: A^2_\alpha\to A^2_\alpha$ follows.

(b) Assume that $|u_n|\to 1$ as $n\to\infty$. Then it is clear that $f_{u_n}(z)$ is a bounded sequence in A^2_α converging to zero uniformly on compacts. Hence

$$\lim_{n\to\infty}\|C_\varphi D^k f_{u_n}\|=0.$$

On the other hand as in (21) it follows that

$$\|C_\varphi D^k f_{u_n}\|^2\ge C\frac{|u_n|^{2(k+1)}}{(1-|u_n|^2)^{\alpha+2+2k}}N_{\varphi,\alpha+2}(u_n),\tag{29}$$

from which (19) holds.

Now assume that (19) holds and that $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in A^2_α converging to zero on compacts of $\mathbb{D}$. Then by Cauchy's inequality it follows that $f'_n\to 0$ uniformly on compacts of $\mathbb{D}$. We also have that for every $\varepsilon>0$ there is an r such that (22) holds where instead of M is ε, whenever $|w|>r$. On the other hand, we have that $|f_n^{(k+1)}(\varphi(0))|\to 0$ as $n\to\infty$, and from (25)

$$\lim_{n\to\infty}\int_{r\mathbb{D}}|f_n^{(k+1)}(w)|^2N_{\varphi,\alpha+2}(w)dm(w)\le C\lim_{n\to\infty}\max_{|w|\le r}|f_n^{(k+1)}(w)|=0.$$

Applying (23) to the function f_n, letting in such obtained inequality $n\to\infty$, and using above mentioned facts, it follows that

$$\limsup_{n\to\infty}\|C_\varphi D^k f_n\|^2\le C\varepsilon.$$

Hence this limit is zero, and consequently the operator $C_\varphi D^k: A^2_\alpha\to A^2_\alpha$ is compact. ■

References

[1] D. Clahane and S. Stević, Norm equivalence and composition operators between Bloch/ Lipschitz spaces of the unit ball, *J. Inequal. Appl.* Vol. 2006, Article ID 61018, (2006), 11 pages.

[2] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1995.

[3] X. Fu and Xiangling, Zhu, Weighted composition operators on some weighted spaces in the unit ball, *Abstr. Appl. Anal.* Vol. 2008, Article ID 605807, (2008), 8 pages.

[4] R. A. Hibschweiler and N. Portnoy, Composition followd by differentiaion between Bergman and Hardy spaces, *Rocky Mountain J. Math.* **35** (2005), 843-855.

[5] S. Li and S. Stević, Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.* **9** (2) (2007), 195-206.

[6] S. Li and S. Stević, Riemann-Stieltjes type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Variables Elliptic Functions* **52** (6) (2007), 495-517.

[7] S. Li and S. Stević, Integral type operators from mixed-norm spaces to α-Bloch spaces, *Integral Transform. Spec. Funct.* **18** (7) (2007), 485-493.

[8] S. Li and S. Stević, Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci. Math. Sci.* **117** (3) (2007), 371-385.

[9] S. Li and S. Stević, Weighted composition operators from α-Bloch space to H^∞ on the polydisk, *Numer. Funct. Anal. Optimization* **28** (7) (2007), 911-925.

[10] S. Li and S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc, *Abstr. Appl. Anal.* Vol. 2007, Article ID 48478, (2007), 12 pages.

[11] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.* **338** (2008), 1282-1295.

[12] S. Li and S. Stević, Products of composition and integral type operators from H^∞ to the Bloch space, *Complex Variables Elliptic Functions* **53** (5) (2008), 463-474.

[13] S. Li and S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to the Zygmund space, *J. Math. Anal. Appl.* **345** (2008), 40-52.

[14] S. Li and S. Stević, Composition followed by differentiation from mixed norm spaces to α-Bloch spaces, *Mat. Sb.* **199** (12) (2008), 117-128.

[15] S. Li and S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwanese J. Math.* **12** (2008) 1625-1639.

[16] S. Li and S. Stević, Products of integral-type operators and composition operator between the Bloch-type spaces, *J. Math. Anal. Appl.* **349** (2009), 596-610.

[17] L. Luo and S. I. Ueki, Weighted composition operators between weighted Bergman spaces and Hardy spaces on the unit ball of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **326** (1) (2007), 88-100.

[18] S. Ohno, Weighted composition operators between H^∞ and the Bloch space, *Taiwanese J. Math.* **5** (2001), 555-563.

[19] S. Ohno, Products of composition and differentiation between Hardy spaces, *Bull. Austral. Math Soc.* **73** (2) (2006), 235-244.

[20] S. Ohno, K. Stroethoff and R. Zhao, Weighted composition operators between Bloch type spaces, *Rocky Mountain J. Math.* **33** (2003), 191-215.

[21] J. Shapiro, The essential norm of a composition operator, *Annals of Math.* **127** (1987), 375-404.

[22] J. H. Shi, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **328** (2) (1991), 619-637.

[23] S. Stević, Weighted integrals of holomorphic functions in $\mathbb{C}^n$, *Complex Variables* **47** (9) (2002), 821-838.

[24] S. Stević, Boundedness and compactness of an integral operator on a weighted space on the polydisc, *Indian J. Pure Appl. Math.* **37** (6) (2006), 343-355.

[25] S. Stević, Composition operators between H^∞ and the α-Bloch spaces on the polydisc, *Z. Anal. Anwendungen* **25** (4) (2006), 457-466.

[26] S. Stević, Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, *Sibirsk. Mat. Zh.* **48** (3) (2007), 694-706.

[27] S. Stević, Continuity with respect to symbols of composition operators on the weighted Bergman space, *Taiwanese J. Math.* **11** (4) (2007), 1177-1188.

[28] S. Stević, Weighted composition operators between mixed norm spaces and H^∞_α spaces in the unit ball, *J. Inequal. Appl.* Vol. 2007, Article ID 28629, (2007), 9 pages.

[29] S. Stević, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, *Util. Math.* **77** (2008) 167-172.

[30] S. Stević, Holomorphic functions on the mixed norm spaces on the polydisc, *J. Korean Math. Soc.* **45** (1) (2008), 63-78.

[31] S. Stević, On a new integral-type operator from the weighted Bergman space to the Bloch-type space on the unit ball, *Discrete Dyn. Nat. Soc.* Vol. 2008, Article ID 154263, (2008), 14 pages.

[32] S. I. Ueki and L. Luo, Compact weighted composition operators and multiplication opera- tors between Hardy spaces, *Abstr. Appl. Anal.* Vol. 2008, Article ID 196498, (2008), 11 pages.

[33] S. Ye, Weighted composition operators between the little α-Bloch space and the logarithmic Bloch, *J. Comput. Anal. Appl.* **10** (2) (2008), 243-252.

[34] X. Zhu, Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, *Indian J. Math.* **49** (2) (2007), 139-149.

[35] X. Zhu, Products of differentiation, composition and multiplication from Bergman type spaces to Bers spaces, *Integral Transform. Spec. Funct.* **18** (3) (2007), 223-231.

Mathematical Institute of the Serbian Academy of Sciences,
Knez Mihailova 36/III, 11000 Beograd, Serbia
E-mail: sstevic@ptt.rs