

Products of modal logics

Valentin Shehtman

Institute for Information Transmission Problems
& Moscow State University
& Higher School of Economics

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- In the early days of modal logic (before 1980s) there was interest in studying multiple particular systems.

Contemporary modal logic also investigates classes of logics and general constructions combining different systems.

- Products were introduced in the 1970s; their intensive study started in the 1990s.

Motivations for studying products of modal propositional logics

- A natural type of combined modal logics
- Connection to first-order classical logic
- Connection to first-order modal logics
- Connection to relation algebras
- Connection to description logics

The main reference for products (BOOK03)

D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyashev. *Many-dimensional Modal Logics: Theory and Applications*. Elsevier, 2003.

PRODUCTS OF FRAMES

Kripke n-frames: (W, R_1, \dots, R_n) (relational system with n binary relations).

Def. The product of two Kripke frames

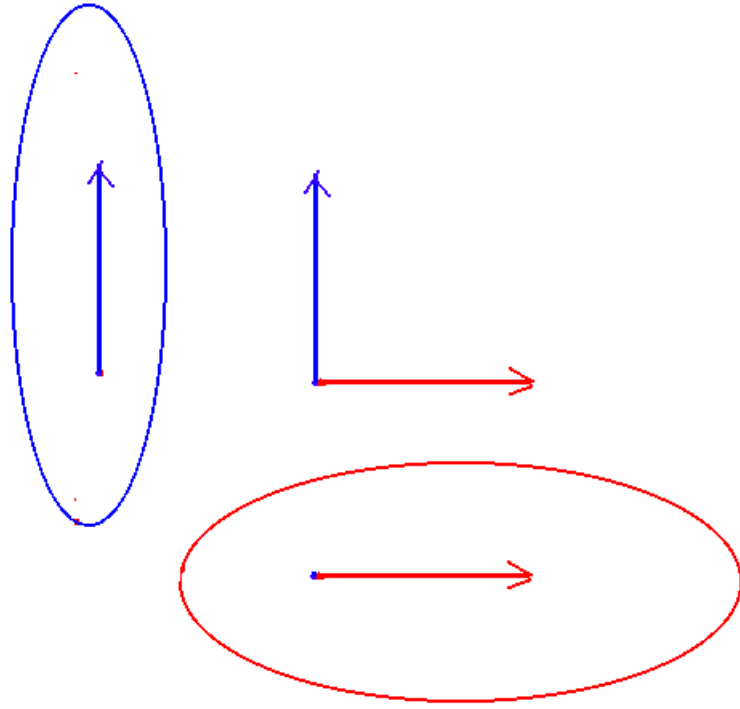
$$(W, R_1, \dots, R_n) \times (V, S_1, \dots, S_m) := (W \times V, R_{11}, \dots, R_{n1}, S_{12}, \dots, S_{m2})$$

where

$$(x_1, y_1) R_{i1} (x_2, y_2) \Leftrightarrow x_1 R_i x_2 \ \& \ y_1 = y_2$$

$$(x_1, y_1) S_{j2} (x_2, y_2) \Leftrightarrow x_1 = x_2 \ \& \ y_1 S_j y_2$$

Multiple products $F_1 \times \dots \times F_n$ are defined in an obvious way. The multiplication is associative up to isomorphism.



PRODUCTS OF MODAL LOGICS

Normal n -modal logics are defined as usual - as sets of modal formulas in the propositional language with unary modal connectives \Box_1, \dots, \Box_n containing the minimal logic and closed under standard rules.

Every Kripke frame is associated with a modal logic – the set of all valid formulas:

$$\mathbf{L}(F) := \{A \mid F \models A\}.$$

Logics of this form are called (Kripke) complete.

If F is finite, $\mathbf{L}(F)$ is called tabular.

For a class of frames \mathcal{C}

$$\mathbf{L}(\mathcal{C}) := \bigcap \{\mathbf{L}(F) \mid F \in \mathcal{C}\}.$$

If all frames in \mathcal{C} are finite, $\mathbf{L}(\mathcal{C})$ has the finite model property (fmp).

A modal logic L defines the class of L -frames

$$\mathbf{V}(L) := \{F \mid F \models L\}.$$

L is called **elementary** if $\mathbf{V}(L)$ is an elementary (first-order definable) class in the classical sense.

Remark L is complete iff $L = \mathbf{L}(\mathbf{V}(L))$.

Some particular complete logics

K_n is the minimal n-modal logic, **K**=**K₁**.

K.t_n is the minimal n-temporal logic, **K.t**=**K.t₁**.

K.t_n -frames are $(W, R_1, (R_1)^{-1}, \dots, R_n, (R_n)^{-1})$.

T = **K**+ $\Box p \rightarrow p$ = **L**(all reflexive frames)

K4 = **K**+ $\Box p \rightarrow \Box \Box p$ = **L**(all transitive frames)

S4 = **K4**+ $\Box p \rightarrow p$ = **L**(all transitive reflexive frames).

K4.3 = **L**(all transitive non-branching frames)

=**L**(all strict linear orders)

S4.3 = **K4.3**+ $\Box p \rightarrow p$ = **L**(all linear orders)

S5 = **S4**+ $\Diamond \Box p \rightarrow p$ = **L**(all equivalence frames)

=**L**(all universal frames)

Grz = **L**(all finite posets)

GL = **L**(all strict finite posets)

Grz3 = **L**(all finite chains)

GL3 = **L**(all strict finite chains)

Def. The product of two modal logics

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \mid F_1 \models L_1, F_2 \models L_2\}).$$

Similarly we can define multiple products

$$L_1 \times \dots \times L_n := \mathbf{L}(\{F_1 \times \dots \times F_n \mid F_1 \models L_1, \dots, F_n \models L_n\}).$$

However, multiplication of logics is probably non-associative (an open problem).

AXIOMATIZATION: FIRST RESULTS

AXIOMATIZATION PROBLEM: *to find axioms of $L_1 \times \dots \times L_n$ given the axioms of L_1, \dots, L_n .*

Theorem 1 (Sh 1987, Gabbay&Sh 1998)

Classes of frames C_1, \dots, C_n are elementary $\Rightarrow \mathbf{L}(C_1 \times \dots \times C_n)$ is RE.

(So, L_1, \dots, L_n are Kripke complete and elementary $\Rightarrow L_1 \times \dots \times L_n$ is RE.)

Corollary 1.1 (Sh 1987) $\mathbf{L}((\mathbf{Q}, <)^2), \mathbf{L}((\mathbf{Q}, \leq)^2)$ are RE.

Def. The fusion of two modal logics with disjoint modalities

$L_1 * L_2 :=$ the smallest logic containing $L_1 \cup L_2$

Remarks on fusions

Fusion of logics preserves many properties:

Theorem 2 (Kracht&Wolter 1991, Fine&Schurz 1996) Fusion preserves Kripke completeness, the fmp, decidability.

Bad news: products do not preserve any of these properties.
Good news: sometimes they still do.

Def. The commutative join of two modal logics with disjoint modalities

\Box_i ($1 \leq i \leq n$), \blacksquare_k ($1 \leq k \leq m$)

is

$[L_1, L_2] := L_1 * L_2 +$

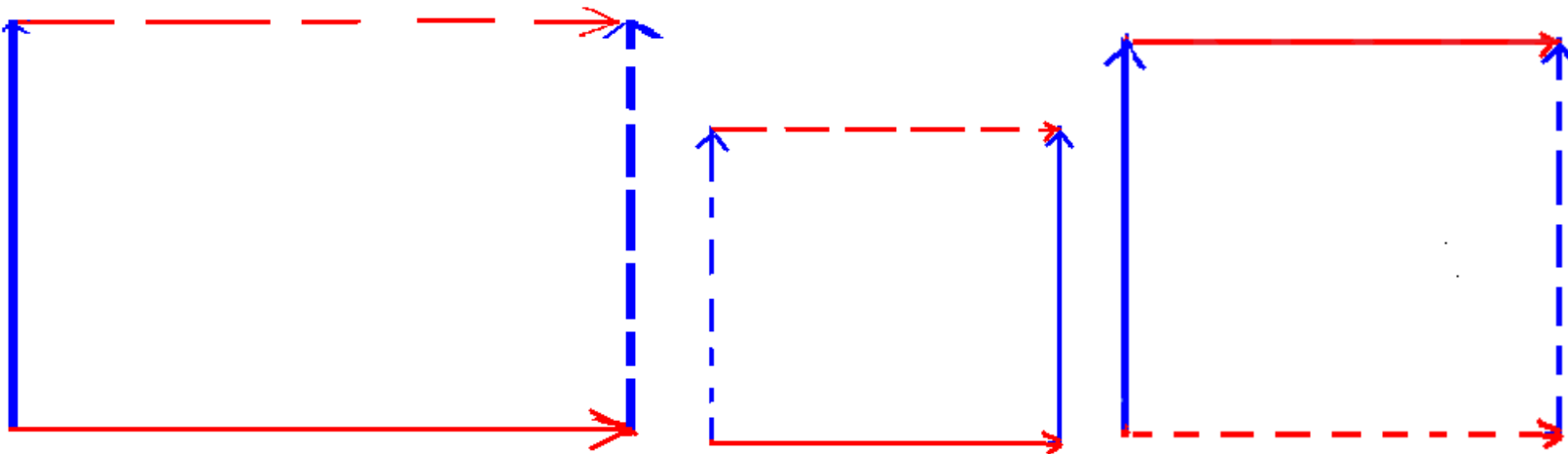
$\Diamond_i \blacksquare_k p \rightarrow \blacksquare_k \Diamond_i p + \Box_i \blacksquare_k p \leftrightarrow \blacksquare_k \Box_i p$ for any i, k

Remark. If the modalities are not disjoint, we rename them.

The additional axioms are Sahlqvist formulas expressing the following properties of the relations in the product frame

$$(R_{i_1})^{-1} \circ S_{k_2} \subseteq S_{k_2} \circ (R_{i_1})^{-1} \text{ (Church - Rosser property)}$$

$$R_{i_1} \circ S_{k_2} = S_{k_2} \circ R_{i_1} \text{ (commutativity)}$$



Def. L_1, L_2 are product matching if $[L_1, L_2] = L_1 \times L_2$

Def. A Horn sentence is a universal first order sentence of the form

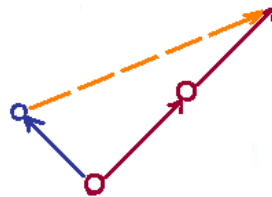
$$\forall x \dots (\varphi(x,y,z) \rightarrow R(x,y)),$$

where φ is positive, $R(x,y)$ is atomic.

A modal formula A is Horn if it corresponds to a Horn sentence (i.e., the class of its frames $\mathbf{V}(A)$ is definable by a Horn sentence).

Example Modal formulas of the form $(\Diamond \dots \Diamond) \Box p \rightarrow (\Box \dots \Box)p$

correspond to Horn sentences



Logics with such axioms are always complete.

Def. A modal logic is **Horn axiomatizable** if it is axiomatizable by formulas that are either variable-free or Horn.

Completeness theorem for products

([Gabbay, Sh 1998] << [BOOK03])

Theorem 3 If L_1, L_2 are Kripke complete and Horn axiomatizable, then they are product matching.

Counterexamples

Theorem 4 [Sh 1987 << Gabbay,Sh 1998]

Let **L** be a nontrivial 1-modal logic containing **Grz**. Then **L** and **S5** are not product-matching.

Theorem 5 [Kurucz & Marcelino 2011]

K4.3 and **S5**, **S4.3** and **S5** are not product-matching.

Stornger counterexamples: finite axiomatizability is not preserved
(see later)

FMP AND PRODUCT FMP

Def. A QTC-logic is axiomatizable by variable-free formulas and formulas or axioms of the form $\diamond_i \Box_j p \rightarrow p$, $\Box_i p \rightarrow (\Box_i)^k p$.

Theorem 6 [Sh 2005] If L_2 is a QTC-logic, then

$\mathbf{K.t}_n \times L_2 = [\mathbf{K.t}_n, L_2]$ has the fmp.

Theorem 7 [Sh 2011] $(\mathbf{K.t}_n)^2 = [\mathbf{K.t}_n, \mathbf{K.t}_n]$ has the product fmp.

THE LACK OF RECURSIVE AXIOMATIZATION

(Reynolds&Zakharyashev 2001 << BOOK03)

Theorem 8 (a) If C_1, C_2 are classes of **K4.3**-frames containing some frames with descending ω -chains and every frame in is Dedekind-complete (i.e., every bounded set has supremum), then $L(C_1 \times C_2)$ is Π_1^1 -hard.

(b) If C_1, C_2 are classes of **K4.3**-frames containing some frames with ascending ω -chains and every frame in C_1 is Dedekind-complete, then $L(C_1 \times C_2)$ is Π_1^1 -hard.

Corollary 8.1 **GL3², Grz3², GL3×Grz3** are Π_1^1 -hard.

Thus products do not preserve any interesting property of modal logics.

Corollary 8.2 $\mathbf{L}(F \times G)$ is Π_1^1 -hard whenever F is \mathbf{R} or ω , F is \mathbf{Q} , \mathbf{R} or ω (with $<$ or \leq).

Theorem 9 If $\mathcal{C}_1, \mathcal{C}_2$ are classes of finite [strict] linear orders of unbounded length, then $\mathbf{L}(\mathcal{C}_1 \times \mathcal{C}_2)$ is Π_1^0 -complete.

Corollary 9.1 $\mathbf{L}((\omega, >)^2), \mathbf{L}((\omega, \geq)^2)$ are Π_1^0 -complete.

PRODUCTS WITH TABULAR LOGICS

Theorem 10 (Sh 2013) (1) If \mathbf{L}_1 has the fmp and \mathbf{L}_2 is tabular, then $\mathbf{L}_1 \times \mathbf{L}_2$ has the product fmp.

(2) If \mathbf{L}_1 is decidable, \mathbf{L}_2 is tabular, then $\mathbf{L}_1 \times \mathbf{L}_2$ is decidable.

TRANSLATION INTO MODAL PREDICATE LOGIC

Consider n-modal predicate formulas with arbitrary predicates, but without equality, constants and function symbols.

Kripke frame semantics with constant domains

Propositional Kripke frames: $F = (W, R_1, \dots, R_n)$

Predicate Kripke frames with constant domains:

$\Phi = (F, D)$, where D is nonempty.

F is the frame of worlds of Φ , D is the set of individuals.

Kripke models over Φ :

$M = (\Phi, V)$, where V is a valuation:

$V(P) \subseteq D^n \times W$ for every n-ary predicate letter P ,

For every formula $A(x_1, \dots, x_n)$ and $d_i \in D$ we construct a D-sentence

$A(d_1, \dots, d_n)$

Forcing relation $M, u \vDash B$ between $u \in W$ and a D-sentence B is defined by induction, in particular:

- $M, u \vDash P(d_1, \dots, d_n)$ iff $(d_1, \dots, d_n, u) \in V(P)$

• $M, u \models \Box_i B$ iff $\forall v \in R_i(u) M, v \models B$

• $M, u \models \forall x B$ iff $\forall d \in D M, u \models [d/x]B$

Def $M \models A(x_1, \dots, x_n)$ iff $\forall u \in W M, u \models \forall x_1 \dots \forall x_n A(x_1, \dots, x_n)$

(validity in a frame) $\Phi \models A$ iff for any M over Φ , $M \models A$

$L(\Phi) := \{A \mid \Phi \models A\}$ is the modal predicate logic of Φ .

Wajsberg-type translation

Wajsberg's translation interprets **S5** in classical first-order logic.

Similarly,

every propositional $(n+1)$ -formula A (with modalities $\Box_1, \dots, \Box_n, [\forall]$)

is translated into

a first-order n -modal formula $A^\#(y)$ with (maybe) a parameter y :

Every $q \in PV$ is associated with a unary predicate letter Q . Then

$$q^\#(y) := Q(y) \text{ (for } q \in PV)$$

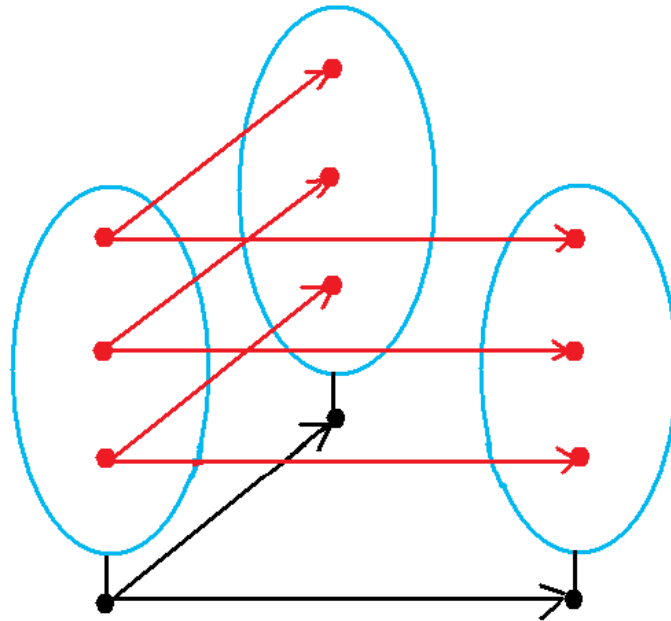
$$\perp^\#(y) := \perp$$

$$(A \rightarrow B)^\#(y) := A^\#(y) \rightarrow B^\#(y)$$

$$(\Box_i A)^\#(y) := \Box_i A^\#(y)$$

$$([\forall]A)^\#(y) := \forall y A^\#(y)$$

Lemma $F_x(D, D \times D) \models A$ iff $(F, D) \models \forall y A^\#(y)$



Theorem 11 Let L_1 be an n-modal propositional logic, $CK(L_1)$ the class of all predicate Kripke frames (F,D) , with $F \models L_1$.

Consider the corresponding predicate modal logic $L(CK(L_1))$. Then $L_1 \times S5$ is (polynomially) reducible to $L(CK(L_1))$: for any $(n+1)$ -modal propositional A

$$L_1 \times S5 \vdash A \text{ iff } \forall y A^\#(y) \in L(CK(L_1))$$

(In other words, $L_1 \times S5$ specifies a fragment within $L(CK(L_1))$)

Completeness theorems for modal predicate logics yield a standard axiomatization of $L(CK(L_1))$ in some cases.

QL_1 is the pure quantified version of L_1 , BF is the conjunction of Barcan schema for all modalities:

$$\forall x \Box_i A \rightarrow \Box_i \forall x A.$$

Theorem 12 (1) (Tanaka&Ono, 1999) If L_1 is complete and $V(L_1)$ is universally axiomatizable (in the classical sense), then

$$L(CK(L_1)) = QL_1 + BF$$

(2) (Ono, 1983 << Gabbay&Skvorstov&Shehtman, 2009)

The same holds if L_1 is tabular.

In all these cases $L_1 \times S5$ is RE, but now from Theorems 11, 12, 3 we obtain

Corollary 12.1 $[L_1, S5] \vdash A$ iff $QL_1 + BF \vdash \forall y A^\#(y)$
whenever L_1 is Horn axiomatizable.

TRANSLATION INTO CLASSICAL PREDICATE LOGIC

This “square translation” resembles the well-known standard translation of modal formulas in the language of $(\mathbf{K}_n)^2$ into classical first-order formulas with relativized quantifiers. Consider the first order language with binary predicate letters $R_1, \dots, R_n, P_1, P_2, \dots$. We associate a binary predicate letter P_i with every proposition letter p_i .

$$(p_i)^2(x,y) := P_i(x,y)$$

$$(A \rightarrow B)^2(x,y) := A^2(x,y) \rightarrow B^2(x,y)$$

$$\perp^2(x,y) := \perp$$

$$(\Box_i A)^2(x,y) := \forall z (R_i(x,z) \rightarrow A^2(z,y))$$

$$(\blacksquare_i A)^2(x,y) := \forall z (R_i(y,z) \rightarrow A^2(x,z))$$

Theorem 13 If L is an elementary modal logic, φ (first-order) axiomatizes $\mathbf{V}(L)$, then

(1) $QCL + \varphi \vdash \forall x \forall y A^2(x,y)$ iff $L^2 \vdash A$.

(Here QCL is the classical first-order theory axiomatized by φ).

(2) If L^2 has the product fmp, then the corresponding "square fragment" of $QCL + \varphi$ with binary predicates has the fmp (in the classical sense).

Axiomatizing some products of non-product-matching logics

Def. A propositional modal logic L is called **locally tabular** if, up to equivalence in L , for any m there are finitely many formulas in m propositional variables.

It is well-known that every locally tabular logic has the fmp.

Def. A propositional 1-modal logic L above **K4** is of finite **depth** $< m$ if all L -frames are of depth $< m$.

L is of **depth** m if it is of depth $< m+1$, but not $< m$.

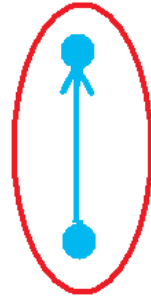
Theorem 14 (Seegerberg, 1971) Every logic of finite depth is locally tabular.

Theorem 15 (Maksimova, 1974) The converse holds for extension of **K4**

Theorem 16 (Sh 2010) If L is of finite depth, then $[↯, \mathbf{S5}]$ is locally tabular.

This allows us to axiomatize products of finite depth logics above **Grz** with **S5** in two ultimate cases: the **catkin formula** ACK is

exactly what is missing. ACK is the Fine – Jankov formula of the following 2-frame (catkin):



Theorem 17 (Sh 2010) If

$L = \mathbf{Grz} + \mathbf{Adepth}_n$ (= \mathbf{L} (all posets of depth n))

or

$L = \mathbf{Grz3} + \mathbf{Adepth}_n$ (= \mathbf{L} (all chains of depth n)),

then $L \times \mathbf{S5} = [L, \mathbf{S5}] + \mathbf{ACK}$.

Corollary 17.1 These logics are decidable.

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References

[GSS09] D. Gabbay, D. Skvortsov, V. Shehtman. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

[GKWZ03] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyashev. Many-dimensional Modal Logics: Theory and Applications. Elsevier, 2003.

[Sano10] K. Sano. Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic? J. Applied Logic, v.8, 2010, pp. 459-474.

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Theorem [Kurucz & Marcelino 2011]

K4.3×K, S4.3×K are not even axiomatizable in finitely many variables

QUESTION. Are the logics

K4.3 × S5, S4.3 × S5

finitely axiomatizable?

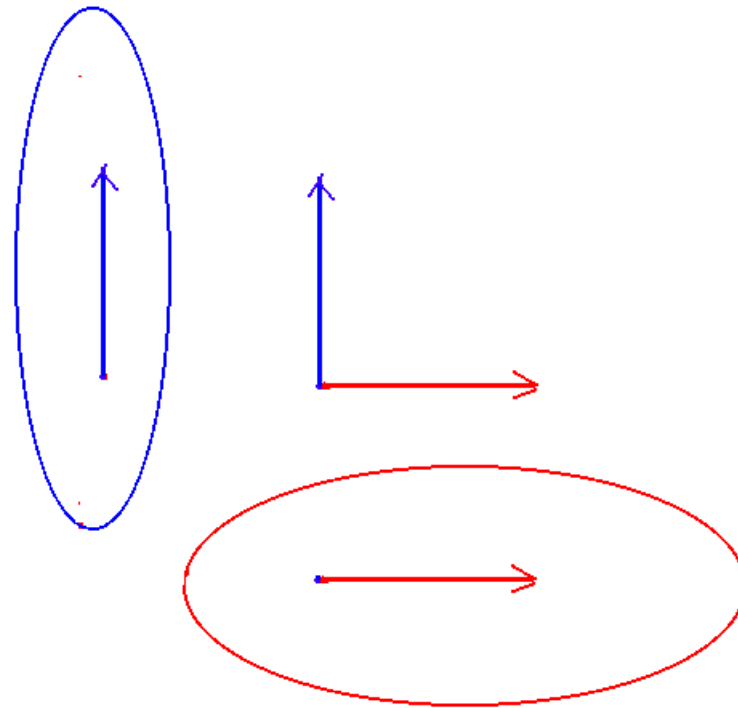
Squares of modal logics with additional connectives

Valentin Shehtman

Institute for Information Transmission Problems
& Moscow State University
& Laboratory Poncelet

London, February 2011

PRODUCTS



Def. The product of two Kripke frames

$$(W, R_1, \dots, R_n) \times (V, S_1, \dots, S_m) :=$$

$$(W \times V, R_{11}, \dots, R_{n1}, S_{12}, \dots, S_{m2}),$$

where

$$(x_1, y_1)R_{i_1}(x_2, y_2) \Leftrightarrow x_1R_i x_2 \ \& \ y_1=y_2$$

$$(x_1, y_1)S_{j_2}(x_2, y_2) \Leftrightarrow x_1=x_2 \ \& \ y_1S_j y_2$$

Def. The product of two modal logics

$$L_1 \times L_2 := L(\{F_1 \times F_2 \mid F_1 \models L_1, F_2 \models L_2\})$$

AXIOMATIZATION PROBLEM: to find axioms of $L_1 \times L_2$ given the axioms of L_1, L_2

Def. The fusion of two modal logics with disjoint modalities

$$L_1 * L_2 := \text{the smallest logic containing } L_1 \cup L_2$$

Def. The commutative join of two modal logics with disjoint modalities

$$\Box_i \ (1 \leq i \leq n), \ \Box_j \ (1 \leq j \leq m)$$

$$[L_1, L_2] := L_1 * L_2 + \Box_i \Box_k p \rightarrow \Box_k \Box_i p + \Box_i \Box_k p \leftrightarrow \Box_k \Box_i p$$

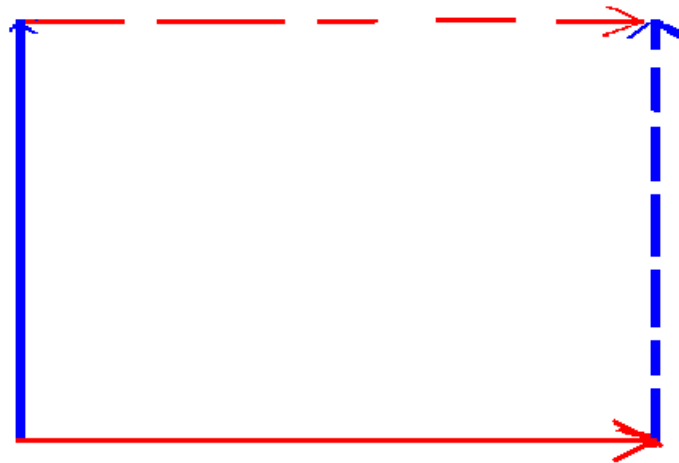
for any i, k

Remark. If the modalities are not disjoint, we can change them.

These are Salqvist formulas expressing the following properties of the relations in the product frame

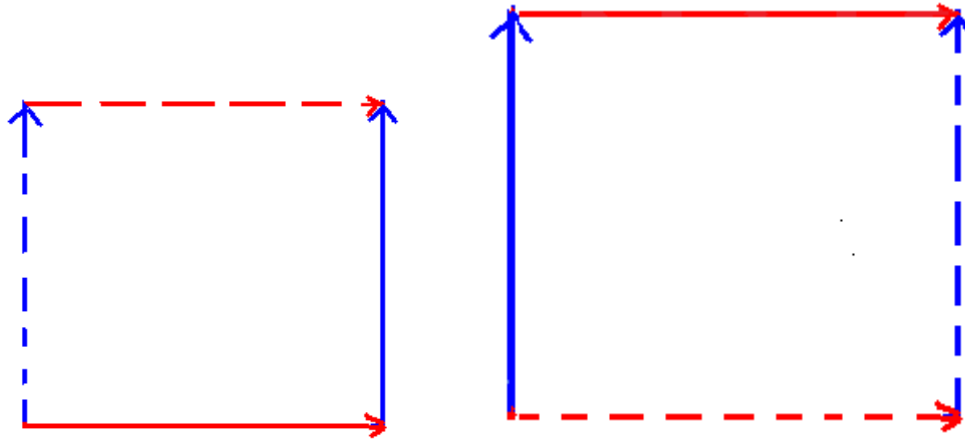
$$\Diamond_i \Box_k p \rightarrow \Box_k \Diamond_i p :$$

$$(R_{i1})^{-1} \circ S_{k2} \subseteq S_{k2} \circ (R_{i1})^{-1} \text{ (Church - Rosser property)}$$



$$\exists_i \forall_k p \leftrightarrow \forall_k \exists_i p:$$

$$R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1} \text{ (commutativity)}$$



Def. L_1, L_2 are product-matching if $L_1 \times L_2 = [L_1, L_2]$

SQUARES

For a class of frames \mathbf{C} put

$$\mathbf{C}^2 := \{F \times F \mid F \in \mathbf{C}\}.$$

For a modal logic Λ put

$$\Lambda^2 := \Lambda \times \Lambda$$

Proposition 1 [Gabbay,Sh 2000]

$$\Lambda^2 = L(\{F \times F \mid F \models \Lambda\}).$$

("Squares of logics are determined by squares of frames".)

Proposition 2 [Gabbay,Sh 2000]

$$L_1 \times L_2 \text{ is embeddable in } (L_1 * L_2)^2.$$

("Products are reducible to squares".)

SEGERBERG SQUARES

These are square frames with additional functions. Krister Segerberg (1973) studied a special type - squares of frames with the universal relation.

He considered the following functions on squares.

$\sigma_{\circ} : (x, y) \mapsto (y, x)$ (diagonal symmetry)

$\sigma_{\ominus} : (x, y) \mapsto (y, y)$ (the first diagonal projection)

$\sigma_{\oplus} : (x, y) \mapsto (x, x)$ (the second diagonal projection)

These functions can be associated with extra modal operators \circ , \ominus , \oplus . So in square frames they are interpreted as follows:

$(x, y) \models \circ A$ iff $(y, x) \models A$

$(x, y) \models \ominus A$ iff $(x, x) \models A$

$(x, y) \models \oplus A$ iff $(y, y) \models A$

Remark. Segerberg used the notation \otimes instead of \circ .

Formally we define the **Segerberg square** of a frame

$F = (W, R_1, \dots, R_n)$ as the $(2n+3)$ -frame

$F^{2\otimes} := (F^2, \sigma_o, \sigma_\ominus, \sigma_\emptyset)$ (where $\sigma_o, \sigma_\ominus, \sigma_\emptyset$ are the functions on W^2 described above).

Respectively, the **Segerberg square** of an n -modal logic Λ is defined the logic of the Segerberg squares of its frames

$$\Lambda^{2\otimes} := L(\{F^{2\otimes} \mid F \models \Lambda\}).$$

TOMORROW (OR SUCCESSOR) LOGIC

$$\mathbf{SL} := \mathbf{K} + \Diamond p \leftrightarrow \Box p$$

(an equivalent form: $\mathbf{K} + \neg \Box p \leftrightarrow \Box \neg p$)

This well-known logic is also due to Segerberg (1967). It is complete w.r.t. the frame



(the successor relation on natural numbers).

Every logic of a frame with a functional accessibility relation is an extension of **SL**.

AXIOMATIZING SEGERBERG SQUARES

Soundness Every Segerberg square validates the following formulas

The corresponding semantic conditions for an arbitrary $(2n+3)$ -frame

$$(V, X_1, \dots, X_n, Y_1, \dots, Y_n, f_{\circ}, f_{\ominus}, f_{\oplus})$$

are in the right column; here fg denotes the composition of functions:

$$(fg)(x) = f(g(x))$$

(I) The **SL**-axioms for the circles \circ , \ominus , \oplus .

(II)

$$(Sg1) \quad \circ \circ p \leftrightarrow p \qquad f_{\circ} f_{\circ} = 1 \text{ (the identity function on } V)$$

The "symmetry" f_{\circ} is an involution.

$$(Sg2) \quad \ominus \ominus p \leftrightarrow \ominus p \qquad f_{\ominus} f_{\ominus} = f_{\circ}$$

$$(Sg2') \quad \Phi \Phi p \leftrightarrow \Phi p \quad \mathbf{f}_\circ \mathbf{f}_\ominus = \mathbf{f}_\circ$$

Both projections \mathbf{f}_\ominus , \mathbf{f}_\circ are idempotent transformations of the square. In fact (Sg2') follows from (Sg1), (Sg2), (Sg3).

$$(Sg3) \quad \circ \Theta p \leftrightarrow \Phi p \quad \mathbf{f}_\ominus \mathbf{f}_\circ = \mathbf{f}_\circ$$

$$(Sg4) \quad \ominus \circ p \leftrightarrow \Theta p \quad \mathbf{f}_\circ \mathbf{f}_\ominus = \mathbf{f}_\ominus$$

In Seegerberg squares (Sg4) means that the image of \mathbf{f}_\ominus consists of self-symmetric points (or: every diagonal point is self-symmetric). But in the general case not all self-symmetric points are in $\mathbf{f}_\ominus[V]$.

(Sg3), (Sg4) imply that

$\mathbf{f}_\circ \mathbf{f}_\ominus \mathbf{f}_\circ = \mathbf{f}_\circ$, i.e., the involution \mathbf{f}_\circ conjugates the projections \mathbf{f}_\ominus and \mathbf{f}_\circ .

(Sg3) shows that Φ is expressible in terms of \circ , \ominus . **It also implies that**
 $\mathbf{f}_\ominus[V] = \mathbf{f}_\circ[V]$.

$$(Sg4') \quad \Phi \circ p \leftrightarrow \Phi p \quad \mathbf{f}_\circ \mathbf{f}_\circ = \mathbf{f}_\circ$$

This conjugate of (Sg4) is derivable from (Sg1), (Sg3), (Sg4).

$$(Sg5) \quad \circ \Xi_i \circ p \leftrightarrow \Xi_i p \quad a R_{i1} b \Leftrightarrow \mathbf{f}_\circ(a) R_{i2} \mathbf{f}_\circ(b)$$

$$(Sg5) \quad \bigcirc \Xi_i \bigcirc p \leftrightarrow \boxplus_i p \quad aR_{i1}b \leftrightarrow f_{\bigcirc}(a)R_{i2}f_{\bigcirc}(b)$$

Symmetry is an isomorphism between R_{i1} and R_{i2}

$$(Sg6) \quad \ominus \Xi_i (\boxplus_i p \rightarrow \bigcirc p) \quad f_{\ominus}(a)R_{i1}b \Rightarrow bR_{i2}f_{\ominus}(b)$$

If $(y,y)R_{i1}(x,y)$ (i.e. $yR_{i1}x$), then $(x,y)R_{i2}(x,x)$.

$$(Sg7) \quad \ominus p \rightarrow \Xi_i \ominus p \quad aR_{i1}b \Rightarrow f_{\ominus}(a) = f_{\ominus}(b)$$

Horizontally accessible points are in the same horizontal row.

$$(Sg8) \quad \boxplus_i \perp \leftrightarrow \ominus \boxplus_i \perp \quad (\exists b aR_{i2}b) \leftrightarrow (\exists b f_{\ominus}(a)R_{i2}b)$$

Vertical seriality is equivalent for (y,y) and (x,y) .

The conjugates of (Sg6)-(Sg8) are derivable, so they are not written here.

Def. For a modal logic Λ , put

$$[\Lambda, \Lambda]^{\otimes} :=$$

$$[\Lambda, \Lambda] + \mathbf{SL} * \mathbf{SL} * \mathbf{SL} \text{ (for } \bigcirc, \ominus, \bigcirc) + \{(Sg1), \dots, (Sg8)\}.$$

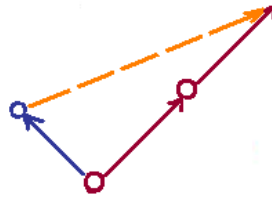
Def. A universal Horn sentence is a first order sentence of the

form $\forall x \dots (\varphi(x,y,z) \rightarrow R(x,y)),$

where φ is positive, $R(x,y)$ is atomic.

Modal formulas corresponding to such sentences are conjunctions of formulas of the form

$$(\diamond \dots \diamond) \Box p \rightarrow (\Box \dots \Box) p$$



Def. A modal logic is Horn axiomatizable if it is axiomatizable by formulas that are either variable-free or correspond to universal Horn sentences.

Completeness theorem for products [Gabbay, Sh 1998]

If L_1, L_2 are Horn axiomatizable, then they are product-matching.

Theorem 1 (Completeness) If a logic Λ is Horn axiomatizable, then $\Lambda^{2^\otimes} = [\Lambda, \Lambda]^\otimes$

Remark Segerberg himself axiomatized the logic **B** of all frames of the form $(W, W \times W)^{2\otimes}$. In this case (Sg8) becomes trivial and (Sg6) should be replaced with a stronger axiom: $\Box p \rightarrow \Diamond p$. So Segerberg's logic is not a Segerberg square in our sense; it is a proper extension of $S5^{2\otimes}$.

Sketch of the proof of Theorem 1

Step 1. $(K_n)^{2\otimes} = [K_n, K_n]^{\otimes}$

Consider the case $n=1$. The logic $L := [K, K]^{\otimes}$ is Sahhqvist, so it has the countable frame property, so it is determined by countable rooted L-frames. Let

$F = (W, R_1, R_2, f_{\circ}, f_{\ominus}, f_{\phi})$ be such a frame.

Now the goal is to construct a p-morphism from a Segerberg square onto F . We use a "rectification game" similar to the one described in [Sh 2005] and originally motivated by the games from [Many-dimensional modal logics, 2003] and [Relation algebras by games, 2002].

Let $T_\omega = (\omega^*, <)$ be the standard countable intransitive irreflexive tree, where

ω^* is the set of all finite sequences in ω ;

$\alpha < \beta$ iff $\exists n \in \omega \beta = \alpha n$.

Let $T_\omega + T_\omega$ be the disjoint union of its two copies:

$\{x\alpha \mid \alpha \in \omega^*\} \cup \{y\alpha \mid \alpha \in \omega^*\}$ with the relation $<$.

Consider the product frame

$$(T_\omega + T_\omega)^2 = ((\omega^* + \omega^*)^2, S_1, S_2).$$

A *network* over F is a partial function from $(T_\omega + T_\omega)^2$ to F

$$h: N \rightarrow V$$

such that

- $\text{dom}(h) = N$ is symmetric:

$$\sigma_\circ[N] = N,$$

$$\sigma_\ominus[N] \subseteq N.$$

- N does not have gaps:

$(\alpha, \beta) \in N \ \& \ (\alpha, \gamma) \in N \ \& \ \beta <^+ \gamma \ \& \ \beta < \beta' \Rightarrow (\alpha, \beta') \in N$

($<^+$ is the transitive closure of $<$)

• h is monotonic:

$a S_i b \Rightarrow h(a) R_i h(b),$

$h(\sigma_o(a)) = f_o(h(a)),$

$h(\sigma_e(a)) = f_e(h(a)).$

The game between **A** and **E** constructs a countable increasing sequence of networks $h_0 \subseteq h_1 \subseteq \dots$

according to the following rules.

1. $N_0 = \{(x, y), (y, x), (x, x), (y, y)\}$, where

$h_0(x, y) = u_0$, the root of F ; then $h_0(y, x)$, $h_0(x, x)$, $h_0(y, y)$ are uniquely determined.

Remark. If u_0 is self-symmetric, we don't need two copies, the game can start from $N_0 = \{(\lambda, \lambda)\}$, where λ is empty.

2. The $(n+1)$ th move of **A** is of two types

Lift enquiry (a, u, j, v) , where $a \in N_n$, $u = h_n(a)$, $u R_j v$

The response of **E** must be a network h_{n+1} extending h_n ch
that $\exists b \in N_{n+1}(aS_jb \ \&$

THE FINITE MODEL PROPERTY

Def. A QT-formula is a modal formula of the form

$\Box_i p \rightarrow \Box_i^k p$ (generalized transitivity)

or

$\Diamond_i \Box_i p \rightarrow p$ (symmetry)

A QTC-logic is axiomatizable by formulas that are either variable-free or QT-formulas.

Notation $\mathbf{K}_{\pm n}$ is the minimal n -temporal logic

(axiomatized by $\diamond_i^{-1} \Box_i p \rightarrow p$, $\diamond_i \Box_i^{-1} p \rightarrow p$)

The fmp for products [Sh 2005]

If L_2 is a QTC-logic, then $\mathbf{K}_{\pm n} \times L_2 = [\mathbf{K}_{\pm n}, L_2]$ has the fmp.

Theorem 2 $(\mathbf{K}_n)^{2\otimes}$ has the fmp.

Conjecture $(\mathbf{K}_{\pm n})^{2\otimes}$ has the fmp.

THE PRODUCT FMP FOR SEGERBERG SQUARES

Def A logic $\Lambda^{2\otimes}$ has the product fmp if it is determined by finite Segerberg squares:

$$\Lambda^{2\otimes} = L(\{F^{2\otimes} \mid F \text{ is finite, } F \models \Lambda\}).$$

The product fmp for products [Gabbay, Sh 2002]

Every logic $\mathbf{K}_{\pm n} \times \mathbf{K}_m$ has the product fmp.

Theorem 3 $(\mathbf{K}_n)^{2\otimes}$ has the product fmp.

Conjecture $(\mathbf{K}_{\pm n})^{2\otimes}$ has the product fmp.

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Ideas for the proofs of Theorems 1,2,3.
Relation algebras

