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PRODUCTS OF TORSION CLASSES OF LATTICE ORDERED GROUPS

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The aim of this note is to prove a conjecture (MARTINEZ [4]) on products of torsion classes of lattice ordered groups.

The notion of a torsion class of lattice ordered groups, the binary operation $\cdot$ (product) and the lattice operations $\bigcap$, $\bigvee$ for torsion classes were defined by Martinez [4]. To each torsion class $\mathscr{T}$ and each lattice ordered group G there corresponds a radical $T(G)$ of G such that $T(G)$ is the greatest convex l-subgroup of G belonging to the class $\mathscr{T}$.

Let $\mathscr{T}$ and $\mathscr{T}_\lambda$ $(\lambda \in \Lambda)$ be torsion classes. Martinez (loc. cit.) proved that

$$\mathscr{T} \cdot (\bigcap_{\lambda\in\Lambda} \mathscr{T}_\lambda) \subseteqq \bigcap_{\lambda\in\Lambda} (\mathscr{T} \cdot \mathscr{T}_\lambda) \tag{1}$$

and conjectured that, for appropriately chosen torsion classes $\mathscr{T}$ and $\mathscr{T}_\lambda$, the classes $\mathscr{T} \cdot (\bigcap_{\lambda\in\Lambda} \mathscr{T}_\lambda)$ and $\bigcap_{\lambda\in\Lambda} (\mathscr{T} \cdot \mathscr{T}_\lambda)$ are distinct. By using the notion of the radical this conjecture can be formulated as follows:

(*) There are torsion classes $\mathscr{T}, \mathscr{T}_\lambda$ $(\lambda \in \Lambda)$ and an l-group G such that, if we denote

$$\mathscr{S}_1 = \mathscr{T} \cdot (\bigcap_{\lambda\in\Lambda} \mathscr{T}_\lambda), \quad \mathscr{S}_2 = \bigcap_{\lambda\in\Lambda} (\mathscr{T} \cdot \mathscr{T}_\lambda)$$

and if $S_1(G)$ and $S_2(G)$ are radicals of G corresponding to the torsion classes $\mathscr{S}_1$ and $\mathscr{S}_2$, respectively, then $S_1(G)$ is a proper subset of $S_2(G)$.

If H is a linearly ordered group and if Λ is a finite set, then $S_1(H) = S_2(H)$ (cf. Lemma 1 below). Thus if (*) is valid then either Λ is infinite or G cannot be linearly ordered.

In this note the following assertions will be proved:

(A) *There exist torsion classes $\mathscr{T}, \mathscr{T}_\lambda$ $(\lambda \in \Lambda = \{1, 2, 3, \ldots\})$ and a linearly ordered group G such that $S_1(G)$ is a proper subset of $S_2(G)$.*

(B) *There exist torsion classes $\mathscr{T}, \mathscr{T}_\lambda$ $(\lambda \in \Lambda = \{1, 2\})$ and a lattice ordered group G such that $S_1(G)$ is a proper subset of $S_2(G)$.*

Each variety of lattice ordered groups is a torsion class. In [4] there are given some examples of torsion classes that are not varieties. The natural question arises: how many torsion classes exist that fail to be varieties? We shall show that the family of torsion classes with this property is very large. To each ordinal α we can assign a torsion class $\mathcal{T}_\alpha$ such that $\mathcal{T}_\alpha$ is not a variety and for any two distinct ordinals α, β we have $\mathcal{T}_\alpha \neq \mathcal{T}_\beta$ (moreover, if $\beta < \alpha$, then $\mathcal{T}_\beta$ is a proper subclass of $\mathcal{T}_\alpha$).

1. PRELIMINARIES

For the terminology, cf. BIRKHOFF [1] and FUCHS [2]. We use the additive notation for the group operation, though we do not suppose it to be abelian. Let G be a lattice ordered group and let $K(G)$ be the system of all convex l-subgroups of G partially ordered by inclusion. Then $K(G)$ is a complete lattice; for $\{H_i\} \subseteqq K(G)$ the lattice operations in $K(G)$ are denoted by $\bigcap H_i$ and $\bigvee H_i$.

For the sake of completeness, let us recall the following notions and results (cf. [4]). Let $\mathcal{T} \neq \emptyset$ be a class of lattice ordered groups such that

(i) if $G \in \mathcal{T}$, then each homomorphic image of G belongs to $\mathcal{T}$;

(ii) if $G \in \mathcal{T}$, then each convex l-subgroup of G belongs to $\mathcal{T}$;

(iii) if G is an l-group and $\{H_i\} \subseteqq K(G)$ such that each H_i belongs to $\mathcal{T}$, then $\bigvee H_i$ belongs to $\mathcal{T}$. Then $\mathcal{T}$ is called a torsion class of lattice ordered groups.

Let $\mathcal{G}$ be the class of all lattice ordered groups and let $T: \mathcal{G} \to \mathcal{G}$ be a mapping such that, for each $G \in \mathcal{G}$, the following conditions are fulfilled:

(i_1) $T(G)$ is an l-ideal of G;

(ii_1) $T(A) = A \cap T(G)$ for each convex l-subgroup A of G;

(iii_1) if $\Phi : G \to H$ is an onto l-homomorphism, then $(T(G))\,\Phi \subseteqq T(H)$.

Under these assumptions T is said to be a torsion radical. The l-ideal $T(G)$ is the T-radical of the l-group G. There is a one-to-one correspondence between torsion classes and torsion radicals that is given by the following rule. If $\mathcal{T}$ is a torsion class and G is a lattice ordered group, then the corresponding T-radical of G is the join $\bigvee H_i$ of all convex l-subgroups H_i of G belonging to $\mathcal{T}$. Conversely, if T is a torsion radical, then the corresponding torsion class $\mathcal{T}$ is the class of all l-groups G such that $T(G) = G$.

Let $\mathcal{A}, \mathcal{B}$ be torsion classes, $G \in \mathcal{G}$. Put

$$H = A(G/B(G))$$

and let H_0 be the set of all $g \in G$ such that $g + B(G) \in H$. Then H_0 is an l-ideal of G and the mapping $C: \mathcal{G} \to \mathcal{G}$ defined by $C(G) = H_0$ is a torsion radical. The corresponding torsion class will be denoted by $\mathcal{C} = \mathcal{A} \cdot \mathcal{B}$.

Let $\mathscr{T}_\lambda$ $(\lambda \in \Lambda)$ be torsion classes. For any $G \in \mathscr{G}$ we put

$$P(G) = \bigcap T_\lambda(G), \quad Q(G) = \bigvee T_\lambda(G).$$

Then P and Q are torsion radicals; the corresponding torsion classes will be denoted by

$$\mathscr{P} = \bigcap \mathscr{T}_\lambda, \quad \mathscr{Q} = \bigvee \mathscr{T}_\lambda.$$

If A, B are torsion radicals corresponding to torsion classes $\mathscr{A}$ and $\mathscr{B}$, then the torsion radical corresponding to the torsion class $\mathscr{A} . \mathscr{B}$ will be denoted by $A . B$. Analogous notations are used for the operations $\bigcap$, $\bigvee$.

2. LINEARLY ORDERED GROUPS

Lemma 1. *Let $\mathscr{T}, \mathscr{T}_\lambda$ $(\lambda \in \Lambda = \{1, 2, \ldots, n\})$ be torsion classes and let G be a linearly ordered group. Let S_1, S_2 be as in $(*)$. Then $S_1(G) = S_2(G)$.*

Proof. It suffices to prove the assertion for $\Lambda = \{1, 2\}$, since then the general case follows by induction. Thus we have to verify that

$$(2) \qquad (T . (T_1 \cap T_2))(G) = (T . T_1)(G) \cap (T . T_2)(G).$$

According to (1),

$$(T . (T_1 \cap T_2))(G) \subseteqq (T . T_1)(G) \cap (T . T_2)(G).$$

Since G is linearly ordered, $K(G)$ is a chain and so we can suppose that

$$T_1(G) \subseteqq T_2(G)$$

is valid. Hence

$$T(G/T_1(G) \cap T_2(G)) = T(G/T_1(G))$$

and therefore

$$(T . (T_1 \cap T_2))(G) = (T . T_1)(G) \supseteqq (T . T_1)(G) \cap (T . T_2)(G).$$

Thus (2) is valid.

We need some auxiliary results on linearly ordered groups.

Let J be a linearly ordered set and let G be an l-group. Assume that, for each $j \in J$, A_j is an l-subgroup of G such that

(a) the group G is a direct sum of its subgroups A_j;

(b) if $0 \neq g \in G$, $g = a_1 + \ldots + a_n$, $0 \neq a_i \in A_{j(i)}$, $j(i) \in J$ for $i = 1, \ldots, n$ and $j(1) < j(2) < \ldots < j(n)$, then $g > 0$ if and only if $a_1 > 0$.

Under these assumptions G is said to be a lexicographic sum of its l-subgroups A_j and we write

$$G = \Gamma^0 A_j \quad (j \in J).$$

If $J = \{1, 2, \ldots, n\}$ with the natural order, then we denote

$$G = A_1 \circ A_2 \circ \ldots \circ A_n .$$

Lemma 2. *Let J be a linearly ordered set and for each $j \in J$ let B_j be a lattice ordered group such that if j is not maximal in J, then B_j is linearly ordered. Then there exists a lattice ordered group $G = \Gamma^0 A_j$ $(j \in J)$ such that A_j is isomorphic to B_j for each $j \in J$.*

This is an easy consequence of [2], p. 41, (d).

Lemma 3. *Let H be a convex l-subgroup of an l-group $G = \Gamma^0 A_j$ $(j \in J)$. For each $j \in J$, $H \cap A_j$ is a convex l-subgroup of A_j and*

$$H = \Gamma^0(A_j \cap H) \ (j \in J).$$

Proof. The first assertion is obvious. Let $0 \neq g \in H$ and let a_i be as in (b) $(i = 1, \ldots, n)$. Then $2|g| \in H$ and

$$-2|g| < a_i < 2|g|$$

holds for $i = 1, \ldots, n$, hence $a_i \in H$. Therefore the conditions (a) and (b) are valid with G, A_j replaced by H, $H \cap A_j$.

Lemma 4. *Let H be an l-ideal of an l-group $G = \Gamma^0 A_j$ $(j \in J)$. Suppose that each A_j is linearly ordered. Then G/H is isomorphic with $\Gamma^0(A_j/H \cap A_j)$ $(j \in J)$.*

Proof. Let $j \in J$. The group $H \cap A_j$ is a normal subgroup of A_j. According to Lemma 3, $H \cap A_j$ is a convex l-subgroup of A_j. Thus $H \cap A_j$ is an l-ideal of A_j and hence we can construct the factor l-group $A_j/H \cap A_j$. Moreover, each l-group $A_j/H \cap A_j$ is linearly ordered. Hence by Lemma 2, the l-group $\Gamma^0(A_j/H \cap A_j)$ $(j \in J) = G'$ does exist.

Let $j(1), \ldots, j(n) \in J$, $j(1) < j(2) < \ldots < j(n)$, and let $a_i, b_i \in A_{j(i)}$ $(i = 1, \ldots, n)$,

$$g = a_1 + \ldots + a_n , \quad g' = b_1 + \ldots + b_n .$$

If $g - g' \in H$, then according to Lemma 3,

$$a_i - b_i \in H \cap A_{j(i)} \quad (i = 1, \ldots, n).$$

Thus the mapping $\varphi : G/H \to G'$ defined by

$$\varphi(g + H) = a_1 + (H \cap A_{j(1)}) + \ldots + a_n + (H \cap A_{j(n)})$$

is correctly defined. φ is a homomorphism of the group G/H onto the group G'. If $\varphi(g + H) = 0$, then $a_i + (H \cap A_{j(i)}) = H \cap A_{j(i)}$ and hence $a_i \in H \cap A_{j(i)}$ for $i = 1, \ldots, n$; thus $g = a_1 + \ldots + a_n \in H$. Therefore φ is an isomorphism of the group G onto the group G'.

Let $g \in G$, $g + H \neq H$. There are elements $j(1), \ldots, j(n) \in J$ with $j(1) < j(2) < \ldots \ldots < j(n)$ and $0 \neq a_i \in A_{j(i)}$ $(i = 1, \ldots, n)$ such that $g = a_1 + \ldots + a_n$. Denote

$$k = \min \{i \in \{1, \ldots, n\} : a_i \text{ non} \in H\} ,$$

$$g' = a_k + a_{k+1} + \ldots + a_n .$$

Then $g' \in g + H$ and hence

$$\varphi(g + H) = a_k + (H \cap A_{j(k)}) + a_{k+1} + (H \cap A_{j(k+1)}) + \ldots + a_n + (H \cap A_{j(n)}).$$

Let $g + H > 0$ in G/H. If $a_k < 0$, then $g' < 0$ and g' non $\in H$, thus $g + H = g' + H < 0$ in G/H, which is a contradiction. Therefore $a_k > 0$ and hence $a_k + (H \cap A_{j(k)}) > 0$ in $A_{j(k)}/H \cap A_{j(k)}$. This implies that $\varphi(g + H) > 0$.

Conversely, let $\varphi(g + H) > 0$. Then $a_k + (H \cap A_{j(k)}) > 0$ and hence $a_k > 0$. From this we obtain $g' > 0$ and so $g + H = g' + H > 0$.

Thus φ is an isomorphism of the linearly ordered group G/H onto $\Gamma^0(A_j/H \cap A_j)$ $(j \in J)$.

If an l-group G is a cardinal sum of its l-subgroups A_i $(i \in I)$, then we denote it by $G = \Sigma A_i$ $(i \in I)$. In the case $I = \{1, \ldots, n\}$ we write $G = A_1 \oplus \ldots \oplus A_n$.

The proof of the following lemma is straightforward.

Lemma 5. *Let H be a convex subgroup of an l-group $G = \Sigma A_i$ $(i \in I)$. Then $H = \Sigma(H \cap A_i)$ $(i \in I)$. If H is an l-ideal of G, then G/H is isomorphic to $\Sigma A_i/H \cap A_i$.*

Let $\mathscr{C}$ be a class of lattice ordered groups that is closed with respect to isomorphisms. We denote by $k(\mathscr{C})$ the class of all lattice ordered groups that can be expressed as cardinal sums of lattice ordered groups belonging to $\mathscr{C}$.

Lemma 6. *Let $\mathscr{C}$ be a class of linearly ordered groups fulfilling* (i) *and* (ii). *Suppose that $\mathscr{C}$ satisfies the condition*

(iii_0) *if G is a linearly ordered group, $\{H_i\} \subseteq K(G)$ such that each H_i belongs to $\mathscr{C}$, then $\bigvee H_i$ belongs to $\mathscr{C}$.*

Then $k(\mathscr{C})$ is a torsion class.

Proof. Let $G \in k(\mathscr{C})$. Then $G = \Sigma A_i$ $(i \in I)$ with $A_i \in \mathscr{C}$ for each $i \in I$.

Let G' be a homomorphic image of G. There exists an l-ideal H_1 of G such that G' is isomorphic to G/H_1. By Lemma 5, G/H_1 is isomorphic to $\Sigma A_i/A_i \cap H_1$. Since $\mathscr{C}$ fulfils (i), $A_i/A \cap H_1 \in \mathscr{C}$ and hence $G' \in k(\mathscr{C})$.

Let H be a convex l-subgroup of G. According to Lemma 5, $H = \Sigma(H \cap A_i)$ $(i \in I)$ and obviously $H \cap A_i$ is a convex l-subgroup of A_i. Thus $H \in k(\mathscr{C})$.

Now let G be any l-group that need not belong to $k(\mathscr{C})$. Let $\mathscr{C}_1$ be the class of all linearly ordered groups. Then $k(\mathscr{C}_1) = \mathscr{T}_1$ is a torsion class (cf. [4]). Hence the T_1-radical $T_1(G)$ of G is a cardinal sum

$$T_1(G) = \Sigma A_i' \quad (i \in I_1) \tag{3}$$

of lineraly ordered group A_i'. Let $i \in I_1$ be fixed and let B_i be the join of all convex l-subgroups of A_i' belonging to $\mathscr{C}$. According to (iii$_0$), B_i belongs to $\mathscr{C}$ and hence the l-subgroup

$$G_0 = \Sigma B_i \quad (i \in I_1)$$

of G belongs to $k(\mathscr{C})$. Since B_i is convex in A_i' for each $i \in I_1$, G_0 is convex in G. Let H be a convex l-subgroup of G belonging to $k(\mathscr{C})$. Then $H \in \mathscr{T}_1$ and hence H is a convex l-subgroup of $T_1(G)$. From Lemma 5 and (3) we obtain

$$H = \Sigma(A_i' \cap H) \quad (i \in I_1).$$

Because $H \in k(\mathscr{C})$, we have $H = \Sigma C_j$ $(j \in J)$ with $C_j \in \mathscr{C}$. Hence according to Thm. 8, [2],

$$H = \sum_{i,j}(A_i' \cap H \cap C_j) \quad (i \in I_1,\ j \in J).$$

Clearly $A_i \cap H \cap C_j \in \mathscr{C}$. Thus $A_i' \cap H \cap C_j \in B_i$ for each $i \in I_1$. Therefore $H \subseteqq G_0$. Thus G_0 is the greatest convex l-subgroup of G belonging to $k(\mathscr{C})$. Hence $k(\mathscr{C})$ is a torsion class.

3. THE CLASSES $\mathscr{T}_0$ AND $\mathscr{T}_n$

We denote by $Z(R)$ the additive group of all integers (all reals) with the natural linear order. Let $\mathscr{C}_n$ be the class of linearly ordered groups G that can be writen as

$$G = A_1 \circ A_2 \circ \ldots \circ A_n,$$

where A_i is isomorphic to some l-subgroup R_i of R for each $i \in \{1, \ldots, n\}$. If B_i is a convex l-subgroup of A_i, then either $B_i = \{0\}$ or $B_i = A_i$. Hence it follows from Lemma 3 and Lemma 4 that the class $\mathscr{C}_n$ fulfils the conditions (i) and (ii).

Let G be any lattice ordered group and let $a, b \in G$. If $na < b$ for each positive integer n, then we write $a \ll b$. For any positive integer n we have:

If $G \in \mathscr{C}_{n+1}$, $G \notin \mathscr{C}_n$, then there are elements $a_1, a_2, \ldots, a_{n+1} \in G$ such that $0 < a_1 \ll a_2 \ll a_3 \ll \ldots \ll a_{n+1}$ and there does not exist any $b \in G$ with $a_{n+1} \ll b$.

Lemma 7. *Each class $\mathscr{C}_n$ fulfils the condition* (iii$_0$).

Proof. We proceed by induction on n. Let G be a linearly ordered group. We denote by S_n the set of all convex l-subgroup of G belonging to $\mathscr{C}_n$. We have to show that each system S_n has a greatest element.

If card $S_1 = 1$, then $\{0\}$ is the greatest element of S_1. Suppose that there is $\{0\} \neq A_1 \in S_1$ and let $B \in S_1$. Then we must have $A_1 \supseteq B$ and hence A_1 is the greatest element of S_1.

Assume that the assertion is proved for n; hence there exists the greatest element A_n of S_n. If $B \subseteqq A_n$ for each $B \in S_{n+1}$, then the assertion holds for $n + 1$. Suppose that B non $\subseteqq A_n$ for some $B \in S_{n+1}$. Then B cannot belong to S_n, hence there are elements $b_1, \ldots, b_{n+1} \in B$ with $0 < b_1 \ll b_2 \ll \ldots \ll b_{n+1}$. If $B_1 \in S_{n+1}$, B_1 non $\subseteqq B$, then $B \subset B_1$ and hence there is $b \in B_1$ with $b_{n+1} \ll b$; this is a contradiction. Therefore B is the greatest element of S_{n+1}.

From Lemma 6 and Lemma 7 we obtain:

Lemma 8. *$k(\mathscr{C}_n)$ is a torsion class for $n = 1, 2, \ldots$.*

We denote $\mathscr{T}_0 = \bigvee k(\mathscr{C}_n)$ $(n = 1, 2, 3, \ldots)$.

Let $P = \{p_1, p_2, \ldots\}$ be the set of all primes. For each positive integer n let $\bar{A}_n$ be the set of all $x \in R$ such that

$$xp_1p_2 \ldots p_n \in Z .$$

Then $\bar{A}_n$ is an l-subgroup of R. For $n \neq m$ the linearly ordered groups $\bar{A}_n$ and $\bar{A}_m$ are not isomorphic.

Lemma 9. *Let $\mathscr{T}_n$ be the class of all l-groups $G \in \mathscr{T}_0$ with the following property: if $H \in K(G)$ and if H_1 is an l-ideal of H, then H/H_1 is not isomorphic to $\bar{A}_n$. Then $\mathscr{T}_n$ is a torsion class.*

This follows from Lemma 8 and [4], Theorem 2.6.

Let $G = \Gamma^0 \bar{A}_j$ $(j \in J = \{1, 2, 3, \ldots\})$ and let n be a positive integer. From the definition of G and from the Lemmas 3, 4 and 5 it follows that

$$T_n(G) = \Gamma^0 \bar{A}_j \quad (j > n) . \tag{4}$$

Thus $G/T_n(G) \in \mathscr{C}_n \subset k(\mathscr{C}_n) \subset \mathscr{T}_0$ and hence

$$(T_0 \,.\, T_n)(G) = G$$

for each positive integer n. Therefore

$$\bigcap_{n=1,2\ldots} (T_0 \,.\, T_n)(G) = G . \tag{5}$$

Moreover we get from (4)

$$\bigcap_{n=1,2,\ldots} T_n(G) = \{0\} ,$$

$$(\bigcap T_n)(G) = \{0\}$$

and thus

$$(T_0 \cdot \cap T_n)(G) = T_0(G). \tag{6}$$

Let $\{0\} \neq H$ be a convex l-subgroup of G. Choose $0 \neq h \in H$. We have

$$|h| = a_{j(1)} + a_{j(2)} + \ldots + a_{j(n)},$$

$a_{j(i)} \neq 0$ for $i = 1, \ldots, n$, $j(1) < j(2) < \ldots < j(n)$. Then $a_{j(1)} > 0$ and hence

$$-2|h| < a_j < 2|h|$$

for each $a_j \in \bar{A}$ with $j > j(1)$. Thus

$$\Gamma^0 \bar{A}_j \ (j > j(1)) \subseteqq H.$$

From this we obtain

$$K_n(G) = \{0\} \quad \text{for} \quad n = 1, 2, \ldots,$$

where K_n is the torsion radical corresponding to the torsion class $k(\mathscr{C}_n)$. Hence

$$T_0(G) = \bigvee K_n(G) = \{0\}. \tag{7}$$

From (6) and (7) we get

$$(T_0 \cdot \cap T_n)(G) = \{0\}. \tag{8}$$

By (5) and (8), the assertion (A) is valid.

Let $\mathscr{Q}_1(\mathscr{Q}_2)$ be the class of all lattice ordered groups that are cardinal sums of linearly ordered groups isomorphic to $R(Z)$. Both $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are torsion classes (cf. [4]). Put $\mathscr{T} = k(\mathscr{C}_2)$. Let $G = A \circ (B \oplus C)$, where A and B are isomorphic to Z, and C is isomorphic to R. Then

$$Q_1(G) = C, \quad Q_2(G) = B, \quad T(G) = B \oplus C,$$

hence $G/Q_1(G)$ is isomorphic to $A \circ B$ and $G/Q_2(B)$ is isomorphic to $A \circ C$. Therefore

$$(T \cdot Q_1)(G) = G = (T \cdot Q_2)(G),$$

$$(T \cdot Q_1 \cap T \cdot Q_2)(G) = G. \tag{9}$$

On the other hand, $(Q_1 \cap Q_2)(G) = \{0\}$, hence

$$(T \cdot (Q_1 \cap Q_2))(G) = T(G) = B \oplus C \neq G. \tag{10}$$

By (9) and (10), the assertion (B) holds.

4. THE CLASSES R_α

Let $\alpha > 1$ be an ordinal and let J_α be an ordered set that is dually isomorphic to the set of all ordinals less than α. Let A_j be a lattice ordered group isomorphic to Z for each $j \in J$ and

$$C_\alpha = \Gamma^0 A_j \quad (j \in J_\alpha).$$

We put $C_1 = \{0\}$. Further let $\mathscr{C}_\alpha$ be the set of all linearly ordered groups C_β with $\beta \leqq \alpha$. Since Z has no convex l-subgroup distinct from $\{0\}$ and Z it follows from Lemma 3 and Lemma 4 that the class $\mathscr{C}_\alpha$ fulfils the conditions (i) and (ii).

Let G be a linearly ordered group. For each ordinal δ we shall define by induction l-subgroups B_δ and D_δ of G such that the following conditions are satisfied:

(a_1) either $B_\delta = \{0\}$ or B_δ is isomorphic to Z;

(a_2) D_δ is a convex l-subgroup of G and

$$D_\delta = \Gamma^0 B_{\varphi(j)} \quad (j \in K_\delta),$$

where K_δ is a linearly ordered set dually isomorphic to the set of all ordinals $\beta \leqq \delta$ and φ is the corresponding isomorphism.

We put $B_1 = D_1 = \{0\}$. Assume that $\gamma > 1$ and that we have defined B_δ, D_δ such that (a_1) and (a_2) are valid for each $\delta < \gamma$. Denote

$$E_\gamma = \bigcup D_\delta \quad (\delta < \gamma).$$

From the condition (a_2) we obtain

$$E_\gamma = \Gamma^0 B_{\psi(j)} \quad (j \in K_\gamma^0),$$

where $K_\gamma^0 = K_\gamma \setminus \{\gamma\}$ and ψ has an analogous meaning as φ with K_γ^0 instead of K_γ.

If $B_\delta = \{0\}$ for some δ with $1 < \delta < \gamma$, then we put $B_\gamma = \{0\}$. Assume that $B_\delta \neq \{0\}$ for each $1 < \delta < \gamma$. If there are l-subgroups H, H_1 of G such that H is a convex l-subgroup of G, $H_1 \neq \{0\}$, H_1 is isomorphic to Z and

$$H = H_1 \circ E_\gamma,$$

then we put $B_\gamma = H_1$, $D_\gamma = H$. If such l-subgroups H, H_1 of G do not exist, we put $B_\gamma = \{0\}$, $D_\gamma = E_\gamma$. Then the conditions (a_1) and (a_2) are valid for the ordinal γ.

From the construction of D_γ it follows, that D_γ is the greatest convex l-subgroup of G that is isomorphic to some lattice ordered group belonging to $\mathscr{C}_\gamma$. Hence $\mathscr{C}_\gamma$ fulfils the condition (iii_0). Therefore according to Lemma 6, $k(\mathscr{C}_\gamma)$ is a torsion class.

If $\alpha < \beta$ are ordinals, then $\mathscr{C}_\alpha \subset \mathscr{C}_\beta$ and hence $k(\mathscr{C}_\alpha) \subseteqq k(\mathscr{C}_\beta)$. But C_β non $\in \mathscr{C}_\alpha$ and hence, because C_β is linearly ordered, C_β non $\in k(\mathscr{C}_\alpha)$. Thus $k(\mathscr{C}_\alpha) \neq k(\mathscr{C}_\beta)$.

Let $\alpha > 2$ and let A, B be lattice ordered groups isomorphic to C_α, $G = A \oplus B$. Then $G \in k(\mathscr{C}_\alpha)$. Both A and B are linearly ordered and nonarchimedean and hence, according to [3], there is an l-subgroup C of G such that C cannot be represented as a cardinal sum of linearly ordered groups. Thus C does not belong to $k(\mathscr{C}_\alpha)$. Therefore the class $k(\mathscr{C}_\alpha)$ is not a variety. If we put $\mathscr{T}_\alpha = k(\mathscr{C}_{\alpha+2})$, then no torsion class $\mathscr{T}_\alpha$ is a variety.

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