# Programming the Finite Element Method

FOURTH EDITION

I. M. Smith University of Manchester, UK

**D. V. Griffiths** Colorado School of Mines, USA



# Contents

Pr	eface			xv
Ac	know	ledgem	ent	xvii
12	Preli	minarie	es: Computer Strategies	1
₹,₽	1.1	Introdu	ction	1
- 144- 	1.2	Hardwa	are	2
14	1.3	Memor	v management	$\frac{-}{2}$
	1.4	Vector	processors	3
- (24)	1.5	Parallel		4
€ł¢	1.6	BLAS	libraries	.4
<u>i</u>	1.7	MPI lil	praries	.5
	1.8	Applica	ations software	5
~		1.8.1	Arithmetic	. 7
		1.8.2	Conditions	. 7
4		1.8.3	Loops	. 8
	1.9	Arrav f	features	9
		1.9.1	Dynamic arrays	9
		1.9.2	Broadcasting	. 10
		1.9.3	Constructors	. 10
		1.9.4	Vector subscripts	10
		1.9.5	Array sections	. 11
		1.9.6	Whole-array manipulations	. 11
		1.9.7	Intrinsic procedures for arrays	12
		1.9.8	Additional Fortran 95 features	. 13
		1.9.9	Subprogram libraries	. 14
		1.9.10	Structured programming	16
	1.10	Conclu	sions	17
÷.,	Refe	rences.		18
2	Spati	ial Disc	retisation by Finite Elements	21
	2.1	Introdu	ction	21
	2.2	Rod ele	ement	21.
		2.2.1	Rod stiffness matrix	21
		2.2.2	Rod mass element	24
	2.3	The eig	genvalue equation	25

#### CONTENTS

	2.4	Beam element	25
		2.4.1 Beam element stiffness matrix	25
		2.4.2 Beam element mass matrix	27
	2.5	Beam with an axial force	28
	2.6	Beam on an elastic foundation	29
	2.7	General remarks on the discretisation process	29
	2.8	Alternative derivation of element stiffness	30
	2.9	Two-dimensional elements: plane strain	
		and plane stress	32
	2.10	Energy approach	35
	2.11	Plane element mass matrix	36
	2.12	Axisymmetric stress and strain	36
	2.13	Three-dimensional stress and strain	38
	2.14	Plate-bending element	40
	2.15	Summary of element equations for solids	43
	2.15	Flow of fluids: Navier-Stokes equations	43
	2.10	Simplified flow equations	45
	2.17	2.17.1. Stoody state	40
		2.17.1 Steady state	47
			49
	<b>a</b> 10		49
	2.18	Further coupled equations: Biot consolidation	50
	2.19	Conclusions	52
	Refe	rences	52
•	<b>n</b>	the Fill to Fill and Channel Advance	
3	Prog	ramming Finite Element Computations	55
3	<b>Prog</b> 3.1	Introduction	<b>55</b> 55
3	<b>Prog</b> 3.1 3.2	Introduction	<b>55</b> 55 55
3	<b>Prog</b> 3.1 3.2	Introduction       Introduction         Local coordinates for quadrilateral elements       Image: Computations         3.2.1       Numerical integration for quadrilaterals	<b>55</b> 55 55 58
3	3.1 3.2	Introduction       Introduction         Local coordinates for quadrilateral elements	<b>55</b> 55 55 58 58
3	3.1 3.2 3.3	gramming Finite Element Computations         Introduction	<b>55</b> 55 58 58 58 60
3	3.1 3.2 3.3	Introduction       Introduction         Local coordinates for quadrilateral elements       Introduction         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements       Integration         3.3.1       Numerical integration for triangles	<b>55</b> 55 55 58 58 60 61
3	3.1 3.2 3.3 3.4	Introduction       Introduction         Local coordinates for quadrilateral elements       Introduction         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements       Integration         3.3.1       Numerical integration for triangles         Multi-element assemblies       Integration	<b>55</b> 55 55 58 58 60 61 62
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64 64
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64 65 66
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements         Josan         Multi-element assemblies         State         Analytical integration definite acustions	<b>55</b> 55 58 58 60 61 62 64 64 65 66
3	3.1 3.2 3.3 3.4 3.5	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements         Local coordinates for triangular elements         Jocal coordinates for triangular elements         Josan         Multi-element assemblies         "Element-by-element" or "Mesh-free" techniques         Josan         Josan         Josan         Solution         Josan         Josan         Integration for triangles         Multi-element assemblies         Solution         Josan         Solution         Josan         Josan         Solution         Josan         Josan         Josan         Integration for triangles         Josan         Solution         Josan         Josan         Josan	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67
3	3.1 3.2 3.3 3.4 3.5	Introduction	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67
3	3.1 3.2 3.3 3.4 3.5 3.6 3.6	Introduction	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67 68
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	Introduction	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67 67 68 70
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67 67 67 70 71
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	<b>gramming Finite Element Computations</b> Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67 67 68 70 71 72
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	<b>gramming Finite Element Computations</b> Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Local coordinates for triangular elements	<b>55</b> 55 58 58 58 60 61 62 64 64 65 66 67 67 67 68 70 71 72 73
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements	<b>55</b> 55 58 58 58 60 61 62 64 64 65 66 67 67 68 70 71 72 73 76
3	3.1 3.2 3.3 3.4 3.5 3.6 3.7	gramming Finite Element Computations         Introduction         Local coordinates for quadrilateral elements         3.2.1       Numerical integration for quadrilaterals         3.2.2       Analytical integration for quadrilaterals         Jocal coordinates for triangular elements	<b>55</b> 55 58 58 60 61 62 64 64 65 66 67 67 68 70 71 72 73 76 77

vi

## CONTENTS

		3.7.7	Mass matrix formation	78
$\{ \boldsymbol{k} \}_{i \in \mathcal{N}_{i}}$		3.7.8	Higher-order 2D elements	79
		3.7.9	Three-dimensional elements	81
		3.7.10	Assembly of elements	86
2.18j	3.8	Solutio	n of equilibrium equations	91
	3.9	Evalua	tion of eigenvalues and eigenvectors	91
. <sup>6</sup> 5		3.9.1	Jacobi algorithm	92
		3.9.2	Lanczos algorithm	92
	3.10	Solutio	n of first order time dependent problems	93
	3.11	Solutio	n of coupled Navier–Stokes problems	96
è	3.12	Solutio	n of coupled transient problems	98
		3.12.1	Absolute load version	99
		3.12.2	Incremental load version	100
	3.13	Solutio	n of second order time dependent problems	100
		3.13.1	Modal superposition	101
		3.13.2	Newmark or Crank-Nicolson method	104
		3.13.3	Wilson's method	105
		3.13.4	Explicit methods and other storage-saving strategies	106
	Refe	rences.		106
Â	Stati	o Fauil	ibrium of Structures	100
•		Introdu	ation	109
	Prom	ram 4.1	One-dimensional analysis of avially loaded elastic rods using	109
	Tiog	2-node	rod elements	110
	Prog	$\frac{2}{1000}$	Analysis of elastic nin-jointed frames using 2-node rod elements	110
	TIOE	in two	or three dimensions	116
	Prog	ram 43	Analysis of elastic hears using 2-node hear elements (elastic	110
	1106	founda	tion optional)	122
	Prog	ram 4.4	Analysis of elastic rigid-jointed frames using 2-node beam/rod	122
	1105	elemen	ts in two or three dimensions	128
	Prog	ram 4 5	Analysis of elastic-enlastic beams or rigid-jointed frames using	120
	1.00	2-node	beam or beam/rod elements in one, two or three dimensions	136
	Prog	ram 4.6	Stability (buckling) analysis of elastic beams using 2-node beam	100
	8	elemen	ts (elastic foundation optional)	145
	Prog	ram 4.7	Analysis of plates using 4-node rectangular plate elements. Homo-	
	0	geneou	s material with identical elements. Mesh numbered in x- or	
		v-direc	tion	148
	4.2	Conclu	ding remarks	153
	4.3	Exercis	Xes	155
	Refe	rences .	······································	164
_	<b>G</b> 4 /*	<b>I2</b> • •		•/-
3	Stati	C Equil	Idrium of Linear Elastic Solids	165
	Prog	ram 5 1	Plane or avisymmetric strain analysis of an elastic solid using 2-	105
	Tiog	6_ 10	$1$ rane of anisymmetric strain analysis of an classic solid using $J^2$ , or 15-node right-angled triangles or $A_1$ , $\mathcal{R}_2$ or 0-node rectangular	
		0-, 10-	, or 13-mode right-angled mangles of 4-, 6-, of 3-mode rectangular aterals. Mash numbered in $r(r)$ , or $y(r)$ direction	166
		quaum	awrais, mesh humbered in $x(r)$ - or $y(z)$ -affection $\ldots$	100

vii

CONTENTS
----------

	Prog	ram 5.2 Non-axisymmetric analysis of an axisymmetric elastic solid using 8-node rectangular quadrilaterals. Mesh numbered in $r$ - or $z$ -direction	184
	Prog	ram 5.3 Three-dimensional analysis of an elastic solid using 8-, 14-, or 20-node brick hexahedra. Mesh numbered in $x$ - $z$ planes then in the	
		y-direction	190
	Prog	ram 5.4 General two- (plane strain) or three-dimensional analysis of elastic solids	195
	Prog	ram 5.5 Three-dimensional strain of an elastic solid using 8-, 14-, or 20- node brick hexahedra. Mesh numbered in $x$ - $z$ planes then in the $y$ -direction. No global stiffness matrix assembly. Diagonally preconditioned conjugate	204
	Prog	ram 5.6 Three-dimensional strain of an elastic solid using 8-, 14-, or 20- node brick hexahedra. Mesh numbered in $x$ - $z$ planes then in the y-direction. No global stiffness matrix assembly. Diagonally preconditioned conjugate	204
		gradient solver. Vectorised version	209
	5.2	Exercises	214
	Refe	rences	222
6	Mate	erial Non-linearity	223
	6.1	Introduction	223
	6.2	Stress-strain behaviour	225
	6.3	Stress invariants	226
	6.4	Failure criteria	228
		6.4.1 Von Mises	228
		6.4.2 Mohr–Coulomb and Tresca	229
	6.5	Generation of body loads	230
	6.6	Viscoplasticity	231
	6.7	Initial stress	233
	6.8	Corners on the failure and potential surfaces	234
	Prog	ram 6.1 Plane strain bearing capacity analysis of an elastic-plastic (von	
		Mises) material using 8-node rectangular quadrilaterals. Viscoplastic strain	
		method	235
	Prog	ram 6.2 Plane strain bearing capacity analysis of an elastic-plastic (von Mises) material using 8-node rectangular quadrilaterals. Viscoplastic strain method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	243
	Prog	ram 6.3 Plane strain slope stability analysis of an elastic-plastic (Mohr- Coulomb) material using 8-node rectangular quadrilaterals. Viscoplastic	, 243
		strain method	248
	Prog	ram 6.4 Plane strain earth pressure analysis of an elastic-plastic (Mohr-	
	8	Coulomb) material using 8-node rectangular quadrilaterals. Initial stress	
		method	253
	6.9	Elasto-plastic rate integration	260
		6.9.1 Forward Euler method	262
		6.9.2 Backward Euler method	263

viii

	iv
	IX
Tangent stiffness approaches	264
6.10.1 Inconsistent tangent matrix	265
6.10.2 Consistent tangent matrix	265
6.10.3 Convergence criterion	266
Mises) material using 8-node rectangular quadrilaterals. Initial stress	
method. Tangent stiffness. Consistent return algorithm	266
<b>Program 6.6</b> Plane strain bearing capacity analysis of an elastic-plastic (von	
Mises) material using 8-node rectangular quadrilaterals. Initial stress	
method. Tangent stiffness. Consistent return algorithm. No global stiffness	
matrix assembly. Diagonally preconditioned conjugate gradient solver	271
6.11 The geotechnical processes of embanking	
and excavation	276
6.11.1 Embanking	276
<b>Program 6.</b> / Plane strain construction of an elastic–plastic (Mohr–Coulomb)	
embankment in layers on a foundation using 8-node quadrilaterals. Vis-	070
coplastic strain method	276
6.11.2 Excavation	283
<b>Trogram 0.8</b> Plane strain construction of an elastic-plastic (Monr-Coulomb)	206
6.12 Undrained analysis	200
<b>0.12</b> Unutatied analysis	293
Coulomb) solid using 8 node rectangular quadrilaterals. Visconlastic strain	
method	295
<b>Program 6</b> 10 Three-dimensional strain analysis of an elastic-plastic (Mohr-	275
Coulomb) slope using 20-node bexabedra. Visconlastic strain method	300
<b>Program 6.11</b> Three-dimensional strain analysis of an elastic-plastic (Mohr-	200
Coulomb) slope using 20-node hexahedra. Viscoplastic strain method. No	
global stiffness matrix assembly. Diagonally preconditioned conjugate gra-	
dient solver	305
6.13 Exercises	314
References	316
Steady State Flow	319
7.1 Introduction $\ldots$	319
Program 7.1 One-dimensional analysis of steady seepage using 2-node line	
elements	320
Program 7.2 Plane or axisymmetric analysis of steady seepage using 4-node	224
rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ - direction	524
<b>Program</b> 1.5 Analysis of plane free-surface flow using 4-node quadrilaterals.	222
Analytical form of element conductivity matrix	332
riogram 7.4 General two- (plane) or inree-dimensional analysis of steady	240
Sucception	540
age No global conductivity matrix assembly. Diagonally preconditioned	
conjugate gradient solver	344
	5.4

CON	TENTS
-----	-------

	7.2 Exercises	350 356
8	Transient Problems: First Order (Uncoupled)	357
	8.1 Introduction	357
	Program 8.1 One-dimensional consolidation analysis using 2-node line elements. Implicit time integration using the "theta" method	358
	Program 8.2 Plane or axisymmetric consolidation analysis using 4-node rect- angular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. Implicit	
	time integration using the "theta" method	363
	8.2 Mesh-free Strategies in Transient Analysis	371
	Program 8.3 Plane or axisymmetric consolidation analysis using 4-node rectan- gular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. Implicit time integration using the "theta" method. No global stiffness matrix assembly.	
	Diagonal preconditioner conjugate gradient solver	371
	Program 8.4 Plane or axisymmetric analysis of the consolidation equation using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction.	
	Explicit time integration using the "theta = 0" method $\ldots$ $\ldots$ $\ldots$	375
	Program 8.5 Plane or axisymmetric analysis of the consolidation equation using	
	4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction.	270
	"theta" method using an element-by-element product algorithm	3/8
	8.5 Comparison of Programs 8.2, 8.3, 8.4, and 8.5	380
	idation equation. Implicit time integration using the "theta" method	382
	Program 8.7 Plane analysis of the diffusion-convection equation using 4-node	
	rectangular quadrilaterals. Implicit time integration using the "theta"	
	method. Self-adjoint transformation	386
	Program 8.8 Plane analysis of the diffusion-convection equation using 4-node rectangular quadrilaterals. Implicit time integration using the "theta"	
	method. Untransformed solution	391
	8.4 Exercises	398
	References	402
9	Coupled Problems	403
	9.1 Introduction	403
	Program 9.1 Analysis of the plane steady state Navier-Stokes equation using	
	8-node rectangular quadrilaterals for velocities coupled to 4-node rectan-	
	gular quadrilaterals for pressures. Mesh numbered in $x$ - or y-direction.	40.4
	Program 9.2 Analysis of the plane steady state Navier–Stokes equation using 8-node rectangular quadrilaterals for velocities coupled to 4-node rectan- gular quadrilaterals for pressures. Mesh numbered in x- or y-direction. Freedoms numbered in the order $u$ - $p$ - $v$ . Element-by-element solution using BiCGStab(l) with no preconditioning. No global matrix assembly	404

х

#### NTENTS

	Program 9.3 Plane strain consolidation analysis of a Biot poro-elastic solid	
	using 8-node rectangular quadrilaterals for displacements coupled to 4-node	
	rectangular quadrilaterals for pressures. Freedoms numbered in the order	
<b>1</b> 8	$u - v - u_w$ . Incremental version	. 416
	Program 9.4 Plane strain consolidation analysis of a Biot poro-elastic-plastic	
	(Mohr-Coulomb) material using 8-node rectangular quadrilaterals for dis-	
÷,	placements coupled to 4-node rectangular quadrilaterals for pressures. Free-	
	doms numbered in the order $u - v - u_w$ . Incremental version. Viscoplastic strain	
R.	method	. 424
Ц.,	Program 9.5 Plane strain consolidation analysis of a Biot poro-elastic solid	
С.	using 8-node rectangular quadrilaterals for displacements coupled to 4-node	
20	rectangular quadrilaterals for pressures. Freedoms numbered in the order	
20	$u-v-u_w$ . Absolute load version. No global stiffness matrix assembly. Diag-	
W.C.	onally preconditioned conjugate gradient solver	. 430
	9.2 Exercises	. 439
<b>4</b> 0:	References	. 440
<b>R</b> ()		
10	Eigenvalue Problems	441
11	10.1 Introduction	. 441
i h	Program 10.1 Eigenvalue analysis of elastic beams using 2-node beam ele-	
ſ.	ments. Lumped mass	. 442
Ċ.	Program 10.2 Eigenvalue analysis of an elastic solid in plane strain using 4- or	
	8-node rectangular quadrilaterals. Lumped mass. Mesh numbered in $x$ - or	
<b>\$</b> .	y-direction	. 446
A	<b>Program</b> 10.3 Eigenvalue analysis of an elastic solid in plane strain using 4-node	
	rectangular quadrilaterals. Lanczos Method. Consistent mass. Mesh num-	
r.	bered in x- or y-direction	. 452
ŗ	<b>Program</b> 10.4 Eigenvalue analysis of an elastic solid in plane strain using 4-node	
c .	rectangular quadrilaterals. Lanczos Method. Lumped mass. Element-by-	
	element formulation. Mesh numbered in $x$ - or $y$ -direction	. 457
₽1.	10.2 Exercises	. 462
	References	. 464
- -	Formed Vibrations	165
EI.	11.1 Introduction	403
	<b>Drogram</b> 11.1 Forced vibration analysis of electic beams using 2 node beam.	. 403
51	elements Consistent mass. Newmork time stenning	ЛББ
	Decorrow 11.2 Forced vibration analysis of an electic solid in plane strain using	. 400
5) 248.	A or 8-node rectangular quadrilaterals. I umned mass. Mash numbered in	
	- or a direction. Model superposition	477
	<b>Decorrow 11.3</b> Forced vibration analysis of an electric solid in plane strain using	. 4/2
	rectangular 8, node quadrilaterals. Lumped or consistent mass. Mech num	
9. 1. 1. <del>1</del>	bered in $x_{-}$ or $y_{-}$ direction. Implicit time integration using the "thete"	
	method	179
	Program 11.4 Forced vibration analysis of an elastic solid in plane strain using	. 470
	rectangular 8-node quadrilaterals. J umped or consistent mass. Mech num-	
1	bered in r- or y-direction. Implicit time integration using Wilson's method	483
5	berea in x - or y-uncerton, implicit time integration using without 8 method	-105

$\mathbf{C}$	0	N	T	E	Ν	T	S
C	U	1.1	r	ч.	1.4	T	J

<ul> <li>Program 11.5 Forced vibration analysis of an elastic solid in plane strain using rectangular uniform size 4-node quadrilaterals. Mesh numbered in the x-or y-direction. Lumped or consistent mass. Mixed explicit/implicit time integration</li></ul>	. 487
gradient solver	. 492
Program 11.7 Forced vibration analysis of an elastic-plastic (von Mises) solid	
in plane strain using rectangular 8-node quadrilateral elements. Lumped	
mass. Mesh numbered in x- or y-direction. Explicit time integration	. 496
11.2 Exercises	. 506
References	. 507
12 Devellet Drocessing of Finite Flowent Analyses	500
12 Paranet Processing of Finite Element Analyses	509
12.1 Infroduction	511
12.2 Differences between parallel and serial programs	511
12.2.1 Parallel inoranes	511
12.2.2 Global variables	512
12.2.5 MPT horary fournes	512
12.2.4 The _pp appendage	512
12.2.5 Reading and writing	512
12.2.6 Problem-specific boundary condition routines	515
12.2.7 rest instead of hi	. 310
12.2.8 Gathering and scattering	. 317
12.2.9 Reindexing	. 517
	. 517
	. 519
Program 12.1 Inree dimensional analysis of an elastic solid. Compare	510
Program 5.5	. 519
Program 12.2 Infee dimensional analysis of an elasto-plastic (Monr-Coulomb)	500
solid. Compare Program 0.11	. 520
Program 12.3 Three dimensional Laplacian now. Compare Program 7.5	. 555
Program 12.4 Infee dimensional transient now-implicit analysis in time. Com-	527
pare Program 8.3	. 537
Program 12.5 Infree dimensional transient now-explicit analysis in time. Com-	5 4 1
pare Program 8.4	. 541
Program 12.6 I free dimensional steady state Navier–Stokes analysis. Compare	517
Program 9.2	. 543
Program 12.7 Infee-dimensional analysis of Biot poro-elastic solid. Compare	551
Program 12.9 Eigenvalue analysis of three dimensional electric selies	. 551
Program 12.6 Engenvalue analysis of three-dimensional elastic solid. Compare	556
Flogram 12.0 Earead vibration analysis of a threa dimensional electic call	. 330
Fiogram 12.9 Forced violation analysis of a infee-dimensional elastic solid.	561
Implicit integration in time. Compare Program 11.4	. 201

xii

### CONTENTS

	Program 12.10 Forced vibration analysis of three-dimensional elasto-plastic	
	solid. Explicit integration in time. Compare Program 11.5	565
	12.3 Performance data for a "Beowulf" PC cluster	569
	12.4 Conclusions	570
	References	576
A	Equivalent Nodal Loads	577
B	Shape Functions and Element Node Numbering	583
С	Plastic Stress-strain Matrices and Plastic Potential Derivatives	591
D	main Library Subroutines	595
E	geom Library Subroutines	605
F	Parallel Library Subroutines	609
Αı	ithor Index	613
Su	bject Index	615