PROGRESS IN STRUCTURAL DYNAMICS WITH STOCHASTIC PARAMETER VARIATIONS: 1987 to 1996

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Abstract

This paper is an update of an earlier paper by Ibrahim (1987) and is aimed at reviewing the papers published during the last decade in the area of vibration of structures with parameter uncertainties. Analytical, computational, and experimental studies conducted on probabilistic modeling of structural uncertainties and free and forced vibration of stochastically defined systems are discussed. The review also covers developments in the areas of statistical modeling of high frequency vibrations and behavior of statistically disordered periodic systems.

1. INTRODUCTION

The dynamic response characterization of structural systems with parameter uncertainties has been the subject of intensive studies in the recent past. This problem is of importance in the assessment of the safety of engineering structures. From a phenomenological point of view, the problem plays a fundamental role in understanding the phenomenon of modal localization in nearly periodic structures. It is also important in the design of large-scale structures subject to broad-band high-frequency excitations. Parameter uncertainties play a significant role in laminated composites. The mathematical modeling of these structures requires a strong background in the theory of random fields, structural mechanics, and finite element and boundary element methods. The analysis and solution of such problems involve discretization of random fields, solutions of algebraic and differential random eigenvalue problems, and inversion of random matrices and differential operators. Difficulties are encountered when one includes interaction between nonlinear and stochastic system characteristics or if one is interested in controlling structural response. The subject is dominated by analytical and numerical investigations, and very few experimental investigations have been conducted for identifying inhomogenieties in elastic, mass, and damping properties of simple structural elements.

The present article is a sequel to an earlier review article (Ibrahim, 1987) on the dynamics of structures with parameter uncertainties. It focuses on reviewing developments in this area over the last decade. Particular attention will be given to the following topics:

- models for structural uncertainties and discretization of random fields
- random eigenvalue problems
- transient and steady state vibrations
- structural nonlinearity and reliability aspects
- stochastic analysis of dynamics of disordered periodic systems
- stochastic aspects of statistical energy analysis of high frequency vibrations
- experimental studies.

In addition to the above topics, the paper discusses different frameworks for structural modeling, namely, finite element method (FEM), transfer matrices, statistical energy analysis (SEA) and boundary element method (BEM). It must be noted that in the recent past several review papers and research monographs have appeared addressing different aspects of this problem. For example, Nakagiri (1987) and Brenner (1991) provided an account of uncertainties of the structural response. Benaroya and Rehak (1988), Shinozuka and Yamazaki (1988), Ghanem and Spanos (1991a), Der Kiureghian *et al.* (1991), Kleiber and Hien (1992), and Liu *et al.* (1992) focused on the role of finite element methods and computational techniques in structural dynamics. Shinozuka (1991) and Shah *et al.* (1992) reported critical assessments of developments pertaining to reliability problems, Benaroya (1992) addressed mathematical aspects of the random eigenvalue problem. Li and Benaroya (1992) Lin and Cai (1995, the last Chapter) reviewed some problems related to modal localization in nearly periodic structures.

Before we proceed further, it is important to clarify that no formal distinction between the terms "random fields" and "random processes" will be made in this review. In some literature the term random processes is reserved to time evolution of random variables and random field for random variables evolving in space. This distinction is perhaps useful if one wishes to differentiate between traditional random vibration problems involving uncertainties only in the time varying loads and more recent studies on the statistics of structures with random spatial variations in stiffness properties. However, when discussing the dynamics of randomly parametered structures under the action of deterministic and/or random dynamic excitations, it becomes difficult to maintain a strict distinction between the two terms. Here the independent parameter in which a response random variable may evolve need not always be space or time: thus, the transfer functions of systems with spatial random inhomogenieties may vary randomly in *frequency* and space. Similarly, the random variations in response need not be in space or time alone as, for example, in the case of impulse response of continuous random systems which evolves randomly in time *and* space. Further, the classical mathematical literature on probability and random processes does not make formal distinctions between random fields and random processes. It is also to be noted that we are not considering studies on time dependent material property variation, as might arise in the study of visco-elastic materials or parametrically excited systems (Ibrahim, 1985).

2. UNCERTAINTY MODELS

Parameter uncertainties in structural dynamics can arise due to several sources. These include variations due to intrinsic material property variability, measurement errors, manufacturing and assembly errors, differences in modeling and solution procedures. From the structural analysis point view, these uncertainties can be classified into two classes. The first is inherent to such system parameter variations such as mass, stiffness and damping properties, while the second belongs to the eigensolutions of constituent subsystems. The first class is more commonly used in structural dynamics analyses, while the latter finds applications in high frequency vibration modeling using SEA formalisms.

The recent study by Brown and Ferri (1996) on combining the component mode synthesis with probabilistic methods revealed some difficulties in correctly identifying the statistical properties of primitive variables such as geometry, stiffness, and mass. Brown and Ferri proposed an alternative method where the measured dynamic properties of substructures were considered as random variables. Other types of uncertainty modeling such as convex modeling and fuzzy set-based approaches are well documented by Elishakoff (1995).

2.1 Models for Material and Geometric Properties

This class of modeling consists of stochastic representation of elastic constants, mass density, and material damping of the structural material. This, in conjunction with possible uncertainties in specifying the geometry and boundary conditions of the structure, defines the structural mass, stiffness, damping matrices, and force vectors. When choosing an appropriate stochastic model for the material properties several questions arise:

1. Can the system uncertainties be adequately described by random variables or is it necessary to use random field models which take into account spatial inhomogenieties?
2. What are the statistical parameters which can adequately describe the model?
3. Are Gaussian models acceptable for strictly positive quantities such as mass and elastic constants? What are the feasible non-Gaussian models and how to describe and simulate them?

Next, one should consider the method for developing the stochastic model of the associated structural matrices. This issue is related to the methods of discretizing a random field and selection of mesh sizes in a finite element/boundary element study. The next section will address this issue.

2.1.1 Gaussian Models

Gaussian random field models with bounded mean squares for elastic constants and mass density have been considered by several authors, see, for example, Bucher and Shinozuka (1988), Karada *et al.* (1989), Spanos and Ghanem (1989), Chang and Yang (1991) and Manohar and Iyengar (1994). In these models, stochastic perturbations are imposed on the corresponding nominal values. The isotropy of the field was assumed for two- or three-dimensional fields. Several models for autocovariance/power spectral density functions of the stochastic perturbation have been used. Thus, for example, Shinozuka (1987) and Shinozuka and Deodatis (1988), in their studies on statistical response variability of randomly parametered skeletal structures, have adopted the following types of models for the power spectral density functions for the deviations of Young's modulus from the mean value:

$$S(\kappa) = \alpha_n \kappa^{2n} \exp[-b|\kappa|] \quad n = 0, 1, ..., 5 \tag{1}$$

$$S(\kappa) = \beta_n \kappa^{2n} \exp[-(\frac{b|\kappa|}{2})^2] \quad n = 0, 1, ..., 5 \tag{2}$$

$$S(\kappa) = \gamma_n \frac{b}{(1 + b^2\kappa^2)^{2n}} \quad n = 1, 2 \tag{3}$$

where $S(\kappa)$ is the power spectral density function, κ is the wave number, the parameters α_n, β_n and γ_n control the variance of the process and the parameter b controls the shape of the power spectral density function. In a similar study, Spanos and Ghanem (1989) have utilized exponential and triangular autocovariance functions of the form

$$R(\xi) = \sigma_s^2 \exp[-c|\xi|] \text{ and}$$

$$R(\xi) = \sigma_s^2(1 - c|\xi|) \tag{4}$$

where $R(\xi)$ is the autocovariance function, ξ is the spatial lag, σ_s^2 is the variance and the parameter c controls the correlation length. In their studies on flexural vibrations of random plates, Bucher and Brenner (1992) modeled Young's modulus and mass density of the plate as independent two dimensional homogeneous isotropic random fields each having covariance functions of the form

$$R(\xi, \zeta) = \sigma_{ff}^2 \exp(\frac{-\sqrt{\xi^2 + \zeta^2}}{l_f}) \tag{5}$$

where ξ and ζ are space lags, $R(\xi, \zeta)$ the autocovariance function of the random field, l_f controls the correlation length and σ_{ff}^2 is the variance. These models have been largely selected with the purpose of illustrating some analytical results. However, no experimental or field data is available to ascertain the relative merits of alternative models in a given context.

2.1.2 Non-Gaussian Models

Gaussian models are not suitable for cases where the parameter experiences large variations or when reliability issues are being examined. Furthermore, Gaussian distributions do not allow information on moments higher than two to enter the model. In view of the first difficulty, Yamazaki *et al.* (1988) and Wall and Deodatis (1994) have imposed a restriction on the variation of samples of Gaussian fields $f(x)$, as follows:

$$-1 + \delta \leq f(x) \leq 1 - \delta; \quad 0 < \delta < 1 \tag{6}$$

The limitation on the upper value establishes symmetry of the stochastic variations about the deterministic values. A similar modification of approximating a Gaussian distribution by a distribution with bounded range has also been made by Iwan and Jensen (1993). A more systematic way of constructing non-Gaussian field models is by making nonlinear memoryless transformations of a specified Gaussian field $\nu(x)$, that is, by considering $f(x) = g[\nu(x)]$, where, g is a 'memoryless' nonlinear function (Grigoriu 1984, Yamazaki and Shinozuka 1988, Der Kiureghian and Liu 1986). This type of transformations enable characterization of $f(x)$ in terms of mean and covariance of $\nu(x)$. Prominent among this type of models is the Nataf model, which can produce any desired marginal distribution for $f(x)$. Let $f(x)$ be a non-Gaussian field with a specified mean $\mu_f(x)$, covariance $\rho_{ff}(x, \tilde{x})$, and first order probability distribution function $F_f(f; x)$. According to Nataf's model the transformed process

$$\nu(x) = \Phi^{-1}\left[F_f(f; x)\right] \tag{7}$$

is taken to be Gaussian, in which Φ = standard Gaussian probability distribution. It can be shown that $\nu(x)$ has zero mean, unit standard deviation and autocovariance $\rho(x,\tilde{x})$ satisfying the integral equation

$$\rho_{ff}(x,\tilde{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\frac{f(x)-\mu_f(x)}{\sigma_f(x)}\}\{\frac{f(\tilde{x})-\mu_f(\tilde{x})}{\sigma_f(\tilde{x})}\}\phi_2\{u,v,\rho(x,\tilde{x})\}\mathrm{d}u\mathrm{d}v \tag{8}$$

where ϕ_2 denotes the bivariate standard normal density. The non-Gaussian random field $f(x)$ can be expressed in terms of the Gaussian field $\nu(x)$ through the relation $f(x) = F_f^{-1}[\Phi\{\nu(x)\}]$. In general $|\rho_{ff}(x,\tilde{x})| \leq |\rho(x,\tilde{x})|$ and for most processes $\rho(x,\tilde{x}) \simeq \rho_{ff}(x,\tilde{x})$. A set of empirical formulae relating $\rho(x,\tilde{x})$ to $\rho_{ff}(x,\tilde{x})$ for common distributions are given by Der Kiureghian *et al.,* (1991). Discussion on the use of this model in structural reliability analysis when random quantities involved are partially specified has been presented by Der Kiureghian and Liu (1986). This model has also been employed by Liu and Der Kiureghian (1991) and Li and Der Kiureghian(1993) in their studies on reliability of stochastic structures. Another example of non-Gaussian field models for material properties can be found in Elishakoff and Shinozuka (1995) and Sobczyk *et al.* (1996). Non-Gaussian random variable models for system parameters in the context of single degree-of-freedom (sdof) systems or individual structural elements of built-up structure have been considered by several authors. Udwadia (1987a,b) considered maximum entropy probability distributions for incompletely specified system parameters of a sdof system. Other examples include lognormal models (Shinozuka and Yamazaki 1988, Cruse *et al.* 1988), beta distributions (Shinozuka and Yamazaki 1988) and ultraspherical random variables (Jensen and Iwan 1991).

2.2 Models for Eigensolutions

The second class of statistical modeling is based on the statistical energy analysis formalisms (Lyon 1975, Lyon and DeJong, 1995, Hodges and Woodhouse 1986, Fahy 1994) developed as tools for response prediction in high frequency regimes. The details of the probabilistic aspects underlying the development of SEA will be discussed in section 5.0. The SEA procedure divides built-up structures into a number of interacting subsystems. The behavior of individual subsystems, when they are uncoupled from the rest of subsystems, is studied first and then they are allowed to interact which results in exchange of vibration energies between the subsystems.

The basic objective of SEA is to characterize the system response by calculating these energy exchanges. These procedures are based on a set of assumptions only valid for system response at high frequencies where a large number of modes participate in vibration. The method avoids dealing with a large amount of information on individual modal contributions from the participating modes. Instead it seeks to average out these contributions in some sense.

A fundamental aspect of this analysis is that the vibrating structure is considered to be drawn from an ensemble of nominally identical systems. This allows for the fact that high frequency vibrations are very sensitive to minor changes in details of system modeling and parameter values. In the traditional SEA procedures this is accomplished by dividing the built-up structure into a collection of energy carrying elements called subsystems and treating the subsystem eigensolutions as having prescribed probability distributions. Specifically, it is assumed that the subsystem natural frequencies constitute a set of Poisson points on the frequency axis which would mean that the natural frequencies are mutually independent and identically distributed uniformly in a given frequency bandwidth. The subsystem mode shapes are approximated to be deterministic. Such a model clearly implies random field models for the variation of mass and stiffness properties within a subsystem. The questions on relations which would exist between the eigensolutions and these system properties are, however, not addressed.

3. RANDOM FIELD DISCRETIZATION

The stochastic finite element method is convenient for built-up structures consisting of spatially random structural elements. An important step of the method requires the replacement of the element property random fields by an equivalent set of a finite number of random variables. This process constitutes the discretization of the random field, and the accuracy of the field representation depends primarily on the size of the element used. The selection of mesh size depends on stress and strain gradients, frequency range of interest, characteristics of the random field, and correlation length of the random fields. Other factors include the tails of the probability density function (pdf), nonhomogeneous random fields, stability of numerical inversion of the probability transformations, and gradient of limit state function.

Other alternative schemes for discretization of random fields have been proposed in the literature. Vanmarcke and Grigoriu (1983) replaced the random field within a finite element by its spatial average. This method of discretization has been used by several authors including Shinozuka and Deodatis (1988), Chang and Yang (1991), Zhu *et al.* (1992) and Anantha Ramu and Ganesan (1992a,b, 1993a,b). Alternatively, one can discretize the random field by assigning its value at the centroid of the finite element (Hisada and Nakagiri 1985, Der Kiureghian and Ke 1988, Yamazaki *et al.* 1988). This method is particularly suitable for discretizing non-Gaussian fields. It has been used for studying the reliability of nonlinear systems with non-Gaussian uncertainties by Liu and Der Kiureghian (1991). When the Nataf model is discretized according to this scheme, the discretized random variables are completely described by the marginal probability density functions and the covariance matrix. In terms of shape functions $\phi_i(x)$, Liu *et al.* (1986) approximated the random field $f(x)$ by the summation

$$f(x) = \sum_{i=1}^{n} \phi_i(x) f_i \tag{9}$$

where f_i are the nodal values of $f(x)$. This representation is equivalent to interpolating $f(x)$ within an element using $\phi_i(x)$ and the nodal values of $f(x)$. Furthermore, the error of discretization is characterized in terms of total mean square difference between the covariance function of the discretized field and the exact covariance function. To achieve computational efficiency, the discretized random variables are transformed into a set of uncorrelated random variables. Accordingly, it is expected that the number of random variables which needs to be retained in subsequent analysis will be significantly less than n. It was also noted that n need not be equal to the degrees of freedom of the finite elements used to discretize the displacement field and $N_i(x)$ need not coincide with the shape functions used for finite element discretization. Lawrence (1987), Spanos and Ghanem (1989), Iwan and Jensen (1993) and Zhang and Ellingwood (1994) constructed series expansions for random fields in terms of a set of deterministic orthogonal functions multiplied by random variables and incorporated them into finite element formulations. The expansion used by Lawrence for the random field $f(\mathrm{x})$ has the form

$$f(\mathrm{x}) = \sum_j C_{0j}\psi_j(\mathrm{x}) + \sum_j \sum_i C_{ij} e_i \psi_j(\mathrm{x}) \tag{10}$$

where e_i is an orthogonal set of random variables with zero mean and unit variance, $\psi_j(\mathrm{x})$ is a set of known orthogonal deterministic functions, such as, for example, Legendre polynomials over a line segment, C_{ij} are unknown deterministic constants to be found by a least square fit to the first and the second moments of $f(\mathrm{x})$. In the study by Spanos and Ghanem (1989), the expansion is based on the Karhunen-Loeve expansion in which $\psi_j(x)$ are obtained as the solutions of the eigenvalue problem

$$\int_L R(x,\xi)\psi_j(\xi)\mathrm{d}\xi = \lambda_j \psi_j(x). \tag{11}$$

where λ_j is the jth eigenvalue. This expansion is mathematically well founded with the expansion guaranteed to converge. In addition, the expansion is optimum in the sense that it minimizes the mean square error resulting from truncating the series at a finite number of terms. For correlated random variables, Zhang and Ellingwood (1994) developed series expansions in terms of arbitrary set of orthogonal functions. Zhang and Ellingwood reported that their method is equivalent to solving equation (11) by a Galerkin approximation in terms of the arbitrarily chosen orthogonal functions. Li and Der

Kiureghian (1993) used optimal linear estimation procedures in representing the random field as linear combination of nodal random variables and a set of unknown shape functions. This means that the random field $f(x)$ is estimated by

$$\bar{f}(x) = a(x) + \sum_{i=1}^{n} b_i(x) f(x_i) \tag{12}$$

where n is the number of nodal points, $a(x)$ is a scalar function of x, $b(x) = [b_i(x)]$ is a vector function of x with element $b_i(x)$. The unknown functions $a(x)$ and $b(x)$ are found by minimizing the variance $[f(x) - \bar{f}(x)]$ under the constraint $\mathrm{E}[f(x) - \bar{f}(x)]$=0. The efficiency of the method is further shown to be improved by employing spectral decomposition of the nodal covariance matrix, which effectively reduces the number of random variables. The shape functions in finite element discretization are usually taken as polynomials in spatial coordinates, and if these functions are used to discretize the random fields then integrals of the form

$$W_n = \int_0^L x^n f(x) \mathrm{d}x; \quad n = 0, 1, 2, ... \tag{13}$$

appear in the expressions for the stiffness coefficients. Clearly W_n are random variables are are referred to as weighted integrals associated with the element. These integrals offer an alternative way of discretizing the random fields. Related studies on static stiffness of one dimensional structural elements were considered by Shinozuka (1987), Bucher and Shinozuka (1988), Karada *et al.* (1989), Takada (1990a), Deodatis (1991), and Deodatis and Shinozuka (1989,1991). Takada (1990b) and Wall and Deodatis (1994) considered the case of two-dimensional elements. Bucher and Brenner (1992) extended the weighted integral approach to dynamic systems. The number of weighted integrals resulting from discretizing a random field for an element depends upon the type of shape function used. If complete polynomials with maximum order n are used, then the number of weighted integrals for one, two and three dimensional elements are given by $(2n + 1)$, $(2n + 1)(n + 1)$ and $(2n + 1)(n + 1)(2n + 3)/3$, respectively.

Deodatis (1991) noted that the displacement field of a stochastic beam element subject to boundary displacements can be derived exactly in terms of the system Green function. This, consequently, leads to the definition of exact *stochastic* shape functions and the exact stochastic static stiffness matrix of the beam element. Note that this result is valid whether or not the random field is Gaussian. Except for this exact solution, all other discretization procedures discussed above lead to discretization errors. Consequently, such error will restrict the size of the element to a fraction of the correlation length of the random field. Zhu *et al.* (1992) observed that the stochastic FEM based on local average discretization with a fewer elements can yield the same accurate results as those provided by the stochastic FEM based on mid-point discretization with more elements. The issue of selection and adaptive refinement of mesh size for random field discretization for reliability studies was examined by Liu and Liu (1993). They recommended that a coarse mesh must be used in the areas where the gradient of the limit state function with respect to the discretized random variables is small, and a fine mesh where the gradients are large.

4. RANDOM EIGENVALUES

Eigensolutions constitute an important descriptor of the dynamics and stability of structural systems. Consequently, the study of probabilistic characterization of the eigensolutions of random matrix and differential operators has emerged as an important research topic in the field of stochastic structural mechanics. In particular, several studies have been conducted on both self adjoint and non-self adjoint (usually encountered with systems involving follower forces, aerodynamic damping, and gyroscopic couples) eigenvalue problems. Other issues include multiplicity of eigenvalues and related problems. A systematic account of perturbational approaches to random eigenvalue problems is well documented in a research monograph by Scheidt and Purkert (1983). The random differential equations of linear discrete systems with proportional damping are usually written in the matrix form

$$M\ddot{X}(\mathrm{t}) \; + \mathrm{C}\dot{\mathrm{X}}(\mathrm{t}) \; + \; \mathrm{KX}(\mathrm{t}) \; = \; 0 \tag{14}$$

where $X(t)$ is the vector of generalized coordinates of the system response, M, C and K are the system mass, damping, and stiffness matrices, respectively. Lee and Singh (1994a) developed a procedure based on a direct product technique based on splitting the random matrices of the system into two components. The first component represents mean values while the second stands for random components with zero mean. The probability distributions of the random components are also assumed to be of the same type and known in advance. Furthermore, the covariances of the random fluctuations are known in the form of cross-correlation matrices. A proportional Rayleigh damping model

$$C \; = \; \alpha_1 \mathrm{M} \; + \; \alpha_2 \mathrm{K}$$

was proposed, where α_1 and α_2 are constant coefficients. Depending on the values of α_1 and α_2 different cases such as uncorrelated, partially correlated and fully correlated with M and/or K can be obtained. The results of Lee and Singh were found to be less accurate as the random fluctuations of the system parameters become very high.

4.1 Applications

The majority of recent studies employed the mean centered first/second order perturbation approach to estimate the first and the second order statistics of eigenvalues and mode shapes. For example, Nakagiri *et al.* (1987) studied the statistics of natural frequencies of simply supported fiber reinforced plastic plates whose stacking sequence is subjected to random fluctuations. They presented case studies on the statistics of first natural frequency of square and rectangular plates using triangular finite elements to discretize the domain into 60-70 elements.

The statistics of the natural frequencies of a three dimensional piping system, shown in Figure1, with uncertain restraint locations were studied by Nakagiri (1987). The lengths of the pipe elements were taken to be uniformly distributed random variables. The mean values of the first four natural frequencies were found to be 44.9, 88.06, 104.59 and 107.88 rad/s, with coefficient of variation up to about 7 percent. It was concluded that the variability in natural frequencies increased with decreasing correlation among member lengths. Mironowicz and Sniady (1987) used a first order perturbational approach to study the vibration of a machine foundation block with random geometry and mass density. They used a resonance index given by

$$\beta = \frac{(\bar{\omega} - \omega_e)}{\sqrt{(\sigma_\omega^2 + \sigma_e^2)}} \tag{15}$$

to characterize the resonance characteristics, where $\bar{\omega}$ and σ_ω are the mean and standard deviation of the natural frequency, respectively, and ω_e and σ_e are the mean and standard deviation of harmonic driving frequency, respectively. This index is akin to the reliability index in structural reliability problems. Nordmann *et al.* (1989) investigated the eigensolution variability of vessel and piping systems in the context of seismic response analysis using response spectrum-based approaches. Zhu *et al.* (1992) used the method of local averages to discretize random fields, in conjunction with a perturbational approach, to study the statistics of the fundamental natural frequency of isotropic rectangular plates. Their formulation also allows for multiplicity of deterministic eigenvalues. The latter issue was also addressed by Zhang and Chen (1991). Song *et al.* (1995) outlined a first order perturbational approach to find the moments of the sensitivity of random eigenvalues with respect to the expected value of specified design variables.

Bucher and Brenner (1992) employed a first order perturbation and, starting from the definition of Rayleigh's quotient for discrete systems

$$E[\lambda_i] = \frac{x_{i0}^{\mathrm{T}} K_0 x_{i0}}{x_{i0}^{\mathrm{T}} M_0 x_{i0}} \tag{16}$$

they showed that

$$\sigma^2_{\lambda_i} = \epsilon^2 \lambda^2_{i0} \mathrm{E}\{(x^{\mathrm{T}}_{i0} K_r x_{i0})^2 + 2(x^{\mathrm{T}}_{i0} K_r x_{i0})(x^{\mathrm{T}}_{i0} M_r x_{i0}) + (x^{\mathrm{T}}_{i0} M_r x_{i0})^2\} \quad (17)$$

where $\epsilon << 1$, the random eigenvalue $\lambda_i = \lambda_{i0} + \lambda_r$, $x_i = x_{i0} + \epsilon x_{ir}$, $K = K_0 + \epsilon K_r$ and $M = M_0 + \epsilon M_r$. Subscripts containing 0 denote deterministic quantities and subscripts with r denote random quantities.

Fang (1995) combined transfer matrix methods with first order second moment approach to analyze the natural frequencies and mode shapes of uncertain beam structures. A computational algorithm based on transfer matrices to compute natural frequencies of a fixed-fixed string with a set of intermediate random spring supports was given by Mitchell and Moini (1992). The spring constants in this study were modeled as a set of independent two-state random variables.

Random eigenvalue problems arising in structural stability were studied by Anantha Ramu and Ganesan (1992a,b, 93a,b), Sankar *et al.* (1993), Zhang and Ellingwood (1995) and Ganesan (1996) using perturbational approaches. Koyluoglu *et al.*(1995b) used Monte Carlo simulation technique in conjunction with weighted integral method of random field discretization. Anantha Ramu and Ganesan and Sankar *et al.* considered several problems associated with stability of beams/rotors with randomly varying Young's modulus and mass density. These include systems with non-self adjoint eigenvalue problems. For example, the determination of whirling speeds of a stochastic spinning shaft is associated with the eigenvalue problem

$$[\bar{K} + K_r]\{x_0\} = \omega^2\{\bar{M} + M_r + (\Omega/\omega)[\bar{G} + G_r]\}\{x_0\} \quad (18)$$

where x_0 is the eigenvector, K is the stiffness matrix, M is the mass matrix, G is the gyroscopic matrix, Ω is the shaft rotational speed, ω is the whirling speed and a bar denotes the expected value. The perturbation of the eigenvalue ω^2 is shown to be given by

$$\mathrm{d}\omega^2_i = \sum_{j=1}^{n}\sum_{s=1}^{n} \frac{\partial \omega^2_i}{\partial k_{js}} \mathrm{d}k_{js} + \sum_{j=1}^{n}\sum_{s=1}^{n} \frac{\partial \omega^2_i}{\partial m^*_{js}} \mathrm{d}m^*_{js} \quad (19)$$

where the symbol of matrix $[M^*]$ (whose elements are m^*_{js}) is used in place of the term $\bar{M} + M_r + \frac{\Omega}{\omega}(\bar{G} + G_r)$. The expressions for gradients of ω^2 with respect to mass and stiffness terms were obtained using the expressions given by Plaut and Huseyin (1973)

$$\frac{\partial \omega_i^2}{\partial k_{js}} = \{y_i\}^T [\frac{\partial(\bar{K} + K_r)}{\partial k_{js}}]\{x_i\} \tag{20}$$

$$\frac{\partial \omega_i^2}{\partial m_{js}} = -\omega_i^2 \{y_i\}^T [\frac{\partial(\bar{M} + M_r)}{\partial k_{js}}]\{x_i\} \tag{21}$$

Here x_i and y_i are, respectively, the right and left eigenvectors defined through the conditions:

$$[\bar{K}+K_r-\omega_i^2(\bar{M}+M_r+(\Omega/\omega)\{\bar{G}+G_r\}]x_i = 0; \quad y_i^T[\bar{K}+K_r-\omega_i^2(\bar{M}+M_r+(\Omega/\omega)\{\bar{G}+G_r\}] = 0 \tag{22}$$

Similarly, the perturbations of elements of x_i and y_i take the form

$$\mathrm{d}x_{ki} = \sum_{j=1}^{n}\sum_{s=1}^{n} \frac{\partial x_{ki}}{\partial k_{js}} \mathrm{d}k_{js} + \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{\partial x_{ik}}{\partial m^*_{js}} \mathrm{d}m^*_{js} \tag{23}$$

where the gradients with respect to stiffness and mass coefficients are available in terms of whirl speeds, x_i and y_i. The covariance structure of the eigensolutions has been obtained using the expressions of $\mathrm{d}\omega^2$ and $\mathrm{d}x_i$.

The flutter of uncertain laminated plates using a perturbation stochastic finite element formulation was studied by Liaw and Yang (1993). They used a 48 dof rectangular plate element. The modulus of elasticity, mass density, thickness, fiber orientation of individual lamina, geometric imperfection of the entire plate and in-plane loads were treated as random variables. The aerodynamic pressure due to supersonic potential flow was modeled using quasi-steady first order piston theory. The governing equation in this case was of the form

$$M\ddot{X} + [K_T + qD]\, X = 0, \tag{24}$$

where q is the aerodynamic pressure parameter, D is the associated matrix to aerodynamic pressure, M is the mass matrix, and K_T is the tangential stiffness matrix which introduces nonlinearity into the problem. This leads to the eigenvalue problem

$$\left[\alpha^2 M + K_T + qD\right] \Delta X = 0 \tag{25}$$

An iterative solution scheme was used to determine the critical aerodynamic pressure which subsequently led to the determination of flutter boundaries. Figure 2 shows the combined effects of parameter uncertainties in the modulus of elasticity E, mass density ρ, thickness h, fiber orientation θ, geometric imperfection of the plate δ and in-plane load ratio P_N/P_{cr} on the structural reliability boundaries. Here reliability is defined as the probability of the critical aerodynamic pressure being greater than a specified aerodynamic pressure. Each uncertainty parameter was assumed to be zero or fully correlated among all the constituent layers. It was found that the random compensation effects among the six parameters with zero correlation tended to increase the reliability.

Iyengar and Manohar (1989) and Manohar and Iyengar (1993, 94) extended the work of Iyengar and Athreya (1975) and studied the free vibration characteristics of systems governed by a second order stochastic wave equation. They considered eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{dx}}[\{1 + \delta g(x)\}\frac{\mathrm{d}y}{\mathrm{dx}}] + \lambda^2[1 + \epsilon f(x)]y = 0 \tag{26}$$

$$y(0) = 0; \quad y(1) = 0 \tag{27}$$

The solution of this stochastic boundary value problem is sought in terms of solutions of an associated inhomogeneous initial value problem which consists of finding the solution of equation (26) under the initial conditions at $x = 0$ given by $y^* = 0$ and $\frac{\mathrm{d}y^*}{\mathrm{d}x} = 1$. Denoting by $Z_n(\lambda)$, the nth zero of $y^*(x, \lambda)$, the eigenvalues of equation (26) can be defined as being the roots of the equation $Z_n(\lambda) = 1$. The study of $Z_n(\lambda)$ is facilitated by the coordinate transformation

$$y(x) = r(x)\sin[\lambda x + \phi(x)] \tag{28}$$

$$[1 + \delta g(x)]\frac{\mathrm{d}y}{\mathrm{dx}} = r(x)\lambda\cos[\lambda x + \phi(x)]. \tag{29}$$

This leads to a pair of nonlinear coupled differential equations in $r(x)$ and $\phi(x)$. The probability distribution of the eigenvalues λ_n is shown to be related to $\phi(x)$ through the identity (Iyengar and Athreya 1975)

$$\mathrm{P}[\lambda_n \leq \lambda] = \mathrm{P}[Z_n(\lambda) \leq 1] = \mathrm{P}[n\pi \leq \phi(1, \lambda)] \tag{30}$$

and, similarly, the joint probability density function of the nth eigenvalue $y_n(x)$ and nth eigenfunction was expressed in the form (Manohar and Iyengar 1994)

$$p_{y_n,\lambda_n}(y, x, \lambda) = \frac{p_{y^*,\phi|\lambda_n}[y, x; n\pi, 1|\lambda_n = \lambda]p_{\lambda_n}(\lambda)}{p_\phi(n, \pi, 1, \lambda)} \tag{31}$$

This would mean that probabilistic characterization of eigensolutions requires the solution of a pair of nonlinear stochastic equations in $r(x)$ and $\phi(x)$. Extension of this formulation to consider other types of boundary conditions, including random boundary conditions, was presented by Manohar and Iyengar (1993). Exact solutions were shown to be possible only under special circumstances (Iyengar and Manohar, 1989, Manohar and Keane, 1993) and, consequently, approximations become necessary. For specific types of mass and stiffness variations, Iyengar and Manohar (1989) and Manohar and Iyengar (1993, 94) have developed solution strategies based on closure, discrete Markov chain approximation, stochastic averaging methods and Monte Carlo simulations. These combined schemes have been employed to estimate probability density functions of the eigensolutions. Figure 3 shows the contours of probability density function of the second eigenfunction as a function of the nondimensional parameter x/L. At $x = 0$ and $x = L$, the rod was held fixed and, consequently, the probability density functions of the eigenfunction at these points degenerated into Dirac's delta functions. The eigenfunction variability was found to be higher at the nodal points than at the antinodal points.

Based on the study of the distribution of zeros of random polynomials, Grigoriu (1992) examined the roots of characteristic polynomials of real symmetric random matrices. These roots identify the most likely values of eigenvalues and the average number of eigenvalues within a specified range. Brown and Ferri (1996) noted the cost effectiveness of component mode synthesis for Monte Carlo simulation of the dynamics of large scale structures. They treated the substructure dynamical properties as the primary random variables and combined the residual flexibility method of component mode synthesis with probabilistic methods.

5. FORCED VIBRATION

5.1 Single-Degree-of-Freedom Systems (SDoF)

The behavior of simple uncertain systems can be studied in terms of damped single-degree-of-freedom systems subjected to stationary white noise excitation. Udwadia (1987a,b) studied the dynamical characteristics of linear sdof systems with random mass, stiffness and damping properties under free and forced vibration states. It was assumed that the probabilistic description of the system parameters is only partially available. Some results on response characteristics were obtained in closed forms. Wall (1987) evaluated the mean and variance of exceedance rate response of random SDoF oscillators subjected to earthquake excitations with Kanai-Tajimi power spectra whose parameters were also assumed to be random. The statistics of the average number of level crossings, average number of maxima and departure from normality were computed by Kotulski and Sobczyk (1987) for the case of random oscillator under white and randomly filtered white noise inputs. Spencer and Elishakoff (1988) investigated the effect of system randomness on first passage failure of linear and nonlinear oscillators. They utilized a discretization of state space of the system random variables and subsequently solved the associated backward Kolmogorov equation using a finite element method to evaluate the first passage statistics.

Branstetter (1988) and Jeong and Branstetter (1991) developed discretized expressions which were numerically evaluated to obtain certain statistical moments of linear oscillators. The stiffness of these oscillators was represented by a random variable. For time integration steps beyond the first, the initial conditions become correlated random variables. Contrary to deterministic linear systems, it was shown that the *finite-time* response variance, velocity variance, and response covariance are initial condition dependent when the oscillator is uncertain. For zero initial conditions, the finite-time displacement variance, velocity variance, and response covariance followed smoother histories with increasing uncertainity. It was also found that these statistics are even functions of a single non-zero initial condition. For either a determinitic or an uncertain oscillator, the non-zero *limiting* displacement variance and velocity variance are independent of initial conditions. The limiting dispacement variance was found to increas as the level of uncertainty increases.

Jensen and Iwan (1991) proposed an expansion technique to analyze random oscillators with random natural frequencies described by the equation

$$\ddot{x} + 2\eta(\bar{\omega} + \mu\omega_r)\dot{x} + (\bar{\omega} + \mu\omega_r)^2 x = f(t) \quad x(0) = 0; \dot{x}(0) = 0 \qquad (32)$$

where η is the damping ratio, μ is a deterministic coefficient, ω_r is a random variable, and $f(t)$ is a random process in time t. The response $x(t)$ was expressed in a series form

$$x(t, \omega_r) = \sum_{j=0}^{n} x_j(t) H_j(\omega_r) \qquad (33)$$

where n is the order of approximation, $x_j(t)$ is an unknown deterministic function of time and $H_j(\omega_r)$ is a set of orthogonal polynomials. Substituting (33) into (32) then multiplying equation (32) by $H_j(\omega_r)$ and using orthogonality and recursion relations satisfied by the polynomials yields an equivalent set of deterministic equations with external random excitations. Koyluoglu *et al.* (1995a) developed similar approach based on transforming the equation with random coefficients to one with deterministic coefficients and random initial conditions. Subsequently, the evolution of the probability density function of the extended response vector can be described by the well-known Liouville equation or by the Fokker-Planck equation. Thus when $f(t)$ is modeled as a white noise process, the transitional pdf of the extended response vector would satisfy the Fokker Planck equation which, in turn, would enable the formulation of equations governing the response moments. However, the equation for mth moment gets coupled to the $(m+1)$th moment which rules out exact solutions. Koyluoglu *et al.,* employed cumulant neglect closure scheme (Ibrahim, 1985) and solved for the first four moments. They obtained an approximate transient solution for a sdof system with random spring and damping coefficients subjected to a nonstationary modulated white noise process. The results were in a good agreement with the exact solution.

5.2 Multi-Degree-of-Freedom Systems (MDoF)

The treatment of MDoF systems is more involved than SDoF systems. The starting point of this discussion is the matrix differential equation

$$M\ddot{X} + C\dot{X} + KX = F(t); \quad X(0) = X_0;\ \dot{X}(0) = \dot{X}_0 \tag{34}$$

where, at least, one of the matrices, M, C or K, is a function of a set of random variables. This equation results from the finite element and random field discretizations of continuous structural models. The set of random variables entering the matrices M, C and K can be taken to be uncorrelated with zero means. The frequency domain representation of the above equation, when admissible, is given by

$$[-\omega^2 M + \mathrm{i}\omega C + K]U(\omega) = P(\omega) \tag{35}$$

Singh and Lee (1993) used a direct product technique to estimate the statistical frequency response of a damped vibratory system. The solution procedure in the time domain consists of analyzing equation (34) using either modal expansion technique or direct numerical integration. In the frequency domain, it involves the inversion of the stochastic matrix $H(\omega) = [-\omega^2 M + \mathrm{i}\omega C + K]$. Both approaches were studied in conjunction with with other techniques such as perturbation, Neuman expansion, optimal series expansions, optimal linearization, and digital simulation methods. Lee and Singh (1994b) assumed the amplitude of the excitation $F(t)$ to be randomly distributed which is multiplied by a time history process. This process can be taken as a deterministic function such as impulse or sinusoidal.

Different versions of perturbation formulations, including those based on the Taylor series expansion and sensitivity vector method have been used in the literature. These methods convert the given equation with stochastic coefficients into a sequence of deterministic equations. Other methods based on perturbations associated with a small parameter $\epsilon << 1$ lead to a sequence of deterministic equations with deterministic operators and random right-hand sides.

Kleiber and Hien (1992) presented a systematic discussion of generalization of the principle of minimum potential energy and the Hamilton principle by including the effect of system stochasticity within the framework of mean-based, second moment, second-order perturbation techniques. The perturbation methods are applicable to a wide range of problems; however, they may be less accurate and suffer computational efficiency and convergence. This is true especially when the system is highly nonlinear, and when the parameters have skewed distributions with high levels of uncertainty. In addition, these methods lack invariance with respect to the formulation of the problem (Igusa and Der Kiureghian 1988, Madsen *et al.,* 1986). Difficulties associated with these methods and their application, especially for transient dynamic problems are discussed in the literature; see, for example, Liu *et al.*

(1992), Kleiber and Hien (1992) and Katafygiotis and Beck (1995). Remedial measures to overcome this limitation have also been suggested (Kleiber and Hien 1992).

Chen *et al.* (1992) used a perturbation approach to assess the relative importance of uncertainty in excitations, geometrical, and material properties by considering examples of randomly driven truss and beam structures. Chang (1993) and Chang and Chang (1994) studied the transient response statistics and reliability of beams with stochastically varying elastic foundation modulus and Young's modulus. The dynamic response of an infinitely long, randomly damped beam resting on a random Winkler's foundation and excited by a moving force was examined by Fryba *et al.* (1993). They utilized a perturbational approach by introducing a new independent variable whose origin moves with the force.

Branstetter and Paez (1986) used a step-by-step expansion to recursively predict the first- and second-order response moments of linear MDOF systems having uncertain stiffness. Their work was based on the assumption that all random variables are Gaussian. This assumption is usually commmon in stochastic finite element methods..

Methods based on orthogonal series expansions for both the system property random fields and response fields were used by Ghanem and Spanos (1990, 91a,b), Jensen and Iwan (1992) and Iwan and Jensen (1993). These methods usually lead to a set of algebraic equations. The size of these equations depends on the finite element discretization of the displace field, the number of random variables entering the formulation, and the order of expansion used in representing the response field. The response power spectra of a beam mounted on a stochastic Winkler's foundation and subjected to a stationary Gaussian random excitation are shown in figures 4. These figures show also the response spectra as estimated using Monte Carlo simulation. The analytical results based on Karhunen-Loeve expansions (shown in figure 4b) are observed to compare well with simulations. On the other hand, Neumann expansion based methods (figure 4c) compared poorly with simulation results, especially, at frequencies near the system natural frequencies.

Jensen and Iwan (1992) considered the case when C and K are functions of a set of zero mean uncorrelated random variables $\{\omega_{ri}\}_{i=1}^{n}$ and $F(t)$ to be a vector of nonstationary excitations. They studied the evolution of the nonstationary covariance matrix in time domain. This was achieved by expanding the response covariance matrix in terms of a set of known orthogonal multidimensional polynomials in ω_{ri}. The resulting deterministic differential equations were integrated numerically. The procedure was illustrated by considering a five-degree-of-freedom system with uncertainty in stiffness/damping parameters and subjected to seismic base excitations. The influence of system uncertainty was shown to be significant in the analysis of tuning and interaction between primary-secondary modes and also in the reliability of secondary modes. Iwan and Jensen (1993) generalized the analysis to continuous stochastic systems. They obtained a set of deterministic ordinary differential equations in time which are integrated numerically. The method was illustrated by considering the response of a stochastic shear beam to seismic base excitation. Figure 5 shows the time history records of the base excitation and response mean and standard deviation. It was observed that the variability in response was about half of the maximum mean, thereby indicating the importance of accounting for the system uncertainties in response calculations. Mahadevan and Mehta (1993) discussed matrix condensation techniques for stochastic finite element methods for reliability analysis of frames. They also computed the sensitivity of the response to the basic random variables by analytical differentiation as applied to the deterministic analysis.

Grigoriu (1991) developed an equivalent linearization approach to study the static equilibrium equation

$$K(\omega_r)U = S(\omega_r) \tag{36}$$

where ω_r is a vector of random variables. The displacement vector was described by the approximate expression $\hat{U} = \alpha\omega_r + \beta$, where α is a matrix and β is a vector with unknown deterministic elements which are determined by minimizing the error $e = E||S - K\hat{U}||^2$. The process yielded a set of deterministic linear algebraic equations in the unknowns α and β. The determination of these unknowns requires the knowledge of probabilistic description of ω_r beyond the second moment. Thus, when K and S are linear functions of ω_r, the determination of α and β requires description of ω_r up to the fourth order moments.

The application of Markov theory-based techniques to problems involving the determination of dynamic stiffness coefficients of structural elements was studied by Manohar (1995). These systems are usually described by a stochastic wave equation. For example the field equation for the axial vibration of a non-homogeneous viscously damped rod element can be written as

$$\frac{\mathrm{d}}{\mathrm{dx}}[\{1+\delta_1 f_1(\mathrm{x})\}\frac{\mathrm{d}y}{\mathrm{dx}}+\mathrm{i}\beta_1\{1+\delta_2 f_2(\mathrm{x})\}\frac{\mathrm{d}y}{\mathrm{dx}}]+\lambda^2[1+\delta_3 f_3(\mathrm{x})]y-\mathrm{i}\beta_2[1+\delta_4 f_4(\mathrm{x})]y = 0 \tag{37}$$

for two sets of inhomogeneous boundary conditions

$$y(0) = y_0; \qquad y(L) = y_L \tag{38}$$

and

$$\frac{dy}{dx}|_{\mathrm{x}=0} = -\frac{P_1}{AE(0)}; \qquad \frac{dy}{dx}|_{\mathrm{x}=L} = -\frac{P_2}{AE(L)} \tag{39}$$

where $\mathrm{i}^2 = -1$ which appears as a result of using complex algebra in solving the original partial differential equation of the rod. P_1 and P_2 are the end loads at $x = 0$ and $\mathrm{x} = L$, respectively. The functions $f_1(\mathrm{x})$ and $f_3(\mathrm{x})$ represent random components of the stiffness and mass, respectively, $f_2(\mathrm{x})$ is the random component of the strain rate dependent viscous damping, and $f_4(\mathrm{x})$ is the random component of the velocity dependent viscous damping. The parameters β_i and λ^2 are functions of the system parameters and driving frequency. It must be noted that the solutions of the above equations do not have Markovian properties even when the stochastic fields $f_i(\mathrm{x})$ arise as filtered white noise processes. This is due to the fact that the solution trajectories have to satisfy boundary conditions at $\mathrm{x} = 0$ and $\mathrm{x} = L$. The approach consists of expressing the solution of the above stochastic boundary value problem as a superposition of two basis solutions which are obtained by solving the field equation (37) under a pair of independent initial conditions. This subsequently enables the application of the Markov process-based approaches to construct the basis solutions. The solutions take into account the mean and power spectral density matrix of the system property random fields.

The problem of seismic wave amplification through stochastic soil layers was studied by Manohar and Shashirekha (1995). Numerical results on the spectra of mean and standard deviation of the amplification factor were found to compare well with corresponding digital simulation results. Sobczyk *et al.* (1996) studied the harmonic response of undamped beams with stochastically varying inertial/elastic foundation moduli. They expressed the governing differential equation of motion by a random integral equation. The integral equation, in turn, was solved using the method of successive approximation. This method avoids the need to compute the stochastic free vibration analysis, and also permits estimation of the error of approximation.

5.3 Nonlinear Systems

The study of structural systems including the effects of system nonlinearity in the presence of parameter uncertainties presents serious challenges and difficulties to designers and reliability engineers. Recent developments in the mathematical theory of random processes and stochastic differential equations have promoted the study of response and stability in structural systems driven by random excitations. However, there is no unique theory that can be generalized to analyze any nonlinear system. Each method has its own limitation with respect to the nature of the excitation, the type of nonlinearity, and the number of degrees of freedom. Moreover, nonlinear modeling allows the designer to predict a wide range of complex response characteristics, such as multiple solutions, jump phenomena, internal resonance, on-off intermittency, and chaotic motion. These phenomena have direct effects on the reliability and safe operation of structural components. Accordingly, the designer must estimate the reliability of nonlinear systems subjected to Gaussian/non-Gaussian random excitations. In this case the engineer has to deal with both the catastrophic type and fatigue type failures. The former is related to the distribution of extreme values of the system response, and the latter is related to the crossing rates at different levels of the system response.

Socha and Soong (1991) and Ibrahim (1991, 95) presented overviews of methods, limitations, and experimental results of treating nonlinear systems under random excitations. Socha and Soong highlighted on the method of statistical and equivalent linearization and its applications to nonlinear systems subjected to stationary and nonstationary random excitations. Some controversies were reported regarding different results obtained by different methods for the same system. In such cases experimental tests are valuable in providing complex phenomena not predicted by the available methods, and can provide guidelines to refine theory. Ibrahim (1995) reported a number of difficult issues and controversies encountered in the development of the nonlinear theory of random vibration. The analysis of nonlinear structural systems with parameter uncertainties is very limited to special cases such as static problems and numerical simulations.

Static problems involving system nonlinearities and stochasticity were studied by Liu *et al* (1986, 88). Liu *et al.* (1987) considered the dynamic response to a step input of an elasto-plastic beam with isotropic hardening whose plastic modulus was modeled as a Gaussian random field. Both displacement and random fields were discretized using 32 elements and, for this purpose, the same finite element shape functions were used. Furthermore, using orthogonalization of random variables, the 32 random variables were replaced by nine transformed random variables. A mean-centered second-order perturbation method in conjunction with direct numerical integration in time was used to study the time evolution of response moments. The transient response of a transversely loaded stochastically inhomogeneous plate on a random nonlinear elastic foundation was studied by Deodatis and Shinozuka (1988). They used Monte Carlo simulation in conjunction with finite element discretization and time integration techniques. They examined the influence of the stochasticity of the elastic modulus and/or stochasticity of a nonlinear foundation, the support conditions and degree of nonlinearity of the foundation (α_0) on the coefficient of variation of the maximum deflection (V_d). Figure 6 shows this coefficient of variation as a function of α_0 for the case of a simply supported rectangular plate. Furthermore, Gaussian and lognormal distributions were shown to provide good fits to the maximum plate deflection. A similar approach was considered by Brenner (1994) to study harmonic forced excitation of a three-dimensional skeletal model of a transmission line tower.

Chang and Yang (1991) considered large amplitude vibration of a beam with randomly varying material and geometric properties. They analyzed the free and forced response to harmonic and random excitations. Their analysis involved discretization of the random field using the method of local averages and a second order perturbation scheme. The forced nonlinear random vibration response was obtained using the equivalent linearization technique. Satisfactory comparisons of analytical results with Monte Carlo simulations were demonstrated. Koyluoglu *et al.,* (1995c) employed weighted integral method of random field discretization in the development of a nonlinear stochastic finite element formulation for stochastic plane frame analysis. The stiffness and damping properties were taken to be random in nature, and the excitations were modeled as stationary random processes. They employed mean centered second order perturbation method to treat system uncertainties and the Gaussian closure method to handle the nonlinearities. Klosner*et al.* (1992) considered the response of nonlinear SDoF and two-dof systems with stochastic parameters under white noise excitation. For more general classes of problems possessing no exact solutions, Klosner*et al.* employed a statistical linearization technique to determine the conditioned response statistics. It should be noted that statistical linearization gives satisfactory results only if the system does not involve secular terms which give rise to internal resonance conditions.

6.0 METHODS OF ANALYSIS

6.1 Statistical Energy Analysis (SEA) and Applications

The framework of SEA is documented in the well known research monograph by Lyon (1975) and recently by Lyon and DeJong (1995). SEA can be viewed as a branch of linear random vibration theory which is applicable to situations in which the response at any frequency consists of small contributions from a large number of modes. The method aims to avoid the detailed calculation of contributions from individual modes and instead to predict response levels which are averaged over these modes in some sense. Participation of a large number of modes in the vibration is thus an essential requirement for the successful application of the method. The method has been used to predict the vibration response of aerospace structures, land based vehicle design, ship dynamics, building acoustics, machinery vibration and, more, recently, in seismic analysis of secondary systems (see, for example, Lai and Soong, 1990 and Hynna *et al.,* 1995). Hodges and Woodhouse (1986), Langley (1989), Keane (1992), Fahy (1994), and Price and Keane (1994) treated different theoretical aspects of the method. As mentioned in Section 2, an important aspect of SEA is the modeling of vibrating systems as being drawn from a statistical ensemble of nominally identical systems. This is done to allow for the high sensitivity of higher eigensolutions to minor changes in system modeling and values of model parameters. Here we focus our attention on issues relevant to stochastic modeling and response statistics prediction in SEA applications. For more details, see the overview by Fahy (1994).

The process of averaging in SEA is carried out for two main reasons. The first is that it accounts for the random nature of the forces acting on most structures. The second is that it accounts for the statistical modeling of the system. Two different forms of averaging are employed in SEA to achieve these goals:

1. Ensemble averaging across realizations of time histories of response.
2. Averaging across a stochastic ensemble of vibrating systems. This involves an integration over the probability distribution of either the random physical parameters of the system or the natural frequencies and mode shapes.

The primary response statistical parameters of interest in SEA are the spectra of steady state average energies stored in the subsystems. These spectra are obtained as integrals over the extent of the subsystems and also over the driving frequency range. The results obtained are clearly dependent on the details of the averaging process, such as the frequency bandwidth and the probability distribution functions assumed for the random quantities. Each form of averaging is accompanied by a reduction in resolution of the response with respect to amplitude, time, space or frequency parameters. In order for the average results to be interpreted properly, it is essential that each averaging process be accompanied by the associated estimates of the measures of dispersion. While it is straightforward to analyze the dispersion associated with averaging across the ensemble of time histories, the study of other forms of averaging is more complicated. This difficulty constitutes a major shortcoming in the application of SEA procedures to practical problems.

According to Lyon (1975), the statistics of temporal mean square velocity at any specified point in the structure are estimated in terms of statistics of coupling loss factors and total steady state energies in the subsystems. Some of the salient features of the analysis are:

1. The natural frequencies of the subsystems are primary stochastic vari-

ables which are distributed as Poisson points on the frequency axis.

2. The mean energy levels are evaluated based on the assumption that the power input and loss factors are uncorrelated. The consideration of higher order statistics require other assumptions yielding to a statistical closure scheme.

3. All modes are equally damped and the contributions to the mean square value from different modes are independent.

4. The temporal mean square velocities are random variables with gamma probability distribution.

It must be noted that these assumptions are *ad hoc* in nature and are made essentially to simplify the calculation pertaining to the statistical aspects of the response. In any given situation, however, it is not possible to distinguish between the fluctuations which arise naturally from statistical variations in system parameters and systematic errors resulting from the assumption made in the dynamical and statistical analyses. This presents considerable difficulties in assessing the assumptions made.

The studies conducted by Davies and Wahab (1981), Davies and Khandoker (1982), Fahy and Mohammed (1992), Manohar and Keane (1993,94), Keane and Manohar (1993), Rebillard and Guyader (1995), and Keane (1996) overcome this difficulty by employing modal expansion-based exact analytical procedures for the dynamic analysis of the system, and Monte Carlo simulation methods for statistical analysis. Approaches based on wave propagation analysis were used in similar contexts by Mace (1992) and by Wester and Mace (1996). The procedures used in these studies are rigorously tractable from the point of view of both dynamics and statistics. However, they lack the simplicity of Lyon's analysis. In these studies the stochastic characterization of the system properties was introduced at the level of mass and stiffness properties rather than at the eigensolution level as was done by Lyon.

Davies and Wahab (1981) considered the statistics of coupling loss factors across the intermediate support of a two-span continuous beam with one of the spans subjected to "rain on the roof" type excitations. The ratio of the two spans was modeled as a random variable with a uniform probability distribution function. The same system under the action of point harmonic forcing was considered by Davies and Khandoker (1982) and the statistics of cross power receptance function were estimated. These studies illustrated the importance of the modal overlap factor, defined as the ratio of average modal bandwidth to the average spacing of natural frequencies, as a parameter influencing the response variability. As may be expected, the variability was found to be higher for lower modal overlap factors. Fahy and Mohammed (1992) considered systems of spring coupled beams, plates and rods. They noted the non-Gaussian nature of the energy flow characteristics at low modal overlap factors. It was concluded that the confidence limits cannot be estimated using the mean and standard deviation alone.

Keane and Manohar (1993) and Manohar and Keane (1993, 94) considered the energy flows in spring coupled beam and rod systems. They examined the effects of the choice of subsystems, damping models, strength of system randomness, type of excitation and types of system randomness on the probabilistic characteristics of power receptance functions. Both Gaussian and non-Gaussian models for the mass, stiffness, and geometrical properties were considered. It was reported that receptance functions are nonstationary random processes due to the occurrence of resonances and variations in mode shapes. However, it was noted in most, but not all cases, that with increases in driving frequency, the receptance functions tended to be stochastically stationary. This indicated that a frequency exists beyond which the receptance can be expected to reach a stochastic 'steady' state. Beyond this frequency, a simplified description of system behavior is possible using such methods as SEA.

Analogous to the definition of the modal overlap factor, a modal statistical overlap factor was defined as the ratio of standard deviation of the natural frequency to the average spacing of the natural frequencies. These two factors were shown to have a significant role in defining the stationarity of mean square responses and other related issues. It was concluded that since both these overlap factors could vary with frequency, precise knowledge of their behavior was a precursor to the successful application of SEA methods. These considerations were further examined by Keane (1996) in the context of vibration energy flow in a pair of line coupled random membranes. Rebillard and Guyader (1995) considered a system of a pair of rectangular plates coupled along edges and executing harmonic flexural and in-plane vibrations. The in-plane and flexural motions are uncoupled only if the connection angle is zero. It was shown that the uncertainty in system response is higher when the nominal angle of connection is small than when the angle is large.

6.2 Stochastic boundary element methods

There are some advantages of BEM over FEM. These include reduction in dimensionality and the ability to handle efficiently problems involving infinite domain and singularities. Few attempts were reported in the literature to formulate random vibration and system stochasticity problems within the framework of boundary element methods. For example, Spanos and Ghanem (1991) proposed a BEM-based method to study vibration of systems with deterministic parameters under surface tractions which are random in space and time. The application of the method needs the discretization of the spatially random surface tractions and other issues similar to those discussed in Section 3.0. Burczynski (1993), Ettouney *et al.* (1993) and Lafe and Cheng (1993) extended the BEM to problems involving randomness in material properties, surface tractions and/or boundary shape. Burczynski (1993) examined the effect of stochastic shape of the boundary on stresses, strains, displacements and natural frequencies using a stochastic sensitivity analysis. A perturbational approach based on small random fluctuations was considered by Ettouney *et al.* (1993) for soil dynamics problems involving random variable models for elastic constants and mass density. Lafe and Cheng (1993) examined the influence of random material inhomogenieties encountered in problems of ground water flow.

6.3 Methods based on interval algebra

When only scant data is available to construct probabilistic models for system uncertainties, probabilistic models become questionable. In such situations one may attempt to estimate bounds on the response variance consistent with the data available (Shinozuka, 1987, and Deodatis and Shinozuka, 1989). The bounds developed do not depend upon the functional form of the autocovariance/psd function of the random fields describing the parameter variations. Other alternative is based on results from interval algebra, proposed by Chen *et al.* (1994), Dimarogonas (1995), Qiu *et al.* (1996a,b) and Koyluoglu *et al.,* (1995d). Here, the uncertain system parameters are modeled through the specification of the lower and upper bounds on their values. The subsequent problems of response analysis consisting of eigenvalue problems, matrix inversions, *etc.* need to be handled within the framework of interval algebra. It may be noted that the rules of interval arithmetic are distinct from those applicable to numbers (Alefeld and Herzberger 1983). A brief introduction to these rules is available as an appendix to the paper by Koyluoglu *et al.* (1995d). The relevant terminology has also been introduced by Dimarogonas (1995).

Elishakoff (1995) discussed non-probabilistic modeling of uncertainties. Thus, in the case of an algebraic eigenvalue problem $Kx = \lambda Mx$, where the elements of the $n \times n$ matrices K and M are uncertain and are partially specified through the constraints $K_{ij}^l \leq K_{ij} \leq K_{ij}^u$ and $M_{ij}^l \leq M_{ij} \leq M_{ij}^u$, where the superscripts l and u represent, respectively, the lower and upper bounds, one has to determine λ_i^l and λ_i^u such that $\lambda_i^l \leq \lambda_i \leq \lambda_i^u$; $i = 1, 2, \cdots n$. Chen *et al.* (1994) used the properties of Rayleigh's quotient and showed that the bounds on the eigenvalues of real symmetric matrix can be obtained by solving a set of three eigenvalue problems. A perturbational approach to estimate the bounds on the eigenvalues was also suggested and it reduces the number of eigenvalues problems to be solved to one.

Dimarogonas (1995) noted the difficulties associated with the interval evaluation of eigenvalues using the interval calculus version of commonly used numerical techniques. He also developed an optimization technique to obtain the minimum-radius intervals of the solution for the eigenvalue sensitivity problem. The problem of linear rotor dynamics with interval bearing properties was studied and the results showed satisfactory agreement between results of interval analysis and Monte Carlo simulation. Qiu, *et al.* (1996a,b) reviewed the results available on interval eigenvalue problems. They also showed that if $K^I = [K^l, K^u] = [K^c - \Delta K, K^c + \Delta K]$ is a semi-definite interval matrix, and $M^I = [M^l, M^u] = [M^c - \Delta M, M^c + \Delta M]$ is a positive definite interval matrix, $\Delta K^I_+ = [0, \Delta K]$ and $\Delta M^I_+ = [0, \Delta M]$ are semi-definite interval matrices, then the eigenvalues of the problem $Kx = \lambda Mx$ where K belongs to K^I and M belongs to M^I, ranges over the interval

$$\lambda^I_i = [\lambda^l_i, \lambda^u_i] \quad i = 1, 2, \cdots n$$

where the lower bound λ^l_i satisfies

$$K^l u^l_i = \lambda^l_i M^u u^l_i \quad i = 1, 2, \cdots n$$

and the upper bound λ^u_i satisfies

$$K^u u^u_i = \lambda^u_i M^l u^u_i \quad i = 1, 2, \cdots n.$$

A numerical example on the natural frequency interval of a 15 member truss which shows that the intervals for higher mode natural frequencies are wider; a measure of relative uncertainty defined as the ratio of interval width to the median value, however is higher for lower natural frequencies and Qiu, *et al.* extended the interval algebra to the matrix perturbation theory to approximately evaluate the interval eigenvalues. Application of interval algebra to the static finite element analysis of skeleton structures with interval parameters has been developed by Koyluoglu *et al.* (1995d).

7.0 LOCALIZATION IN DISORDERED PERIODIC STRUCTURES

Weak structural irregularities may result in the occurrence of the well known phenomenon of modal localization. Dynamics of spatially periodic structures, such as, turbomachinery blade assemblies, multi-span structures and aircraft fuselages, is characterized by alternating sequence of frequency bands which pass and stop traveling waves with the system natural frequencies occurring in clumps within the pass bands. The presence of irregularities in these structures, which destroys system periodicity, can confine vibration to specific parts of the structure. Due to wave reflections at the interfaces of non-identical elements, wave propagation in a disordered structure is attenuated even if the damping is absent. This effect is solely due to system disorder, not to energy dissipation, and is more pronounced in higher frequency regions. This phenomenon was originally observed by Anderson (1958, 78, 86) in solid state physics. Weaver (1993) and Weaver and Burkhardt (1994) studied Anderson localization phenomenon in acoustic wave propagation. In a special issue of AMR, a series of review articles by Lin (1996), Pierre, et al. (1996), Vakakis (1996), Photiadis (1996), and Weaver (1996) provided a wide spectrum of overviews of modal localization problems in periodic structures, truss beams, localization of nonlinear continuous systems, fluid-loaded structures, and other related problems. In a recent research monograph, Vakakis,*et al.* (1996) documented the problem of localization in discrete systems, coupled beams and other continuous systems.

These studies provided a basic understanding of factors influencing localization such as degree of disorder, strength of coupling, number of coupling paths, and variation of localization effects with respect to frequency. In this section we focus on developments and results which only deal with probabilistic descriptions of mode localization and consequent response analyses.

More realistic modeling of structural disordered can be described as a random field. The studies by Kissel (1988, 91), Cai and Lin (1991), Lin and Cai (1995), Xie and Ariaratnam (1994, 96a,b) and Ariaratnam and Xie (1995) treated the problem within the framework of wave propagation analysis using transfer matrices under the assumption that spatial disorder is modeled as an ergodic random process. Attention was focused on determining the localization factor which is defined as the average rate of exponential decay of the wave per periodic unit with respect to the propagation distance attributable to the disorder. The reciprocal of this quantity gives a measure of length over which the propagating waves extend in the structure. In a multi-coupled system, corresponding to each of the coupling wave types, one can define an associated localization factor. The reciprocal of the smallest value of these factors defines the largest distance over which localization occurs.

Kissel (1988, 91) discussed the relevance of some theorems dealing with properties of products of random matrices. These theorems, originally developed by Furstenberg (1963) and Oseledec (1968), can be used in modeling disordered of single- and multi-wave one-dimensional periodic structures using random matrix products. Kissel considered three systems: i) a chain of spring coupled masses, ii) a rod in axial compression with attached resonators, and iii) multispan Euler-Bernoulli beams. Disorder is characterized in terms of a set of independent uniformly distributed random variables. In the study of mono-coupled disordered systems, the system behavior is characterized in terms of products of random transfer matrices. After some transformation of the transfer matrices, the random matrix product, in terms of the 2×2 wave transfer matrices $\bar{W}_j$, takes the form

$$\Pi_{j=1}^{n}\bar{W}_j = \begin{bmatrix} \frac{1}{\tau_n} & \frac{-\rho_n}{\tau_n} \\ \frac{-\rho_n^*}{\tau_n^*} & \frac{1}{\tau_n^*} \end{bmatrix} \tag{40}$$

where n is the number of repetitive units, τ_n is the transmission coefficient, ρ_n is the reflection coefficient, a star * denotes complex conjugate and

$$|\tau_n|^2 + |\rho_n|^2 = 1. \tag{41}$$

Furthermore, with the assumption that $\bar{W}_1, \bar{W}_2, \cdots, \bar{W}_n$, form a sequence of independent, identically distributed random matrices and applying the

Furstenberg theorem, it was shown that, with probability 1,

$$\gamma = -\lim_{n\to\infty} \frac{1}{n} \ln|\tau_n|, \quad \gamma > 0 \tag{42}$$

where γ is the localization factor which is dependent on frequency and properties of the structure. Alternatively, relation (42) can be written in the form

$$\lim_{n\to\infty} |\tau_n|^2 \to \exp[-2\gamma n] \tag{43}$$

Relation (43) implies that the transmitted energy decays exponentially as number of repetitive units become large. It is apparent that the localization factor is related to the largest Lyapunov exponent which characterizes the average rate of exponential growth per bay of the structure. Lyapunov exponents provide useful information on localization phenomena in multi-coupled periodic structures.

Several analytical and computational procedures for approximate computation of the localization factors have been developed. These include the method of multiple reflections, the method of invariant distribution, and the method of Lyapunov exponents. Lin and Cai (1995) provided a systematic account of these techniques. Localization in multi-wave periodic structures was studied by Kissel (1988,1991) and Xie and Ariaratnam (1994). In order to find the largest length over which localization occurs, the smallest Lyapunov exponent needs to be determined. Furstenberg theorem does not help in this case since it leads to only the highest Lyapunov exponent. Kissel discussed the application of the theorem due to Oseledec (1968) and derived the multiwave localization factor. Xie and Ariaratnam (1994) modified the algorithm of Wolf *et al.* (1985) to determine all the Lyapunov exponents and studied the localization in beam-like lattice trusses shown in Figure 7. The lengths of the members of the lattice truss were modeled as mutually independent uniformly distributed random variables. Figure 8 shows a plot of the Lyapunov exponents λ_1 and λ_2 as a function of driving frequency. Note that λ_2 is the lowest exponent and $\lambda_3 = -\lambda_2$ and $\lambda_4 = -\lambda_1$.

Calculation of localization factors for disordered cyclic periodic structures using transfer matrices and Green's function formulation was presented by Xie and Ariaratnam (1996a,1996b). Pierre (1990) considered an assembly of coupled random oscillators in which the strength of coupling could be varied. He studied the mode localization using a perturbational approach in which the coupling parameter is also treated as a small parameter. Both wave and modal approaches were used. The results on localization factor, obtained analytically, was compared with Monte Carlo simulation results. The localization factor was shown to depend upon the disorder to coupling strength ratio and the excitation frequency. Cha and Pierre (1991) and Pierre *et al.* (1994) extended this study to the case of mono-coupled multimodal systems. These studies have demonstrated that severe vibration confinement is unavoidable at high frequencies.

Finite element method was used to study modal localization in cyclic and multi-span disordered structures by Cornwell and Bendiksen (1992) and Lust *et al.,* (1995). These studies proposed a length scale in terms of system modeshapes as an indicator of degree of localization of a given mode. It was shown that modes within the same mode group can have significantly different degrees of localization.

Hodges and Woodhouse (1989a,b) investigated localization of propagating disturbances in disordered coupled oscillators and in beams. They used different forms of ensemble averaging by considering the spatial response variability in a set of coupled random oscillators. They showed dramatic different results between linear and geometric averages of the response. It was also shown that the commonly used linear averaging is significantly affected by those realizations which are far removed from the average. Consequently, the linear average was concluded to be a poor guide for understanding the typical system behavior. On the other hand, the geometric mean, which is not very sensitive to very high or very low valued realizations, was shown to be a better measure of a typical system behavior.

The application of SEA procedures to periodic and near-periodic systems were considered by Hodges and Woodhouse (1986) and Keane and Price (1989). The clumping of system natural frequencies in such systems, and the possibility of localization of mode shapes, cast serious doubts on the applicability of SEA procedures to these systems. This is because the SEA assumes that the subsystem's natural frequencies are Poisson points on the frequency axis and also it altogether ignores the mode shape variability. Keane and Price (1989) considered a system of two periodic/near periodic subsystems which are coupled by a discrete spring. Within a frequency band, the subsystem natural frequencies were taken to be independent and identically distributed. The probability distribution of the individual natural frequencies was taken to have a piecewise uniform variation. It was assumed that the probability of finding a natural frequency in the pass bands of the basic repetitive unit was higher than the probability of finding it in the stop bands. This modification was shown to yield significantly better results than those predicted by traditional SEA procedures.

8.0 EXPERIMENTAL RESULTS

The study of structural dynamics with parameter uncertainties has been dominated by analytical and computational techniques. Nevertheless, the data needed for validation of models and specification of model parameters must originate from experimental measurements. Furthermore, analytical and numerical results of system response, dependence on initial conditions, and mode localization need experimental verification. Ibrahim (1991, 95) presented comprehensive overviews of random experimental tests, observed complex phenomena, difficulties encountered in generating random excitations, system modeling and data acquisition. A few studies were reported in the literature with varying degrees of success. The works of Hodges and Woodhouse on disordered strings and that of Pierre, Tang, and Dowell on a two-span disordered beam were already discussed in the previous review by Ibrahim (1987). Recently, few experimental studies were conducted by King, *et al.* (1995) and Aubrecht, *et al.* (1996) to verify the existence of localized nonlinear normal modes in fixtures of coupled cantilever beams.

Recently, Fahy (1993) has demonstrated the variability in high frequency response which one might expect in an ensemble of engineered structures by studying the dynamic response of an ensemble of empty beer cans. Figure 9 shows a set of measurements made as a part of this experiment and one can judge from this figure the difficulty which might arise in analyzing this type of structures by using finite element type of approaches. An ingenious attempt to produce a large ensemble of vibrating systems within a laboratory setup has been reported by Lenaghan and Fahy (1992,1993). This study consisted of producing acoustical analogues for a stochastic ensemble of a pair of axially vibrating rods which were coupled through a linear spring. The experimental setup consisted of two, end coupled acoustic pipes with the first of these pipes being driven acoustically by a loud speaker at one end and the second pipe terminating in a water tank whose depth was under computer control. By suitable adjustment of water level in the tank, ensembles of data for similar, but not identical, subsystems were generated and tested without manual intervention. Correlation of these experimental results with computational models for axially vibrating rods was discussed by Keane *et al.* (1994).

Keane and Bright (1996) conducted experimental tests on disorder as a means of passive vibration control in lattice structures. They constructed two trusses each consisting of forty members. One truss had a regular geometry. Disordered geometry was deliberately introduced into the other. The configuration for the disordered truss was predetermined analytically and was optimized to display certain isolation characteristics. The experimental study demonstrated that it is possible to build lightweight, lightly damped aluminum truss structures that have up to 50 dB of energy transmission isolation between their ends, without using additional damping materials or active control.

9. CONCLUSIONS

A broad based review of advances made over the last decade in the field of structural dynamics with parameter uncertainties has been presented. These advances have been dominated by new developments of analytical and computational tools for random eigenvalue analysis, response prediction of linear and nonlinear systems and reliability analysis. Phenomenological features associated with dynamic stability, mode localization and high frequency response have also received notable attention. The following are some of the needs for future research:

1. Influence of parameter uncertainties on the nonlinear modal interaction of dynamic systems in the neighborhood of internal resonance conditions.

2. Probabilistic foundations of SEA formalisms, with many of the assumptions made, lack rigorous justification. Thus, for example, for most types of subsystems, the basic assumption of natural frequencies being Poisson points on frequency axis is questionable. Development of methods based on stochastic finite element (SFEM) techniques to estimate confidence in SEA is needed.

3. Methods for experimental measurement of disorder parameters and development of stochastic models for system disorders.

4. More research is needed in the area of optimum design sensitivity in reliability-based design under multilevel reliability constraints to evaluate the significance of various uncertainties on the optimum solutions.

5. The problem of identification of spatial inhomogenieties should be extended based on experimental measurements.

6. Study of interaction between disorder and nonlinearity in nearly periodic systems. Studies by Vakakis and others have established the influence of system nonlinearity as a source of periodicity breaking disorder and the consequent occurrence of localization.

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Figure Captions

Figure 1. Three dimensional piping system whose restraint locations are uncertain (from Nakagiri 1987).

Figure 2. Reliability of the simply supported graphite/epoxy laminated $[90/\pm 45/0]_s$ square plate in supersonic flow with uncertain parameters (from Liaw and Yang 1993).

Figure 3 Contours of the probability density functions of the second eigenfunction; $\epsilon = 0.2$, $\alpha = 20.8$; (from Manohar and Iyengar 1994).

Figure 4 (a) Beam on random elastic foundation subjected to a random dynamic excitation; exponential covariance model; (b) Spectral density of the displacement at the end of the beam; (c) Spectral density of displacement at the end of the beam; (from Ghanem and Spanos 1990).

Figure 5 Seismic response of a stochastic shear beam; (a) Earthquake-like base excitation; (b) Mean value response; (c) Standard deviation response; (d) variability parameter; (from Iwan and Jensen 1993).

Figure 6 Coefficient of variation of the maximum deflection at the center (V_d) as a function of the nonlinearity parameter α_0, for the case of simply supported plate (from Deodatis and Shinozuka 1988).

Figure 7 Beam like lattice truss (from Xie and Ariaratnam 1994).

Figure 8 Lyapunov exponents for Timoshenko beam (from Xie and Ariaratnam 1994).

Figure 9 Scatter in the vibrational acceleration response at one point on a beer can generated by broad band acoustic excitation (41 cans); (from Fahy 1993).