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PROJECTABILITY AND WEAK HOMOGENEITY OF PSEUDO EFFECT ALGEBRAS*

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Abstract. In this paper we deal with a pseudo effect algebra $\mathscr{A}$ possessing a certain interpolation property. According to a result of Dvurečenskij and Vettterlein, $\mathscr{A}$ can be represented as an interval of a unital partially ordered group G. We prove that $\mathscr{A}$ is projectable (strongly projectable) if and only if G is projectable (strongly projectable). An analogous result concerning weak homogeneity of $\mathscr{A}$ and of G is shown to be valid.

Keywords: pseudo effect algebra, unital partially ordered group, internal direct factor, polar, projectability, strong projectability, weak homogeneity

MSC 2010: 06D35, 06F20

1. Introduction

Pseudo effect algebras were introduced by Dvurečenskij and Vetterlein in [3], [4], [5]. In [4] it was proved that if $\mathscr{A}$ is a pseudo effect algebra having an interpolation property denoted as (RDP_1), then there exists a unital partially ordered group (G, u) such that $\mathscr{A}$ can be represented as the interval $[0, u]$ of G (for detailed definitions, cf. Section 2 below).

We denote by $\mathscr{D}$ the class of all pseudo effect algebras satisfying the condition (RDP_1). The class of all MV-algebras (cf. Cignoli, D'Ottaviano and Mundici [1]) and the class of all pseudo MV-algebras (cf. Georgescu and Iorgulescu [6], [7] and Rachůnek [11]) are subclasses of the class $\mathscr{D}$. Assume that $\mathscr{A} \in \mathscr{D}$.

A series of results on the relations between the properties of $\mathscr{A}$ and of G were proved in [5].

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The relations between internal direct product decompositions of $\mathscr{A}$ and those of G were dealt with by the author [10].

The notion of a polar in $\mathscr{A}$ or in G is defined analogously to the case of lattice ordered groups, where it was extensively studied in several papers.

In accordance with the terminology applied in the theory of lattice ordered groups we say that G is projectable (strongly projectable) if every principal polar (or every polar, respectively) of G is an internal direct factor of G. An analogous definition is used for $\mathscr{A}$.

By using some results of [10] we prove that $\mathscr{A}$ is projectable (strongly projectable) if and only if G is projectable (strongly projectable).

The underlying set of $\mathscr{A}$ will be denoted by A. An interval of a partially ordered set is called trivial if it is a one-element set.

We say that $\mathscr{A}$ is weakly homogeneous if any two nontrivial intervals of $\mathscr{A}$ have the same cardinality. The weak homogeneity of G (and of other partially ordered sets) is defined analogously. We prove the following result:

Assume that $\operatorname{card} A > 2$. Then $\mathscr{A}$ is weakly homogeneous if and only if G is weakly homogeneneous.

The notions of projectability and strong projectability of lattice ordered groups were studied by several authors in an extensive series of papers; for detailed references cf., e.g., the monograph Darnel [2].

Further we recall that the weak homogeneity of Boolean algebras was investigated by Sikorski [12]; the weak homogeneity of MV-algebras and of lattice ordered groups was dealt with by the author [8], [9].

2. Preliminaries

For the sake of completeness, we recall some definitions.

A partial algebra $\mathscr{A} = (A; +, 0, 1)$ where $+$ is a partial binary operation and 0, 1 are constants is a *pseudo effect algebra* if for all $a, b, c \in A$ the following conditions are satisfied:

(i) suppose that $a+b$ exists; then $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist and in this case $(a+b)+c = a+(b+c)$;

(ii) there is exactly one $d \in A$ and exactly one $e \in A$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, then there are $d, e \in A$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a = 0$.

We put $a \leqslant b$ iff there exists $c \in A$ with $a+c = b$. Then $\leqslant$ is a relation of partial order on A and $0 \leqslant a \leqslant 1$ for each $a \in A$.

The group operation in a partially ordered group is denoted additively though it is not assumed to be commutative. Let G be a partially ordered group, $0 \leqslant u \in G$.

Put $A = [0,u]$. Consider the partial operation $+$ on A defined by restricting the group operation to the set A. We denote $\Gamma(G,u) = (A;+,0,u)$. Then $\Gamma(G,u)$ is a pseudo effect algebra.

Let $\mathscr{A}$ be a pseudo effect algebra; consider the following condition:

(RDP$_1$) For any $a_1, a_2, b_1, b_2 \in A$ with $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in A$ such that

(i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_3 + d_4 = b_2$;

(ii) for each $d_2', d_3' \in A$ with $d_2' \leqslant d_2$ and $d_3' \leqslant d_3$ we have $d_2' + d_3' = d_3' + d_2'$.

An element u of a partially ordered group G is a *strong* unit if for each $g \in G$ there exists a positive integer n such that $g \leqslant nu$; the pair (G,u) is called a *unital* partially ordered group.

In view of [4], for each pseudo effect algebra $\mathscr{A}$ satisfying the condition (RDP$_1$) there exists a unique unital partially ordered group (G,u) such that $\mathscr{A} = \Gamma(G,u)$.

The condition (RDP$_1$) implies the validity of the following conditions for $\mathscr{A}$:

(RIP) For any $a_1, a_2, b_1, b_2 \in A$ such that $a_i \leqslant b_j$ $(i,j = 1,2)$ there is $c \in A$ with $a_i \leqslant c \leqslant b_j$ $(i,j = 1,2)$.

(RDP$_0$) For any $a, b_1, b_2 \in A$ with $a \leqslant b_1 + b_2$ there are $d_1, d_2 \in A$ such that $d_1 \leqslant b_1$, $d_2 \leqslant b_2$ and $a = d_1 + d_2$.

3. Polars of $\mathscr{A}$ and of G

Polars in vector lattices, lattice ordered groups, partially ordered groups and in some other partially ordered algebraic structures were extensively investigated in several papers.

As above, we suppose that $\mathscr{A}$ is a pseudo effect algebra belonging to the class $\mathscr{D}$ and that $\mathscr{A} = \Gamma(G,u)$. The underlying partially ordered set of $\mathscr{A}$ will be denoted by $\ell(\mathscr{A})$.

If $a_1, a_2, b \in A$, then b is the join of the set $\{a_1, a\}$ in $\ell(\mathscr{A})$ if and only if b is the join of $\{a_1, a_2\}$ in G; in such case we write $a_1 \vee a_2 = b$. The corresponding dual assertion is also valid; in this case we write $a_1 \wedge a_2 = b$.

As usual, G^+ denotes the positive cone of G; i.e., $G^+ = \{x \in G \colon x \geqslant 0\}$.

Let $\emptyset \neq X \subseteq G^+$. We put

$$X^{\delta_0} = \{g \in G^+ \colon g \wedge x = 0 \text{ for each } x \in X\}, \quad X^{\delta} = \bigcup_{t \in X^{\delta_0}} [-t, t].$$

The set X^{δ} is a *polar of* G.

Further, let $\emptyset \neq Y \subseteq A$. We set

$$Y^{\delta_1} = \{a \in A\colon\ a \wedge y = 0 \ \text{ for each } \ y \in Y\}.$$

We call Y^{δ_1} a *polar of* $\mathscr{A}$.

The system of all polars of G will be denoted by $P(G)$; the symbol $P(\mathscr{A})$ has an analogous meaning concerning $\mathscr{A}$. Also, we put

$$P(G^+) = \{X^{\delta_0}\colon\ \emptyset \neq X \subseteq G^+\}.$$

Each of the systems $P(G)$, $P(\mathscr{A})$ and $P(G^+)$ is partially ordered by the set-theoretical inclusion.

For each $Z \in P(G)$ we put $\chi_1(Z) = Z \cap G^+$. From the definition of $P(G)$ and $P(G^+)$ we immediately obtain

Lemma 3.1. *χ_1 is an isomorphism of $P(G)$ onto $P(G^+)$.*

Let $a, x \in G^+$. There exists a positive integer n with $a \leqslant nu$. Then according to (RDP_0) and by induction we obtain that there exist $a_1, \ldots, a_n \in G^+$ such that $a = a_1 + \ldots + a_n$ and $a_i \leqslant u$ for $i = 1, 2, \ldots, n$. Under this notation we have

Lemma 3.2. *$x \wedge a = 0$ if and only if $x \wedge a_i = 0$ for $i = 1, 2, \ldots, n$.*

P r o o f. Assume that $x \wedge a = 0$ and let $i \in \{1, 2, \ldots, n\}$. Then $0 \leqslant a_i \leqslant a$, whence $x \wedge a_i = 0$. Conversely, suppose that $x \wedge a_i = 0$ for $i = 1, 2, \ldots, n$. Further, assume that $z \in G$, $0 \leqslant z \leqslant x$, $z \leqslant a$. Applying (RDP_0) and induction we get that there exist $z_1, \ldots, z_n \in G^+$ such that $z_i \leqslant a_i$ for $i = 1, 2, \ldots, n$ and $z = z_1 + \ldots + z_n$. Hence $z_i \leqslant x$ and thus $z_i = 0$ for $i = 1, 2, \ldots, n$. Therefore $z = 0$.

Further, assume that $t \in G$, $t \leqslant x$, $t \leqslant a$. In view of $0 \leqslant x$, $0 \leqslant a$ and according to (RIP) there exists $z \in G$ with $0 \leqslant z \leqslant x$, $t \leqslant z \leqslant a$. We have already verified that in this case $z = 0$. Hence $t \leqslant 0$. Therefore $x \wedge a = 0$. □

For each $\emptyset \neq X \subseteq G^+$ we denote by X_0 the set of all $x_0 \in [0, u]$ such that there is $x \in X$ with $x_0 \leqslant x$. From 3.1 we immediately obtain

Lemma 3.3. *For each $\emptyset \neq X \subseteq G^+$ we have $X^\delta = (X_0)^\delta$.*

Let $\emptyset \neq Y \subseteq A$. The definitions of Y^{δ_0} and Y^{δ_1} imply

$$Y^{\delta_1} = Y^{\delta_0} \cap A. \tag{$*$}$$

Hence if $\emptyset \neq Y_1 \subseteq A$, then

$$Y^{\delta_0} = Y_1^{\delta_0} \Rightarrow Y^{\delta_1} = Y_1^{\delta_1}. \tag{1}$$

We put

$$\chi_2(Y^{\delta_1}) = Y^{\delta_0}.$$

Lemma 3.4. *χ_2 is an isomorphism of $P(\mathscr{A})$ onto $P(G^+)$.*

Proof. In view of (1), χ_2 is a correctly defined mapping of $P(\mathscr{A})$ into $P(G^+)$; moreover, χ_2 is a monomorphism. According to 3.3, χ_2 is surjective. Hence χ_2 is a bijection. Also, if Y and Y_1 are nonempty subsets of A, then

$$Y^{\delta_1} \subseteq Y_1^{\delta_1} \Leftrightarrow \chi_2(Y^{\delta_1}) \subseteq \chi_2(Y_1^{\delta_1}).$$

Therefore χ_2 is an isomorphism. □

Lemma 3.5. *Let $Z \in P(G^+)$. Then $\chi_2^{-1}(Z) = Z \cap A$.*

Proof. In view of 3.4, there exists $\emptyset \neq Y \subseteq A$ such that $Z = Y^{\delta_0}$. Then $Z = \chi_2(Y^{\delta_1})$, whence $\chi_2^{-1}(Z) = Y^{\delta_1}$. Thus according to $(*)$, $\chi_2^{-1}(Z) = Z \cap A$. □

Let $y \in A$; we put $Y = \{y\}$. Then $Y^{\delta_1\delta_1}$ is a *principal polar of* $\mathscr{A}$ (generated by y).

Further, let $x \in G^+$ and $X = \{x\}$. The set $X^{\delta_0\delta_0}$ is a *principal polar of* G^+ (generated by x), and $\chi_1^{-1}(X^{\delta_0\delta_0})$ is said to be a principal polar of G.

Consider the isomorphisms

$$P(\mathscr{A}) \xrightarrow{\chi_2} P(G^+) \xrightarrow{\chi_1^{-1}} P(G).$$

For each $X \in P(\mathscr{A})$ we put $\chi_3(X) = \chi_1^{-1}(\chi_2(X))$. In view of the definition of χ_1 and according to 3.5 we have

Lemma 3.5.1. *Let $Z \in P(G)$. Then $\chi_3^{-1}(Z) = Z \cap A$.*

Lemma 3.6. *Let Z be a principal polar of $\mathscr{A}$. Then $\chi_3(Z)$ is a principal polar of G.*

Proof. This is an immediate consequence of the definition of χ_3. □

4. Internal direct factors; strong projectability

Internal direct product decompositions of pseudo effect algebras satisfying the condition (RDP_1) were investigated in [10]. For the present purposes it suffices to consider only finite direct product decompositions.

Again, let $\mathscr{A}$ be a pseudo effect algebra belonging to the class $\mathscr{D}$ and let (G, u) be a unital partially ordered group such that $\mathscr{A} = \Gamma(G, u)$. Further, let $u_1 \in A$; the convex subgroup of G generated by u_1 will be denoted by G_1. Hence u_1 is a strong unit of G_1. Put $\mathscr{A}_1 = \Gamma(G_1, u_1)$. We say that $\mathscr{A}_1$ is an *interval subalgebra* of $\mathscr{A}$. We denote $A_1 = [0, u_1]$.

Now suppose that $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are interval subalgebras of $\mathscr{A}$ such that the following conditions are satisfied (we put $I = \{1, 2, \ldots, n\}$):

(i) for each $a \in A$ there are uniquely determined elements a_i $(i \in I)$ such that $a_i \in A_i$ and $a = a_1 + \ldots + a_n$;

(ii) if a is as above and $a' \in A$, $a' = a'_1 + \ldots + a'_n$ with $a'_i \in A_i$ $(i \in I)$, then $a + a'$ exists in $\mathscr{A}$ iff for each $i \in I$, $a_i + a'_i$ exists in $\mathscr{A}_i$; in that case we have $a + a' = (a_1 + a'_1) + \ldots + (a_n + a'_n)$;

(iii) if a and a' are as in (ii), then $a \leqslant a'$ iff $a_i \leqslant a'_i$ for each $i \in I$.

Under these assumptions we say that $\mathscr{A}$ is an internal direct product of $\mathscr{A}_1, \ldots, \mathscr{A}_n$ and we write

$$\mathscr{A} = (\mathrm{int})\mathscr{A}_1 \times \ldots \times \mathscr{A}_n. \tag{1}$$

The relation (1) is called an internal direct product decomposition of $\mathscr{A}$ and $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are called internal direct factors of $\mathscr{A}$.

For $a \in A$ and $i \in I$, a_i is the *component* of a in $\mathscr{A}_i$; we denote it also by $a(\mathscr{A}_i)$.

An analogous definition can be applied for a partially ordered group G; instead of interval subalgebras of $\mathscr{A}$ we take now convex subgroups of G.

Also, in the same way we define the internal direct product decomposition of the partially ordered semigroup G^+; here, instead of interval subalgebras of $\mathscr{A}$ we consider convex subsemigroups of G^+ containing the element 0.

Under the above mentioned conditions, G is an *internal direct product* of $G_1, \ldots, G_n$; we write

$$G = (\mathrm{int})G_1 \times \ldots \times G_n. \tag{2}$$

The convex subgroups $G_1, \ldots, G_n$ are *internal direct factors* of G; for $g \in G$ the *component* of G in G_i is denoted by g_i or by $g(G_i)$.

Analogous terminology and notation is applied for the partially ordered semigroup G^+. It is easy to verify that if (2) is valid, then

$$G^+ = (\mathrm{int})G_1^+ \times \ldots \times G_n^+. \tag{2'}$$

It can be proved that (iii) is a consequence of (i) and (ii). We will not apply this implication.

We denote by $DF(\mathscr{A})$ the system of all internal direct factors $\mathscr{A}$; the symbols $DF(G)$ and $DF(G^+)$ have analogous meanings with respect to G and G^+.

Let $\mathscr{A}_1$ be an element of $DF(\mathscr{A})$. Then there exists $\mathscr{A}_2 \in DF(\mathscr{A})$ and an internal direct product decomposition

$$\mathscr{A} = (\mathrm{int})\mathscr{A}_1 \times \mathscr{A}_2. \tag{3}$$

For $i \in \{1,2\}$ let A_i be the underlying set of $\mathscr{A}_i$. (Below we apply this convention also in the case when an interval subalgebra of $\mathscr{A}$ is denoted by another symbol).

Lemma 4.1. *Let* (3) *be valid and let $a \in A$. Then the component a_1 of a in $\mathscr{A}_1$ is the greatest element of the set $\{x \in A_1 \colon\ x \leqslant a\}$.*

P r o o f. Under the notation as above, we have $a = a_1 + a_2$. This yields $a_1 \leqslant a_2$. Let $x \in A$, $x \leqslant a$. Under similar notation, $x = x_1 + x_2$. Assume that $x \in A_1$. Then $x_1 = x$ and $x_2 = 0$. Since $x_1 \leqslant a_1$, we obtain $x \leqslant a_1$. □

In view of the condition (ii) above, A_2 is the set of all $t \in A$ such that there exists $a \in A$ with $a = a_1 + t$. Thus we obtain

Lemma 4.2. *If an element $\mathscr{A}_1$ of $DF(\mathscr{A})$ is given and if* (3) *is valid then $\mathscr{A}_2$ is uniquely determined.*

Let us denote by G_1 the convex subgroup of G which is generated by the set A_1, i.e.,

$$G_1 = \bigcup_{n \in \mathbb{N}} [-nu_1, nu_1],$$

where u_1 is the greatest element of A_1. In fact, $u_1 = u(\mathscr{A}_1)$.

We put $\psi_0(\mathscr{A}_1) = G_1$. Then ψ_0 is a mapping of $DF(\mathscr{A})$ into $DF(G)$. Applying 4.2 and [10 (Theorem 6.8, Lemma 7.2 and the construction performed in Section 6)] we conclude

Lemma 4.3. *ψ_0 is an isomorphism of $DF(\mathscr{A})$ onto $DF(G)$. For each $Z \in DF(G)$ we have $\psi_0^{-1}(Z) = Z \cap A$.*

Lemma 4.4. *Let $G = (\mathrm{int})G_1 \times G_2$. Then $G_1^+, G_2^+ \in P(G^+)$ and $(G_1^+)^{\delta_0} = G_2^+$.*

P r o o f. Let $x \in G_1^+$, $y \in G_2^+$. Then in view 3.1 and 4.3 of [10], $x \wedge y = 0$. This yields $G_2^+ \subseteq (G_1^+)^{\delta_0}$. Assume that $z \in (G_1^+)^{\delta_0}$. Put $z_1 = z(G_1)$, $z_2 = z(G_2)$. Hence $z_1 \geqslant 0$ and $z_2 \geqslant 0$; also, $z = z_1 + z_2$. Then $z_1 \in G_1^+$, $z_1 \leqslant z$. In view of the assumption, $z \wedge z_1 = 0$. Since $z \wedge z_1 = z_1$ we obtain $z_1 = 0$, thus $z = z_2 \in G_2$. Summarizing, $(G_1^+)^{\delta_0} = G_2^+$. Hence $G_2^+ \in P(G^+)$. Similarly, $G_1^+ \in P(G^+)$. $\square$

According to 4.4, G_2^+ is an element of $P(G^+)$. Similarly, $G_1^+ \in P(G^+)$. Hence by the definition of $P(G)$ we have

Corollary 4.4.1. *Let $G = (\mathrm{int})G_1 \times G_2$. Then G_1 and G_2 belong to $P(G)$.*

The proof of the next lemma is analogous to that of 4.4.

Lemma 4.5. *Let $\mathscr{A} = (\mathrm{int})\mathscr{A}_1 \times \mathscr{A}_2$. Then $A_1, A_2 \in P(\mathscr{A})$ and $(A_1)^{\delta_1} = A_2$.*

From the fact that χ_3^{-1} is a bijection of $P(G)$ onto $P(\mathscr{A})$ and from 3.5 we obtain

Lemma 4.6. *If $Z_1, Z_2 \in P(G)$ and $Z_1 \cap A = Z_2 \cap A$, then $Z_1 = Z_2$.*

Lemma 4.7. *Let $\mathscr{A}_1 \in DF(\mathscr{A})$. Then $\psi_0(\mathscr{A}_1) = \chi_3(A_1)$.*

P r o o f. Denote $\psi_0(\mathscr{A}_1) = G_1$ and $\chi_3(A_1) = Z$. In view of 4.4.1, $G_1 \in P(G)$. According to 4.3, $A_1 = G_1 \cap A$. Further, 3.5 yields $A_1 = Z \cap A$. Then in view of 4.6, $G_1 = Z$. $\square$

Theorem 4.8. *Let $\mathscr{A}$ be a pseudo effect algebra belonging to the class $\mathscr{D}$. Let (G, u) be a unital partially ordered group with $\mathscr{A} = \Gamma(G, u)$. Then the following conditions are equivalent:*
(i) *$\mathscr{A}$ is strongly projectable;*
(ii) *G is strongly projectable.*

P r o o f. Let (i) be valid and let P be a polar of G. Put $Q = \chi_3^{-1}(P)$. Then $Q \in P(\mathscr{A})$. In view of (i), there exists $\mathscr{A}_1 \in DF(\mathscr{A})$ such that $A_1 = Q$. Denote $\psi_0(\mathscr{A}_1) = G_1$. Hence $G_1 \in DF(G)$. According to 4.6, $P = G_1$. Thus (ii) is valid.

Conversely, assume (ii) holds. Let $Q \in P(\mathscr{A})$. Put $P = \chi_3(Q)$. Hence $P \in P(G)$ and in view of (ii), $P \in DF(G)$. According to 4.3 and 4.7 we have $Q = P \cap A$ and there is $\mathscr{A}_1 \in DF(\mathscr{A})$ such that $A_1 = Q$. Therefore (i) is satisfied. $\square$

Assume that $\mathscr{A}$ and (G, u) are as in 4.6. In the present section we will consider the conditions

(i_0) $\mathscr{A}$ is projectable;

(ii_0) G is projectable.

Lemma 5.1. (ii_0) $\Rightarrow$ (i_0).

Proof. Let (ii_0) be valid and let P be a principal polar in $\mathscr{A}$. Put $Q = \chi_3(P)$. Then in view of 3.6, A is a principal polar of G. According to (ii_0), $Q \in DF(G)$. Hence 4.3 and 4.7 yield that there exists $\mathscr{A}_1 \in DF(\mathscr{A})$ with $A_1 = P$. Therefore (i_0) is valid. □

Let Q be a principal polar of G. The question whether $\chi_3^{-1}(Q)$ is a principal polar of $\mathscr{A}$ remains open. Therefore, to prove the implication (ii_0) $\Rightarrow$ (i_0) we cannot apply the method analogous to that of 4.8.

Let $\mathscr{A}_1 \in \mathscr{D}$, $\mathscr{A}_1 = \Gamma(G_1, u_1)$ for some unital partially ordered group (G_1, u_1) and let $x \in G_1^+$. Let n_1 be the least positive integer with $x \leqslant n_1 u_1$. Then we put

$$n_1 = n(x, \mathscr{A}_1, G_1, u_1).$$

For $0 \leqslant y \in G_1$ we denote

$$\{y\}^{\delta(G_1)} = \{t \in G_1 \colon\ t \wedge y = 0\}$$

where the meaning of the relation $t \wedge y = 0$ is taken with respect to the partially ordered group G_1.

Lemma 5.2. *Under the notation as above and under the assumption that $\mathscr{A}$ is projectable there exists an internal direct product decomposition*

$$G_1 = (\mathrm{int}) G_0 \times G'_{02} \tag{1}$$

such that $\{x\}^{\delta(G_1)} = G'_{02}$.

Proof. The case $x = 0$ being trivial we may suppose that $x > 0$. Then x can be expressed in the form $x = a_1 + \ldots + a_{n_1}$ with $0 \neq a_i \in A$ for $i = 1, 2, \ldots, n_1$. We proceed by induction with respect to n_1. To $\mathscr{A}_1$ and G_1 we apply the notation as introduced above for $\mathscr{A}$ and G.

a) First assume that $n_1 = 1$. Then $x = a_1$. We have

$$\{a_1\}^{\delta(G_1)} = \chi_3(\{a_1\}^{\delta_1(\mathscr{A}_1)}),$$

where the meaning of $\delta_1(\mathscr{A}_1)$ is analogous to that of $\delta(G_1)$. Since $\mathscr{A}_1$ is projectable, $\{a_1\}^{\delta_1(\mathscr{A}_1)}$ belongs to $DF(\mathscr{A}_1)$. Then 4.3 and 4.7 yield that $\chi_3(\{a_1\}^{\delta_1(\mathscr{A}_1)})$ is an element of $DF(G_1)$. Thus there exists $G_0 \in DF(G_1)$ such that (1) is valid, where $G'_{02} = \{a_1\}^{\delta(G_1)}$.

b) Now suppose that $n_1 > 1$. Considering the element a_1 instead of x and using the method from a) we obtain that there exists an internal direct product decomposition

$$G_1 = (\text{int})G_{01} \times G'_{01} \tag{2}$$

such that $\{a_1\}^{\delta(G_1)} = G'_{01}$. This yields $a_1 \in G_{01}$ and hence $a_1(G'_{01}) = 0$. From (2) and from 6.8 in [10] we obtain

$$\mathscr{A}_1 = (\text{int})\mathscr{A}_{01} \times \mathscr{A}'_{01}, \tag{3}$$

where $\mathscr{A}_{01}$ and $\mathscr{A}'_{01}$ are interval subalgebras of $\mathscr{A}$ with the underlying sets $A_{01} = A_1 \cap G_{01}$ and $A'_{01} = A_1 \cap G'_{01}$, respectively.

Put $y = x(G'_{01})$, $y_2 = a_2(G'_{01})$, …, $y_n = a_{n1}(G'_{01})$. We have $y = y_2 + \ldots + y_{n_1}$ and according to (3), all elements $y_2, \ldots, y_{n_1}$ belong to A'_{01}.

Let u_{01} be the component of u_1 in G'_{01}. Hence we obtain

$$n(y, \mathscr{A}'_{01}, G'_{01}, u_{01}) \leqslant n_1 - 1.$$

Thus from the induction hypothesis we get that there exists an internal direct product decomposition

$$G'_{01} = (\text{int})G_{02} \times G'_{02} \tag{4}$$

such that

$$\{y\}^{\delta_{01}} = G'_{01}, \tag{5}$$

where the meaning of δ_{01} is analogous to that of δ with the distinction that instead of G_1 we now have G'_{01}.

The relations (2) and (4) yield

$$G_1 = (\text{int})G_{01} \times G_{02} \times G'_{02}. \tag{6}$$

Thus we obtain

$$G_1 = (\text{int})G_0 \times G'_{02},$$

where $G_0 = (\mathrm{int})G_{01} \times G_{02}$. We have to verify that the relation

$$\{x\}^{\delta(G_1)} = G'_{02} \tag{7}$$

is valid.

In view of (2) we have

$$x = x(G_{01}) + x(G'_{01}).$$

Hence

$$x = x(G_{01}) + y, \quad y \in G'_{01}. \tag{8}$$

Then (4) yields

$$y = y(G'_{01}) = y(G_{02}) + y(G'_{02}).$$

In view of (5) we obtain $y(G'_{02}) = 0$, whence $y = y(G_{02})$ and so $y \in G_{02}$. Thus $y \in G_0$. Also, $x(G_{01}) \in G_{01} \subseteq G_0$. Therefore according to (8), $x \in G_0$.

We have already shown that (1) is valid, whence $x \wedge t = 0$ in G_1 for each $t \in (G'_{02})^+$. We obtain

$$G'_{02} \subseteq \{x\}^{\delta(G_1)}. \tag{9}$$

Now let $z \in \{x\}^{\delta(G_1)}$, $z \geqslant 0$. By way of contradiction, assume that $z \notin G'_{02}$. Then in view of (1) $z(G_0) > 0$. Put $z(G_0) = z_0$. Since $z_0 \in G_0$, we get

$$z_0 = z_0(G_{01}) + z_0(G_{02}).$$

Therefore either $z_0(G_{01}) > 0$ or $z_0(G_{02}) > 0$.

Suppose that $z_0(G_{01}) > 0$. According to the relation between a_1 and G_{01} and in view of the construction described in Section 6 of [10] we have

$$G_{01} = \bigcup_{n \in \mathbb{N}} [-na_1, na_1].$$

Hence there exists $n \in \mathbb{N}$ with

$$0 < z_0(G_{01}) \leqslant na_1.$$

Applying (RDP_0) and induction we obtain that there exists $0 < z^1 \in G$ with $z^1 \leqslant z_0(G_{01})$, $z^1 \leqslant a_1$. This yields $z^1 \leqslant z$ and $z^1 \leqslant x$, hence $z \notin \{x\}^{\delta(G_1)}$, which is a contradiction. Therefore $z_0(G_{01}) = 0$.

Then we have $z_0 \in G_{02} \subseteq G'_{01}$. Since $z_0 \leqslant z$, we get $z_0 \in \{x\}^{\delta(G_1)}$. Moreover, because of $z \leqslant x$ we obtain $z_0 \in \{y\}^{\delta_{01}}$, where the meaning of δ_{01} is as in (5). But then $z_0 \in G_0 \cap G'_{02} = \{0\}$, which is a contradiction. This completes the proof. $\square$

If (1) is valid, then $(G'_{02})^{\delta(G_1)} = G_0$. From this and from 5.2 we obtain

Corollary 5.3. *Under the assumption and notation as in* 5.2 *we have*

$$\{x\}^{\delta(G_1)\delta(G_1)} = G_0.$$

As an immediate consequence we get

Theorem 5.4. *Let $\mathscr{A}$ be a pseudo effect algebra belonging to the class $\mathscr{D}$. Suppose that $\mathscr{A}$ is projectable and that $\mathscr{A} = \Gamma(G, u)$. Then the partially ordered group G is projectable.*

We conclude that under the notation as in Lemma 5.1, the relation $(\mathrm{ii}_0) \Rightarrow (\mathrm{i}_0)$ is valid.

6. Weak homogeneity

Similarly to the previous sections, we assume that $\mathscr{A}$ is a pseudo effect algebra belonging to the class $\mathscr{D}$ and that $\mathscr{A} = \Gamma(G, u)$ for some unital partially ordered group (G, u).

Example. Let G be the set of all real functions on a finite set T; the operation $+$ and the partial order in G are defined componentwise. Let $u \in G$ such that $u(t) = 1$ for each $t \in T$. Put $\mathscr{A} = \Gamma(G, u)$. Both G and $\mathscr{A}$ are weakly homogeneous.

From the definition of weak homogeneity we immediately obtain

Lemma 6.1. *Assume that G is weakly homogeneous. Then $\mathscr{A}$ is weakly homogeneous.*

Example. Let Z have the usual meaning; put $u = 1$ and $\mathscr{A} = \Gamma(G, u)$. Hence $A = \{0, 1\}$ and thus $\mathscr{A}$ is weakly homogeneous. On the other hand, G fails to be weakly homogeneous.

We will prove that there are no examples of this type in the case when $\operatorname{card} A > 2$; i.e., we have

Theorem 6.2. *Let $\mathscr{A} \in \mathscr{D}$, $\mathscr{A} = \Gamma(G, u)$, $\operatorname{card} A > 2$. Then $\mathscr{A}$ is weakly homogeneous if and only if G is weakly homogeneous.*

We need some auxiliary results.

Lemma 6.3. *Let* $[p, q]$ *be an interval in* G, $\operatorname{card}[p, q] = \alpha > 1$. *Then there exists* $0 < a \in A$ *such that* $\operatorname{card}[0, a] \leqslant \alpha$.

P r o o f. We have $\operatorname{card}[p, q] = \operatorname{card}[0, q-p]$ and $q-p > 0$. There exists a positive integer n such that $q - p \leqslant nu$. Hence there are $x_1, x_2, \ldots, x_n \in [0, u]$ such that $q - p = x_1 + x_2 + \ldots + x_n$. Without loss of generality we can assume that $x_1 > 0$. Put $x_1 = a$. We have $0 < a \in A$ and $a \leqslant q-p$. Hence $\operatorname{card}[0, a] \leqslant \operatorname{card}[0, q-p] = \alpha$. □

Lemma 6.4. *The following conditions are equivalent:*
(i) *all nontrivial intervals in* G *are infinite;*
(ii) *all nontrivial intervals in* $\mathscr{A}$ *are infinite.*

P r o o f. The validity of the inplication (i) $\Rightarrow$ (ii) is obvious. The relation (ii) $\Rightarrow$ (i) is a consequence of 6.3. □

Lemma 6.5. *Assume that* $\operatorname{card} A > 2$ *and that* $\mathscr{A}$ *is weakly homogeneous. Then each nontrivial interval of* $\mathscr{A}$ *is infinite.*

P r o o f. By way of contradiction, assume that $[p, q]$ is a finite nontrivial interval in $\mathscr{A}$. Put $q' = q - p$. Then $0 < q' \in A$ and the interval $[0, q']$ of $\mathscr{A}$ is finite. Hence there exists an atom q_1 in $[0, q']$. In view of the weak homogeneity we obtain $2 = \operatorname{card}[0, q_1] = \operatorname{card}[0, u] = A$, which is a contradiction. □

Lemma 6.6. *Let* α *be an infinite cardinal. Assume that each nontrivial interval of* $\mathscr{A}$ *has cardinality* α. *Then each nontrivial interval of* G *has cardinality* α *as well.*

P r o o f. a) We have $\operatorname{card}[0, u] = \alpha$. Denote $\operatorname{card}[0, 2u] = \beta$. Since $[0, u] \subseteq [0, 2u]$ we get $\alpha \leqslant \beta$. If $x_1, x_2 \in [0, u]$, then $x_1 + x_2 \in [0, 2u]$. Let $x \in [0, 2u]$. There are $x_1, x_2 \in [0, u]$ such that $x_1 + x_2 = x$. Consider the mapping $\varphi\colon\ [0, u] \times [0, u] \to [0, 2u]$ defined by $\varphi(x_1, x_2) = x_1 + x_2$. Hence φ is surjective. This yields

$$\alpha = \alpha^2 = \operatorname{card}([0, u] \times [0, u]) \geqslant \operatorname{card}[0, 2u] = \beta.$$

Therefore $\beta = \alpha$. By the obvious induction we then obtain $\operatorname{card}[0, nu] = \alpha$ for each $n \in \mathbb{N}$.

b) Let $[p, q]$ be a nontrivial interval in G; put $q' = q-p$. Let $\operatorname{card}[p, q] = \beta$. Hence $\operatorname{card}[0, q'] = \beta$. There exists $n \in \mathbb{N}$ with $q' \leqslant nu$. Then $[0, q'] \subseteq [0, nu]$ and thus, in view of a), $\beta \leqslant \alpha$. Further, according to 6.3 we obtain $\alpha \leqslant \beta$. Therefore $\alpha = \beta$. □

Lemma 6.7. *Let* card $A > 2$ *and assume that* $\mathscr{A}$ *is weakly homogeneous. Then* G *is weakly homogeneous.*

P r o o f. In view of 6.5, there is an infinite cardinal α such that each nontrivial interval of $\mathscr{A}$ has cardinality α. According to 6.6, the same holds for G; hence G is weakly homogeneous. $\square$

In view of 6.7 and 6.1 we conclude that 6.2 is valid.

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