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PROJECTABLE ALMOST COMPLEX CONTACT STRUCTURES

BY D. E. BLAIR, S. ISHIHARA AND G.D. LUDDEN

A complex manifold of complex dimension $2m+1$ is said to be a *complex contact manifold* if it admits an open covering $\{u_\alpha\}$ such that on each u_α there is a holomorphic 1-form ω_α with $\omega_\alpha \wedge (d\omega_\alpha)^m \neq 0$ on $u_\alpha \cap u_\beta \neq \emptyset$, $\omega_\beta = f\omega_\alpha$ for some non-vanishing holomorphic function f. In general such a structure is not given by a global 1-form ω; in fact this is the case for a compact complex manifold if and only if its first Chern class vanishes [6]. However, a complex contact manifold is the base space of a principal fibre bundle with 1-dimensional fibres and real contact structure. Homogeneous complex contact manifolds were studied by Boothby in [3].

It is also shown in [6] that the structural group of the tangent bundle of a Hermitian contact manifold M is reducible to $(Sp(m)\cdot Sp(1))\times U(1)$ where $Sp(m)\cdot Sp(1)=Sp(m)\times Sp(1)/\{\pm I\}$ and hence equivalently M admits the following local structure tensors. Let F denote the almost complex structure and g the Hermitian metric on M. Then each coordinate neighborhood admits tensor fields G, H of type (1, 1) and vector fields U, V with covariant forms u and v such that (G, U, V, g) and (H, U, V, g) are metric f-structures with complemented frames (see e.g. [1]), $FU=V$ and $GH=-HG=F+v\otimes U-u\otimes V$. In the overlap of coordinate neighborhoods we have

$$\begin{aligned} G' &= aG+bH, & u' &= au+bv, \\ H' &= -bG+aH, & v' &= -bu+av \end{aligned} \tag{0.1}$$

with $a^2+b^2=1$. Such a structure is called an *almost complex contact structure* [5] and our first project here will be to given an equivalent definition in terms of global tensor fields.

A standard example of a complex contact manifold is the odd-dimensional complex projective space PC^{2m+1}. It is also well known that PC^{2m+1} is a fibre space over the quaternionic projective space PH^m with fibres $S^2 \approx PC^1$. In sections 3 and 4 we generalize this situation to a projectable almost complex contact structure on a Kählerian manifold.

§ 1. Almost Complex Contact Structures

In terms of the above local tensor fields G, H, U, V we can define global

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tensor fields Σ of type (1, 3) and S of type (1, 1). For local vector fields X, Y, Z set

$$\Sigma_{XY}Z=g(GX, Y)GZ+g(HX, Y)HZ \tag{1.1}$$

and

$$SX=u(X)U+v(X)V\,. \tag{1.2}$$

It is then easy to check using equations (0.1) that Σ and S are globally defined. Note also that S is a projection tensor field of rank 2, i.e. $S^2=S$. For a unit vector $A\in T_pM$ with $SA=0$, let

$$\sigma_A=\{B\in T_pM\,|\,g(A, B)=0,\ \|B\|=1,\ \underline{\Sigma}(A, B, A, B)=1\}\,,$$

where $\underline{\Sigma}(X, Y, Z, W)=g(\Sigma_{XY}Z, W)$ and $[\sigma_A]$ the subspace of T_pM generated by σ_A.

The following properties of Σ and S are now easily deduced. 1)-8) are straightforward computations using equations (1.1) and (1.2) and elementary properties of metric f-structures. For 9) given A set $B=GA$ and it is easy to see that $B\in\sigma_A$.

1) $SF=FS$
2) $\Sigma_{XY}=-\Sigma_{YX}$
3) $\Sigma^2_{XY}=\underline{\Sigma}(X, Y, X, Y)(-I+S)$
4) $\Sigma_{XY}S=S\Sigma_{XY}=0$
5) $\Sigma_{XYF}=-F\Sigma_{XY}$
6) $\Sigma_{XFY}F=\Sigma_{XY}$
7) $\underline{\Sigma}(X, Y, Z, W)=\underline{\Sigma}(Z, W, X, Y)$
8) $\Sigma_{X\Sigma_{YZ}X}W=g(X, (I-S)X)\Sigma_{YZ}W$
9) $\sigma_A\neq\emptyset$ for any unit vector A with $SA=0$ and at any point p of M.

Conversely we will show that an almost Hermitian manifold M with structure tensors (F, g) admitting global tensor fields Σ and S satisfying 1)-9) is an almost complex contact manifold. We first give several lemmas.

LEMMA 1.1. *For $B\in\sigma_A$, $\Sigma_{AB}A=B$, $SB=0$ and σ_A is invariant under F.*

Proof. Since $\underline{\Sigma}(A, B, A, B)=g(\Sigma_{AB}A, B)=1$ to show that $\Sigma_{AB}A=B$ it suffices to show that $\Sigma_{AB}A$ is a unit vector.

$$g(\Sigma_{AB}A, \Sigma_{AB}A)=-g(\Sigma^2_{AB}A, A)=-g(-A+SA, A)=1$$

by 2), 7) and 3), since A is a unit vector and $SA=0$. Now $SB=S\Sigma_{AB}A=0$ by 4). Finally for the invariance by F,

$$\Sigma_{AFB}A=-\Sigma_{AFB}F^2A=F\Sigma_{AB}A=FB\,,$$

from which $\underline{\Sigma}(A, FB, A, FB)=1$ and $g(FB, A)=g(\Sigma_{AFB}A, A)=0$.

LEMMA 1.2. *For any unit vector $B\in\sigma_A$ set $C=FB\in\sigma_A$, then*

$$\Sigma_{AB}\Sigma_{AC}=F-SF\,.$$

Proof. If we take an arbitrary vector $D\in T_pM$ then, using (3) and (6), we have

$$\Sigma_{AB}\Sigma_{AC}D=\Sigma_{AB}\Sigma_{AFB}D=-\Sigma_{AB}\Sigma_{AFB}F^2D$$
$$=-\Sigma_{AB}\Sigma_{AB}FD=(I-S)FD\,.$$

LEMMA 1.3. *For any orthonormal pair* $\{B, C\}\in\sigma_A$,

$$\Sigma_{AB}\Sigma_{AC}=-\Sigma_{AC}\Sigma_{AB}\,.$$

Proof. First, using (3), we have

$$\Sigma^2_{AB+C}=\Sigma(A, B+C, A, B+C)(-I+S)$$
$$=2(-I+S)\,.$$

On the other hand, we obtain

$$\Sigma^2_{AB+C}=(\Sigma_{AB}+\Sigma_{AC})^2$$
$$=2(-I+S)+(\Sigma_{AB}\Sigma_{AC}+\Sigma_{AC}\Sigma_{AB})\,.$$

Thus we have $\Sigma_{AB}\Sigma_{AC}+\Sigma_{AC}\Sigma_{AB}=0$.

LEMMA 1.4. $\dim[\sigma_A]=2$.

Proof. Take B and C as in Lemma 1.1 and assume that there is a unit vector $D\in[\sigma_A]$ such that D is orthogonal to B and C. They by Lemmas 1.2 and 1.3 we have

$$\Sigma_{AB}\Sigma_{AC}\Sigma_{AD}=\Sigma_{AD}\Sigma_{AB}\Sigma_{AC}\,,$$

and so

$$(F-SF)\Sigma_{AD}=\Sigma_{AD}(F-SF)\,.$$

Thus, using (1) and (4), we obtain

$$F\Sigma_{AD}=\Sigma_{AD}F\,,$$

which contradicts (5). Therefore, $[\sigma_A]$ is necessarily of dimension 2.

LEMMA 1.5. *For any vectors* $B, C\in T_pM$, *satisfying* $\underline{\Sigma}(B, C, B, C)=1$, $\Sigma_{BC}A\in\sigma_A$.

Proof. Using (8), we have

$$\Sigma_{A\Sigma_{BC}A}A=\Sigma_{BC}A\,,$$

from which it follows that $\Sigma_{BC}A\in\sigma_A$.

LEMMA 1.6. *Take a unit vector $A \in T_pM$ with $SA=0$ and a unit vector $B \in \sigma_A$. Put $C=FB \in \sigma_A$. The $\Sigma_{AB}D$ and $\Sigma_{AC}D$ are orthonormal, where D is an arbitrary unit vector at p such that $SD=0$.*

Proof. $g(\Sigma_{AB}D, \Sigma_{AB}D)=-g(\Sigma^2_{AB}D, D)=g(D-SD, D)=1$ and similarly $\Sigma_{AC}D$ is also a unit vector. Finally

$$g(\Sigma_{AB}D, \Sigma_{AC}D)=-g(\Sigma_{AB}D, \Sigma_{AFB}F^2D)=-g(\Sigma_{AB}D, \Sigma_{AB}FD)$$
$$=g(\Sigma_{AB}D, F\Sigma_{AB}D)=0\,.$$

Summing up Lemmas 1.4, 1.5 and 1.6, we have

PROPOSITION 1. *Take a unit vector $A \in T_pM$ such that $SA=0$ and a unit vector $B \in \sigma_A$. Put $C=FB \in \sigma_A$. Then, for any unit vector $D \in T_pM$ with $SD=0$, $\Sigma_{AB}D$ and $\Sigma_{AC}D$ form an orthonormal basis of $[\sigma_D]$.*

LEMMA 1.7. *Take A, B and C as in Proposition 1. Then, for any $D, E \in T_pM$,*

$$\Sigma_{DE}=\underline{\Sigma}(A, B, D, E)\Sigma_{AB}+\underline{\Sigma}(A, C, D, E)\Sigma_{AC}\,.$$

Proof. When D (or E) satisfies $SD=D$ (or $SE=E$), then both sides of the equation above vanish because of (4). So, D and E may be assumed to satisfy $SD=SE=0$ and also that D and E are unit. First, we consider the case in which E is orthogonal to σ_D. Linearizing (3) we have $\Sigma_{XY}\Sigma_{XZ}+\Sigma_{XZ}\Sigma_{XY}=2\underline{\Sigma}(X, Y, X, Z)(-I+S)$. Thus if $Y \in \sigma_D$, Σ_{DE} anti-commutes with Σ_{DY} and Σ_{DFY} and hence Σ_{DE} commutes with $\Sigma_{DY}\Sigma_{DFX}$ which by Lemma 1.2 is equal to $F-SF$. Therefore using (1) and (4)

$$F\Sigma_{DE}=(F-SF)\Sigma_{DE}=\Sigma_{DE}(F-SF)=\Sigma_{DE}F\,,$$

from which by (5) and the non-singularity of F we have $\Sigma_{DE}=0$ and again both sides of the above equation vanish.

Finally we consider the case where $E \in \sigma_D$. For simplicity set $a=g(\Sigma_{AB}D, E)$ and $b=g(\Sigma_{AC}D, E)$. Then as $\{\Sigma_{AB}D, \Sigma_{AC}D\}$ is an orthonormal basis of $[\sigma_D]$,

$$E=a\Sigma_{AB}D+b\Sigma_{AC}D\,.$$

Using (8) we have

$$\Sigma_{DE}A=a\Sigma_{D\Sigma_{AB}D}A+b\Sigma_{D\Sigma_{AC}D}A$$
$$=a\Sigma_{AB}A+b\Sigma_{AC}A$$
$$=aB+bC\,.$$

Using (8) again

$$\Sigma_{DE}=\Sigma_{A\Sigma_{DE}A}=a\Sigma_{AB}+b\Sigma_{AC}\,,$$

which is the desired formula.

Take a suitable coordinate neighborhood u of an arbitrary point p of M and a unit vector field A in u. Then there is in u a unit vector field B belonging to σ_A at each point of u. On putting $C=FB\in\sigma_A$ we define locally in u two tensor fields G and H of type (1, 1) respectively by

$$G=\Sigma_{AB}, \quad H=\Sigma_{AC}.$$

Then setting $F^H=F-FS$ and using (3) and (4) and Lemma 1.3, we have

$$\begin{aligned}&(F^H)^2=G^2=H^2=-I+S,\\ &GH=-HG=F^H,\ HF^H=-F^HH=G,\ F^HG=-GF^H=H,\\ &F^HS=SF^H=GS=SG=HS=SH=0.\end{aligned} \tag{1.3}$$

Next, (1), (2), and (7) imply

$$\begin{aligned}&g(F^HX, Y)=-g(F^HY, X),\\ &g(GX, Y)=-g(GY, X),\ g(HX, Y)=-g(HY, X),\end{aligned}$$

for all X and Y. By Lemma 1.7, a local expression for Σ_{XY} in u is the following

$$\Sigma_{XY}=g(GX, Y)G+g(HX, Y)H. \tag{1.4}$$

We now take another coordinate neighborhood $u'(u\cap u'\neq\emptyset)$ and define G' and H' as in u, say $G'=\Sigma_{A'B'}$ and $H'=\Sigma_{A'C'}$. By the formula of Lemma 1.7

$$\begin{aligned}&\Sigma_{A'B'}=\underline{\Sigma}(A, B, A', B')\Sigma_{AB}+\underline{\Sigma}(A, C, A', C')\Sigma_{AC}.\\ &\Sigma_{A'C'}=\underline{\Sigma}(A, B, A', C')\Sigma_{AB}+\underline{\Sigma}(A, C, A', C')\Sigma_{AC}.\end{aligned}$$

Setting $a=\underline{\Sigma}(A, B, A', B')$ and $b=\underline{\Sigma}(A, C, A', B')$ we have that

$$1=g(\Sigma_{A'B'}A', B')=\underline{\Sigma}(A, B, A', B')^2+\underline{\Sigma}(A, C, A', B')^2=a^2+b^2$$

and

$$\begin{aligned}\underline{\Sigma}(A, C, A', C')&=-g(\Sigma_{AFB}F^2A', FB')=-g(\Sigma_{AB}FA', FB')\\ &=g(F\Sigma_{AB}A', FB')=g(\Sigma_{AB}A', B')=a,\end{aligned}$$

$$\begin{aligned}\underline{\Sigma}(A, B, A', C')&=-g(\Sigma_{AFB}FA', F^2C')=-g(F\Sigma_{AC}A', FB')\\ &=-g(\Sigma_{AC}A', B')=-b,\end{aligned}$$

so that $G'=aG+bH$ and $H'=-bG+aH$.

THEOREM 1. *Let (M, G, F) be an almost Hermitian manifold. Then M is an almost complex contact manifold if and only if M admits a global tensor field Σ of type* (1, 3) *and a projection tensor field S of rank* 2 *satisfying* 1)-9).

§ 2. Horizontal and Vertical Tensors

Given a vector field X on an almost Hermitian manifold (M, g, F) with almost complex contact structure (g, F, Σ, S), $X^V=SX$ and $X^H=X-X^V$ will be called the *vertical* and the *horizontal parts* of X, respectively. For a 1-form ω, $\omega^V=\omega\circ S$ and $\omega^H=\omega-\omega^V$ will be called the *vertical* and the *horizontal parts* of ω, respectively. We now define, for a function $f, f^H=f^V=f$. Then we easily have

$$(fX+hY)^H=f^HX^H+h^HY^H,\ (fX+hY)^V=f^VX^V+h^VY^V, \tag{2.1}$$
$$(f\omega+h\pi)^H=f^H\omega^H+h^H\pi^H,\ (f\omega+h\pi)^V=f^V\omega^V+h^V\pi^V,$$

where f, h are arbitrary functions and ω, π are arbitrary 1-forms.

We now define the *horizontal* part T^H of an arbitrary tensor field T. Assume that the operation of taking the horizontal part satisfies

$$(P+Q)^H=P^H+Q^H,\ (P\otimes U)^H=P^H\otimes U^H, \tag{2.2}$$

where P and Q are arbitrary tensor fields of the same type and U another arbitrary tensor field, then by using (2.1) we can inductively define the horizontal part T^H of an arbitrary tensor field T on M.

§ 3. Almost Complex Contact Structures which are Projectable

The Riemannian connection is denoted by ∇ in a Kählerian manifold M with almost complex contact structure (g, F, Σ, S). We define a tensor field P of type (1, 2) by

$$P_XY=((\nabla_YS)X)^H. \tag{3.1}$$

Note that

$$SP_X=0. \tag{3.2}$$

Next, differentiating covariantly $S^2=S$ we have

$$P_{SX}=P_X \tag{3.3}$$

and differentiating covariantly (1)

$$P_{FX}=FP_X. \tag{3.4}$$

LEMMA 3.1. *When $P=0$, a Kählerian manifold M of complex dimension $2m+1$ with almost complex contact structure (g, F, Σ, S) is locally a product of two Kählerian manifolds of complex dimensions $2m$ and 1 respectively.*

Proof. If $P=0$, (3.1) implies

$$(\nabla_Y(SX))^H=((\nabla_YS)X)^H=0,$$

which means that the distribution determined by S and its complement are parallel. This with $SF=FS$ proves the lemma.

We now consider the following conditions:

(P1) for any vector $A\in T_pM$, there are two vectors $B, C\in T_pM$ such that $P_A=\Sigma_{BC}$, $\underline{\Sigma}(B, C, B, C)=ag(SA, SA)$ with constant a;

(P2) $(\nabla_{SX}\underline{\Sigma})^H=0$.

When an almost complex contact structure (g, F, Σ, S) satisfies the conditions (P1) and (P2), it is said to be *projectable.*

In this section, the almost complex contact structure (g, F, Σ, S) is assumed to be projectable. Then 3)-4) and (P1) imply

$$P_X{}^2=ag(SX, SX)(-I+S) \tag{3.5}$$

for some a and

$$P_XS=0 \tag{3.6}$$

is equivalent to

$$S(\nabla_{SY}S)=\nabla_{SY}S\,. \tag{3.7}$$

Thus we now have from (1) and (3.7)

PROPOSITION 2. *In a Kählerian manifold M with almost complex contact structure (g, F, Σ, S) which satisfies* (P1), *the distribution determined by S is integrable and each of its integral submanifolds is totally geodesic and holomorphic.*

Since (P1) is satisfied, restricting ourselves to a coordinate neighborhood u in which (1.4) is established, we find

$$P_X=c(u(X)G+v(X)H) \tag{3.8}$$

with local 1-forms u and v defined in u, where the associated vector fields U of u and V of v satisfy $\|U\|^2=\|V\|^2=1$, $g(U, V)=0$, i. e.,

$$S=u\otimes U+v\otimes V\,. \tag{3.9}$$

(3.8) implies that

$$(\nabla(SX))^H=c(u(x)G+v(x)H)\,. \tag{3.10}$$

The fundamental 2-form Φ of the Kählerian manifold (M, g, F) is defined by $\Phi(X, Y)=g(FX, Y)$. We now define in M a tensor field $\underline{\Lambda}$ of type (0, 4) by

$$\underline{\Lambda}=\Phi^H\otimes\Phi^H+\underline{\Sigma}\,, \tag{3.11}$$

which is horizontal, that is, $\underline{\Lambda}^H=\underline{\Lambda}$. Then, using (1.3) and (3.8), we can verify that in u

$$P_U\cdot\underline{A}=0\,,\quad P_V\cdot\underline{A}=0\,,$$

where $P_X\cdot$ denotes the action of P_X as a derivation. Thus, using (3.9), we obtain

$$P_X\cdot\underline{A}=0\,. \tag{3.13}$$

Since $\nabla F=0$, we find

$$(\nabla_{SX}\underline{A})^H=0 \tag{3.14}$$

as a consequence of (P2). As is well known, the Lie derivative $\mathcal{L}_{SX}\underline{A}$ is given by

$$\mathcal{L}_{SX}\underline{A}=\nabla_{SX}\underline{A}+P_X\cdot\underline{A}$$

(See e.g. Yano [8]). Thus we have

$$(\mathcal{L}_{X^V}\underline{A})^H=0\,. \tag{3.15}$$

LEMMA 3.2. *If an almost complex contact structure (g, F, Σ, S) is projectable, then*

$$(\mathcal{L}_{X^V}\underline{A}^H)^H=0\,.$$

On the other hand, by Proposition 2, each integral submanifold of the distribution determined by S is totally geodesic. Thus we have (see Ishihara and Konishi [5])

LEMMA 3.3. *If an almost complex contact structure (g, F, Σ, S) is projectable, then*

$$(\mathcal{L}_{X^V}g^H)^H=0\,.$$

We now put

$$A=\Phi\otimes F+\Sigma\,.$$

Then Lemmas 3.2 and 3.3 imply

LEMMA 3.4. *If an almost complex contact structure (g, F, Σ, S) is projectable, then*

$$(\mathcal{L}_{X^V}A^H)^H=0\,.$$

§ 4. Submersion of a Kählerian Manifold with Almost Complex Contact Structure

Let (M, G, F) be a Kählerian manifold of complex dimension $2m+1$ with almost complex contact structure (g, F, Σ, S), which is projectable, and $\tilde{M}$ a manifold of real dimension $4m$. Suppose that there is a differential mapping $\pi: M\to\tilde{M}$ which is of rank $4m$ everywhere and satisfies $\pi(M)=\tilde{M}$ and that for each point p of $\tilde{M}$, $\pi^{-1}(p)$ is a connected integral submanifold of the distribution determined by S. In such a case, the Kählerian manifold M with almost complex contact structure is said to have a *fibred Riemannian structure* $\pi: M \to \tilde{M}$ and $\tilde{M}$ is called the *base space*. When M is compact and the distribution $\mathcal{D}$ determined by S is regular, M has a fibred Riemannian structure if $\tilde{M}$ is defined as the set of all maximal integral submanifolds of $\mathcal{D}$, $\pi: M \to \tilde{M}$ being defined by $\pi(p) = \mathcal{D}p$, $p \in M$, where $\mathcal{D}p$ is the maximal integral submanifold passing through p, and $\tilde{M}$ is naturally topologized.

Consider a Kählerian manifold M with almost complex contact structure (g, F, Σ, S), which is projectable, and with fibred Riemannian structure $\pi: M \to \tilde{M}$. Then, taking account of arguments developed in [5], we see by Lemma 3.4 that the tensor field Λ is projectable in M and its projection is a tensor field $\tilde{\Lambda}$ of type $(1, 3)$ in the base space $\tilde{M}$. The metric tensor g in M is, by Lemma 3.3, projectable and its projection $\tilde{g}$ defines a Riemannian structure on $\tilde{M}$. Thus, (2)-(9) implies that $(\tilde{g}, \tilde{\Lambda})$ is an almost quaternionic structure in the base space $\tilde{M}$ (see Blair and Showers [2]). Thus, summing up, we have

THEOREM 2. *Suppose that (M, g, F) is a Kählerian manifold with almost complex contact structure (g, F, Σ, S), which is projectable. Assume moreover that (M, g, F) has a fibred Riemannian structure $\pi: M \to \tilde{M}$. Then $(\tilde{g}, \tilde{\Lambda})$ is an almost quaternionic structure in the base space $\tilde{M}$, where $\tilde{g}$ and $\tilde{\Lambda}$ are the projections of g and Λ, respectively.*

If in a Kählerian manifold M satisfying the conditions given in Theorem 2

$$(\nabla \Lambda^H)^H = 0$$

holds, then the projection $\tilde{\Lambda}$ of Λ in $\tilde{M}$ is covariantly constant. Thus in such a case $(\tilde{g}, \tilde{\Lambda})$ is a quaternionic Kählerian structure (see Ishihara [4]). Thus we have

THEOREM 3. *If, an a Kählerian manifold M satisfying the conditions given in Theorem 2, $(\nabla \Lambda^H)^H = 0$, then $(\tilde{g}, \tilde{\Lambda})$ is a quaternionic Kählerian structure in the base space $\tilde{M}$.*

Taking account of Lemma 3.1, we easily have

PROPOSITION 3. *If a Kählerian manifold M of complex dimension $2m+1$ with almost complex contact structure (g, F, Σ, S), which is projectable, satisfies the condition $P=0$, then M is locally a product of Kählerian manifolds (M_1, g_1, F_1) of complex dimension $2m$ and (M_2, g_2, F_2) of complex dimension 1, where M_1 admits quaternion structure (g_1, Λ_1).*

PROPOSITION 4. *If, in a Kählerian manifold M satisfying the conditions given in Proposition 3 $(\nabla \Lambda^H)^H = 0$ then M_1 admits a quaternionic Kählerian structure (g_1, Λ_1) with vanishing Ricci tensor (see Ishihara [4]).*

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