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PROJECTED PRODUCTS OF POLYGONS

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Abstract. It is an open problem to characterize the cone of f-vectors of 4-dimensional convex polytopes. The question whether the "fatness" of the f-vector of a 4-polytope can be arbitrarily large is a key problem in this context.

Here we construct a 2-parameter family of 4-dimensional polytopes $\pi(P_n^{2r})$ with extreme combinatorial structure. In this family, the "fatness" of the f-vector gets arbitrarily close to 9; an analogous invariant of the flag vector, the "complexity," gets arbitrarily close to 16.

The polytopes are obtained from suitable deformed products of even polygons by a projection to $\mathbb{R}^4$.

1. Introduction

1.1. ***f*-vectors.** The combinatorial structure of a d-dimensional convex polytope is given by the incidences between its k-dimensional faces, for $0 \le k \le d-1$. In particular one looks at the faces of dimensions 0, 1, $d-2$, and $d-1$, known as the *vertices*, *edges*, *ridges*, and *facets*, respectively.

To get an overview over enumerative and extremal properties of the multitude of combinatorial types of d-polytopes, one tries to classify their *f-vectors*, that is, the d-tuples

$$f(P) \ := \ (f_0, f_1, \dots, f_{d-2}, f_{d-1}) \ \in \mathbb{Z}^d,$$

where f_k denotes the number of k-dimensional faces of the d-polytope P. The f-vector of any d-polytope is a point in $\mathbb{R}^d$, but due to the Euler–Poincaré relation

$$f_0 - f_1 \pm \dots + (-1)^{d-1} f_{d-1} \ = \ 1 - (-1)^d$$

the set of all f-vectors of d-polytopes

$$\mathcal{F}_d \ := \ \big\{ f = (f_0, f_1, \dots, f_{d-1}) \in \mathbb{Z}^d : f = f(P) \text{ for a } d\text{-polytope } P \big\}$$

has dimension $d-1$ [7, Chapter 8].

The f-vector (and more so the flag vector of a d-polytope, as discussed below) not only provides numerical data: it also encodes various extremal properties. So any attempt to characterize the f-vectors of polytopes is closely linked to the analysis and construction of extremal polytopes.

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For example, a d-polytope is simplicial if and only if its f-vector satisfies $f_{d-2} = \frac{d}{2} f_{d-1}$. Indeed, each facet of a d-polytope is bounded by at least d ridges, while every ridge is in exactly 2 facets. Hence $2f_{d-2} \leq d\, f_{d-1}$ is valid for all polytopes, and the constraint is tight exactly if all facets are simplices. Thus simplicial polytopes are *extremal* in the sense that they maximize a (linear) quantity among all f-vectors: they maximize $2f_{d-2} - d\, f_{d-1}$.

In the study of f-vectors of d-polytopes, one tries to find all such linear inequalities for the f-vectors, and to understand the extremal polytopes for which these inequalities are tight.

My lecture notes [19] contain a more extensive discussion of the interplay between f-vector theory and constructions of extremal polytopes that arises from this.

1.2. **Flag vectors.** Additional combinatorial information is contained in the *flag vector* of a d-polytope, which in addition to the components f_k of the f-vector also records the numbers $f_{k,\ell}$ of incidences of k-faces with ℓ-faces (that is, the numbers of pairs (F, G) consisting of a k-face F contained in an ℓ-face G, for $k < \ell$), the numbers $f_{k,\ell,m}$ of chains of three faces $F \subset G \subset H$ of dimensions $k < \ell < m$, etc. The flag vector is an integral vector with $2^d - 1$ components; nevertheless, due to a multitude of linear relations, the *generalized Dehn-Sommerville relations* [4], the set of all flag-vectors has dimension only $F_d - 1$, where F_d denotes the d-th Fibonacci number. In particular, for $d \leq 3$ there is no additional information contained in the flag vector, while for 4-polytopes the set of f-vectors is 3-dimensional, but the set of flag vectors is 4-dimensional. So, there is indeed extra information contained in, say, f_{03}, while all the other components of the flag vector can be recovered from $\widetilde{f} = (f_0, f_1, f_2, f_3; f_{03})$.

It makes sense to treat the f-vector problem for each dimension separately. This starts at $d = 2$, where the trivial answer is $\mathcal{F}_2 = \{(n, n) : n \geq 3\}$.

1.3. ***f*-vectors of 3-polytopes.** According to Steinitz' paper [16] of 1906, the f-vectors of 3-polytopes are all the integral vectors that satisfy

$$f_2 \leq 2f_0 - 4 \qquad \text{and} \qquad f_0 \leq 2f_2 - 4$$

in conjunction with Euler's formula, $f_1 = f_0 + f_2 - 2$; see also [7, Section 10.3]. Since both inequalities are tight for the 3-simplex, with $f(\Delta_3) = (4, 6, 4)$, this implies that $\mathcal{F}_3$ is the set of all integral points in a 2-dimensional polyhedral cone with apex $f(\Delta_3)$, which is pictured in Figure 1.

Here the extreme cases, polytopes whose f- or flag vectors lie on the boundary of the cone, are given by the simplicial polytopes (for which Steinitz' first inequality is tight) and the simple polytopes (second inequality tight). One can thus say that "all (f-vectors of) 3-polytopes lie between the extremes of simple and of simplicial polytopes."

1.4. ***f*-vector cones.** For 4-dimensional polytopes, such a complete and simple answer is not to be expected. Indeed, $\mathcal{F}_d$ is not just the set of all integral points in a convex set, since some of the constraints, such as $f_1 \leq \binom{f_0}{2}$, are concave rather than convex. Also, some of the 2-dimensional coordinate projections of $\mathcal{F}_4$ show "holes" that cannot be explained by such systematic inequalities; compare Grünbaum [7, Section 10.4], Bayer [3], and Höppner and Ziegler [8].

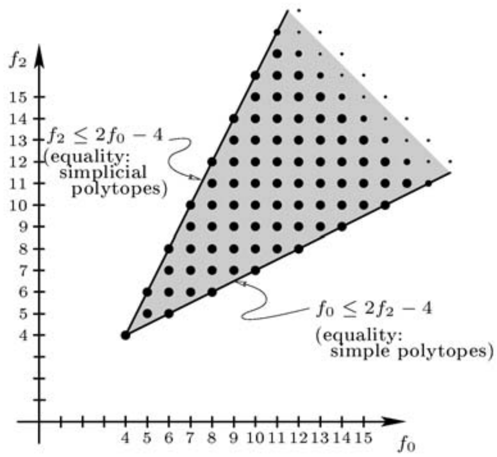

FIGURE 1. The f-vectors of 3-polytopes, graphed in the (f_0, f_2)-plane.

Thus, one heads for a more modest goal, the complete description of the closed convex cone, with apex $f(\Delta_d)$, that is spanned by the f-vectors of d-polytopes:

$$\text{cone}(\mathcal{F}_d) := \text{topological closure of}$$

$$\Big\{ f(\Delta_d) + \sum_{i=1}^{N} \lambda_i (f(P_i) - f(\Delta_d)) : N \geq 0,\ \lambda_1, \ldots, \lambda_N \geq 0, \\ P_1, \ldots, P_N \ d\text{-polytopes} \Big\}.$$

Equivalently, this is the solution set for the system of all *linear* inequalities that are valid for the f-vectors of d-polytopes and are tight for the f-vector of the d-simplex.

1.5. **The f-vector cone for 4-polytopes.** The f-vector cone for 4-polytopes, $\text{cone}(\mathcal{F}_4)$, is a 3-dimensional convex cone. One can visualize it in terms of an intersection with an affine hyperplane, which yields a 2-dimensional convex set; equivalently, one can introduce projective coordinates for the cone, that is, suitable ratios of linear quantities which vanish at the apex of the cone.

Here we use the projective coordinates introduced in [18],

$$\varphi_0 \ := \ \frac{f_0 - 5}{f_1 + f_2 - 20} \qquad \text{and} \qquad \varphi_3 \ := \ \frac{f_3 - 5}{f_1 + f_2 - 20}.$$

In terms of these quantities, we describe (and picture) our knowledge about the f-vector cone of 4-polytopes. The known necessary conditions can be written as

$$(1) \qquad \varphi_0 \geq 0, \quad \varphi_3 \geq 0, \quad \varphi_0 + 3\varphi_3 \leq 1, \quad 3\varphi_0 + \varphi_3 \leq 1, \quad \text{and} \quad \varphi_0 + \varphi_3 \leq \tfrac{2}{5}.$$

The first two conditions are trivial, the second two have simplicial, resp. simple, polytopes as extreme cases, and the last condition is a non-trivial bound that Bayer [3] derived from a flag vector inequality, which in terms of the "toric g-vector" of Stanley [15] reads "$g_2^{\text{tor}} \geq 0$"; a rigidity-theoretic proof was obtained by Kalai [10].

Figure 2 represents the pentagon described by the five linear inequalities (1). As indicated in the figure, four of its vertices are spanned by the (φ_0, φ_3)-pairs of known classes of polytopes.

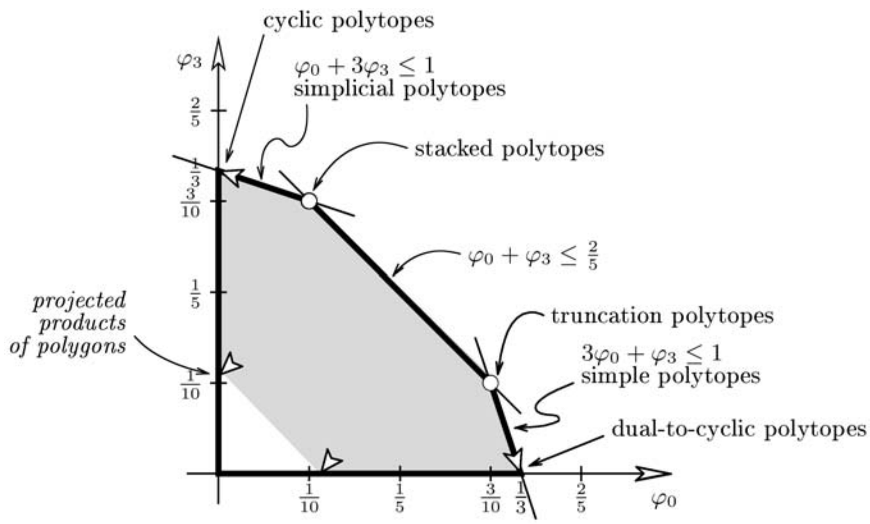

FIGURE 2. The f-vector cone of 4-polytopes, graphed in projective coordinates, (φ_0, φ_3).

There is some hope that the five linear inequalities of (1) represent a *complete* description of cone($\mathcal{F}_4$). This is true if and only if there exist polytopes whose (φ_0, φ_3)-pairs approach the fifth vertex of the pentagon, that is, for which the sum $\varphi_0 + \varphi_3$ is arbitrarily small. Equivalently, we want the *fatness* parameter

$$F(P) \;:=\; \frac{1}{\varphi_0 + \varphi_3} \;=\; \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$$

arbitrarily large [18]. This observation has sparked a certain race for "fat" 4-dimensional polytopes. The following table summarizes the main steps. Most of the examples that appear there are 2-simple and 2-simplicial, with a symmetric f-vector; the first infinite family of such polytopes was constructed by Eppstein, Kuperberg and Ziegler [5]; a simple construction appears in Paffenholz and Ziegler [12]. We call a polytope "even" if its 1-skeleton is a bipartite graph.

4-polytopes	fatness	property	reference	date
simple or simplicial	< 3			
24-cell	4.526	2-simple, 2-simplicial	Schläfli [13]	1852
dipyramidal 720-cell	5.020	2-simple, 2-simplicial	Gevay [6]	1991
neighborly cubical	$5 - \varepsilon$	even	Joswig and Ziegler [9]	2000
[E-construction]	5.048	2-simple, 2-simplicial	Eppstein et al. [5]	2003
$E(C_m \times C_n)$	$6 - \varepsilon$	2-simple, 2-simplicial	Paffenholz [11]	2004
Projected products	$9 - \varepsilon$	even	here	

For example, the neighborly cubical 4-polytopes C_4^n for $n \to \infty$ yield the point $(0, \frac{1}{5})$ in the (φ_0, φ_3)-plane, and thus fatness arbitrarily close to 5.

The concept of "strictly preserving a face" used in the following theorem will be explained in Section 3. (Compare the concept of faces in the "shadow boundary" of a projection, e.g. in [2].)

Theorem 1.1 (Projected Products of Polygons)**.** *Let $n \geq 4$ be even and $r \geq 2$. Then there is a $2r$-polytope $P_n^{2r} \subset \mathbb{R}^{2r}$, combinatorially equivalent to a product of r n-gons, $P_n^{2r} \cong (C_n)^r$, such that the projection $\pi : \mathbb{R}^{2r} \to \mathbb{R}^4$ to the last four coordinates strictly preserves the 1-skeleton as well as all the "polygon 2-faces" of P_n^{2r}.*

Theorem 1.2. *The polytopes $\pi(P_n^{2r})$ have the f-vectors*

$$\left(\frac{4}{r}, 4, 5 - \frac{6}{r} + \frac{4}{n}, 1 - \frac{2}{r} + \frac{4}{n}\right) \cdot \frac{1}{4} r\, n^r.$$

For $n, r \to \infty$ they yield the point $(0, \frac{1}{9})$, in the (φ_0, φ_3)-plane, while their duals yield $(\frac{1}{9}, 0)$. In particular, for each $\varepsilon > 0$ there are polytopes of fatness larger than $9 - \varepsilon$.

Thus the known polytopes now span a hexagon, which is shaded in Figure 2.

A flag vector parameter that is similar to fatness, called *complexity* [18], is defined by

$$C(P) \;:=\; \frac{f_{03} - 20}{f_0 + f_3 - 10}.$$

All 4-polytopes satisfy $C(P) \geq 3$. Fatness and complexity are roughly within a factor of 2: $C(P) \leq 2F(P) - 2$ and $F(P) \leq 2C(P) - 2$. In particular, it is not known whether $C(P)$ can be arbitrarily large. Previously, the polytopes with the largest known complexity were the "neighborly cubical polytopes" of Joswig and Ziegler [9], of complexity $8 - \varepsilon$. Our present construction yields "neighborly cubical polytopes" for $n = 4$, but for $n, r \to \infty$ it yields complexity as large as $16 - \varepsilon$.

In the following two sections, we review the main ingredients for the construction of $\pi(P_n^{2r})$. The construction that yields Theorem 1.1 is described in Section 4, with a sketch of the proof for its correctness. The flag vectors of the polytopes $\pi(P_n^{2r})$ are computed in Section 5, which yields Theorem 1.2. Detailed proofs, the combinatorial characterization of the resulting polytopes, possible extensions, further remarkable aspects (such as the polyhedral surfaces of high genus embedded in the 2-skeleta of the resulting 4-polytopes; cf. [14]) as well as necessity of the restrictions (e.g., that n must be even) are topics of current research and will be presented later.

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2. Products and deformed products

The combinatorial structure of the products of polygons $(C_n)^r$ is easy to describe. These are simple $2r$-polytopes, with $f_0 = n^r$ vertices, $f_1 = rn^r$ edges, and $f_{2r-1} = rn$ facets. In general, its non-empty faces are products of non-empty faces of the polygons, so $\sum_{i=0}^{2r} f_i t^i = (n + nt + t^2)^r$.

The 2-dimensional faces of $(C_n)^r$, and thus of any polytope combinatorially equivalent to $(C_n)^r$, may be split into two classes. There are rn^{r-1} faces that are n-gons, to which we refer as *polygons*; they arise as products of one of the n-gons with a vertex from each of the other factors. There are also $\binom{r}{2}n^r$ *quadrilaterals* that (in $(C_n)^r$) arise as products of edges from two of the factors with vertices from the others. Thus, in total $(C_n)^r$ has $f_2 = rn^{r-1} + \binom{r}{2}n^r$ 2-faces.

In the case $n = 4$, the polygon 2-faces of $(C_n)^r$ are 4-gons, but we nevertheless treat the $r4^{r-1}$ polygons and the $\binom{r}{2}4^r$ quadrilaterals separately also in this case.

An inequality description for such a product polytope may be obtained as

$$\begin{pmatrix} V & & & & \\ & V & & 0 & \\ & & V & & \\ & 0 & & \ddots & \\ & & & & V \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{b} \\ \boldsymbol{b} \\ \vdots \\ \boldsymbol{b} \end{pmatrix},$$

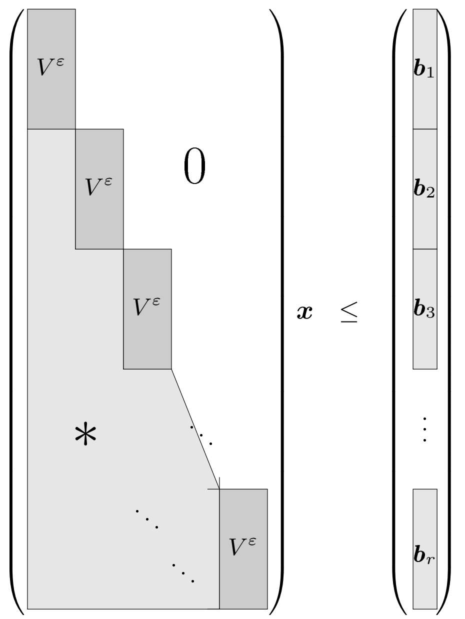
assuming that $V\boldsymbol{x} \leq \boldsymbol{b}$ is a correct description for an n-gon. For this it is necessary and sufficient that the row vectors $\boldsymbol{v}_i$ of V are non-zero and distinct and that they positively span $\mathbb{R}^2$, that the components b_i of $\boldsymbol{b}$ are positive, and that the rescaled vectors $\frac{1}{b_i}\boldsymbol{v}_i$ are in convex position (the vertices of the polar of the polygon).

For this we say that a finite set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^d$ *positively spans* if it satisfies the following equivalent conditions:

(i) every vector $\boldsymbol{x} \in \mathbb{R}^d$ is a linear combination of the vectors $\boldsymbol{v}_i$, with non-negative coefficients;

(ii) every vector $\boldsymbol{x} \in \mathbb{R}^d$ is a linear combination of the vectors $\boldsymbol{v}_i$, with positive coefficients;
(iii) the vectors $\boldsymbol{v}_i$ span $\mathbb{R}^d$, and $\mathbf{0} \in \mathbb{R}^d$ is a linear combination of the vectors $\boldsymbol{v}_i$, with positive coefficients (that is, the vectors $\boldsymbol{v}_i$ are *positively dependent*).

In the following, we will need "deformed products." (The deformations are more general than the "rank 1" deformations as described in Amenta and Ziegler [1].) For this, we look at systems of the form



Given any such left-hand side matrix for such a system, we can adapt the right-hand side so that the resulting polytope is *combinatorially equivalent* to $(C_n)^r$. For this all components of $\boldsymbol{b}_k$ have to be sufficiently large compared to $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k-1}$, for $k = 2, 3, \ldots, r$. (Compare [1] and [9].)

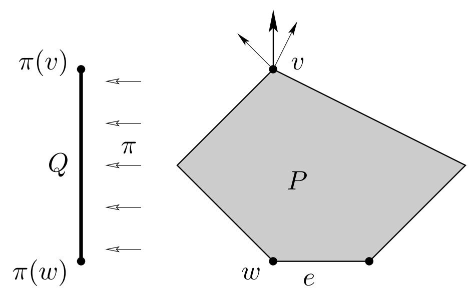
3. Projections

We will work with a rather restrictive concept of faces "being preserved under projection."

Definition 3.1 (Strictly preserving faces under projection)**.** Let $\pi : P \to Q = \pi(P)$ be a projection of polytopes. Then a face $G \subseteq P$ is *strictly preserved* if

(i) its image $\pi(G)$ is a face of Q,
(ii) the map $G \to \pi(G)$ is a bijection, and
(iii) the preimage of the image is G, that is, $\pi^{-1}(\pi(G)) = G$.

In the definition, conditions (ii) and (iii) are both needed. Indeed, in the projection "to the second coordinate" displayed in our figure, the vertex v is strictly preserved, but the vertex w and the edge e are not: for w condition (iii) fails, while for e condition (ii) is violated.



For simplicity, the following characterization result is given only in a coordinatized version, for the projection "to the last d coordinates."

We say that a vector $\boldsymbol{c}$ *defines* the face $G \subseteq P$ given by all the points of P that have maximal scalar product with $\boldsymbol{c}$. This describes exactly all the vectors in the relative interior of the *normal cone* of G. If P is full-dimensional, this interior of the normal cone consists of all the positive combinations of outer facet normals $\boldsymbol{n}_F$ to the facets $F \subset P$ that contain G. (Compare [17, Sections 2.1, 3.2, 7.1].)

Proposition 3.2. *Let $\pi : \mathbb{R}^{e+d} \to \mathbb{R}^d$, $(\boldsymbol{x}', \boldsymbol{x}'') \mapsto \boldsymbol{x}''$ be the projection to the last d coordinates, and let $P \subset \mathbb{R}^{e+d}$ be an $(e+d)$-dimensional polytope, and let G be a face of P. Then the following three conditions are equivalent:*

1. *G is strictly preserved by the projection $\pi : P \to \pi(P) = Q$.*
2. *Any $\boldsymbol{c}' \in \mathbb{R}^e$ arises as the first e components of a vector $(\boldsymbol{c}', \boldsymbol{c}'')$ that defines G.*
3. *The vectors $\boldsymbol{n}'_F$, given by the first e components of the normal vectors $(\boldsymbol{n}'_F, \boldsymbol{n}''_F) = \boldsymbol{n}_F$ to facets F of P that contain G, positively span $\mathbb{R}^e$.*

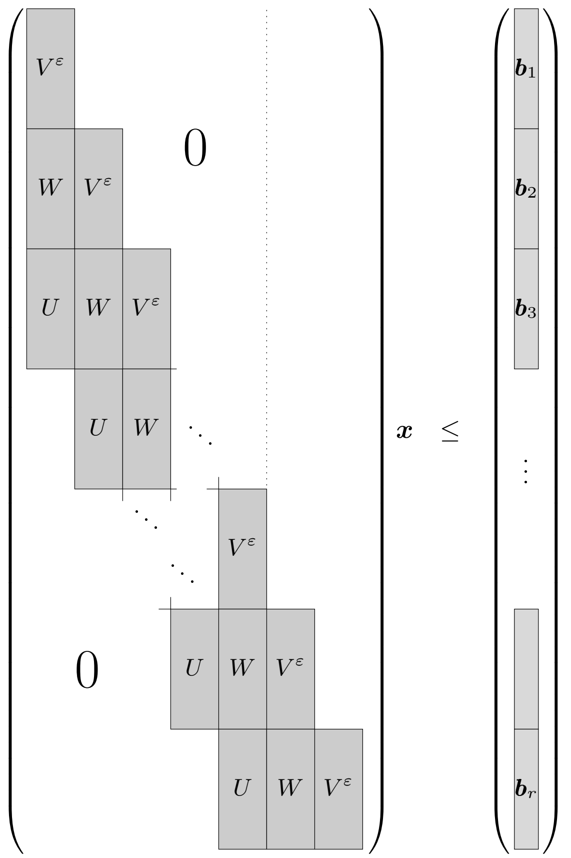
Proof. Here we only establish "(3) $\Rightarrow$ (1)," which is used in the following.

If the vectors $\boldsymbol{n}'_F$ are positively dependent, then some positive combination of the vectors $(\boldsymbol{n}'_F, \boldsymbol{n}''_F) = \boldsymbol{n}_F$ yields $(\boldsymbol{0}, \boldsymbol{c}'') =: \boldsymbol{c}$. A point $\boldsymbol{x} \in P$ lies in the face $G \subseteq P$ if and only if its scalar product with each facet normal $\boldsymbol{n}_F$ is maximal. This happens if and only if $\boldsymbol{c}^t\boldsymbol{x}$ is maximal, that is, iff $(\boldsymbol{c}'')^t\boldsymbol{x}''$ is maximal under the restriction $\boldsymbol{x}'' \in \pi(P)$. Thus we have established that under the assumption (3), $\pi(G) =: \bar{G}$ is a face of $\pi(P)$, and $\pi^{-1}(\pi(G)) = \pi^{-1}(\bar{G}) = G$; that is, conditions (i) and (iii) of Definition 3.1 are satisfied.

For part (ii) of Definition 3.1, we have to show that $G \to \pi(G)$ is injective. Assume that $\boldsymbol{x} = (\boldsymbol{x}', \boldsymbol{x}'')$ and $\boldsymbol{y} = (\boldsymbol{y}', \boldsymbol{y}'')$ are points in G with $\pi(\boldsymbol{x}) = \pi(\boldsymbol{y})$, that is, $\boldsymbol{x}'' = \boldsymbol{y}''$. For each normal vector $\boldsymbol{n}_F = (\boldsymbol{n}'_F, \boldsymbol{n}''_F)$ we have $\boldsymbol{n}_F{}^t\boldsymbol{x} = \boldsymbol{n}_F{}^t\boldsymbol{y}$ and $(\boldsymbol{n}''_F)^t\boldsymbol{x}'' = (\boldsymbol{n}''_F)^t\boldsymbol{y}''$, which implies $(\boldsymbol{n}'_F)^t\boldsymbol{x}'' = (\boldsymbol{n}'_F)^t\boldsymbol{y}''$. But the vectors $\boldsymbol{n}'_F$ that correspond to the various facets F that contain G are positively spanning in $\mathbb{R}^e$, which implies $\boldsymbol{x}' = \boldsymbol{y}'$. $\square$

4. Construction

Proposition 4.1 (Construction for the proof of Theorem 1.1)**.** *For $n \geq 4$ even and $r \geq 2$, let P_n^{2r} be defined by the linear inequality system*



Here the left-hand side coefficient matrix $A^\varepsilon_{n,r} \in \mathbb{R}^{rn\times 2r}$ contains blocks of size $n \times 2$, where

$$V = \begin{pmatrix} \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_0 \\ \boldsymbol{v}_1 \\ \vdots \end{pmatrix} \longrightarrow V^\varepsilon = \begin{pmatrix} \boldsymbol{v}_0^\varepsilon \\ \boldsymbol{v}_1^\varepsilon \\ \boldsymbol{v}_2^\varepsilon \\ \boldsymbol{v}_3^\varepsilon \\ \vdots \end{pmatrix}, \qquad W = \begin{pmatrix} \boldsymbol{w}_0 \\ \boldsymbol{w}_1 \\ \boldsymbol{w}_0 \\ \boldsymbol{w}_1 \\ \vdots \end{pmatrix}, \qquad U = \begin{pmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \\ \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \\ \vdots \end{pmatrix} \in \mathbb{R}^{n\times 2},$$

with

$$\boldsymbol{v}_0 = (1,0),\ \boldsymbol{v}_1 = (0,0) = \mathbf{0},\ \boldsymbol{w}_0 = (0,1),\ \boldsymbol{w}_1 = (-3,-\tfrac{2}{3}),$$
$$\boldsymbol{u}_0 = (-\tfrac{31}{4},\tfrac{1}{2}),\ \boldsymbol{u}_1 = (9,-\tfrac{2}{3}).$$

The block V^ε arises from V by an ε-perturbation:

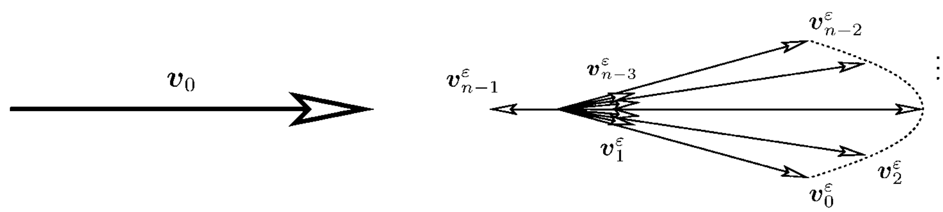
$$\boldsymbol{v}_i^\varepsilon = \begin{cases} \big(1-\varepsilon(n-2-2i)^2, \varepsilon(n-2-2i)\big) & \text{for } i = 0,2,4,\dots,n-2, \\ \varepsilon\big(1-\varepsilon(n-2-2i)^2, \varepsilon(n-2-2i)\big) & \text{for } i = 1,3,5,\dots,n-3, \\ \varepsilon(-1,0) = (-\varepsilon,0) & \text{for } i = n-1 \end{cases}$$

for a sufficiently small $\varepsilon > 0$. All entries of $A^\varepsilon_{n,r}$ outside the $r+(r-1)+(r-2) = 3r-3$ blocks of types V^ε, W and U are zero.

Let the right-hand side vector be such that $\boldsymbol{b}_1$ is given by $b_{1,i} = 1$ for even i, and $b_{1,i} = \varepsilon$ for odd i, and by $\boldsymbol{b}_k = M^{k-1}\boldsymbol{b}_1$ for sufficiently large M.

Then P_n^{2r} has the properties claimed by Theorem 1.1. In particular, it is a deformed product of r n-gons, and all its polygon 2-faces survive the projection to the last 4 coordinates.

Proof. The rows $\boldsymbol{v}_i^\varepsilon$ of V^ε are indeed in cyclic order:



Moreover, rescaled as $\frac{1}{b_{k,i}}\boldsymbol{v}_i^\varepsilon = \frac{1}{M^{k-1}\varepsilon}\boldsymbol{v}_i^\varepsilon$ for odd i and as $\frac{1}{b_{k,i}}\boldsymbol{v}_i^\varepsilon = \frac{1}{M^{k-1}}\boldsymbol{v}_i^\varepsilon$ for even i they are in convex position, if ε is small; so $V^\varepsilon\boldsymbol{x} \le \boldsymbol{b}_i$ defines a convex n-gon. Thus for sufficiently small ε and sufficiently large M, the polytope P_{2r} is indeed a deformed product of polygons, as discussed in Section 2.

Now we show that for sufficiently small ε, all the polygon 2-faces of P_n^{2r} survive the projection to the last 4 coordinates. For this, we verify that the left-hand side matrix with V-blocks instead of V^ε-blocks, which we denote by $A_{n,r}^0 = A_{n,r}$, satisfies the linear algebra condition dictated by Proposition 3.2(3). This is sufficient, since the "positively spanning" condition is stable under perturbation by a small ε.

Any polygon 2-face G of the simple $2r$-polytope P_n^{2r} is defined by the facet normals to the $2r-2$ facets that contain G. The facet normals correspond to the rows of the inequality system, and thus for the facet normals of a polygon 2-face one has to choose two cyclically adjacent rows from each block (corresponding to a vertex from each factor polygon), except from one of the blocks no row is taken. Moreover, due to the structure of the matrices U, V, and W, in which rows alternate, any choice of two cyclically-adjacent rows from a block yields the same pair of rows (only the order is not clear, but it also does not matter).

Thus, to apply Proposition 3.2(3) we have to show:

If one of the r pairs of rows is deleted from the reduced matrix

$$A'_{n,r} = \begin{pmatrix} \boldsymbol{v}_0 & & & & & & \\ \boldsymbol{w}_0 & \boldsymbol{v}_0 & & & & & \\ \boldsymbol{w}_1 & & & & & & \\ \boldsymbol{u}_0 & \boldsymbol{w}_0 & \boldsymbol{v}_0 & & & & \\ \boldsymbol{u}_1 & \boldsymbol{w}_1 & & & & & \\ & \boldsymbol{u}_0 & \boldsymbol{w}_0 & \boldsymbol{v}_0 & & & \\ & \boldsymbol{u}_1 & \boldsymbol{w}_1 & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \boldsymbol{u}_0 & \boldsymbol{w}_0 & \boldsymbol{v}_0 \\ & & & \boldsymbol{u}_1 & \boldsymbol{w}_1 & \\ & & & & \boldsymbol{u}_0 & \boldsymbol{w}_0 \\ & & & & \boldsymbol{u}_1 & \boldsymbol{w}_1 \\ & & & & & \boldsymbol{u}_0 \\ & & & & & \boldsymbol{u}_1 \end{pmatrix} \in \mathbb{R}^{2r\times(2r-4)},$$

then the remaining $2r-2$ rows
(a) *span $\mathbb{R}^{2r-4}$, and*
(b) *have a linear dependence with strictly positive coefficients.*

Let us establish (b) first. For this, let

$$\alpha_k \;:=\; 2^k + 2^{-k} - 2 \qquad \text{and} \qquad \beta_k \;:=\; 2^k + \tfrac{5}{4}2^{-k} - \tfrac{9}{4}.$$

These sequences are designed to be non-negative, $\alpha_k, \beta_k \geq 0$ for all $k \in \mathbb{Z}$, with equality only for $k = 0$. Thus for (b) it suffices to verify that

For any $1 \leq t \leq r$, the rows of $A'_{n,r}$ are positively dependent with coefficients α_{k-t} for the even-index row from the k-th block, and β_{k-t} for the odd-index row from the k-th block,

since the (two) vectors in the t-th block thus get zero coefficients, so they may be deleted from any linear dependence (with otherwise positive coefficients). Thus we are led to the condition

$$\alpha_{k-1}\boldsymbol{v}_0 + \alpha_k\boldsymbol{w}_0 + \beta_k\boldsymbol{w}_1 + \alpha_{k+1}\boldsymbol{u}_0 + \beta_{k+1}\boldsymbol{u}_1 \;=\; \boldsymbol{0},$$

which is needed to hold for $k \leq |r-2|$, but which we impose for all $k \in \mathbb{Z}$. The choice of vectors $\boldsymbol{v}_0, \boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{u}_0, \boldsymbol{u}_1$ is designed to satisfy this condition. Indeed, except for the choice of a basis, which we took to be $\boldsymbol{v}_0 = (1,0)$ and $\boldsymbol{w}_0 = (0,1)$, the configuration of five vectors $\boldsymbol{v}_0, \boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{u}_0, \boldsymbol{u}_1$ is uniquely determined by the condition.

For property (a), we have to show that if one of the r pairs of rows is deleted from the matrix $A'_{n,r}$, then the resulting matrix still has full rank. If the first or the second pair of rows is deleted, then we still have the last $2r-4$ rows, and they form a block upper triangular matrix, which has full rank since its diagonal block

$$\begin{pmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{pmatrix}$$

is non-singular. If a later pair of rows is deleted, then we are faced with the task to show that the $2k \times 2k$ matrices M_k of the form

$$M_k \quad := \quad \begin{pmatrix} \boldsymbol{w}_0 & \boldsymbol{v}_0 & & & \\ \boldsymbol{w}_1 & & & & \\ \boldsymbol{u}_0 & \boldsymbol{w}_0 & \boldsymbol{v}_0 & & \\ \boldsymbol{u}_1 & \boldsymbol{w}_1 & & & \\ & \boldsymbol{u}_0 & \boldsymbol{w}_0 & \boldsymbol{v}_0 & \\ & \boldsymbol{u}_1 & \boldsymbol{w}_1 & & \\ & & \ddots & \ddots & \ddots \\ & & & \boldsymbol{u}_0 & \boldsymbol{w}_0 \\ & & & \boldsymbol{u}_1 & \boldsymbol{w}_1 \end{pmatrix} \quad \in \; \mathbb{R}^{2k\times 2k}$$

are non-singular. To verify this (without proving explicitly that $\det M_k = \frac{(2^k-1)^2}{3^k}$, which might need combinatorial ingenuity) we use our knowledge about row combinations of M_k. Indeed, if we sum the rows of M_k with coefficients $(\alpha_0, \beta_0, \alpha_1, \ldots, \alpha_{k-1}, \beta_{k-1})$, then this will result in the linear combination of the three rows of the matrix

$$H_3 \quad = \quad \begin{pmatrix} \boldsymbol{v}_0 & & & \\ & & & \boldsymbol{u}_0 \\ & & & \boldsymbol{u}_1 \end{pmatrix} \; \in \; \mathbb{R}^{3\times 2r}$$

with the coefficients $(-\alpha_{-1}, -\alpha_k, -\beta_k)$, since $\boldsymbol{v}_1 = \boldsymbol{0}$. Similarly, if we sum the rows of M_k with coefficients $(\alpha_1, \beta_1, \alpha_2, \ldots, \alpha_k, \beta_k)$, then we get a linear combination of the same three rows, with coefficients $(-\alpha_0, -\alpha_{k+1}, -\beta_{k+1})$. And if we use coefficients $(\alpha_2, \beta_2, \alpha_3, \ldots, \alpha_{k+1}, \beta_{k+1})$ to sum the rows of M_k, then the result will be a sum with coefficients $(-\alpha_1, -\alpha_{k+2}, -\beta_{k+2})$. The coefficient matrix

$$\begin{pmatrix} -\alpha_{-1} & -\alpha_k & -\beta_k \\ -\alpha_0 & -\alpha_{k+1} & -\beta_{k+1} \\ -\alpha_1 & -\alpha_{k+2} & -\beta_{k+2} \end{pmatrix}$$

is non-singular for $k \geq 0$: its determinant is $\frac{3}{8}(2^k - 1 + 2^{-k-2})$. Thus the full row space of H_3 is contained in the row space of M_k. In particular, we find the unit vectors $\boldsymbol{e}_{2k-1}, \boldsymbol{e}_{2k} \in \mathbb{R}^{2k}$ in the row space of H_3, and thus of M_k, and this allows us to complete the argument by induction. □

5. Flag vectors

The following result includes Theorem 1.2. Note that its proof relies only on the properties of $\pi(P_n^{2r})$ that are guaranteed by the statement of Theorem 1.1; it does not refer to the specific combinatorial type of the polytopes constructed in Section 4, in our proof of Theorem 1.1.

Theorem 5.1. *The* 4*-polytope* $\pi(P_n^{2r})$ *has the flag vector*

$$\begin{aligned}(f_0, f_1, f_2, f_3; f_{03}) &= (n^r, rn^r, \tfrac{5}{4}rn^r - \tfrac{3}{2}n^r + rn^{r-1}, \tfrac{1}{4}rn^r - \tfrac{1}{2}n^r + rn^{r-1}; 4rn^r - 4n^r) \\ &= (4n, 4rn, 5rn - 6n + 4r, rn - 2n + 4r; 16rn - 16n) \cdot \tfrac{1}{4}n^{r-1}.\end{aligned}$$

Proof. We obtain $f_0 = n^r$ and $f_1 = rn^r$ from the products $(C_n)^r$, which are simple $2r$-polytopes with n^r vertices. With the abbreviation $N := \frac{1}{4}n^{r-1}$ this yields $f_0 = 4nN$ vertices and $f_1 = 4rnN$ edges for $\pi(P_n^{2r})$.

The products $(C_n)^r$ have $P := rn^{r-1} = 4rN$ polygon 2-faces. In the projection, all these are preserved, in addition to some of the quadrilateral 2-faces.

The projected polytope has two types of facets. There are "prism" facets, which involve two of the polygons, as well as "cube" facets, which in $(C_n)^r$ arise as products of three edges and $r - 3$ vertices, but contain no polygon 2-faces. Thus each prism facet is bounded by two polygons, and each polygon lies in two prism facets. Hence there are $P = 4rN$ prism facets, as well as some number $C \geq 0$ of cube facets.

Now double counting of ridges yields $6C + (n + 2)P = 2f_2$. Thus with the Euler equation we get $C = \frac{1}{4}(r - 2)n^r = (rn - 2n)N$. Finally, counting the vertex-facet incidences according to facets yields $f_{03} = 8C + 2nP = (8rn - 16n + 8rn)N$. □

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