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PROJECTION AND ITERATED PROJECTION METHODS FOR NONLINEAR INTEGRAL EQUATIONS*

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Dedicated to Werner C. Rheinboldt on the occasion of his 60th birthday.

Abstract. Consider the nonlinear operator equation $x=\mathcal{K}(x)$ with $\mathcal{K}$ a completely continuous mapping of a domain in the Banach space $\mathcal{X}$ into $\mathcal{X}$; and let x^* denote an isolated fixed point of $\mathcal{K}$. Let $\mathcal{X}_n$, $n\geqq 1$, denote a sequence of finite dimensional approximating subspaces, and let P_n be a projection of $\mathcal{X}$ onto $\mathcal{X}_n$. The projection method for solving $x=\mathcal{K}(x)$ is given by $x_n=P_n\mathcal{K}(x_n)$, and the iterated projection solution is defined as $\tilde{x}_n=\mathcal{K}(x_n)$. We analyze the convergence of x_n and $\tilde{x}_n$ to x^*, giving a general analysis that includes both the Galerkin and collocation methods. A more detailed analysis is then given for a large class of Urysohn integral operators in one variable, showing the superconvergence of $\tilde{x}_n$ to x^*.

Key words. nonlinear integral equation, Galerkin method, collocation method

AMS(MOS) subject classifications. 45L10, 65R20, 65J15

1. Introduction. Consider the nonlinear operator equation

$$x=\mathcal{K}(x), \tag{1.1}$$

where $\mathcal{K}$ is a completely continuous operator defined on the closure $\bar{D}$ of an open subset D of a Banach space $\mathcal{X}$. An example of such an operator $\mathcal{K}$ is the Urysohn integral operator

$$\mathcal{K}(x)(t)=\int_\Omega K(t,s,x(s))\,ds,\quad t\in\Omega,\quad x\in D, \tag{1.2}$$

with Ω a closed bounded region in $\mathbb{R}^m$, some $m\geqq 1$. It will be examined in more detail in § 3. We are interested in the evaluation of fixed points x^* of $\mathcal{K}$, and we will investigate the use of projection methods to approximate such fixed points.

For $\mathcal{X}_n$ a finite dimensional approximating subspace of $\mathcal{X}$, let P_n be a projection of $\mathcal{X}$ onto $\mathcal{X}_n$. The projection method consists of solving

$$x_n=P_n\mathcal{K}(x_n). \tag{1.3}$$

This method was analyzed in Krasnoselskii (1964, Chap. 3, § 3), and results on the rate of convergence of $\{x_n\}$ to x^* were obtained. We will give additional such results, including improved convergence rates for some widely used subspaces $\mathcal{X}_n$.

The iterated projection solution is defined by

$$\tilde{x}_n=\mathcal{K}(x_n). \tag{1.4}$$

When $\mathcal{X}_n$ is a Hilbert space and P_n is an orthogonal projection, the sequence $\{\tilde{x}_n\}$ always converges more rapidly than does $\{x_n\}$, as shall be seen in § 2; and this is also true for some other projection methods. We will give an analysis of the convergence of $\{\tilde{x}_n\}$, including results on uniform and pointwise convergence.

Projection methods have been studied extensively, as is indicated in Krasnoselskii, Vainikko et al. (1972) and Krasnoselskii and Zabreiko (1984, p. 327). We will generalize to the nonlinear case the results of Chatelin and Lebbar (1984) for projection methods for linear integral equations. This will include detailed convergence results for the use

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of spaces $\mathscr{X}_n$ of piecewise polynomial functions, for both Galerkin and collocation projection methods. In addition, we obtain estimates for the order of convergence of the derivatives of the approximate solutions given by the iterated Galerkin and iterated collocation methods. Related results in the linear case have been given by Sloan and Thomée (1985).

Section 2 contains a general framework for the convergence analysis of projection and iterated projection methods. Section 3 gives mapping properties for a class of Urysohn integral operators, and § 4 gives preliminary approximation results for piecewise polynomial functions. The Galerkin method for Urysohn integral operators on an interval $\Omega = [a, b]$ is analyzed in § 5, and similar results for the collocation method are given in § 6. Numerical examples are given in § 7.

Although projection methods are widely used, there are severe practical problems in using them to solve nonlinear integral equations. In future papers we will discuss modifications of these methods, to make them into more efficient and practical methods.

2. The projection and iterated projection method. We assume a slightly different setting for $\mathscr{K}$ than indicated in the Introduction. Let $\mathscr{X}$ be a Banach space, and let $\mathscr{Y}$ be a closed subspace. Assume $\mathscr{K}$ is a completely continuous operator defined on $\bar{D} \subset \mathscr{X}$, D an open set, and assume the values $\mathscr{K}(x) \in \mathscr{Y}$ for all $x \in \bar{D}$. The main application is to let $\mathscr{X} = L^\infty(\Omega)$, $\mathscr{Y} = C(\Omega)$, with Ω a closed bounded region in $\mathbb{R}^m$, some $m \geqq 1$.

Let $\{\mathscr{X}_n\}$ be a sequence of finite dimensional subspaces of $\mathscr{X}$, such that

$$\inf_{x \in \mathscr{X}_n} \|y - x\| \to 0 \quad \text{as } n \to \infty \text{ for all } y \in \mathscr{Y}. \tag{2.1}$$

Let $\{P_n\}$ be a sequence of projections associated with $\{\mathscr{X}_n\}$:

$$P_n : \mathscr{X} \xrightarrow[\text{onto}]{} \mathscr{X}_n, \qquad n \geqq 1. \tag{2.2}$$

Assume that when restricted to $\mathscr{Y}$, the projections are uniformly bounded:

$$\sup_n \|P_n | \mathscr{Y}\| \leqq p < \infty. \tag{2.3}$$

By the principle of uniform boundedness, (2.1) and (2.3) are equivalent to assuming

$$\|P_n y - y\| \to 0 \quad \text{as } n \to \infty \text{ for all } y \in \mathscr{Y}. \tag{2.4}$$

The projection method for solving (1.1) is

$$x_n = P_n \mathscr{K}(x_n), \tag{2.5}$$

or equivalently,

$$P_n(x_n - \mathscr{K}(x_n)) = 0, \qquad x_n \in \mathscr{X}_n. \tag{2.6}$$

In the literature, the name "Galerkin method" is used in the case P_n is an orthogonal projection. In case P_n is an interpolation operator, (2.5) is called a collocation method. For these methods applied to linear operator equations, see Atkinson (1976, Part II, Chap. 2).

The iterated projection method is given by

$$\tilde{x}_n = \mathscr{K}(x_n). \tag{2.7}$$

It was first introduced by Ian Sloan for linear integral equations; for example, see Sloan (1976). From (2.7), it is immediate that

$$P_n \tilde{x}_n = x_n, \tag{2.8}$$

and hence $\tilde{x}_n$ satisfies

$$\tilde{x}_n = \mathcal{K}(P_n \tilde{x}_n). \tag{2.9}$$

An analysis of the existence and convergence of $\{\tilde{x}_n\}$ can be given using Atkinson (1973) or Weiss (1974). Under suitable assumptions on $\mathcal{K}$ and the fixed point x^*, it can be shown that $\tilde{x}_n$ exists for all sufficiently large n. In addition,

$$\|x^* - \tilde{x}_n\| \leqq c \|\mathcal{K}(x^*) - \mathcal{K}(P_n x^*)\|. \tag{2.10}$$

The convergence of $\{x_n\}$ will follow by using (2.8) to write

$$x^* - x_n = [x^* - P_n x^*] + P_n[x^* - \tilde{x}_n]. \tag{2.11}$$

Instead of using this approach, we will use one based on first considering the projection solution x_n.

The following major result for the existence and convergence of $\{x_n\}$ is due to Krasnoselskii.

THEOREM 2.1. *Suppose that $x^* \in D$ is a fixed point of nonzero index for $\mathcal{K}$. Then for all sufficiently large n, the equation* (2.5) *has at least one solution $x_n \in \mathcal{X}_n \cap D$ such that*

$$\operatorname*{limit}_{n\to\infty} \|x_n - x^*\| = 0. \tag{2.12}$$

Proof. See Krasnoselskii (1964, Chap. 3, § 3) or Krasnoselskii and Zabreiko (1984, p. 326). □

To simplify the notation, we will suppose (2.5) has a solution x_n for all $n \geqq 1$.

Assume $\mathcal{K}(x)$ is Fréchet differentiable about x^*, and let $L = \mathcal{K}'(x^*)$. Define

$$r_n = \frac{\|\mathcal{K}(x_n) - \mathcal{K}(x^*) - L(x_n - x^*)\|}{\|x_n - x^*\|}. \tag{2.13}$$

From (2.12) and the definition of L,

$$r_n \to 0 \quad \text{as } n \to \infty. \tag{2.14}$$

In addition, assume $\mathcal{K}'(x)$ is Lipschitz continuous in some neighborhood V of x^*:

$$\|\mathcal{K}'(x) - \mathcal{K}'(y)\| \leqq q\|x - y\|, \qquad x, y \in V \tag{2.15}$$

for some constant q. For example, q could be a bound on $\mathcal{K}''(x)$ over V, if the second Fréchet derivative exists. Then easily,

$$r_n \leqq \tfrac{1}{2} q \|x_n - x^*\|; \tag{2.16}$$

for example, see Potra and Pták (1984, p. 21).

It is known that $\mathcal{K}$ being completely continuous implies that $L = \mathcal{K}'(x^*)$ is compact on $\mathcal{X}$ (see Krasnoselskii and Zabreiko (1984, p. 77)). Also, $\mathcal{K}(D) \subset \mathcal{Y}$ implies Range $(L) \subset \mathcal{Y}$. Using (2.4) with these facts, we have (see Atkinson (1976, pp. 53–54))

$$a_n := \|(I - P_n)L\| \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

We will also need to consider the sequence

$$b_n := \|L(I - P_n)|\mathcal{Y}\|. \tag{2.18}$$

It is uniformly bounded:

$$b_n \leqq b, \qquad n \geqq 1; \tag{2.19}$$

and for some cases (for example, if $\mathcal{X}$ is a Hilbert space and P_n is orthogonal), we have $b_n \to 0$ as $n \to \infty$, as will be seen later in Theorem 2.3.

If 1 is not an eigenvalue of L, then $(I-L)^{-1}$ is a bounded operator on $\mathscr{X}$ to $\mathscr{X}$. (This will also imply that x^* is of nonzero index as a fixed point of $\mathscr{K}$ (see Krasnoselskii (1964, p. 136)), and that x_n will be the unique solution of (2.5) within some sufficiently small neighborhood of x^*.) As further notation, let

$$a=\|I-L\|, \quad c=\|(I-L)^{-1}\|, \quad d=a^{-1}.$$

THEOREM 2.2. *If* 1 *is not an eigenvalue of* $L=\mathscr{K}'(x^*)$, *then there are two nonnegative sequences* $\{\varepsilon_n\}$, $\{\delta_n\}$, *convergent to zero, such that*

$$d(1-\varepsilon_n)\|P_nx^*-x^*\| \leqq \|x^*-x_n\| \leqq c(1+\delta_n)\|P_nx^*-x^*\|. \tag{2.20}$$

Proof. Using the identity

$$(I-L)(x_n-x^*)=(P_n-I)L(x_n-x^*)+(P_n-I)x^*+P_n[\mathscr{K}(x_n)-\mathscr{K}(x^*)-L(x_n-x^*)],$$

and bounding from above and below, we obtain (2.20). The constants are given by

$$\delta_n=\frac{c(a_n+pr_n)}{1-c(a_n+pr_n)}, \qquad \varepsilon_n=\frac{d(a_n+pr_n)}{1+d(a_n+pr_n)}.$$

This result is given in Krasnoselskii and Zabreiko (1984, p. 326), without the values of the constants being given. □

In the case that $\tilde{x}_n \to x^*$ more rapidly than $x_n \to x^*$, the constants c and d in (2.20) can be replaced by 1. This follows from (2.11). The result (2.20) shows that the speed of convergence of x_n to x^* is exactly the same as that of P_nx^* to x^*. Thus it does not depend explicitly on $\mathscr{K}$, but only on the approximation properties of $\mathscr{X}_n$ and P_nx^*. (The convergence does depend on $\mathscr{K}$ implicitly, since the smoothness of x^* depends on the properties of $\mathscr{K}$.)

The iterated Galerkin method in Hilbert spaces. Let $\mathscr{X}$ be a Hilbert space and let P_n be the orthogonal projection of $\mathscr{X}$ onto $\mathscr{X}_n$. We will show that $\{\tilde{x}_n\}$ converges more rapidly than $\{x_n\}$. To this end, we introduce

$$s_n := \frac{\|\tilde{x}_n-x^*\|}{\|x_n-x^*\|}. \tag{2.21}$$

We say the method is superconvergent if

$$s_n \to 0 \quad \text{as } n\to\infty.$$

THEOREM 2.3. *Assume that* 1 *is not an eigenvalue of* $L=\mathscr{K}'(x^*)$. *In addition, let* $\mathscr{X}$ *be a Hilbert space and let* P_n *be the orthogonal projection of* $\mathscr{X}$ *onto* $\mathscr{X}_n$. *Then* $\tilde{x}_n$ *is superconvergent to* x^*; *more precisely, for some constant* $s>0$,

$$s_n \leqq s\cdot \operatorname{Max}\{b_n, r_n\}, \tag{2.22}$$

and $b_n\to 0$, *so that* $s_n \to 0$ *as* $n\to\infty$.

Proof. By the orthogonality of P_n and the argument of Sloan (1976, Thm. 1),

$$b_n=\|(I-P_n)^*L^*\|=\|(I-P_n)L^*\|\to 0. \tag{2.23}$$

For the convergence of $\tilde{x}_n$ to x^*, use the identity

$$(I-L)(\tilde{x}_n-x^*)=[I-L(I-P_n)][\mathscr{K}(x_n)-\mathscr{K}(x^*)-L(x_n-x^*)] \\ -L(I-P_n)(L-I)(x_n-x^*).$$

Multiplying by $(I-L)^{-1}$, and then bounding the right side, we find that

$$\|\tilde{x}_n-x^*\| \leqq c[(1+b_n)r_n+ab_n]\|x_n-x^*\|. \tag{2.24}$$

Inequality (2.22) follows easily. Note that this latter inequality gives a bound for $\|\tilde{x}_n - x^*\|$, showing the dependence on b_n and r_n. □

We have seen in (2.16), assuming (2.15), that

$$r_n = O(\|x_n - x^*\|) = O(\|P_n x^* - x^*\|).$$

We will see that for integral operators with sufficiently smooth kernel functions that $b_n = O(\|P_n x^* - x^*\|)$. Thus for such operators, we have

$$\|\tilde{x}_n - x^*\| = O(\|P_n x^* - x^*\|^2) = O(\|x_n - x^*\|^2). \tag{2.25}$$

This can make a dramatic difference in the convergence, as is illustrated with the examples in § 7. Theorem 2.3 gives convergence in an L^2 sense; the applications to Urysohn integral operators are left until later, following (2.32).

The iterated projection method in Banach spaces. Results on the uniform convergence of the Galerkin method will follow from the results given here.

THEOREM 2.4. *Assume $\mathscr{X}$ is a Banach space, and assume* 1 *is not an eigenvalue of $L = \mathscr{K}'(x^*)$. Then $\{\tilde{x}_n\}$ is superconvergent if and only if*

$$e_n := \frac{\|L(I - P_n)x^*\|}{\|(I - P_n)x^*\|} \to 0 \quad \text{as } n \to \infty. \tag{2.26}$$

Moreover, in this case there is a constant $\hat{s}$ such that

$$s_n \leqq \hat{s} \cdot \operatorname{Max}\{a_n b_n, r_n, e_n\}. \tag{2.27}$$

Proof. The following identity is easily verified:

$$\begin{aligned}(I - L)(\tilde{x}_n - x^*) = {} & [I - L(I - P_n)][\mathscr{K}(x_n) - \mathscr{K}(x^*) - L(x_n - x^*)] \\ & - L(I - P_n)^2 L(x_n - x^*) - L(I - P_n)x^*.\end{aligned} \tag{2.28}$$

Then

$$\|\tilde{x}_n - x^*\| \geqq d\|L(I - P_n)x^*\| - d[(1 + b_n)r_n + a_n b_n]\|x_n - x^*\|.$$

Hence

$$s_n \geqq d f_n e_n - d[(1 + b_n)r_n + a_n b_n]$$

with $f_n = \|(I - P_n)x^*\| / \|x_n - x^*\|$. From Theorem 2.2, f_n is bounded above and bounded away from zero, as $n \to \infty$. Thus if $s_n \to 0$, which means $\{\tilde{x}_n\}$ is superconvergent; then the convergence to zero of $\{r_n\}$ and $\{a_n\}$ implies that $e_n \to 0$ as $n \to \infty$.

For the converse statement, use (2.28) to obtain

$$\|\tilde{x}_n - x^*\| \leqq c[e_n f_n + (1 + b_n)r_n + a_n b_n]\|x_n - x^*\|. \tag{2.29}$$

This will show that $e_n \to 0$ implies $s_n \to 0$. □

COROLLARY 2.5. *With the same hypothesis, there is $\hat{g} > 0$ such that*

$$\|\tilde{x}_n - x^*\| \leqq \hat{g} \operatorname{Max}\{a_n b_n, r_n, e_n\}\|P_n x^* - x^*\|. \tag{2.30}$$

Note that

$$e_n = \frac{\|L(I - P_n)^2 x^*\|}{\|(I - P_n)x^*\|} \leqq b_n. \tag{2.31}$$

Then (2.27) *implies* (2.30).

Urysohn integral operators. Let Ω be a compact subset of $\mathbb{R}^m$ with nonempty interior, or let it be a bounded, closed, piecewise smooth surface in $\mathbb{R}^m$. We will consider the Urysohn operator

$$\mathcal{K}(x)(t) = \int_\Omega K(t, s, x(s))\, ds \tag{2.32}$$

where the kernel $K(t, s, u)$ is a measurable function from $\Omega \times \Omega \times \mathbb{R}$ into $\mathbb{R}$.

From Krasnoselskii et al. (1976) we can deduce various conditions on K which will imply $\mathcal{K}$ is completely continuous on $L^2(\Omega)$ or $L^\infty(\Omega)$. The same reference also contains various types of sufficient conditions for the Fréchet differentiability of $\mathcal{K}$. If any of those conditions are satisfied, then the Fréchet derivative $\mathcal{K}'(x)$ is the linear integral operator

$$(\mathcal{K}'(x)h)(t) = \int_\Omega \frac{\partial}{\partial u} K(t, s, x(s))h(s)\, ds. \tag{2.33}$$

Let $\{\mathcal{X}_n\}$ be a sequence of finite dimensional subspaces of $L^2(\Omega)$, satisfying (2.1) on a closed subspace $\mathcal{Y}$ of $L^2(\Omega)$, with $\mathcal{Y}$ containing the range of $\mathcal{K}$. Let P_n be the orthogonal projection of $L^2(\Omega)$ onto $\mathcal{X}_n$. Then the results in and following Theorem 2.3 will apply, and we obtain the superconvergence of the iterated Galerkin method along with the associated error bounds in the L^2 norm.

However, as is often the case in numerical applications, we would like similar results in the uniform norm. In order to apply Theorem 2.4 and its corollary, we must estimate e_n of (2.26) for the L^∞ norm.

We have $L = \mathcal{K}'(x^*)$, given by (2.33). Let us denote

$$l_t(s) = \frac{\partial}{\partial u} K(t, s, x^*(s)). \tag{2.34}$$

Then

$$\begin{aligned} |[L(I-P_n)x^*](t)| &= |((I-P_n)x^*, \bar{l}_t)| \\ &= |((I-P_n)x^*, (I-P_n)\bar{l}_t)| \\ &\leqq \|(I-P_n)x^*\|_2 \|(I-P_n)\bar{l}_t\|_2 \\ &\leqq [\text{Meas}\,(\Omega)]^{1/2} \|(I-P_n)x^*\|_\infty \|(I-P_n)l_t\|_2. \end{aligned}$$

It follows that

$$e_n \leqq [\text{Meas}\,(\Omega)]^{1/2} \sup_{t\in\Omega} \|(I-P_n)l_t\|_2. \tag{2.35}$$

Under certain mild smoothness assumptions on the kernel of Urysohn's operator $\mathcal{K}$, the family $\{l_t\colon t \in \Omega\}$ is precompact in $L^2(\Omega)$. From the pointwise convergence of P_n, it will then follow that $e_n \to 0$ as $n \to \infty$, thus implying superconvergence of the iterated Galerkin method in the uniform norm. In order to obtain bounds on the rate of uniform convergence for the approximate x_n, $\tilde{x}_n$ given by the Galerkin and iterated Galerkin methods, we only have to estimate the rates of convergence to zero of $\|P_n x^* - x^*\|_\infty$, a_n, b_n, e_n and r_n.

In §§ 5 and 6, the above schema will be applied to the case $\Omega = [0, 1]$ and $\mathcal{X}_n$ a space of piecewise polynomial functions on some partition of Ω. For $\mathcal{X}_n$ a space of piecewise polynomial functions of degree $\leqq r$, we will show that

$$\|x^* - x_n\|_\infty = O\left(\frac{1}{n^{r+1}}\right), \qquad \|x^* - \tilde{x}_n\|_\infty = O\left(\frac{1}{n^{2r+2}}\right)$$

provided the integrand $K(s, t, u)$ is sufficiently smooth; see Theorems 5.2 and 6.1.

The above schema can also be applied to multivariate problems, but that will be deferred to another paper in which we look at discrete Galerkin methods (in analogy with Atkinson and Bogomolny (1987) for linear integral equations).

3. Urysohn integral operators of class $\mathscr{G}_1(\alpha, \gamma)$. In this and the following sections, we will consider a special class of Urysohn integral operators

$$\mathscr{K}(x)(t) = \int_0^1 K(t, s, x(s))\, ds. \tag{3.1}$$

A theory will be presented that closely parallels that of Chatelin and Lebbar (1984) for linear integral equations.

Let α and γ be integers with $\alpha \geqq \gamma$, $\alpha \geqq 0$, $\gamma \geqq -1$. We will assume that the kernel K has the following properties.

(G_1) The partial derivative

$$l(t, s, u) = \frac{\partial K(t, s, u)}{\partial u} \tag{3.2}$$

exists for all $(t, s, u) \in \Psi \equiv [0, 1] \times [0, 1] \times \mathbb{R}$.

(G_2) Define

$$\Psi_1 = \{(t, s, u) \mid 0 \leqq s \leqq t \leqq 1, u \in \mathbb{R}\},$$

$$\Psi_2 = \{(t, s, u) \mid 0 \leqq t \leqq s \leqq 1, u \in \mathbb{R}\}.$$

There are functions $l_i \in C^\alpha(\Psi_i)$, $i = 1, 2$, with

$$l(t, s, u) = \begin{cases} l_1(t, s, u), & (t, s, u) \in \Psi_1, \quad t \neq s, \\ l_2(t, s, u), & (t, s, u) \in \Psi_2. \end{cases} \tag{3.3}$$

(G_3) If $\gamma \geqq 0$, then $l \in C^\gamma(\Psi)$. If $\gamma = -1$, then l may have a discontinuity of the first kind along the line $s = t$.

The class of kernels satisfying (G_1)–(G_3) will be denoted by $\mathscr{G}_1(\alpha, \gamma)$. The assumption that the variable u ranges over all of $\mathbb{R}$ can be weakened to having u belong to a bounded set; but then the arguments will be more complicated in their details, without any essential difference in the form of the final results. If $K \in \mathscr{G}_1(\alpha, \gamma)$, then it is easily shown that

(G_4) There are two functions $K_i \in C^\alpha(\Psi_i)$, $i = 1, 2$, such that

$$K(t, s, u) = \begin{cases} K_1(t, s, u), & (t, s, u) \in \Psi_1, \quad t \neq s, \\ K_2(t, s, u), & (t, s, u) \in \Psi_2. \end{cases} \tag{3.4}$$

As additional notation, we will let $L^p = L^p(0, 1)$, $1 \leqq p \leqq \infty$, and $C^k = C^k[0, 1]$, $k \geqq 0$. For $x \in C^k$, define

$$\|x\|_{k,p} = \sum_{i=0}^{k} \|x^{(i)}\|_p, \qquad 1 \leqq p \leqq \infty.$$

We will write $\|x\|_p$ for $\|x\|_{0,p}$.

THEOREM 3.1. *Suppose that Urysohn's operator* (3.1) *has a kernel* $K \in \mathscr{G}_1(\alpha, \gamma)$. *Then*

(a) *$\mathscr{K}$ is a completely continuous operator from L^∞ into C^ν, for $\nu = 0, 1, \cdots, \gamma_1$, with $\gamma_1 \equiv \text{Min}\{\gamma + 1, \alpha\}$.*

(b) *For $\alpha \geqq 1$, $\mathscr{K}$ is a continuous operator from C^ν into $C^{\nu+1}$, $\nu = 0, 1, \cdots, \alpha - 1$.*

Proof. (i) We will first show (a) for $\gamma_1 = 0$.

Let B_a be the closed ball in L^∞, of radius a and centered at 0. Define

$$\Psi_i(a)=\{(t, s, u): (t, s, u)\in\Psi_i, |u|\leqq a\}, \qquad i=1,2.$$

Denote

$$M_i=\text{Max}\,\{|K_i(t, s, u)|: (t, s, u)\in\Psi_i(a)\},$$
$$M=\text{Max}\,\{M_1, M_2\}. \tag{3.5}$$

The function K_i is uniformly continuous on $\Psi_i(a)$. Thus for any $\varepsilon>0$, there is $\delta_i>0$ such that

$$|K_i(t+\Delta t, s, u)-K_i(t, s, u)|\leqq\frac{\varepsilon}{2} \tag{3.6}$$

whenever (t, s, u), $(t+\Delta t, s, u)\in\Omega_i(a)$ and $|\Delta t|\leqq\delta_i$. Take

$$\delta=\text{Min}\,\{\delta_1, \delta_2, \varepsilon/(2M_1+2M_2)\}.$$

Let $x\in B_a$, $y=\mathcal{K}(x)$. We wish to show $y\in C$. Given $\varepsilon>0$, if $|\Delta t|\leqq\delta$, then

$$\begin{aligned}|y(t+\Delta t)-y(t)|\leqq&\int_0^{t-\delta}|K_1(t+\Delta t, s, x(s))-K_1(t, s, x(s))|\,ds\\&+\int_{t-\delta}^{t+\delta}|K(t+\Delta t, s, x(s))-K(t, s, x(s))|\,ds\\&+\int_{t+\delta}^{1}|K_2(t+\Delta t, s, x(s))-K_2(t, s, x(s))|\,ds.\end{aligned} \tag{3.7}$$

(For the endpoints $t=0$ and $t=1$, we will have to suitably modify this argument.) The sum of the first and third terms is majorized by $(1-2\delta)\varepsilon/2$.

For the second integral in (3.7), over $[t-\delta, t+\delta]$, let $\Delta t>0$. The integral is majorized by

$$\begin{aligned}&\int_{t-\delta}^{t}|K_1(t+\Delta t, s, x(s))-K_1(t, s, x(s))|\,ds\\&\quad+\int_{t}^{t+\Delta t}|K_1(t+\Delta t, s, x(s))-K_2(t, s, x(s))|\,ds\\&\quad+\int_{t+\Delta t}^{t+\delta}|K_2(t+\Delta t, s, x(s))-K_2(t, s, x(s))|\,ds\\&\leqq\frac{\varepsilon}{2}\,\delta+(M_1+M_2)\Delta t+\frac{\varepsilon}{2}(\delta-\Delta t).\end{aligned}$$

A similar estimate holds if $\Delta t<0$. Thus for $|\Delta t|<\delta$, we have

$$|y(t+\Delta t)-y(t)|\leqq\varepsilon. \tag{3.8}$$

This shows that $y=\mathcal{K}(x)\in C$.

To show compactness of $\mathcal{K}$, note that (3.8) shows that $\mathcal{K}(B_a)$ is an equicontinuous family of functions. In addition, it is straightforward that

$$x\in B_a\Rightarrow\|\mathcal{K}(x)\|_\infty\leqq M.$$

Thus $\mathcal{K}(B_a)$ is a precompact set, by the Arzela-Ascoli theorem.

The continuity of $\mathcal{K}$, from L^∞ into C, is easily proven from (G_1) and (G_2). Thus $\mathcal{K}$ is completely continuous as an operator from L^∞ into C.

(ii) Consider now the case $\gamma_1 \geqq 1$. Let $x \in B_a$, $y = \mathcal{K}(x)$. We can then show that $y \in C^1$, with

$$y'(t) = \int_0^1 \frac{\partial K(t, s, x(s))}{\partial t}\, ds, \qquad 0 \leqq t \leqq 1. \tag{3.9}$$

The argument is much the same as in (i). In fact, the argument in (i) when applied to (3.9) will show that $\mathcal{K}$ is completely continuous from L^∞ to C^1 (with norm $\|x\|_{1,\infty} = \|x\|_\infty + \|x'\|_\infty$).

The general case is obtained in a similar way, using

$$y^{(\nu)}(t) = \int_0^1 \frac{\partial^\nu K(t, s, x(s))}{\partial t^\nu}\, ds, \qquad \nu = 0, 1, \cdots, \gamma_1. \tag{3.10}$$

(iii) To prove (b), consider $\gamma \geqq -1$, $\alpha \geqq 1$. The possibility that $\gamma = -1$ requires us to modify (3.9). For $x \in C \cap B_a$, let $y = \mathcal{K}(x)$. Then we can show, much as in (i) or (ii), that

$$y'(t) = \int_0^1 \frac{\partial K(t, s, x(s))}{\partial t}\, ds + K_1(t, t, x(t)) - K_2(t, t, x(t)). \tag{3.11}$$

Using this formula, we can easily prove that $\mathcal{K}$ is a continuous operator from C into C^1.

If $\gamma \geqq -1$, $\alpha \geqq 2$, and $x \in C^1$, then

$$\begin{aligned} y''(t) = \int_0^1 \frac{\partial^2 K(t, s, x(s))}{\partial t^2}\, ds &+ 2\frac{\partial K_1(t, t, x(t))}{\partial t} - 2\frac{\partial K_2(t, t, x(t))}{\partial t} \\ &- \sum_{i=1}^{2} (-1)^i \left[\frac{\partial K_i(t, t, x(t))}{\partial s} + \frac{\partial K_i(t, t, x(t))}{\partial u} x'(t)\right]. \end{aligned}$$

This shows $\mathcal{K}$ is continuous from C^1 into C^2, and we have proven (b) for $\alpha \leqq 2$. The general case is obtained in a similar manner. □

COROLLARY 3.2. *Let K be of class $\mathcal{G}_1(\alpha, \gamma)$, and consider the Urysohn integral operator $\mathcal{K}$ of* (3.1). *Then if x^* is a fixed point of $\mathcal{K}$, we have $x^* \in C^\alpha$.*

4. The approximating subspaces $\mathcal{P}_{r,\Delta_n}$. For the Urysohn integral operator $\mathcal{K}$ of (3.1), we intend to apply the results of § 2, with $\mathcal{X} = L^\infty$ and $\mathcal{Y} = C$. In this section, we will define the approximating subspaces and will give some results on their approximation properties. The analysis of Galerkin's method will be given in the following section.

Let $\Delta^{(n)}$ denote a partition of $[0, 1]$:

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)} = 1. \tag{4.1}$$

Define $\mathcal{P}_{r,\Delta^{(n)}}$ to be the set of functions that are polynomials of degree $\leqq r$ on each of the subintervals $[t_{i-1}^{(n)}, t_i^{(n)}]$. The space $\mathcal{P}_{r,\Delta^{(n)}}$ is a subspace of L^∞, but not of C. For the partition $\Delta^{(n)}$, define

$$\begin{aligned} h_i^{(n)} = t_i^{(n)} - t_{i-1}^{(n)}, &\qquad h^{(n)} = \mathrm{Max}\, h_i^{(n)}, \\ q^{(n)} = \mathop{\mathrm{Max}}_{1 \leqq i,j \leqq m_n} &\frac{h_i^{(n)}}{h_j^{(n)}}. \end{aligned} \tag{4.2}$$

We choose $P_n : L^\infty \to \mathcal{P}_{r,\Delta^{(n)}}$ to be the restriction to L^∞ of the orthogonal projection of L^2 onto $\mathcal{P}_{r,\Delta^{(n)}}$. We will assume that (2.1) and (2.3) (or equivalently (2.4)) are satisfied. This will certainly be true if $\Delta^{(n)}$ is quasiuniform, i.e.,

$$\mathop{\mathrm{Limit}}_{n \to \infty} m_n = \infty, \qquad \mathop{\mathrm{Sup}}_{n} q^{(n)} < \infty. \tag{4.3}$$

We note that (4.3) implies $h^{(n)} \to 0$ as $n \to \infty$. For more general conditions, see Güsmann (1980).

In order to simplify notation, we will omit the index n in denoting the partition and its elements. Thus we will write $\Delta = \Delta^{(n)}$, $t_i = t_i^{(n)}$, $m = m_n$, $h_i = h_i^{(n)}$, $h = h^{(n)}$, $q = q^{(n)}$. In addition, we will let $\Delta_i = \Delta_i^{(n)} = [t_{i-1}, t_i]$. The dependence on n will be given by retaining the notation P_n for the projection and x_n for the projection solution in (2.5).

The subspace $\mathcal{P}_{r,\Delta}$ is embedded in the space

$$C_\Delta^\nu = \{y \in L^\infty \colon y \,|\, \Delta_i \in C^\nu(\Delta_i),\ i = 1, \cdots, m\}.$$

For $y \in C_\Delta^\nu$, we will write $y_i = y \,|\, \Delta_i$,

$$\|y\|_{\nu,p,\Delta_i} = \|y_i\|_{\nu,p}, \quad \|y\|_{p,\Delta_i} = \|y_i\|_{0,p}, \quad 1 \leqq p \leqq \infty$$

for $1 \leqq i \leqq m$, $\nu \geqq 0$. It is easily seen that for $z \in L^\infty$,

$$(P_n z) \,|\, \Delta_i = P_{n,i} z_i$$

where $P_{n,i}$ is the orthogonal projection of $L^2(\Delta_i)$ onto $\mathcal{P}_{r,\Delta_i}$, the polynomials of degree $\leqq r$ on Δ_i.

We generalize Theorem 3.1 to $\mathcal{K}$ acting on C_Δ^ν.

THEOREM 4.1. *Assume* $K(s, t, u)$ *is of class* $\mathcal{G}_1(\alpha, \gamma)$. *Then the Urysohn operator* (3.1) *is a continuous operator on* C_Δ^ν *into* $C_\Delta^{\gamma_{\nu+2}}$, $\nu \geqq 0$, *with* $\gamma_{\nu+2} = \operatorname{Min}\{\gamma + \nu + 2, \alpha\}$. (*Note that* γ_1 *was defined in Theorem* 3.1.)

Proof. We first prove the case $\nu = 0$. Let $x \in C_\Delta$, and set $y = \mathcal{K}(x)$. From Theorem 3.1, $y \in C^{\gamma_1}$ and

$$y^{(\mu)}(t) = \int_0^1 \frac{\partial^\mu K(t, s, x(s))}{\partial t^\mu}\, ds, \qquad 0 \leqq \mu \leqq \gamma_1. \tag{4.4}$$

If $\alpha = \gamma_1$, the proof is finished.

Suppose $\alpha > \gamma_1$, and let $0 < t < 1$, $t \notin \Delta$. Let us calculate the limit as $\Delta t \to 0$ of

$$\frac{1}{\Delta t} \int_{t-\Delta t}^{t+\Delta t} \left[\frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K(t + \Delta t, s, x(s)) - \frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K(t, s, x(s)) \right] ds. \tag{4.5}$$

Let $\Delta t > 0$. For Δt sufficiently small, $(t - \Delta t, t)$ and $(t, t + \Delta t)$ do not contain any element from Δ. Then (4.5) can be written as

$$\frac{1}{\Delta t} \int_{t-\Delta t}^{t} \left[\frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_1(t + \Delta t, s, x(s)) - \frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_1(t, s, x(s)) \right] ds$$
$$+ \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \left[\frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_1(t + \Delta t, s, x(s)) - \frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_2(t, s, x(s)) \right] ds.$$

Taking the limit as $\Delta t \to 0$, we obtain the value

$$\frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_1(t, t, x(t)) - \frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_2(t, t, x(t)).$$

The same limit is obtained for $\Delta t < 0$, as $\Delta t \to 0$.

This result can be combined with a proof such as that given in Theorem 3.1, to show that $y \in C_\Delta^{\gamma_1+1}$. More explicitly, if $t \notin \Delta$, then

$$y^{(\gamma_1+1)}(t) = \int_0^1 \frac{\partial^{\gamma_1+1}}{\partial t^{\gamma_1+1}} K(t, s, x(s))\, ds - \sum_{i=1}^{2} (-1)^i \frac{\partial^{\gamma_1}}{\partial t^{\gamma_1}} K_i(t, t, x(t)). \tag{4.6}$$

By our assumptions on K_1 and K_2, the values $y^{(\gamma_1+1)}(t+0)$ and $y^{(\gamma_1+1)}(t-0)$ will exist, using limits in (4.6), for all $t \in \Delta$. This completes the proof for the case $\nu = 0$.

The general case is obtained in a similar way. For example, if $x \in C^1_\Delta$ and if $\gamma+3 \leqq \alpha$, then for $t \notin \Delta$, we have

$$(4.7)\quad \begin{aligned} y^{(\gamma_1+2)}(t) &= \int_0^1 \frac{\partial^{\gamma_1+2}}{\partial t^{\gamma_1+2}} K(t,s,x(s))\,ds - 2\sum_{i=1}^{2} (-1)^i \frac{\partial^{\gamma_1+1}}{\partial t^{\gamma_1+1}} K_i(t,t,x(t)) \\ &\quad - \sum_{i=1}^{2} (-1)^i \left[\frac{\partial^{\gamma_1+1}}{\partial s\, \partial t^{\gamma_1}} K_i(t,t,x(t)) + \frac{\partial^{\gamma_1+1}}{\partial u\, \partial t^{\gamma_1}} K_i(t,t,x(t))x'(t) \right]. \quad \square \end{aligned}$$

In the remainder of this section, we give some approximating properties of $\mathcal{P}_{r,\Delta}$ which will be used in the proof of our main theorem, in § 5. First, we give a slight improvement of Theorem 6 of Chatelin and Lebbar (1984). To obtain explicit estimates of the constants appearing in the L^p error bounds of the remainder, we approximate $f \in C^\alpha[a,b]$ by its Taylor polynomial of degree $\beta-1$,

$$(4.8)\quad [\mathcal{T}_\beta(f)](t) = \sum_{j=0}^{\beta-1} \frac{f^{(j)}(a)}{j!}(t-a)^j$$

where $\beta := \min\{\alpha, r+1\}$.

For $1<p<\infty$ and $0 \leqq j < \beta$, let $q>1$ satisfy $1/p+1/q=1$. Then define

$$(4.9)\quad c(\beta,j,p) = p^{-1/p}\left[\frac{\beta-j}{(\beta-j-1)q+1}\right]^{1/q}.$$

Also define

$$(4.10)\quad \begin{aligned} &c(\beta,\beta,p)=1, \qquad 1 \leqq p \leqq \infty, \\ &c(\beta,j,1)=c(\beta,j,\infty)=1, \qquad 0 \leqq j \leqq \beta. \end{aligned}$$

Using this notation, we have the following proposition.

PROPOSITION 4.2. *If $f \in C^\alpha[a,b]$ and $1 \leqq p \leqq \infty$, then*

$$(4.11)\quad \|f^{(j)} - \mathcal{T}_\beta(f)^{(j)}\|_p \leqq c(\beta,j,p)\frac{(b-a)^{\beta-j}}{(\beta-j)!}\|f^{(\beta)}\|_p.$$

Proof. Using the integral form of the remainder, for $0 \leqq j < \beta$,

$$f^{(j)}(t) - \mathcal{T}_\beta f^{(j)}(t) = \frac{1}{(\beta-j-1)!}\int_a^t (t-s)^{\beta-j-1} f^{(\beta)}(s)\,ds.$$

Let $1<p<\infty$. Applying Hölder's inequality, this gives

$$\|f^{(j)} - \mathcal{T}_\beta f^{(j)}\|_p \leqq \frac{\|f^{(\beta)}\|_p}{(\beta-j-1)!}\left\{\int_a^b \left[\int_a^t (t-s)^{(\beta-j-1)q}\,ds\right]^{p/q} dt\right\}^{1/p}.$$

Carry out the exact integration, and then simplify to get (4.11) and (4.9). The cases $p=1$ and $p=\infty$ are straightforward; and moreover $c(\beta,j,p)$ approaches 1 as $p \to 1$ or ∞. The case $j=\beta$ is trivial, since then $\mathcal{T}f^{(\beta)} \equiv 0$. $\square$

Following Chatelin and Lebbar (1984), we obtain the following.

COROLLARY 4.3. *Let $\{\pi_n\}$ be a sequence of projections from C_Δ onto $\mathcal{P}_{r,\Delta}$ such that*

$$\operatorname{Sup} \|\pi_n\|_p < \infty$$

for some $1 \leqq p \leqq \infty$. Then there is a constant c_p such that for any $g \in C^\alpha_\Delta$,

$$(4.12)\quad \|(I-\pi_n)g\|_p \leqq c_p h^\beta \|g^{(\beta)}\|_p$$

where $\beta = \operatorname{Min}\{\alpha, r+1\}$.

Suppose the Urysohn kernel $K(t, s, u)$ is of class $\mathcal{G}_1(\alpha, \gamma)$, and consider the function

$$l_*(t, s) \equiv l_t(s) = \frac{\partial K(t, s, x^*(s))}{\partial u} \tag{4.13}$$

where x^* is a fixed point of the Urysohn operator (3.1). Using Corollary 3.2, it follows that l_* belongs to the class $\mathcal{G}(\alpha, \gamma)$ defined in Chatelin and Lebbar (1984):

A function $w:[0, 1]\times[0, 1]\to\mathbb{C}$ is of class $\mathcal{G}(\alpha, \gamma)$ (with $\alpha \geqq \gamma$, $\alpha \geqq 0$, $\gamma \geqq -1$) if and only if

$$w(t, s) = \begin{cases} w_1(t, s), & 0 \leqq s < t \leqq 1, \\ w_2(t, s), & 0 \leqq t \leqq s \leqq 1 \end{cases}$$

with $w_1 \in C^\alpha(\{0 \leqq s \leqq t \leqq 1\})$, $w_2 \in C^\alpha(\{0 \leqq t \leqq s \leqq 1\})$ and $w \in C^\gamma([0, 1]\times[0, 1])$ for $\gamma \geqq 0$. In case $\gamma = -1$, w may have a discontinuity of the first kind on $\{s = t\}$.

Let us denote

$$\begin{aligned} \beta = \operatorname{Min}\{\alpha, r+1\}, \qquad \beta_1 = \operatorname{Min}\{\beta, \gamma+1\} = \operatorname{Min}\{\alpha, r+1, \gamma+1\}, \\ \beta_2 = \operatorname{Min}\{\beta, \gamma+2\} = \operatorname{Min}\{\alpha, r+1, \gamma+2\}. \end{aligned} \tag{4.14}$$

Following Chatelin and Lebbar (1984), we have the following two results.

COROLLARY 4.4. *Let $w \in \mathcal{G}(\alpha, \gamma)$ and set $w_t(s) = w(t, s)$, $0 \leqq t, s \leqq 1$. Let $\{\pi_n\}$ be a sequence of projections as in Corollary 4.3. Then there are constants c_p', c_p'' such that*

$$\|(I - \pi_n)w_t\|_p \leqq c_p' h^\beta \quad \text{for all } t \in \Delta,$$

$$\|(I - \pi_n)w_t\|_p \leqq c_p'' h^{\beta_1} \quad \text{for all } t \in [0, 1]\backslash\Delta.$$

PROPOSITION 4.5. *Let $w \in \mathcal{G}(\alpha, \gamma)$ and set $w_t(s) = w(t, s)$. Let P_n be the orthogonal projection of C_Δ onto $\mathcal{P}_{r,\Delta}$, $n \geqq 1$. Let $x \in C_\Delta^\alpha$. Then there are constants c', c'' such that*

$$|((I - P_n)x, \overline{w_t})| \leqq c' h^{2\beta}, \qquad t \in \Delta,$$

$$|((I - P_n)x, \overline{w_t})| \leqq c'' h^{\beta+\beta_2}, \qquad t \in [0, 1]\backslash\Delta.$$

5. The order of convergence of the Galerkin method. In this section, we will establish the order of convergence of the Galerkin and the iterated Galerkin methods for solving $x = \mathcal{K}(x)$, where $\mathcal{K}$ is the Urysohn integral operator (3.1). To this end, we will apply the results of § 2 with $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_n$, P_n defined as in the beginning of § 4.

The Fréchet derivative of $\mathcal{K}$ at x is given by

$$\mathcal{K}'(x)g(t) = \int_0^1 l(t, s, x(s))g(s)\, ds \tag{5.1}$$

with l defined in (3.2). As before, we denote $L = \mathcal{K}'(x^*)$, where x^* is the fixed point of $\mathcal{K}$ which we want to calculate. We have

$$(Lg)(t) = (g, \bar{l}_t) \tag{5.2}$$

where k_t is defined in (4.13) and $(\cdot, \cdot)$ denotes the inner product of L^2. As in § 2, we assume 1 is not an eigenvalue of L. Since L is compact, this means that $(I - L)^{-1}$ exists and is bounded on $\mathcal{X}$. Hence the operator

$$M := (I - L)^{-1}L$$

is compact. In the proof of our main theorem, we will need the following result.

LEMMA 5.1. *There is a function* $m_* \in \mathscr{G}(\alpha, \gamma)$ *such that*

$$Mg(t) = (g, \bar{m}_t), \qquad u \in L^2 \tag{5.3}$$

where

$$m_t(s) = m_*(t, s), \qquad 0 \leqq s \leqq 1. \tag{5.4}$$

Proof. The existence of the function $m_* \in L^2([0,1]\times[0,1])$ satisfying (5.3) follows from Riesz and Sz.-Nagy (1955, p. 158). We must prove $m_* \in \mathscr{G}(\alpha, \gamma)$.

For any $x \in L^2$, let $y = Mx$. Then $Lx = (I-L)y$. This implies that for any $x \in L^2$,

$$\int_0^1 l_*(t,s)x(s)\,ds = \int_0^1 \left[m_*(t,s) - \int_0^1 l_*(t,v)m_*(v,s)\,dv \right] x(s)\,ds.$$

If we denote

$$m_{*s}(t) = m_*(t,s), \qquad l_{*s}(t) = l_*(t,s), \tag{5.5}$$

m_{*s} satisfies the equation $m_{*s} = Lm_{*s} + l_{*s}$. It follows by a straightforward argument that for any fixed s, $0 \leqq s \leqq 1$,

$$m_{*s}|[0,s] \in C^\alpha[0,s], \qquad m_*|[s,1] \in C^\alpha[s,1], \qquad m_{*s} \in C^\gamma. \tag{5.6}$$

Similarly, for any $u \in L^2$, let $w = M^*u$. Then $L^*u = (I - L^*)w$ can be written as

$$\int_0^1 \bar{l}_*(t,s)u(t)\,dt = \int_0^1 \left[\bar{m}_*(t,s) - \int_0^1 \bar{l}_*(v,s)\bar{m}_*(t,v)\,dv \right] u(t)\,dt.$$

This leads to

$$m_t(s) = \int_0^1 l_*(v,s)m_t(v)\,dv + l_*(t,s).$$

By a similar argument to that for m_{*s}, it follows that

$$m_t|[0,t] \in C^\alpha[0,t], \qquad m_t|[t,1] \in C^\alpha[t,1], \qquad m_t \in C^\gamma. \tag{5.7}$$

Combining (5.6) and (5.7) implies $m_* \in \mathscr{G}(\alpha, \gamma)$. □

In order to obtain convenient estimates for the numbers r_n of (2.13), we want the Fréchet derivative of $\mathscr{K}$ in (3.1) to be Lipschitz continuous in a neighborhood V of x^*, i.e., to satisfy (2.15). There are rather general sufficient conditions to ensure (2.15). However, in order to simplify the presentation, we will assume the stronger condition

$$\frac{\partial^2 K_i}{\partial u^2} \in C(\Psi_i), \qquad i = 1, 2. \tag{5.8}$$

This guarantees that (2.15) is satisfied, in both L^2 and L^∞ norms, where V can be any bounded subset of L^∞. If $K \in \mathscr{G}_1(\alpha, \gamma)$ and satisfies (5.8), we will say that K is of class $\mathscr{G}_2(\alpha, \gamma)$. We now state our main result.

THEOREM 5.2. *Assume* $K \in \mathscr{G}_2(\alpha, \gamma)$; *and let* x^* *be a fixed point of the Urysohn operator* $\mathscr{K}$ *of* (3.1), *with* 1 *not an eigenvalue of* $\mathscr{K}'(x^*)$. *Then for n sufficiently large, the Galerkin solution* x_n *of* (1.3) *and the iterated Galerkin solution* $\tilde{x}_n$ *of* (1.4), *corresponding to* x^*, *will satisfy*

$$\|x_n - x^*\|_\infty = O(h^\beta), \tag{5.9}$$

$$\|\tilde{x}_n - x^*\|_\infty = O(h^{\beta+\beta_2}), \tag{5.10}$$

$$\operatorname*{Sup}_{t \in \Delta^{(n)}} |\tilde{x}_n(t) - x^*(t)| = O(h^{2\beta}). \tag{5.11}$$

Recall $\beta = \operatorname{Min}\{\alpha, r+1\}$, $\beta_2 = \operatorname{Min}\{\alpha, r+1, \gamma+2\}$.

Proof. From Corollaries 3.2 and 4.3, we have $\|P_n x^* - x^*\|_\infty = O(h^\beta)$; and using Theorem 2.2, we have (5.9).

Applying $(I-L)^{-1}$ to (2.28), we get

$$\tilde{x}_n - x^* = [I + MP_n][\mathcal{K}(x_n) - \mathcal{K}(x^*) - L(x_n - x^*)] \\ - M(I-P_n)L(x_n - x^*) - M(I-P_n)x^*. \tag{5.12}$$

From (2.3) with $\mathcal{Y} = C$, (2.13), (2.16), (5.9), (5.2) and (5.3), it follows that

$$\tilde{x}_n(t) - x^*(t) = \varphi_n(t) + \psi_n(t) + O(h^{2\beta}), \qquad 0 \leqq t \leqq 1 \tag{5.13}$$

with

$$\varphi_n(t) = -((I-P_n)L(x_n - x^*), \bar{m}_t), \tag{5.14}$$

$$\psi_n(t) = -((I-P_n)x^*, \bar{m}_t). \tag{5.15}$$

If we use Corollary 3.2 and Proposition 4.5, it follows that

$$\operatorname*{Sup}_{0\leqq t\leqq 1} |\psi_n(t)| = O(h^{\beta+\beta_2}), \qquad \operatorname*{Sup}_{t\in\Delta^{(n)}} |\psi_n(t)| = O(h^{2\beta}). \tag{5.16}$$

To bound $\varphi_n(t)$, begin with

$$|\varphi_n(t)| = \sum_{j=1}^{n} |((I-P_n)L(x_n - x^*), (I-P_n)\bar{m}_t)_j|$$

where the subscript j denotes that the inner product is over $\Delta_j = [t_{j-1}, t_j]$. In the sum, use the Cauchy-Schwarz inequality to obtain

$$|\varphi_n(t)| = \sum_{j=1}^{n} \|(I-P_n)L(x_n - x^*)\|_{2,j} \|(I-P_n)\bar{m}_t\|_{2,j}.$$

In this sum,

$$\begin{aligned} \|(I-P_n)L(x_n - x^*)\|_{2,j} &\leqq h_j^{1/2} \|(I-P_n)L(x_n - x^*)\|_{\infty,j} \\ &\leqq h_j^{1/2} \|(I-P_n)L\| \, \|x_n - x^*\|_\infty \\ &\leqq c h_j^{1/2} h^{\beta_1+\beta}. \end{aligned}$$

The result $\|(I-P_n)L\| = O(h^{\beta_1})$ follows as in the proof of Chatelin and Lebbar (1984, Cor. 8). Also, using the proof in Lemma 9 of the same paper, if $t \in \Delta_i$, then

$$\|(I-P_n)\bar{m}_t\|_{2,j} = \begin{cases} O(h_j^{\beta+1/2}), & j \neq i, \\ O(h_i^{\beta_1+1/2}), & j = i. \end{cases}$$

If $t \in \Delta^{(n)}$, then

$$\|(I-P_n)\bar{m}_t\|_{2,j} = O(h_j^{\beta+1/2}), \qquad j = 1, \cdots, n.$$

When we combine these results,

$$\begin{aligned} &\operatorname*{Max}_{0\leqq t\leqq 1} |\varphi_n(t)| \leqq O(h^{2\beta+\beta_1}) + O(h^{2\beta_1+\beta+1}) \leqq O(h^{\beta+\beta_1+\beta_2}), \\ &\operatorname*{Max}_{t\in\Delta^{(n)}} |\varphi_n(t)| \leqq O(h^{2\beta+\beta_1}). \end{aligned} \tag{5.17}$$

Combining (5.13), (5.16) and (5.17) proves (5.10) and (5.11). □

The above theorem shows that the sequence $\{\tilde{x}_n\}$ obtained via the iterated Galerkin method converges faster to the solution of $x = \mathcal{K}(x)$ than does the sequence $\{x_n\}$ obtained from the original Galerkin method. This is further illustrated, sometimes dramatically, in the numerical examples of § 7.

In addition to the convergence of $\{x_n\}$ and $\{\tilde{x}_n\}$, it also turns out that the first γ_2 (see Theorem 4.1) derivatives of $\tilde{x}_n$ converge to the corresponding derivatives of x^*. Before giving a precise statement of this fact, note that

$$\tilde{x}_n \in C^{\gamma_1} \cap C^{\alpha}_{\Delta}. \tag{5.18}$$

This follows from Theorems 3.1 and 4.1, the definition $\tilde{x}_n = \mathcal{K}(x_n)$, and the fact that $x_n \in C^{\infty}_{\Delta}$. Thus in general, $\tilde{x}_n^{(\gamma_2)}(t)$ may not exist at the grid points (i.e. $t \in \Delta^{(n)}$). Nonetheless, we will be able to prove that

$$\operatorname*{Max}_{t\in[0,1]\backslash\Delta} |\tilde{x}_n^{(\gamma_2)}(t) - x^{*(\gamma_2)}(t)| = O(h^{\beta}). \tag{5.19}$$

The other derivatives of $\tilde{x}_n$ (i.e. $\tilde{x}_n^{(\nu)}$ with $\gamma_2 < \nu \leqq \alpha$) may not converge to the corresponding derivative of x^*.

THEOREM 5.3. *Assume the kernel of the Urysohn integral operator* (3.1) *is of class* $\mathcal{G}_2(\alpha, \gamma)$. *Then* (5.19) *holds for sufficiently large n. Moreover, for any* $0 \leqq \nu \leqq \gamma_1$,

$$\operatorname*{Sup}_{t\in[0,1]} |\tilde{x}_n^{(\nu)}(t) - x^{*(\nu)}(t)| = O(h^{\beta+\beta_{2,\nu}}) \tag{5.20}$$

with

$$\beta_{2,\nu} = \operatorname{Min}\{r+1, \alpha-\nu, \gamma+2-\nu\}.$$

Proof. (a) We first show (5.19). If $\gamma_1 = \gamma_2$, then use (4.4) to examine $\tilde{x}_n^{(\gamma_2)}(t) - x^{*(\gamma_2)}(t)$; and if $\gamma_2 = \gamma_1 + 1$, use (4.6). In the latter case let $x = x^*$ and $x = x_n$ in (4.6), and then use the mean-value theorem to prove

$$|\tilde{x}_n^{(\gamma_2)}(t) - x^{*(\gamma_2)}(t)| \leqq C\|x_n - x^*\|_\infty = O(h^{\beta})$$

where

$$C \leqq \sum_{i=1}^{2} \left\{ \operatorname*{Sup}_{\Omega_i(a)} \left| \frac{\partial^{\gamma_1+2} K_i}{\partial u\, \partial t^{\gamma_1+1}} \right| + \operatorname*{Sup}_{\Omega_i(a)} \left| \frac{\partial^{\gamma_1+1} K_i}{\partial u\, \partial t^{\gamma_1}} \right| \right\}$$

with $a > \operatorname{Sup}_n \|x_n\|_\infty$. In the case $\gamma_1 = \gamma_2$, the proof is essentially the same.

(b) To show (5.20), fix a as above, let $0 \leqq \nu \leqq \gamma_1$, and let $\mathcal{N}$ denote the Urysohn integral operator with kernel $\partial^\nu K/\partial t^\nu$. From (4.4),

$$\tilde{x}_n^{(\nu)} - x^{*(\nu)} = \mathcal{N}(x_n) - \mathcal{N}(x^*). \tag{5.21}$$

Since the kernel of $\mathcal{K}$ is in $\mathcal{G}_2(\alpha, \gamma)$, it follows that the kernel of $\mathcal{N}$ is of class $\mathcal{G}_2(\alpha-\nu, \gamma-\nu)$. In particular, $\mathcal{N}$ is Fréchet differentiable and its Fréchet derivative is Lipschitz continuous in any bounded neighborhood of x^* in L^∞. It follows that

$$\tilde{x}_n^{(\nu)} - x^{*(\nu)} = \mathcal{N}'(x^*)(x_n - x^*) + z_n \tag{5.22}$$

with

$$\|z_n\|_\infty \leqq c\|x_n - x^*\|_\infty^2 = O(h^{2\beta}). \tag{5.23}$$

Let us denote $G = \mathcal{N}'(x^*)$. By a simple manipulation,

$$G(x_n - x^*) = GP_n(\tilde{x}_n - x^*) + G(P_n - I)x^*. \tag{5.24}$$

By Theorem 5.2,

$$\|GP_n(\tilde{x}_n - x^*)\|_\infty \leqq \|GP_n\|\,\|\tilde{x}_n - x^*\| \leqq O(h^{\beta+\beta_2}). \tag{5.25}$$

For the last term in (5.24), use (5.16) with L replaced by G. Then

$$\|G(P_n - I)x^*\|_\infty = O(h^{\beta+\beta_{2,\nu}}).$$

Combining (5.22)-(5.25) proves (5.20). □

6. The collocation method. In what follows, we make use of the notation introduced in §§ 3 and 4. On each subinterval $\Delta_i = [t_{i-1}, t_i]$, let the nodes $\{\tau_j^i\}$,

$$t_{i-1} < \tau_0^i < \cdots < \tau_r^i < t_i \tag{6.1}$$

denote the Gauss-Legendre nodes relative to Δ_i. We define the projection operator

$$Q_n : C_\Delta \to \mathcal{P}_{r,\Delta}$$

by having

$$Q_{n,i}y \equiv (Q_n y)|\Delta_i, \qquad y \in C_\Delta$$

be the polynomial of degree $\leqq r$ that interpolates y at the $r+1$ Gaussian points of (6.1). We assume that the partitions $\Delta^{(n)}$ are such that

$$\operatorname*{Sup}_n \|Q_n|C_\Delta\|_\infty < \infty. \tag{6.2}$$

Let us consider the nonlinear equations

$$y = \mathcal{K}(y), \tag{6.3}$$

$$y = Q_n\mathcal{K}(y), \tag{6.4}$$

$$y = \mathcal{K}(Q_n y), \tag{6.5}$$

with $\mathcal{K}$ the Urysohn operator of (3.1). As before, we assume x^* is a solution of (6.3), with 1 not an eigenvalue of $L = \mathcal{K}'(x^*)$. From (6.2) and Theorem 2.1, it follows that for sufficiently large n, (6.4) has a solution y_n that is unique within some fixed neighborhood of x^* and for which $\|y_n - x^*\|_\infty \to 0$. Further, $\tilde{y}_n = \mathcal{K}(y_n)$ is a solution of (6.5), and it will converge to x^* at least as rapidly as $\{y_n\}$. The approximating equation (6.4) is called the *collocation method*, and (6.5) is called the *iterated collocation method.* The order of convergence of these methods is given by the following.

THEOREM 6.1. *If the kernel of the Urysohn operator* (3.1) *is of class* $\mathcal{G}_2(\alpha, \gamma)$, *then*

$$\|y_n - x^*\|_\infty = O(h^\beta). \tag{6.6}$$

In addition, if $\alpha \geqq r+1$, *then*

$$\|\tilde{y}_n - x^*\|_\infty = O(h^{\omega_1}), \tag{6.7}$$

$$\operatorname*{Sup}_{t \in \Delta^{(n)}} |\tilde{y}_n(t) - x^*(t)| = O(h^{\omega_2}) \tag{6.8}$$

with

$$\omega_1 = \operatorname{Min}\{\alpha, 2r+2, r+\gamma+3\}, \qquad \omega_2 = \operatorname{Min}\{\alpha, 2r+2\}. \tag{6.9}$$

In proof, we will use some results from the preceding sections, as well as the following results of Chatelin and Lebbar (1984). Let

$$\eta = \alpha - r - 1, \quad \tilde{\beta} = \operatorname{Min}\{\eta, r+1\}, \quad \tilde{\beta}_2 = \operatorname{Min}\{\tilde{\beta}, \gamma+2\}.$$

Then also

$$\omega_1 = r+1+\tilde{\beta}_2, \qquad \omega_2 = r+1+\tilde{\beta}.$$

LEMMA 6.2. *For* $f, g \in C(\Delta_i)$,

$$((I - Q_{n,i})g, \bar{f})_i = ((I - P_{n,i})f\delta^{r+1}g, v_i)_i \tag{6.10}$$

where P_n *is the orthogonal projection of* § 5,

$$v_i(s) = \prod_{j=0}^{r} (s - \tau_j^i),$$

$\delta^{r+1}g(s)$ is the divided difference of g at $\{\tau_0^i, \cdots, \tau_r^i, s\}$ and $(\cdot,\cdot)_i$ denotes the inner product of $L^2(\Delta_i)$.

COROLLARY 6.3. *Let η be a nonnegative integer. If $f \in C^\eta(\Delta_i)$ and $g \in C^\alpha(\Delta_i)$, then*

$$|((I-Q_{n,i})g, \bar{f})_i| \leqq ch^{r+\tilde{\beta}+2}\|f\|_{\tilde{\beta},\infty,\Delta_i}\|g\|_{\tilde{\beta}+r+1,\infty,\Delta_i}. \qquad (6.11)$$ □

Proof of Theorem 6.1. From Corollaries 3.2 and 4.3,

$$\|x^*-Q_nx^*\|_\infty = O(h^\beta), \qquad \beta = \text{Min}\{\alpha, r+1\}, \qquad (6.12)$$

and then (6.6) follows from Theorem 2.2.

To examine $\tilde{y}_n - x^*$, take Q_n, y_n, $\tilde{y}_n$ in place of P_n, x_n, $\tilde{x}_n$ in the derivation of (5.13):

$$\tilde{y}_n(t) - x^*(t) = \tilde{\varphi}_n(t) + \tilde{\psi}_n(t) + O(h^{2\beta}) \qquad (6.13)$$

with

$$\tilde{\varphi}_n(t) = ((Q_n - I)L(y_n - x^*), \bar{m}_t), \qquad \tilde{\psi}_n(t) = ((Q_n - I)x^*, \bar{m}_t). \qquad (6.14)$$

If $t \in \Delta$, then apply Corollaries 3.2 and 6.3 to obtain

$$|((Q_{n,i} - I)x^*, \bar{m}_t)_i| \leqq ch_i^{r+2+\tilde{\beta}}, \qquad t \in \Delta \qquad (6.15)$$

where c is independent of i and n. Using

$$(x, y) = \sum_1^n (x, y)_i, \qquad (6.16)$$

we deduce that

$$\sup_{t\in\Delta} |\tilde{\psi}_n(t)| \leqq \tilde{c}h^{r+1+\tilde{\beta}}. \qquad (6.17)$$

Now we want to bound $\tilde{\psi}(t)$ over $[0, 1]$. Let $t_{j-1} < t < t_j$. For $i \neq j$, the bound in (6.15) is still valid. For $i = j$, we must take account of the fact that $m_t \in c^\gamma(\Delta_j)$. By carefully bounding the quantities in

$$\begin{aligned} |((I-Q_{n,j})x^*, \bar{m}_t)_j| &= |((I-P_{n,j})m_t\delta^{r+1}x^*, v_j)_j| \\ &\leqq \|(I-P_{n,j})m_t\delta^{r+1}x^*\|_{2,\Delta_j}\|(I-P_{n,j})v_j\|_{2,\Delta_j} \end{aligned}$$

we can show, using Chatelin and Lebbar (1984, Lemma 9), that

$$|((I-Q_{n,j})x^*, \bar{m}_t)_j| \leqq ch_j^{r+1+\tilde{\beta}_2}. \qquad (6.18)$$

Combining (6.18) with (6.16) and (6.15), $i \neq j$, we obtain

$$\sup_{t\in[0,1]} |\tilde{\psi}_n(t)| \leqq ch^{r+1+\tilde{\beta}_2}. \qquad (6.19)$$

We now consider the term $\tilde{\varphi}_n(t)$ in (6.14). By Lemma 6.2,

$$\begin{aligned} |((I-Q_{n,i})L(y_n-x^*), \bar{m}_t)_i| &= |((I-P_{n,i})m_t\delta^{r+1}L(y_n-x^*), v_i)_i| \\ &\leqq \|(I-P_{n,i})m_t\delta^{r+1}L(y_n-x^*)\|_{2,\Delta_i}\|(I-P_{n,i})v_i\|_{2,\Delta_i} \\ &\leqq ch_i^{r+2}\|y_n-x^*\|_\infty \leqq ch_i^{r+2+\beta}. \end{aligned}$$

Using (6.16), we have

$$|\tilde{\varphi}_n(t)| \leqq ch^{r+1+\beta}. \qquad (6.20)$$

Combining (6.13), (6.17) and (6.20) yields (6.8); and combining (6.13), (6.19) and (6.20) yields (6.7). □

Concerning the convergence of the derivatives of $\tilde{y}_n$, we have the following result.

THEOREM 6.4. *Assume that the kernel of the Urysohn integral operator* (3.1) *is of class* $\mathscr{G}_2(\alpha, \gamma)$. *Then* $\tilde{y}_n \in C^{\gamma_1} \cap C^{\alpha}_{\Delta}$, *and*

$$\|\tilde{y}_n^{(\nu)} - x^{*(\nu)}\|_\infty = O(h^\beta), \qquad \nu = 1, 3, \cdots, \gamma_2. \tag{6.21}$$

In addition, if $\alpha \geqq r+1$, *then*

$$\|\tilde{y}_n^{(\nu)} - x^{*(\nu)}\|_\infty = O(h^{\omega_{1,\nu}}), \qquad 0 \leqq \nu \leqq \tilde{\mu} \tag{6.22}$$

where

$$\tilde{\mu} = \mathrm{Min}\,\{\alpha - r - 1, \gamma + 1\}, \qquad \omega_{1,\nu} = r + 1 + \tilde{\beta}_{2,\nu},$$

$$\tilde{\beta}_{2,\nu} = \mathrm{Min}\,\{\alpha - r - 1 - \nu, \gamma - \nu + 2\}.$$

Proof. The proof of (6.21) is exactly the same as that given for (5.19) in Theorem 5.3. To prove (6.22), we also follow the proof of Theorem 5.3. Use (5.22)-(5.24), with x_n, $\tilde{x}_n$, P_n replaced by y_n, $\tilde{y}_n$, Q_n, respectively. Then

$$\tilde{y}_n^{(\nu)} - x^{*(\nu)} = G(y_n - x^*) + z_n, \qquad G = \mathscr{N}'(x^*), \tag{6.23}$$

$$\|z_n\|_\infty \leqq c\|y_n - x^*\|_\infty^2 = O(h^{2\beta}), \tag{6.24}$$

$$G(y_n - x^*) = GQ_n(\tilde{y}_n - x^*) + G(Q_n - I)x^*. \tag{6.25}$$

From (6.7),

$$\|GQ_n(\tilde{y}_n - x^*)\|_\infty = O(h^{\omega_1}). \tag{6.26}$$

The linear integral operator $G = \mathscr{N}'(x^*)$ has the kernel function

$$g(t, s) = \frac{\partial^\nu l(t, s, x^*(s))}{\partial t^\nu}.$$

From the assumptions on K, it follows that $g(t, s)$ is the Chatelin and Lebbar class $\mathscr{G}(\alpha - \nu, \gamma - \nu)$. For the last term in (6.25), we have

$$\|G(Q_n - I)x^*\|_\infty = \sup_{0 \leqq t \leqq 1} |((Q_n - I)x^*, g_t)|.$$

From the derivation used to obtain (6.19), we have

$$\|G(Q_n - I)x^*\|_\infty \leqq ch^{r+1+\tilde{\beta}_{2,\nu}} \tag{6.27}$$

with

$$\tilde{\beta}_{2,\nu} = \mathrm{Min}\,\{\alpha - \nu - r - 1, r + 1, \gamma - \nu + 2\}. \tag{6.28}$$

Combine (6.23)-(6.28) to conclude the proof of (6.22). □

7. Numerical examples. We illustrate the convergence results that were given in Theorem 5.2 for the Galerkin and iterated Galerkin methods. We give results for two integral equations.

Our first equation is

$$x(t) = \int_0^1 \frac{ds}{t + s + x(s)} + y(t), \qquad 0 \leqq t \leqq 1 \tag{7.1}$$

where y is so chosen that

$$x^*(t) = \frac{1}{t+c}, \qquad c > 0 \tag{7.2}$$

is a solution of (7.1). The function K is given by

$$K(t, s, u) = \frac{1}{t+s+u} + y(t).$$

In this case, the constants α and γ can be chosen as large as desired. Therefore the constants of (4.14) are given by

$$\beta = \beta_1 = \beta_2 = r + 1.$$

From Theorem 5.2,

$$\|x^* - x_n\|_\infty = O(h^{r+1}), \qquad \|x^* - \tilde{x}_n\|_\infty = O(h^{2r+2}). \tag{7.3}$$

The results for $c = 1$ are given in Tables 1 and 2, for degrees $r = 1$ and $r = 2$. The analogous results for $c = .1$ are given in Tables 3 and 4. The latter are worse because x^* becomes more ill behaved as $c \to 0$. In both cases, the rates predicted by (7.3) are confirmed approximately by the numerical results. Of special note is the great improvement in accuracy given by the iterated Galerkin solution $\tilde{x}_n$ over that of the Galerkin solution x_n. We will let n denote the number of (equal) subdivisions of [0, 1], as given in (4.1). The number of equations used in solving for x_n is denoted by n_e; and $h = 1/n$.

TABLE 1
$x^* = 1/(t+1)$; $r = 1$.

n	n_e	$\|x^*-x_n\|_\infty$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio
2	4	2.51E−2		4.02E−6	
4	8	7.92E−3	3.17	7.83E−7	5.13
8	16	2.26E−3	3.50	5.88E−8	13.3
16	32	6.05E−4	3.74	3.82E−9	15.4

TABLE 2
$x^* = 1/(t+1)$; $r = 2$.

n	n_e	$\|x^*-x_n\|_\infty$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio
2	6	3.03E−3		1.05E−6	
4	12	5.28E−4	5.74	1.86E−8	56.5
8	24	7.96E−5	6.63	2.90E−10	64.1
16	48	1.10E−5	7.24	4.58E−12	63.3

TABLE 3
$x^* = 1/(t+.1)$; $r = 1$.

n	n_e	$\|x^*-x_n\|_\infty$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio
2	4	3.36		1.12E−2	
4	8	1.93	1.74	1.19E−3	9.41
8	16	.910	2.12	8.95E−5	13.3
16	32	.353	2.58	5.99E−6	14.9

TABLE 4
$x^* = 1/(t+.1)$; $r=2$.

n	n_e	$\|x^*-x_n\|_\infty$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio
2	6	1.65		1.69E−3	
4	12	.688	2.40	7.94E−5	21.3
8	24	.216	3.19	2.18E−6	36.4
16	48	.0509	4.24	4.39E−8	49.7

There are integrals in setting up the nonlinear system for $x_n = P_n K(x_n)$ and in evaluating $\tilde{x}_n$. These integrals were evaluated numerically to high accuracy, to imitate exact integration. In a later paper, we will consider the effects of approximate integration.

Our second example is

$$x(t) = \int_0^1 G(t,s)[f(s,x(s)) + z(s)]\,ds, \tag{7.4}$$

$$G(t,s) = \begin{cases} -(1-t)s, & s \leq t, \\ -(1-s)t, & t \leq s \end{cases} \tag{7.5}$$

with $z(s)$ so chosen that

$$x^*(t) = \frac{t(1-t)}{t+c}, \qquad c > 0. \tag{7.6}$$

The integral equation (7.4) is a reformulation of the boundary value problem

$$\begin{gathered} x''(t) = f(t, x(t)) + z(t), \qquad 0 < t < 1, \\ x(0) = x(1) = 0. \end{gathered} \tag{7.7}$$

We consider the particular example

$$f(t,u) = \frac{1}{1+t+u}. \tag{7.8}$$

For this equation, $\gamma = 0$ and α can be chosen as large as desired. From (4.14), for $r \geq 1$,

$$\beta = r+1, \quad \beta_1 = 1, \quad \beta_2 = 2.$$

From Theorem 5.2,

$$\begin{gathered} \|x^* - x_n\|_\infty = O(h^{r+1}), \qquad \|x^* - \tilde{x}_n\|_\infty = O(h^{r+3}), \\ E_n \equiv \operatorname*{Max}_{t \in \Delta^{(n)}} |x^*(t) - \tilde{x}_n(t)| = O(h^{2r+2}). \end{gathered} \tag{7.9}$$

The set $\Delta^{(n)}$ is given by $\{i/n : 0 \leq i \leq n\}$.

The results for $c = 2$ are given in Tables 5 and 6 for degrees $r = 1$ and $r = 2$; and the analogous results for $c = .4$ are given in Tables 7 and 8. The rates (7.9) are confirmed approximately by the numerical results; and again, the iterate $\tilde{x}_n$ is a great improvement over x_n.

TABLE 5
$x^* = t(1-t)/(t+2)$; $r=1$.

n	$\|x^*-x_n\|$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio	E_n	Ratio
2	2.37E−2		1.13E−4		2.76E−5	
4	6.77E−3	3.50	1.05E−5	10.8	2.52E−6	11.0
8	1.81E−3	3.74	7.98E−7	13.2	1.78E−7	14.2
16	4.70E−4	3.85	5.47E−8	14.6	1.16E−8	15.3

TABLE 6
$x^* = t(1-t)/(t+2)$; $r=2$.

n	$\|x^*-x_n\|$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio	E_n	Ratio
2	1.58E−3		2.91E−6		4.11E−7	
4	2.39E−4	6.61	1.23E−7	23.7	1.20E−8	34.3
8	3.30E−5	7.24	4.43E−9	27.8	2.22E−10	54.1
16	4.34E−6	7.60	1.48E−10	29.9	3.66E−12	60.7

TABLE 7
$x^* = t(1-t)/(t+.4)$; $r=1$.

n	$\|x^*-x_n\|$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio	E_n	Ratio
2	1.27E−1		3.91E−4		1.10E−4	
4	4.95E−2	2.57	5.77E−5	6.78	2.03E−5	5.42
8	1.63E−2	3.04	6.17E−6	9.35	1.98E−6	10.3
16	4.77E−3	3.42	5.17E−7	11.9	1.52E−7	13.0
32	1.30E−3	3.67	3.75E−8	13.8	1.00E−8	15.2

TABLE 8
$x^* = t(1-t)/(t+.4)$; $r=2$.

n	$\|x^*-x_n\|$	Ratio	$\|x^*-\tilde{x}_n\|_\infty$	Ratio	E_n	Ratio
2	3.02E−2		3.84E−5		8.40E−6	
4	7.12E−3	4.24	3.03E−6	12.7	6.92E−7	12.1
8	1.32E−3	5.39	1.63E−7	18.6	2.40E−8	28.8
16	2.07E−4	6.38	6.82E−9	23.9	5.57E−10	45.5
32	2.93E−5	7.06	2.47E−10	27.6	9.32E−12	56.2

These examples were computed on a PRIME 850 (in double precision) and on the CRAY X-MP (in single precision). The PRIME is located at the University of Iowa, and the CRAY X-MP at the University of Illinois.

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