Projection based model reduction of multi-agent systems using graph partitions

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Abstract—In this paper we establish a projection based model reduction method for multi-agent systems defined on a graph. Reduced order models are obtained by clustering the vertices (agents) of the underlying communication graph by means of suitable graph partitions. In the reduction process the spatial structure of the network is preserved and the reduced order models can again be realized as multi-agent systems defined on a graph. The agents are assumed to have single-integrator dynamics and the communication graph of the original system is weighted and undirected. The proposed model reduction technique reduces the number of vertices of the graph (which is equal to the dynamic order of the original multi-agent system) and yields a reduced order multi-agent system defined on a new graph with a reduced number of vertices. This new graph is a weighted symmetric directed graph. It is shown that if the original multi-agent system reaches consensus, then so does the reduced order model. For the case that the clusters are chosen using an almost equitable partition of the graph, we obtain an explicit formula for the $\mathcal{H}_2$-norm of the error system obtained by comparing the input-output behaviors of the original model and the reduced order model. We also prove that the error obtained by taking an arbitrary partition of the graph is bounded from below by the error obtained by using the largest almost equitable partition finer than the given partition. The proposed results are illustrated by means of a running example.

Index Terms—Model reduction, Clustering, Graph partitions, Networks of autonomous agents, Multi-agent systems.

I. INTRODUCTION

Multi-agent systems and distributed control of networks of dynamic agents have received compelling attention in the last decade. In particular, reaching an agreement among agents in a network has been widely studied in terms of consensus and synchronization; see e.g. [10], [19], [21], [13], [23], [24]. Among numerous research directions in this area we mention formation control, flocking, placement of mobile sensors, and controllability analysis of networks; see e.g. [7], [18], [6], [5], [20], [27].

In order to analyze or control a large-scale system, model reduction techniques are highly advantageous. Clearly, lower order models admit easier analysis and provide a better understanding of the system behavior. Various model reduction techniques, such as balanced truncation, Hankel-norm

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approximation and Krylov projection are available in the literature; see e.g. [1]. Naively, one may think of exploiting these model reduction techniques to deal with analysis or control of large-scale networks. However, a major drawback here is that the spatial structure of the network may collapse by direct application of classical model reduction tools. Of course, related to this issue, structure preserving model reduction techniques have been established in the literature. In particular, preservation of the Lagrangian structure, the second-order structure, and the interconnection structure of several subsystems have been studied in [11], [12], and [22]. Nevertheless, multi-agent systems and dynamical networks have their own structural characteristics and this motivates us to study the model reduction problem for this class of systems in a more focused manner. Obviously, the key structure needed to be preserved in the reduced order model is the network topology. Some recent work in this direction is [4] and [9], where clustering based algorithms are proposed for asymptotically stable networks. In particular, a notion of (relaxed) cluster reducibility is used in [9] that is closely related to the notions of *leader symmetry* and *leader-invariant equitable partitions* ; see [20], [15], [27].

In the present paper we consider multi-agent systems defined on weighted undirected graphs, and we propose a projection based technique to obtain reduced order models for these systems. The projection used is formulated in terms of the characteristic matrix of a graph partition. The reduction procedure preserves the spatial structure of the network, meaning that the reduced order model is realized as a new multi-agent system. The communication graph of the reduced order multi-agent system is weighted, symmetric and directed, and has a reduced number of nodes.

Observe that the Laplacian matrix of the communication graph serves as the state matrix in the model of a typical multi-agent system with a consensus based feedback protocol. Thus, inevitably, the system is *not* asymptotically stable, and hence most of the aforementioned existing results do not directly apply to this case. Another issue which is relevant here is the preservation of consensus in the reduced order model. As we will observe, clustering the agents does not jeopardize the consensus property of the original multi-agent system.

In [17] a model reduction scheme was established in which the dynamic order of the agents is reduced, but the communication graph remains unchanged. As a counterpart of [17], in the present paper we consider single integrator dynamics with a consensus type of protocol, and we aim at reducing the size of the underlying communication graph. Note that the problem under study in this paper is inherently different to

that of [17] as in the current paper we seek for reduction in the communication graph, whereas in the latter the dynamic order of the agents is being reduced.

To further justify our work, we mention here two notable advantages of preserving the network structure in the reduction process. First, note that the reduced order models obtained are again realized as new multi-agent systems. Therefore, analysis and design methods that have already been established in the literature for this class of systems are still applicable to the reduced order systems. Consequently, if the approximated models are "close" to the original models in a certain cense, one can perform the analysis/design techniques to the reduced order systems, and expect that the performed analysis/design would be still "valid" for the original model. Second, as we use graph partitions in our proposed model reduction method, there is a tangible relationship between the reduced order system and the original one. In particular, the agents in the reduced order system approximate the behavior of the clusters of the original network. This is particularly interesting in cases where intra-cluster behaviors are not of crucial importance, but one is interested rather in a hyper level of behaviors, namely the interaction between clusters; see e.g. [25] and [26].

An important challenge is to compare the input-output behavior of the reduced and original network. In this paper, we work with a leader-follower set up, meaning that some agents, often called *leaders*, may receive an external command, a disturbance, or a reference signal. Moreover, as outputs we essentially consider the differences among the states of the communicating agents, as these differences play a crucial role in the context of distributed control. Then, for the case where the proposed projection corresponds to an almost equitable partition, we will establish an explicit expression for the exact model reduction error in the sense of the $\mathcal{H}_2$-norm. The expression provided for the associated model reduction error is simple, easy to compute, and can be derived directly from the graph partition involved in obtaining the reduced order model. Moreover, by using the notion of partial ordering of partitions, we will show how the established model reduction error can be used to obtain a lower bound on the model reduction error associated to arbitrary graph partitions, not necessarily almost equitable.

Note that the underlying idea of the proposed model reduction technique is to define a projection, based on clustering the vertices (agents) of the graph, in order to obtain a reduced order model. Hence, as the proposed projection acts on the communication graph and not on the dynamics of the agents, the proposed idea is potentially applicable to other classes of multi-agent systems where the agents may have general linear dynamics, or follow a different type of protocol; see e.g. [13], [14]. In the same vein, the proposed method is potentially applicable to multi-agent systems with discrete-time dynamics; see e.g. [10]. However, in order to derive more explicit results, we will restrict our attention in this paper to the case of continuous-time leader-follower Laplacian based dynamics.

The structure of the paper is as follows. In Section II, we review some basic notions and preliminaries that are needed in the rest of the paper. The proposed model reduction scheme is discussed in Section III. The input-output behaviors of the reduced order and the original multi-agent system are compared in Section IV. Finally, Section V concludes the paper.

II. PRELIMINARIES

In this section, we will provide some preliminaries and basic material needed in the sequel. In particular, we will discuss some basic notions from graph theory, describe the model used in this paper for multi-agent systems, and finally recap the notion of Petrov-Galerkin projection.

A. Graph Theory

In this paper we consider both weighted undirected graphs and weighted directed graphs. A weighted *undirected* graph is a triple $G = (V, E, A)$ where $V = \{1, 2, \ldots, n\}$ is the vertex set, E is the edge set, and $A = [a_{ij}]$ is the adjacency matrix, with nonnegative elements a_{ij} called the weights. The edge set of G is a set of unordered pairs $\{i, j\}$ of distinct vertices of G. Similarly, a weighted *directed* graph is a triple $G = (V, E, A)$ where $V = \{1, 2, \ldots, n\}$ is the vertex set, E is the arc set, and $A = [a_{ij}]$ is the adjacency matrix with nonnegative elements a_{ij}, again called the weights. The arc set of G is a set of ordered pairs (i, j) of distinct vertices of G. For an arc $(i, j) \in E$, we say i is the tail, and j is the head of the arc. In this paper we consider *simple* graphs meaning that self-loops and multiple edges (multiple arcs in the same direction) between one particular pair of vertices are not permitted. We have $a_{ij} > 0$ whenever there is an edge between i and j (an arc from j to i). Clearly, $a_{ij} = a_{ji}$ for undirected graphs. A directed graph is called *symmetric* if whenever (i, j) is an arc also (j, i) is. We note that for symmetric directed graphs the weights a_{ij} and a_{ji} can be distinct. Clearly any weighted undirected graph can be identified with a symmetric directed graph in which the weights satisfy $a_{ij} = a_{ji}$.

Both for undirected and directed graphs the degree matrix of G is the diagonal matrix, denoted by $D = \text{diag}(d_1, d_2, \ldots, d_n)$, with

$$d_i = \sum_{j=1}^{n} a_{ij}.$$

Note that, for directed graphs, the definition above corresponds to the so-called *in-degree matrix* of a graph (see e.g. [16, p. 26]). The Laplacian matrix of G is defined as $L = D - A$. For directed graphs, the incidence matrix of G, denoted by $R = [r_{ij}]$, is defined as

$$r_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of arc } j \\ -1 & \text{if vertex } i \text{ is the tail of arc } j \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, where k is the total number of arcs. In order to obtain an incidence matrix for a given undirected graph, we first assign an arbitrary orientation to each of the edges and next take the incidence matrix of the

corresponding directed graph (see [16, p.21]). Corresponding to the incidence matrix R, let

$$W = \mathrm{diag}(w_1, w_2, \ldots, w_k) \tag{2}$$

be a $k \times k$ matrix such that w_j indicates the weight associated to the edge (arc) j, for each $j = 1, 2, \ldots, k$. For undirected graphs, the relationship between the incidence matrix and the Laplacian matrix is then captured by the following equality:

$$L = RWR^\top. \tag{3}$$

B. Multi-agent systems

Let $G = (V, E, A)$ be a weighted undirected graph where $V = \{1, 2, \ldots, n\}$. Let $V_L = \{v_1, v_2, \ldots, v_m\}$ be a subset of V, and let $V_F = V \setminus V_L$. By a leader-follower multi-agent system, we mean the following dynamical system:

$$\dot{x}_i = \begin{cases} z_i & \text{if } i \in V_F \\ z_i + u_\ell & \text{if } i = v_\ell \in V_L \end{cases} \tag{4}$$

where $x_i \in \mathbb{R}$ denotes the state of agent i, $u_\ell \in \mathbb{R}$ is the external input applied to agent v_ℓ, and $z_i \in \mathbb{R}$ is the coupling variable for the agent i which is given by

$$z_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{5}$$

Let $x = \mathrm{col}(x_1, x_2, \ldots, x_n)$, $u = \mathrm{col}(u_1, u_2, \ldots, u_m)$, and the matrix $M \in \mathbb{R}^{n \times m}$ be defined as

$$M_{i\ell} = \begin{cases} 1 & \text{if } i = v_\ell \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then we can write the above leader-follower linearly diffusively coupled multi-agent system associated with the graph G in a compact form as

$$\dot{x} = -Lx + Mu, \tag{7}$$

where L is the Laplacian matrix of G, and M is given by (6).

C. Petrov-Galerkin projections

Consider the input/state/output system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{8}$$

where $x = \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y = \mathbb{R}^p$ is the output of the system. Let $\mathbf{W}, \mathbf{V} \in \mathbb{R}^{n \times r}$ such that $\mathbf{W}^\top \mathbf{V} = I$. By using the projection $\Gamma = \mathbf{V}\mathbf{W}^\top$, a reduced order model (projected model) is obtained as

$$\begin{aligned} \dot{\hat{x}} &= \mathbf{W}^\top A \mathbf{V} \hat{x} + \mathbf{W}^\top B u \\ y &= C \mathbf{V} \hat{x} \end{aligned} \tag{9}$$

where $\hat{x} \in \mathbb{R}^r$ denotes the state of the reduced model. This projection is called a Petrov-Galerkin projection. Note that Γ defines a projection onto the image of $\mathbf{V}$ and along the kernel of $\mathbf{W}^\top$. In case that $\mathbf{W}$ is equal to $\mathbf{V}$, the projection Γ is orthogonal and is called a Galerkin projection. The Petrov-Galerkin projection is a rather general reduction framework
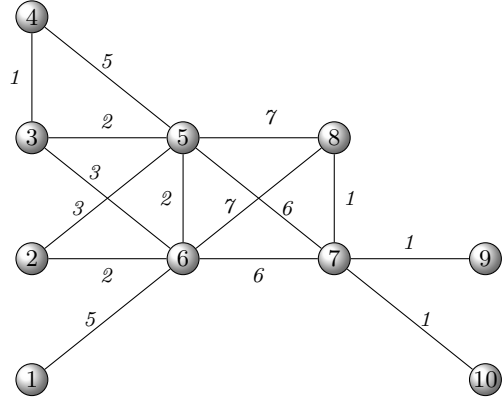

Fig. 1: $G = (V, E, A)$

meaning that many of the model reduction techniques including Krylov based and truncation methods essentially use this projection with appropriate choice of matrices $\mathbf{V}$ and $\mathbf{W}$. In particular, depending on the application, one can choose the matrix $\mathbf{V}$, and consequently $\mathbf{W}$, to preserve stability, passivity, or to match certain moments and Markov parameters (see [1] for more details).

III. PROJECTION BY GRAPH PARTITIONS

It is not hard to see that a direct application of Petrov-Galerkin projection will, in general, destroy the spatial structure of the network. In particular, the relationship between the reduced order network and the original one is not transparent, the structure of the Laplacian matrix may be lost, and the reduced order model may not be in the form of a leader-follower multi-agent system as given by (7). Therefore, we propose to use graph partitions in order to preserve the structure of the network in the reduced order model. First, we need to recap the notions of cells and graph partitions.

Let $V = \{1, 2, \ldots, n\}$ be the vertex set of a graph G. We call any nonempty subset of V a *cell* of V. We call a collection of cells, given by $\pi = \{C_1, C_2, \ldots, C_r\}$, a *partition* of V if $\cup_i C_i = V$ and $C_i \cap C_j = \varnothing$ whenever $i \neq j$. With a little abuse of notation, we say π is a partition of $G = (V, E)$, or shortly G, meaning that π is a partition of V. We say a vertex v is a *cellmate* of a vertex w in π if v and w belong to the same cell of π. For a cell $C \subseteq V$, the *characteristic vector* of C is defined as the n-dimensional column vector $p(C)$ with

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

For a partition $\pi = \{C_1, C_2, \ldots, C_r\}$, we define the *characteristic matrix* of π as

$$P(\pi) = \begin{bmatrix} p(C_1) & p(C_2) & \cdots & p(C_r) \end{bmatrix}. \tag{11}$$

Example 1 As an example, consider the graph G depicted in Figure 1.
Then,

$$\pi = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\} \tag{12}$$

is a partition of $V = \{1, 2, \ldots, 10\}$, and its characteristic matrix is given by

$$P(\pi) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^\top . \tag{13}$$

$\square$

Now, consider again in general the multi-agent system (7) with associated graph $G = (V, E, A)$. Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be a partition of V, and $P(\pi)$ be the characteristic matrix of π. Recall the Petrov-Galerkin projection discussed in Section II-C. We propose the following choices for the matrices $\mathbf{V}$ and $\mathbf{W}$:

$$\mathbf{W} = P(\pi)(P^\top(\pi)P(\pi))^{-1} \tag{14a}$$

$$\mathbf{V} = P(\pi). \tag{14b}$$

Note that the columns of $P(\pi)$ are orthogonal, thus the matrix $P^\top(\pi)P(\pi)$ is a diagonal matrix. Moreover, its ith diagonal element is equal to the number of vertices in cell C_i of π. Hence, $P^\top(\pi)P(\pi)$ is invertible. Also note that $\mathbf{W}^\top\mathbf{V} = I$, and the projection $\Pi = \mathbf{V}\mathbf{W}^\top$ is orthogonal since $\mathrm{im}\,\mathbf{V} = (\ker \mathbf{W}^\top)^\perp$. Then, by applying the Petrov-Galerkin projection to (7), with the choices of $\mathbf{V}$ and $\mathbf{W}$ given by (14), we obtain the reduced order system

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{15}$$

where $\hat{x} \in \mathbb{R}^r$ is the state of the reduced order model, and the matrices $\hat{L}$ and $\hat{M}$ are given by

$$\hat{L} = (P^\top P)^{-1} P^\top L P, \tag{16}$$

$$\hat{M} = (P^\top P)^{-1} P^\top M, \tag{17}$$

where $P(\pi)$ is denoted shortly by P.

Next, we show that the reduced model (15) is associated with a leader-follower multi-agent system defined on a graph, in a similar form as (7). First, observe that $\hat{M}$ has a similar structure as M. More precisely, each column of $\hat{M}$ contains exactly one nonzero-element, indicating a leader. The only difference is that the non-zero elements do not need to be 1 anymore. This can be interpreted by saying that input signals are now weighted.

It is easy to observe that the matrix $\hat{L}$ is equal to the Laplacian matrix of a weighted *directed* graph, say $\hat{G} = (\hat{V}, \hat{E}, \hat{A})$. In fact, as a consequence of the aforementioned projection, the underlying graph G is mapped to the graph $\hat{G}$. In particular, each cell of π in G is mapped to a vertex in $\hat{G}$. Hence, the number of vertices in $\hat{G}$ is equal to the cardinality of π, i.e. the number of cells in π. Moreover, there is an arc from vertex p to vertex q in $\hat{G}$ if and only if there exist $i \in C_p$ and $j \in C_q$ with $p \neq q$ such that $\{i, j\} \in E$. Therefore, $\hat{G}$ is a *symmetric directed* graph, i.e. $(i, j) \in \hat{E} \Leftrightarrow (j, i) \in \hat{E}$. For the relationship between the matrices A and $\hat{A} = [\hat{a}_{pq}]$, we have

$$\hat{a}_{pq} = \frac{1}{|C_p|} \sum_{i \in C_p, j \in C_q} a_{ij}, \tag{18}$$

for $p \neq q$, where $|.|$ denotes the cardinality of a set. Observe that the row sums of $\hat{L}$ are indeed zero as $P(\pi)\mathbb{1} = \mathbb{1}$ and $L\mathbb{1} = 0$, where $\mathbb{1}$ denotes the vector of ones of appropriate dimension. Note that the matrix $\hat{L}$ is not necessarily symmetric, as the number of vertices may differ from cell to cell in π. However, $\hat{L}$ is similar to the symmetric matrix $(P^\top P)^{-\frac{1}{2}} P^\top L P (P^\top P)^{\frac{1}{2}}$, thus $\hat{L}$ inherits nice properties of L, like diagonalizability and having real eigenvalues.

As observed, the reduced order model (15) is associated with a new multi-agent system where the diffusive coupling rule is defined based on the graph $\hat{G}$. The idea behind the proposed projection is that the partition π clusters some vertices (agents) together, and these vertices are mapped to a single vertex in the reduced order (projected) model. In addition, note that the components of the reduced state $\hat{x}$ approximate the averages of the states of the agents that are cellmates in π. In case the agents that are cellmates in π have a "similar" interconnection to the rest of the network, then this approximation tends to be exact. We will clarify what we mean by "similar" in the next section.

Example 2 We will now return to our example in Figure 1. Suppose that agents (vertices) 6 and 7 are leaders. Then the multi-agent system associated with the graph G is given by:

$$\dot{x} = -Lx + Mu \tag{19}$$

where

$$L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 & 0 & 0 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -6 & 15 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $P(\pi)$, given by (13), be denoted in short by P. Then, the reduced order model obtained by clustering the agents according to π, given by (12), is given by

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{20}$$

where $\hat{L}$ and $\hat{M}$ are computed as

$$\hat{L} = (P^\top P)^{-1} P^\top L P = \begin{bmatrix} 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$$\hat{M} = (P^\top P)^{-1} P^\top M = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$
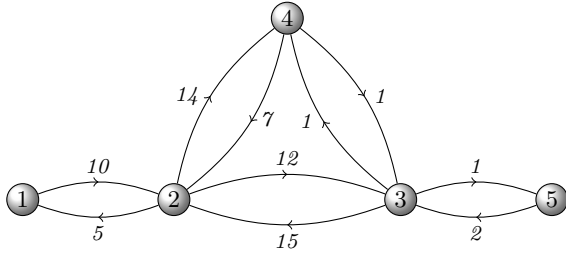
Fig. 2: $\hat{G} = (\hat{V}, \hat{E}, \hat{A})$

The graph $\hat{G}$ associated with the reduced order system (20) is shown in Figure 2.

Observe that $\hat{G}$ has 5 vertices, each of which corresponds to a cell in π. For instance, vertex 1 in $\hat{G}$ corresponds to the cell $C_1 = \{1, 2, 3, 4\}$, and vertex 2 corresponds to the cell $C_2 = \{5, 6\}$. Then the arcs $(1, 2)$ and $(2, 1)$ of $\hat{G}$ account for the coupling between C_1 and C_2 in the graph G given by Figure 1. In particular, the weight associated to the arc $(2, 1) \in \hat{E}$ is indeed equal to the average of the weights of the edges $\{i, j\} \in E$ with $i \in C_1$ and $j \in C_2$, as given by (18). Observe that the input weights indicated by $\hat{M}$ depend on the cardinality of the cells in π. For instance, $\hat{x}_2$ receives half of u_1 in the reduced order model (20). This value indeed indicates the average of the input signals received by the agents in $C_2 = \{5, 6\}$. □

Before proceeding, we point out another useful relationship between the Laplacian matrices L and $\hat{L}$. We need to recap the notion of interlacing first. Let X be a real symmetric $n \times n$ matrix, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of X in an increasing order. Also let Y be a real symmetric $m \times m$ matrix, where $m \leqslant n$. Moreover, let $\mu_1, \mu_2, \ldots, \mu_m$ denote the eigenvalues of Y in an increasing order. Then we say that the eigenvalues of Y *interlace* the eigenvalues of X if

$$\lambda_i \leqslant \mu_i \leqslant \lambda_{n-m+i} \qquad (21)$$

for each $i = 1, 2, \ldots, m$.

The eigenvalues of $\hat{L}$, given by (16), interlace the eigenvalues of L as stated in the following lemma.

Lemma 3 *Let L be a symmetric matrix, and let $\hat{L}$ be given by* (16) *for a given partition π. Then the eigenvalues of $\hat{L}$ interlace the eigenvalues of L.*

Proof. Recall that $P^\top P$ is a diagonal matrix with strictly positive diagonal elements. Clearly, the matrix $\hat{L}$ is similar to the matrix $F^\top L F$ for $F = P(P^\top P)^{-\frac{1}{2}}$. Now, noting that $F^\top F = I$, the result immediately follows by [8, Thm. 9.5.1]. ∎

Next, we discuss consensus and convergence rate preservation in the reduced order model. Roughly speaking, consensus means that the agents agree on a certain quantity of interest. Consensus is defined in the absence of the external input, thus we deal with the following multi-agent system:

$$\dot{x}_i = \sum_{j=1}^n a_{ij}(x_j - x_i). \qquad (22)$$

We say that the multi-agent system (22) reaches consensus if for any arbitrary initial condition we have:

$$\lim_{t \to \infty} x_i(t) - x_j(t) = 0 \text{ for all } i, j \in V \qquad (23)$$

Now, suppose that the original multi-agent system reaches consensus, and hence (23) holds. Then, consensus is preserved in the reduced order system:

Theorem 4 *Consider the multi-agent system* (7) *with $u = 0$, i.e.*

$$\dot{x} = -Lx, \qquad (24)$$

and suppose that consensus is reached for this system. For any given partition π, the reduced order multi-agent system

$$\dot{\hat{x}} = -\hat{L}\hat{x} \qquad (25)$$

also reaches consensus, where $\hat{L}$ is given by (16).

Proof. Clearly, as the multi-agent system (24) reaches consensus, zero is a simple eigenvalue of the Laplacian matrix L. In addition, note that $\hat{L}\mathbb{1} = 0$. By Lemma 3, we conclude that zero is also a simple eigenvalue of $\hat{L}$, and the rest of the eigenvalues are real and strictly positive. This implies that the reduced order model (25) reaches consensus. ∎

Note that, because of the interlacing property provided in Lemma 3, the rate of convergence in the reduced order model is at least as fast as that of the original model.

IV. INPUT-OUTPUT APPROXIMATION OF MULTI-AGENT SYSTEMS

We have observed that by applying an appropriate projection to the original multi-agent system defined on G, we obtain a reduced-order model that represents a multi-agent system defined on a new graph $\hat{G}$. Moreover, consensus and the convergence rate are preserved by this model reduction. In this section, we discuss appropriate choices of partitions such that the behavior, in particular the input-output behavior, of the reduced and the original multi-agent system are "close" in a certain sense. Without loss of generality, assume that graph G is connected. Obviously, in case G is not connected, one can apply the proposed model reduction technique on disconnected components of G, individually.

We first include some output variables in system (7). Note that in the context of distributed control, differences of the states of the agents play a crucial role. In fact, these differences reflect the disagreement among the agents, and the network reaches consensus if this disagreement vanishes as time evolves. Observe that the differences of the states of communicating agents are embedded in the incidence matrix. Therefore, we choose the output variables as

$$y = W^{\frac{1}{2}} R^\top x, \qquad (26)$$

where W is given by (2). Hence, the disagreement in the states of a pair of agents is reflected in the output variables (26) in accordance with the weight of the edge connecting those agents (vertices). Furthermore note that, as G is connected, the multi-agent system (7) reaches consensus if and only

if $\lim_{t\to\infty} y(t) = 0$ for all initial states $x(0)$. It is also worth mentioning that, by (3), we have $\|y\|^2 = x^\top L x = \frac{1}{2}\sum_{i,j} a_{ij}(x_i - x_j)^2$ which is a measure of group disagreement (see e.g. [19]).

Consequently, we obtain the following input/state/output model for the original multi-agent system defined on the graph G:

$$\dot{x} = -Lx + Mu, \tag{27a}$$

$$y = W^{\frac{1}{2}} R^\top x, \tag{27b}$$

where $x \in \mathbb{R}^n$, L is the Laplacian, and R is the incidence matrix of G. Now, let again π be a partition of G. Then, the input/state/output model for the reduced order (projected) model is obtained as

$$\dot{\hat{x}} = -\hat{L}\hat{x} + \hat{M}u, \tag{28a}$$

$$y = W^{\frac{1}{2}} \hat{R}^\top \hat{x}, \tag{28b}$$

where $\hat{x} \in \mathbb{R}^r$ with $r \leqslant n$, $\hat{L}$ is given by (16), $\hat{M}$ is given by (17), and $\hat{R} = P^\top R$.

Recall that $\hat{L}$ is the Laplacian matrix of the weighted symmetric directed graph $\hat{G}$. It is worth mentioning that the matrix $\hat{R}$ is closely related to the incidence matrix of the graph $\hat{G}$, which we denote by R'. Indeed, it can be shown that each column of $\hat{R}$ is either equal to zero or is equal to a column of R'. Consequently, the output equation (28b) captures the (weighted) differences of the states of communicating agents in the reduced order multi-agent system (28). Note that the zero columns of $\hat{R}$ indeed correspond to the difference of the states of cellmate agents, which are approximated to be identical in deriving the reduced order model.

Clearly, different choices of graph partitions lead to different reduced order models, and one may think of choosing an appropriate partition to approximate the behavior of the original multi-agent system relatively well. Note that we have two trivial partitions here, one is taking each vertex (agent) as a singleton and the other one is $\pi = \{V\}$. In the first case, no order reduction occurs and the corresponding model reduction error is zero. In the latter case, the network topology is neglected, and the reduced model is a single agent with a zero transfer matrix from u to y. Thus, these two trivial partitions indicate the finest and the coarsest approximation by graph partitions. Clearly, similar to model reduction in ordinary linear systems, this leads to a compromise between the order of the reduced model and the accuracy of the approximation.

Recall that the dynamics of the individual nodes are the same. So, an appropriate partitioning (clustering) decision solely depends on the graph topology. Hence, in order to achieve a better approximation, it is expected that the agents (vertices) that are connected to the rest of the network in a "similar" fashion should be clustered in one cell. In order to formalize this heuristic idea, in what follows we distinguish a class of partitions, namely *almost equitable partitions*, from other partitions. An easily computable model reduction error in the sense of the $\mathcal{H}_2$-norm will be provided for this class of partitions. The notion of almost equitability is recapped next.

Let $G = (V, E)$ be an unweighted undirected graph. For a given cell $C \subseteq V$, we write $N(i, C) = \{j \in C \mid \{i, j\} \in E\}$. We call a partition $\pi = \{C_1, C_2, \ldots, C_r\}$ an *almost equitable partition* (AEP) of G if for each $p, q \in \{1, 2, \cdots, r\}$ with $p \neq q$ there exists an integer d_{pq} such that $|N(i, C_q)| = d_{pq}$ for all $i \in C_p$.

An almost equitable partition, say π, has the key property that $\operatorname{im} P(\pi)$ is L-invariant (see e.g. [27, Lem. 2]). Note that we call a subspace $\mathcal{X} \subseteq \mathbb{R}^n$ $\mathbf{A}$-*invariant* if $\mathbf{A}\mathcal{X} \subseteq \mathcal{X}$ where $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$. To incorporate the case of weighted graphs, the notion of almost equitability can be extended as follows.

Let $G = (V, E, A)$ be a weighted undirected graph. Recall that a_{ij} indicates the weight associated to the edge $\{i, j\}$. We call a partition $\pi = \{C_1, C_2, \ldots, C_r\}$ an *almost equitable partition* (AEP) of G if for each $p, q \in \{1, 2, \cdots, r\}$ with $p \neq q$ there exists an integer d_{pq} such that $\sum_{j \in N(i, C_q)} a_{ij} = d_{pq}$ for all $i \in C_p$.

As an example consider the graph G in Figure 1. It is easy to verify that π given by (12) is an AEP of G.

Note that the definition of almost equitability for weighted graphs includes the case of unweighted graphs as a special case. In fact, for unweighted graphs, $a_{ij} = 1$ whenever $\{i, j\} \in E$. Hence, the quantity $\sum_{j \in N(i, C_q)} a_{ij}$ coincides with the cardinality of $N(i, C_q)$. Moreover, the L-invariance property remains valid for weighted graphs, as stated in the following lemma.

Lemma 5 *Let π be a partition of a weighted undirected graph G, and let L denote the Laplacian matrix of G. Then π is almost equitable if and only if $\operatorname{im} P(\pi)$ is L-invariant, i.e.*

$$L \operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi) \tag{29}$$

Proof. The proof is analogous to that of [3, Prop. 1]. ∎

Now, assume that $\pi = \{C_1, C_2, \ldots, C_r\}$ is an AEP of a weighted undirected graph G, and suppose that the reduced order model (28) is obtained from (27) by choosing the partition π. Moreover, recall that $V_L = \{v_1, v_2, \ldots, v_m\}$, and for each $i = \{1, 2, \ldots, m\}$, let k_i be an integer such that $v_i \in C_{k_i}$. Then, the (normalized) model reduction error involved in obtaining the reduced order model (28) is provided in the following theorem.

Theorem 6 *Let G be a weighted undirected graph, and assume G is connected. Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be an almost equitable partition of G. Suppose that the reduced order multi-agent system (28) is obtained from (27) by choosing the partition π. Also, let S and $\hat{S}$ denote the transfer matrices from u to y in (27) and (28), respectively. Let $\Xi(\pi) := \frac{\|S - \hat{S}\|_2^2}{\|S\|_2^2}$ denote the normalized model reduction error associated to π. Then, we have*

$$\Xi(\pi) = \frac{\sum_{i=1}^m (1 - \frac{1}{|C_{k_i}|})}{m(1 - \frac{1}{n})}, \tag{30}$$

where n is the number of vertices (agents) in G, m is the number of leaders, and the integers k_is are defined as before.

Proof. Note that the columns of $P(\pi)$ form an orthogonal set of vectors in $\mathbb{R}^n$. We complete this set to an orthogonal basis for $\mathbb{R}^n$. In particular, we construct a matrix T as $T = \begin{bmatrix} P & Q \end{bmatrix}$, where $P(\pi)$ is denoted in short by P, and Q is an $n \times (n-k)$ matrix such that the columns of T are orthogonal. Observe that we have

$$P^\top Q = 0. \tag{31}$$

Now, we apply the state space transformation $x = T\tilde{x}$ to system (27). Consequently, we obtain the following input/state/output system:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = - \begin{bmatrix} (P^\top P)^{-1} P^\top L P & (P^\top P)^{-1} P^\top L Q \\ (Q^\top Q)^{-1} Q^\top L P & (Q^\top Q)^{-1} Q^\top L Q \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} (P^\top P)^{-1} P^\top M \\ (Q^\top Q)^{-1} Q^\top M \end{bmatrix} u \tag{32a}$$

$$y = \begin{bmatrix} W^{\frac{1}{2}} R^\top P & W^{\frac{1}{2}} R^\top Q \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{32b}$$

Clearly, the transfer matrices from u to y in (27) and (32) are identical. Moreover, observe that the reduced order model (28) is the system obtained by truncating the state components $\tilde{x}_2$ in (32). Since π is an AEP of G, $\operatorname{im} P$ is L-invariant by Lemma 5. Thus, there exists a matrix X such that $LP = PX$. Hence, we obtain

$$Q^\top L P = 0. \tag{33}$$

Therefore, the transfer matrices S and $\hat{S}$ of the original system (27) and its reduced order model (28), respectively, are related by:

$$S(s) = \hat{S}(s) + \Delta(s), \tag{34}$$

where

$$\Delta(s) = W^{\frac{1}{2}} R^\top Q(sI + (Q^\top Q)^{-1} Q^\top L Q)^{-1} (Q^\top Q)^{-1} Q^\top M. \tag{35}$$

By using (3) and (33), we have $\hat{S}^\top(-s)\Delta(s) = 0$. Hence, we have

$$\|S\|_2^2 = \|\hat{S}\|_2^2 + \|\Delta\|_2^2. \tag{36}$$

Now, let the matrices $X_1 \in \mathbb{R}^{n \times n}$ and $Y_1 \in \mathbb{R}^{r \times r}$ be defined as:

$$X_1 = \int_0^\infty e^{-Lt} L e^{-Lt} dt, \tag{37}$$

$$Y_1 = \int_0^\infty e^{-\hat{L}^\top t} P^\top L P e^{-\hat{L}t} dt, \tag{38}$$

where

$$\hat{L} = (P^\top P)^{-1} P^\top L P.$$

Of course, one should address the issue of convergence of these improper integrals. Since the original model reaches consensus we have $e^{-Lt} L e^{-Lt} \to 0$ as $t \to \infty$. Hence, since the components of $e^{-Lt} L e^{-Lt}$ are products of polynomials and exponentials, the integral defining X_1 exists. Similarly, by Theorem 4 the reduced order model (25) also reaches consensus and therefore the integral defining Y_1 exists as well.

Let $L = U \Lambda U^\top$ be a spectral decomposition of the Laplacian, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n$ are the eigenvalues of L and U is an orthogonal matrix. Note that $\lambda_2 > 0$, due to the connectedness of G. Moreover, the first column of the matrix U is equal to the normalized vector of ones, i.e. $\frac{1}{\sqrt{n}} \mathbb{1}$. Thus X_1 is computed as

$$X_1 = \int_0^\infty e^{-Lt} L e^{-Lt} dt = \int_0^\infty L e^{-2Lt} dt$$

$$= -\frac{1}{2} e^{-2Lt} |_0^\infty dt = -\frac{1}{2} U e^{-2\Lambda t} U^\top |_0^\infty = \frac{1}{2} I_n - \frac{1}{2n} \mathbb{1} \mathbb{1}^\top. \tag{39}$$

In addition, with T given as above, we have

$$T^\top X_1 T = \int_0^\infty T^\top e^{-Lt} L e^{-Lt} T dt$$

$$= \int_0^\infty e^{-(T^{-1} L T)^\top t} T^\top L T e^{-T^{-1} L T t} dt$$

$$= \int_0^\infty e^{-\begin{bmatrix} \hat{L}^\top & 0 \\ 0 & * \end{bmatrix} t} \begin{bmatrix} P^\top L P & 0 \\ 0 & * \end{bmatrix} e^{-\begin{bmatrix} \hat{L} & 0 \\ 0 & * \end{bmatrix} t} dt,$$

where we have used (33) to derive the last equality, and where "$*$" denotes values that are not of interest. Hence, we obtain that

$$T^\top X_1 T = \begin{bmatrix} Y_1 & 0 \\ 0 & * \end{bmatrix}.$$

This yields, $Y_1 = P^\top X_1 P$. Therefore, by (39), Y_1 is computed as

$$Y_1 = \frac{1}{2} P^\top P - \frac{1}{2n} P^\top \mathbb{1} \mathbb{1}^\top P.$$

Next we compute the values $\|S\|_2^2$ and $\|\hat{S}\|_2^2$. From the definition of the $\mathcal{H}_2$-norm, it readily follows that

$$\|S\|_2^2 = \operatorname{trace} M^\top X_1 M.$$

Hence, by (39), $\|S\|_2^2$ is computed as

$$\|S\|_2^2 = \operatorname{trace} M^\top X_1 M = \frac{1}{2} \operatorname{trace} M^\top (I_n - \frac{1}{n} \mathbb{1} \mathbb{1}^\top) M$$

$$= \frac{1}{2} \operatorname{trace} M M^\top (I_n - \frac{1}{n} \mathbb{1} \mathbb{1}^\top).$$

Note that $M M^\top$ is a diagonal matrix, where the diagonal elements are either zero or 1. In particular, the ith diagonal element is equal to 1 if $i \in V_L$, and is equal to zero otherwise. Thus, we conclude that

$$\|S\|_2^2 = \frac{m}{2}(1 - \frac{1}{n}), \tag{40}$$

where m is the cardinality of V_L as before. Moreover, we have

$$\|\hat{S}\|_2^2 = \operatorname{trace} M^\top P (P^\top P)^{-1} Y_1 (P^\top P)^{-1} P^\top M$$

$$= \frac{1}{2} \operatorname{trace} M^\top P (P^\top P)^{-1} (P^\top P - \frac{1}{n} P^\top \mathbb{1} \mathbb{1}^\top P)(P^\top P)^{-1} P^\top M$$

$$= \frac{1}{2} \operatorname{trace} M M^\top (P(P^\top P)^{-1} P^\top - \frac{1}{n} P(P^\top P)^{-1} P^\top \mathbb{1} \mathbb{1}^\top P(P^\top P)^{-1} P^\top)$$

It is easy to verify that $P(P^\top P)^{-1} P^\top \mathbb{1} = \mathbb{1}$. Hence, we obtain that

$$\|\hat{S}\|_2^2 = \frac{1}{2} \operatorname{trace} M M^\top (P(P^\top P)^{-1} P^\top - \frac{1}{n} \mathbb{1} \mathbb{1}^\top) \tag{41}$$

Recall the diagonal structure of $MM^\top$. Also recall that $v_i \in C_{k_i}$ for each $v_i \in V_L$. Then it is straightforward to check that (41) yields

$$\|\hat{S}\|_2^2 = \frac{1}{2}\sum_{i=1}^{m}\frac{1}{|C_{k_i}|} - \frac{m}{2n}. \tag{42}$$

Therefore, by (36) and (40) we obtain that

$$\|\Delta\|_2^2 = \frac{m}{2}(1-\frac{1}{n}) - \frac{1}{2}\sum_{i=1}^{m}\frac{1}{|C_{k_i}|} + \frac{m}{2n} = \frac{1}{2}\sum_{i=1}^{m}(1-\frac{1}{|C_{k_i}|}). \tag{43}$$

This together with (40) completes the proof. ∎

Theorem 6 provides a simple and easily computable expression for the model reduction error in case π is an AEP of G. Depending on the structure of π, the normalized model reduction error (30) takes a value between 0 and 1. Moreover, for a given multi-agent system, this value is determined by the *population*, i.e. cardinality, of those cells of π containing the leaders. Clearly, the less populated these cells are, the less model reduction error we have. An interesting case is when all the leaders appear as singleton in π. Then, the corresponding model reduction error is zero by (30). This is formally formulated in the following corollary.

Corollary 7 *Let $V_L = \{v_1, v_2, \ldots, v_m\}$ as before. Suppose that π^* is an almost equitable partition of G with the property that $\{v_i\} \in \pi^*$ for each $i = 1, 2, \ldots, m$. Then we have $\Xi(\pi^*) = 0$, where $\Xi(\pi^*)$ is the normalized model reduction error corresponding to π^*, as defined in the statement of Theorem 6.*

Note that almost equitable partitions with each leader appearing as a singleton are typically studied in the context of controllability of multi-agent systems (see e.g. [15] and [27]). In particular, by [27, Thm. 3], it is easy to observe that any π^* in Corollary 7 yields an upper bound for the reachable subspace of (7). Therefore the proposed model reduction technique in this case, in fact, corresponds to removing uncontrollable modes. Consequently, the input-output behavior remains unchanged, which is in accordance with the model reduction error being zero in (30).

Example 8 As an example, consider again the multi-agent system (19) corresponding to the graph G given in Figure 1, and the reduced order model (20) obtained from the partition π given by (12). For system (19), include output variables as in (27b). The output equations for the reduced order system (20) are given in (28b). Recall that π is an AEP of G. Also recall that the leader set is $\{6, 7\}$ in this case. Clearly, we have $6 \in C_2 = \{5, 6\}$ and $7 \in C_3 = \{7\}$. As in the statement of Theorem 6, let $\Xi(\pi)$ denote the normalized model reduction error corresponding to the partition π. Then, by Theorem 6, the normalized model reduction error $\Xi(\pi)$ in our example is computed as:

$$\Xi(\pi) = \frac{(1-\frac{1}{|C_2|}) + (1-\frac{1}{|C_3|})}{2(1-\frac{1}{10})} = 0.2778.$$

Remark 9 As mentioned in Subsection III, the reduced order model (15) approximates the dynamics of the average of the states of the agents that are cellmates in π. By (32a) and (33), it can be observed that this approximation is exact in case π is an almost equitable partition of G. That is, (15) indeed describes the dynamics of the average of the states of cellmates agents.

Previously, we have established an explicit formula for the model reduction error in case clustering is performed with respect to an almost equitable partition. Of course we are also interested in computing or estimating the errors associated with *arbitrary* partitions of the graph.

In order to attack this issue, we will first compare the model reduction error corresponding to an almost equitable partition, say π_0, to that of an arbitrary, not necessarily almost equitable, partition, say π. We restrict our attention to the case in which the partitions π and π_0 are *comparable* in the sense that one is *finer* than the other.

Given two partitions π_1 and π_2 of the graph G, we call π_1 *finer* than π_2 if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1 \leqslant \pi_2$. Alternatively, π_2 is called *coarser* than π_1. It is immediate that

$$\pi_1 \leqslant \pi_2 \iff \operatorname{im} P(\pi_2) \subseteq \operatorname{im} P(\pi_1). \tag{44}$$

Now, we have the following result.

Theorem 10 *Let π_0 be an almost equitable partition of G. Then for every partition π that is coarser than π_0 we have*

$$\Xi(\pi_0) \leqslant \Xi(\pi).$$

Proof. Suppose that π is a partition of G and $\pi \geqslant \pi_0$. Let S_0 and $\tilde{S}$ denote the transfer matrices from u to y in the reduced order model (28) corresponding to the partitions π_0 and π, respectively. Also let S denote the transfer matrix from u to y in (27), as before. Then, clearly, it suffices to show that

$$\|S - \tilde{S}\|_2^2 \geqslant \|S - S_0\|_2^2. \tag{45}$$

We have

$$\begin{aligned}\|S - \tilde{S}\|_2^2 &= \|S - S_0 + S_0 - \tilde{S}\|_2^2 \\ &= \|S - S_0\|_2^2 + \|S_0 - \tilde{S}\|_2^2 + 2\langle S - S_0, S_0 - \tilde{S}\rangle \\ &\geqslant \|S - S_0\|_2^2 + 2\langle S - S_0, S_0\rangle - 2\langle S - S_0, \tilde{S}\rangle,\end{aligned}$$

where $\langle S_1, S_2\rangle = \int_{-\infty}^{\infty} \operatorname{trace} S_1^\top(-i\omega)S_2(i\omega)d\omega$ is the inner product in $\mathcal{H}_2$.

Since π_0 is an AEP of G, by Lemma 5 we have

$$L \operatorname{im} P(\pi_0) \subseteq \operatorname{im} P(\pi_0). \tag{46}$$

In addition, as $\pi \geqslant \pi_0$, the subspace inclusion $\operatorname{im} P(\pi) \subseteq \operatorname{im} P(\pi_0)$ holds. Hence, by (46), we obtain that $L \operatorname{im} P(\pi) \subseteq L \operatorname{im} P(\pi_0) \subseteq \operatorname{im} P(\pi_0)$. Therefore, there exist matrices X and Y such that

$$LP(\pi_0) = P(\pi_0)X \tag{47}$$

and

$$LP(\pi) = P(\pi_0)Y. \tag{48}$$

Now, recall that $S(s) - S_0(s) = \Delta(s)$ where Δ is given by (35). Then, by using (3), (31), (47), and (48), it is easy to verify that $\langle S - S_0, S_0 \rangle = 0$ and $\langle S - S_0, \tilde{S} \rangle = 0$. Hence, (45) holds which completes the proof. ∎

As a consequence of the above, if, starting from a given AEP, we choose an arbitrary partition that is coarser than this AEP and perform the reduction based upon the latter, then the error will be at least as big as the error associated with the AEP.

Using the previous result, we are now able to estimate a lower bound for the error associated with an arbitrary, not necessarily almost equitable, partition. Let π be an arbitrary partition of G. Consider the set

$$\Pi_{\mathrm{AEP}}(\pi) = \{\pi_0 \mid \pi_0 \text{ is an AEP of } G \text{ and } \pi_0 \leqslant \pi \}$$

of all almost equitable partitions of G that are finer than π. As shown in [27], this set contains a unique maximal element, which we will call *the maximal almost equitable partition finer than* π, and that will be denoted by $\pi_{\mathrm{AEP}}^*(\pi)$. In fact, we have $\pi_{\mathrm{AEP}}^*(\pi) = \vee\Pi_{\mathrm{AEP}}(\pi)$, where $\vee\Pi_{\mathrm{AEP}}(\pi)$ denotes the least upper bound of the set $\Pi_{\mathrm{AEP}}(\pi)$ and is identified by the following property:

$$\pi_0 \leqslant \vee\Pi_{\mathrm{AEP}}(\pi), \text{ for all } \pi_0 \in \Pi_{\mathrm{AEP}}(\pi) \quad (49a)$$

$$\exists \tilde{\pi} \in \Pi \text{ s.t. } \pi_0 \leqslant \tilde{\pi} \text{ for all } \pi_0 \in \Pi_{\mathrm{AEP}}(\pi) \Rightarrow \vee\Pi_{\mathrm{AEP}}(\pi) \leqslant \tilde{\pi} \quad (49b)$$

with Π denoting the set of all partitions of G. This least upper bound always exists since, with the partial ordering "$\leqslant$", the set Π becomes a complete lattice (see [2]) meaning that every subset of Π has both its greatest lower bound and least upper bound within Π. In addition, the uniqueness of $\vee\Pi_{\mathrm{AEP}}(\pi)$ readily follows from (49b), and the fact that $\vee\Pi_{\mathrm{AEP}}(\pi) \in \Pi_{\mathrm{AEP}}(\pi)$ is shown in [27, Lemma 4].

It follows from Theorem 10 that a lower bound on the model reduction error associated with a given partition π is obtained by taking the error associated with the maximal almost equitable partition finer than π, as stated by the following corollary.

Corollary 11 *Let π be a partition of G, and $\pi_{\mathrm{AEP}}^*(\pi)$ denote the maximal element of the set $\Pi_{\mathrm{AEP}}(\pi)$. Then, we have*

$$\Xi(\pi_{\mathrm{AEP}}^*(\pi)) \leqslant \Xi(\pi).$$

An algorithm to actually compute $\pi_{\mathrm{AEP}}^*(\pi)$ for a given partition π was given in [27]. Note that $\Xi(\pi_{\mathrm{AEP}}^*(\pi))$ can then be computed using Theorem 6.

Remark 12 The problem of finding all almost equitable partitions of a given graph is in general a very difficult problem and computationally expensive, typically requiring an exhaustive search in the set of all partitions. In contrast, given an initial partition π, the almost equitable partition $\pi_{\mathrm{AEP}}^*(\pi)$ can be computed in an efficient manner (see [27]). In particular, the algorithm proposed in [27] generates a sequence of partitions converging to π_{AEP}^* in $n-m$ steps, where n is the number of agents and m is the number of leaders. Note that the partition $\pi_{\mathrm{AEP}}^*(\pi)$ is, in fact, a "finer" almost equitable approximation of π.

V. Conclusions

In this paper, by means of graph partitions we have established a projection based model reduction method for multi-agent systems defined on a graph. Reduced order models are obtained by clustering the vertices (agents) of the underlying communication graph in accordance with suitable graph partitions. In particular, the states of the vertices that are clustered together are approximated to be identical in deriving the reduced order models. As observed, the spatial structure of the network is preserved in this reduction process, and the reduced order models are realized as multi-agent systems defined on a new graph of smaller size. We have shown that if the original multi-agent system reaches consensus, then so does the reduced order model. As observed the underlying intuitive idea is to cluster together the vertices (agents) which are connected to the rest of the network in a similar fashion. This heuristic idea is formally formulated in terms of almost equitable partitions. Corresponding to an almost equitable partition, an explicit formula for the $\mathcal{H}_2$-norm of the error system has been provided. The proposed formula is simple, easy to compute, and can be derived directly from the graph partition involved in the reduction procedure. We also have shown that the error obtained by taking an arbitrary partition of the graph is at least as big as the one obtained by using the maximal almost equitable partition finer than the given partition. We have adopted a running example for illustration of the proposed results.

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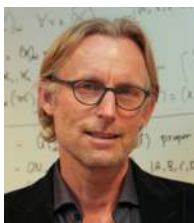
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