Projection Methods in Nonlinear Numerical Functional Analysis

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1. Introduction. The main purpose of this paper is to present a general and self-contained theory of the projection method for the solution of equations in a reflexive Banach space involving nonlinear operators acting in the same space. (Let us remark that with a suitable modification of our arguments the theory presented in this paper remains also valid for equations involving nonlinear operators from one reflexive Banach space into another reflexive Banach space.) Another purpose is to develop the theory in such a way as to unify the recent results on nonlinear equations obtained by Zarantonello [27], Minty [15], Browder [1, 2, 3, 4], Kachurovsky [8], Shinbrot [25], the author [19, 20] and others with the earlier results on linear equations obtained by Kravchuk [9], Mikhlin [14], Polsky [22, 23], Sobolevsky [24], Medvedev [13], Martyniuk [11, 12], Shalov [26], Lashko [10], Céa [6], Hildebrandt and Wienholtz [7], the author [17, 18, 21] and others. The projection method is used here to give a *constructive* proof of the existence and the uniqueness of solutions for a general class of nonlinear operator equations, the so-called *equations which are projectionally and strongly* (*weakly*) *solvable.*

Let us note that the class of operator equations studied in this paper is quite general. In fact, on the one hand, it includes almost all linear equations investigated in [7, 12, 13, 14, 17, 18, 22, 23, 24] for which the projection method and its actual realizations (*i.e.*, the methods of Ritz, Galerkin, and moments [9, 10, 11]) are known to be applicable as approximation methods and, on the other hand, it includes a class of nonlinear equations involving continuous, demicontinuous, and weakly continuous monotone, complex monotone, J-monotone, and other operators recently studied (from the point of view of the existence of solutions) by a number of authors [1, 3, 5, 8, 15, 19, 25, 27]. Consequently, from our two main theorems, Theorem I in Section 3 and Theorem II in Section 4, we deduce in a constructive fashion, as corollaries, some new special results as well as a number of known important results of the above mentioned authors both for linear and nonlinear equations. Furthermore, our study seems to indicate

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at the same time the natural position and the role played in the development of the theory of projection methods by the concepts of continuity, demicontinuity, and boundedness of nonlinear operators.

In writing this paper we were primarily concerned with its application to numerical analysis and applied mathematics and very often, therefore, we have preferred to forego the generality for the sake of construction and the simplicity and clarity of arguments. Finally we make no claim that our references exhaust the numerous contributions to the field (*i.e.*, projection and direct methods) under investigation since only those papers are included to which a direct reference is made.

2. Preliminaries and the formulation of the problem. Let X be a real or complex reflexive Banach space with the property that there exists a sequence $\{X_n\}$ of finite dimensional subspaces X_n of X, a sequence of linear projections $\{P_n\}$ defined on X, and a constant $C > 0$ such that

$$(1) \qquad P_nX = X_n, \qquad X_n \subset X_{n+1}, \qquad n = 1, 2, 3, \cdots, \quad \overline{\bigcup_n^{\infty} X_n} = X$$

$$(2) \qquad \|P_n\| \leqq C, \qquad n = 1, 2, 3, \cdots, \qquad P_nP_j = P_j \qquad \text{for } n \geqq j.$$

Some authors (*e.g.*, [5]) refer to X, having properties (1) and (2), as a space with *property* $(\pi)_C$. Note that if X is a separable Hilbert space the property $(\pi)_C$ of X with $C = 1$ follows from (1) alone; if X is a Banach space, then (1) and (2) are satisfied if X possesses a Schauder basis and in this case X_n is a subspace determined by the n first elements of the basis. In what follows we assume that X has property $(\pi)_C$, X^* denotes the conjugate space, and use the symbols "$\to$" and "$\rightharpoonup$" to denote the *strong* and the *weak* convergence in X, respectively.

Let A be a nonlinear mapping from X to X and let $\{A_n\}$ be a sequence of nonlinear mappings defined by $A_n = P_nAP_n$, $n = 1, 2, 3, \cdots$. In this paper we consider the problem of giving a constructive proof of the existence and the uniqueness of solutions of the "exact" equation

$$(3) \qquad Ax = f, \qquad f \in X,$$

as either the strong limit or the weak limit of solutions $x_n \in X_n$ of "approximate" equations

$$(4) \qquad A_nx_n = P_nf.$$

To make the nature of our investigation more precise we need the following definition.

Definition 1. Eq. (3) is said to be *projectionally and strongly* (*weakly*) *solvable* or *PS-solvable* (*PW-solvable*) if there exist an integer $N > 0$ such that for each $n \geqq N$ and each given f in X Eq. (4) has a unique solution $x_n \in X_n$ such that $x_n \to x(x_n \rightharpoonup x)$ in X and x is the unique solution of Eq. (3).

3. Equations which are projectionally and strongly solvable. In this section we give a constructive proof of existence and uniqueness of solutions of Eq. (3) as strong limits of solutions of approximate equations (4). The main result of this section is the following theorem.

Theorem I. Suppose that there exists an integer $N > 0$ *and a continuous monotonically increasing real function* $\alpha(r)$ *defined for* $r \geqq 0$ *with* $\alpha(0) = 0$ *and* $\lim_{r\to\infty} \alpha(r) = \infty$ *such that the following conditions are satisfied:*

(Ia) A_n *is continuous in* X_n *for each* $n \geqq N$ *and* $A_n x \to Ax$ *for each* x *in* X.

(Ib) *For each* $n \geqq N$ *and all* x *and* y *in* X_n

$$||A_n x - A_n v|| \geqq \alpha(||x - y||).$$

(Ic) *If* $\{X_m\}$ *is any sequence of subspaces of* X *which satisfies* (1) *and* (2) *and* $\{x_m\}$ *is a sequence in* X *with* $x_m \in X_m$ *such that* $x_m \rightharpoonup x$ *and* $A_m x_m \to g$, *then* $Ax = g$.

Then Eq. (3) *is PS-solvable.*

We remark that in the hypothesis (Ic) it is sufficient to assume that $\{X_m\}$ is an arbitrary infinite subsequence of $\{X_n\}$. The same remark applies to the hypothesis (IIc) of Theorem II below.

Proof. (*Uniqueness*). Suppose that $u \neq v$ and $Au = Av$. Then, since $P_n^2 = P_n$, (Ib) implies that for all $n \geqq N$

$$||A_n u - A_n v|| \geqq \alpha(||P_n u - P_n v||).$$

Since $P_n x \to x$ and, by (Ia), $A_n x \to Ax$ for each x in X the passage to the limit in the above inequality yields the inequality

$$0 = ||Au - Av|| \geqq \alpha(||u - v||)$$

which contradicts the fact that $\alpha(r)$ is monotonically increasing with $\alpha(0) = 0$. Hence $u = v$ and A is one-to-one.

(*Existence*). First note that for each $n \geqq N$, the range $R(A_n)$ is a closed set in X_n. In fact, if $\{y_m\} \subset R(A_n)$ and $y_m \to y$, then there exists a sequence $\{x_m\} \subset X_n$ such that $y_m = A_n x_m$ and by (Ib)

$$||y_e - y_k|| = ||A_n x_e - A_n x_k|| \geqq \alpha(||x_e - x_k||) \qquad (n \geqq N).$$

Since $\{A_n x_m\}$, $m = 1, 2, 3, \cdots$, is a Cauchy sequence, $\alpha(||x_e - x_k||) \to 0$ as $e, k \to \infty$. If $\beta(r)$ denotes the continuous monotonically increasing function defined for $r \geqq 0$ with $\beta(0) = 0$ which is the inverse of $\alpha(r)$, then

$$||x_e - x_k|| \leqq \beta(||A_n x_e - A_n x_k||) \to 0, \qquad (e, k \to \infty).$$

Thus there exists an element x in X_n such that $x_m \to x$ and $A_n x_m \to A_n x = y \in R(A_n)$, *i.e.*, $R(A_n)$ is closed. On the other hand, A_n is clearly one-to-one and continuous for each $n \geqq N$. Hence, by Brouwer theorem on invariance of domain, A_n is an open mapping and $R(A_n)$ is an open set in X_n. Since X_n is connected and $R(A_n)$ is a nonempty set in X_n, which is both open and closed, it follows that $R(A_n) = X_n$.

Hence for each $n \geqq N$ and each given f in X there exists a unique element $x_n \varepsilon X_n$ such that $A_n x_n = P_n f$. For the sequence $\{x_n\}$ thus determined, (2) and (Ib) imply that

$$||A_n x_n|| \geqq ||A_n x_n - A_n(0)|| - ||A_n(0)|| \geqq \alpha(||x_n||) - C\,||A(0)||$$

whence, in view of (4),

$$\alpha(||x_n||) \leqq ||P_n f|| + C\,||A(0)|| \leqq C(||f|| + ||A(0)||).$$

Thus, for all $n \geqq N$,

$$||x_n|| \leqq \beta(C\,||f|| + C\,||A(0)||) \equiv M.$$

Since X is reflexive, the closed ball of radius M about the origin in X is weakly compact. Hence there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some element x in X. Furthermore, in view of (4), $A_{n_i} x_{n_i} = P_{n_i} f \to f$, as $n_i \to \infty$; therefore, (Ic) implies that $Ax = f$. Since, as was shown above, x is the unique solution of Eq. (3), we conclude *a posteriori* that the selection of the subsequence was not necessary and, consequently, the entire sequence $\{x_n\}$ converges weakly to x.

Moreover, $x_n \to x$ since (Ia) and (Ib) and $P_n^2 = P_n$ imply that

$$\alpha(||x_n - P_n x||) \leqq ||A_n x_n - A_n P_n x|| = ||P_n f - A_n x|| \to 0,$$

as $n \to \infty$. Hence the properties of the function $\beta(r)$ imply that

$$||x_n - P_n x|| \leqq \beta(||P_n f - A_n x||) \to 0,$$

as $n \to \infty$, and consequently

$$\begin{aligned} ||x_n - x|| &\leqq ||x_n - P_n x|| + ||P_n x - x|| \\ &\leqq \beta(||P_n f - A_n x||) + ||P_n x - x|| \to 0, \end{aligned}$$

as $n \to \infty$. Thus x is also the strong limit of x_n and hence Theorem I is proved.

Remark 1. In the proof of Theorem I we did not use the fact that $X_n \subset X_{n+1}$ for each n and that $P_n P_j = P_j$ for $n \geqq j$. Consequently, Theorem I and Theorem 1 below remain valid if $\{X_n\}$ is any sequence of finite dimensional subspaces of X which is *projectionally complete* (see [23]), *i.e.*, $\{X_n\}$ is such that $P_n X = X_n$ and $P_n x \to x$ for each x in X, for each n.

Remark 2. If X has a Schauder basis $\{\varphi_i\}$, then for each x in X

$$x = \sum_{i=1}^{\infty} a_i \varphi_i\,, \qquad a_i = (\phi_i\,, x)\;\varepsilon\; X^*, \qquad \text{and} \qquad (\phi_i\,, \varphi_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

where for any w in X^* and y in X we denote the value of w at y by (w, y). Let P_n denote the projection of X onto the space X_n determined by $\{\varphi_1\,, \cdots, \varphi_n\}$. Then

$$P_n x = \sum_{i=1}^{n} \phi_i(x)\varphi_i , \qquad P_n^2 = P_n , \qquad ||P_n|| \leqq C.$$

Now if the n-th order Galerkin approximation x_n is taken in the form

$$x_n = \sum_{i=1}^{n} c_i^n \varphi_i , \qquad n = 1, 2, 3, \cdots , \tag{5}$$

then the coefficient $\{c_i^n\}$ are determined by the conditions

$$(\phi_j , Ax_n - f) = 0 \qquad \text{or} \qquad (\phi_j , Ax_n) = (\phi_j , f), \qquad 1 \leqq j \leqq n. \tag{6}$$

It is easy to see that the nonlinear algebraic system (6) is equivalent to Eq. (4). In case X is a Hilbert space H, Eq. (6) takes the form

$$(Ax_n , \varphi_j) = (f, \varphi_j), \qquad 1 \leqq j \leqq n. \tag{6$_0$}$$

Consequently, our Theorem I includes also the corresponding results concerning the methods of Ritz and Galerkin. For the extensive earlier contributions to the theory of the latter two methods and their applications to the approximate solution of linear differential equations see the monographs of Kravchuk [9], Mikhlin [14] and Polsky [22].

3.1 Special cases of equations which are *PS*-solvable. In this section we give some examples of linear and non-linear equations in Hilbert and Banach spaces which are PS-solvable. It will be shown that for linear equations our Theorem I includes almost every known theorem concerning the convergence of the projection method studied in [7, 11, 13, 17, 23, 24]. At the same time we also obtain some new results even for linear equations. In particular, for bounded linear operators, we give simple necessary and sufficient conditions for Eq. (3) to be PS-solvable. Furthermore for nonlinear equations we deduce from our constructive Theorem I the fundamental and important theorems of Minty [15] (see also Browder [1, 2]) and Zarantonello [27] concerning the strongly monotone and complex monotone operators. Some additional results for P-compact operators in X are also obtained. The latter class of operators was studied by the author in [20].

(i) Applications to equations involving bounded linear operators. Let A be a bounded linear mapping of X into X. Then conditions (Ia) and (Ic) are automatically satisfied while (Ib) reduces to the requirement

$$||A_n x|| \geqq \alpha(||x||) \quad \text{for all} \quad x \text{ in } X_n \quad \text{and all} \quad n \geqq N. \tag{1b}$$

In fact in this case we can (and will) assume that the function $\alpha(r)$ is of the form $\alpha(r) = cr$ for some constant $c > 0$. Hence in this case Theorem I reduces to

Corollary 1. *Suppose that there exists an integer $N > 0$ and a function $\alpha(r) = cr$ with $c > 0$ such that* (1b) *holds. Then Eq.* (3) *is PS-solvable.*

It is not hard to see that in this case the converse is also true, *i.e.*, if Eq. (3) is PS-solvable, then there exists an integer $N > 0$ and a constant $c > 0$ such that (1b) holds with $\alpha(r) = cr$. To see this note that, by PS-solvability of Eq. (3), there exists an integer $N > 0$ such that for each $n \geqq N$ and each f in X, Eq. (4) has a unique solution $x_n = A_n^{-1}P_nf$ in X_n, $x_n \to x$ and x is the unique solution of Eq. (3). This implies that the null space $N(A) = \{0\}$, $R(A) = X$, and, by uniform boundedness principle, there exists a constant $M > 0$ such that $\|A_n^{-1}P_n\| \leqq M$ for all $n \geqq N$. The latter fact clearly implies condition (1b) with $\alpha(r) = M^{-1}r$.

Thus, Corollary 1 and the above discussion imply the validity of the following interesting results.

Theorem 1. *Eq.* (3) *is PS-solvable if and only if there exists an integer* $N > 0$ *and a constant* $c > 0$ *such that* (1b) *holds with* $\alpha(r) = cr$.

In case X is a separable Hilbert space H and A has a bounded inverse A^{-1}, the necessary and sufficient conditions for Eq. (3) to be PS-solvable were first given by Polsky [23]. In our terminology and for operators treated here these conditions can be stated as follows.

Theorem 1P. *Let A be a bounded linear operator in H with bounded inverse A^{-1}. Suppose further that for $n \geqq N_0$*

$$\tau_n = \min_{x \in H_n} \left\{ \frac{\|P_nAx\|}{\|Ax\|} \right\} > 0. \tag{7}$$

Then Eq. (3) *is PS-solvable if and only if A satisfies condition* (A):

$$\liminf_{n \to \infty} \tau_n = \tau > 0. \tag{8}$$

Let us remark that Polsky did not prove directly that condition (A) is also necessary for Eq. (3) to be PS-solvable. The necessity of condition (A) was shown by him (in his thesis which is not accessible here) that when $\tau = 0$ then one can construct an element f in H such that the norms of the corresponding projectional approximations x_n increase indefinitely and, consequently, do not converge to x.

Proposition 1. *Theorems 1 and 1P are equivalent.*

Proof. Suppose that the conditions of Theorem 1P hold. Now, since A^{-1} is bounded, there exists a constant $m > 0$ such that $\|Ax\| \geqq m \|x\|$ for all x in H. This and (7) imply that

$$\|A_nx\| = \|P_nAx\| \geqq m\tau_n \|x\| \text{ for all } x \text{ in } H_n \text{ and } n \geqq N_0 .$$

Since $\tau > 0$, the definition of τ implies that, for any given $\epsilon > 0$ with $(\tau - \epsilon) > 0$, there exists an integer $N(\epsilon)$ such that $m\tau_n > m(\tau - \epsilon)$ for all $n \geqq N(\epsilon)$. Thus our condition (1b) is satisfied with $N = \max(N(\epsilon), N_0)$ and $\alpha(r) = m(\tau - \epsilon)r$. Hence, Theorem 1P follows from Theorem 1.

Converse. Suppose now that conditions of Theorem 1 hold. Then, since $P_n^2 = P_n$, $||A_n x|| \geqq c\, ||P_n x||$ for each x in X and each $n \geqq N$. Since $P_n x \to x$ and $A_n x \to Ax$, the passage to the limit in the above inequality yields $||Ax|| \geqq c\, ||x||$ for all x in X, *i.e.*, A^{-1} exists and is bounded. Furthermore, (1b) and $||Ax|| \leqq ||A||\, ||x||$ imply that

$$\frac{||P_n Ax||}{||Ax||} \geqq \frac{||P_n Ax||}{||Ax||\, ||x||} \geqq \frac{c}{||A||} > 0 \quad \text{for all} \quad x \text{ in } H_n \quad \text{and all} \quad n \geqq N.$$

This implies the validity of (7) for $n \geqq N$ and (8) with $\tau \geqq c\, ||A||^{-1}$. Consequently, Theorem 1 follows from Theorem 1P.

Corollary 2. *Eq.* (3) *involving a bounded linear operator A in H is PS-solvable if A satisfies anyone of the following conditions*:

(a) *A is definite, i.e., there exists a constant $\alpha_1 > 0$ such that*

$$|(Ax, x)| \geqq \alpha_1\, ||x||^2 \quad \textit{for all} \quad x \textit{ in } H. \tag{9}$$

(b) $N(A) = \{0\}$ *and there exists a constant $\delta > 0$ such that for every weakly convergent (to zero) sequence $\{x_n\} \subset H$ with $||x_n|| = 1$*

$$\delta \leqq \liminf_{n \to \infty} |(Ax_n, x_n)| \tag{10}$$

(c) $A = T + S$, $N(A) = \{0\}$, *S is completely continuous and T satisfies either* (*a*) *or* (*b*).

Proof. In view of Corollary 1, it suffices to show that our conditions imply the validity of (1b) for a certain function $\alpha(r)$.

(a) Clearly (9) implies (1b) with $N = 1$ and $\alpha(r) = \alpha_1 r$.

(b) Suppose that (1b) is not true. Then we can find a sequence $\{x_{n_i}\} \subset H$ with $x_{n_i} \varepsilon H_{n_i}$ and $||x_{n_i}|| = 1$ such that

$$x_{n_i} \rightharpoonup x \quad \text{and} \quad ||P_{n_i} A x_{n_i}|| \to 0, \quad \text{as} \quad n_i \to \infty. \tag{11}$$

Since $x_{n_i} \rightharpoonup x$ implies that $P_{n_i} Ax_{n_i} \rightharpoonup Ax$, (11) shows that $Ax = 0$. Hence $x = 0$ since $N(A) = \{0\}$. Furthermore, (11) also shows that

$$|(Ax_{n_i}, x_{n_i})| = |(P_{n_i} Ax_{n_i}, x_{n_i})| \leqq ||P_{n_i} Ax_{n_i}|| \to 0 \qquad (n_i \to \infty)$$

in contradiction to (10).

(c) As in (b), if (1b) were not true, there would exist a sequence. $\{x_{n_i}\} \subset H$ with $x_{n_i} \varepsilon H_{n_i}$ and $||x_{n_i}|| = 1$ such that

$$x_{n_i} \rightharpoonup x \quad \text{and} \quad P_{n_i} A x_{n_i} \to 0, \quad \text{as} \quad n_i \to \infty.$$

First, the complete continuity of S implies that $P_{n_i} Sx_{n_i} \to Sx$ and

$$P_{n_i} Tx_{n_i} = P_{n_i} Ax_{n_i} - P_n Sx_{n_i} \to -Sx, \quad \text{as} \quad n_i \to \infty. \tag{12}$$

But, $P_{n_i} Tx_{n_i} \rightharpoonup Tx$. Consequently, $Tx = -Sx$ or $Ax = 0$.

Suppose first that T satisfies (a). Then $x_{n_i} \to x$, since by (9) and (12),

$$\alpha_1 \|x_{n_i} - x\|^2 \leqq |(Tx_{n_i} - Tx, x_{n_i} - x)|$$
$$= |(Tx_{n_i}, x_{n_i}) - (Tx, x_{n_i}) - (Tx_{n_i}, x) + (Tx, x)|$$
$$= |(P_{n_i}Tx_{n_i}, x_{n_i}) - (Tx, x_{n_i}) - (x_{n_i}, T^*x) + (Tx, x)| \to 0.$$

Hence $\|x\| = 1$ and $Ax = 0$ in contradiction to $N(A) = \{0\}$.

If, on the other hand, T satisfies (b) then $x = 0$ since $N(A) = \{0\}$ and, therefore, $Sx = 0$, *i.e.*, $P_{n_i} Tx_{n_i} \to 0$. Hence

$$|(Tx_{n_i}, x_{n_i})| = |(P_{n_i}Tx_{n_i}, x_{n_i})| \leqq \|P_{n_i}Tx_{n_i}\| \to 0, \quad \text{as} \quad n_i \to \infty,$$

in contradiction to (10). This yields the proof of Corollary 2 as an immediate consequence of Corollary 1 or Theorem 1.

Remark 3. Corollary 2(a) in its present form was first proved by Polsky [23]. A more general but similar result for bounded bilinear functionals $B(x, y)$ which are definite were independently derived by Hildebrandt and Wienholtz [7] and the author [18]. A still more general result for such bilinear functionals $B(x, y)$ were obtained by Céa [6]. (The author is grateful to the referee for calling his attention to the interesting paper of Céa.) Corollary 2(b) was established by Medvedev [13] while Corollary 2(c) was independently derived by Medvedev [13] for T satisfying (b) and by Hildebrandt and Wienholtz for T satisfying (a). When T is self-adjoint and positive definite, Corollary 2(c) was first proved by Polsky [23].

(ii) Application to equations involving unbounded linear operators. In this section we consider two general classes of linear equations involving unbounded densely defined operators in a Hilbert space H. These classes of equations involve the operators forming an acute angle investigated by Sobolevsky [24] and the *K-p.d.* and non-*K-p.d.* operators studied by the author [17, 21]. The above classes of equations contain a number of particular equations for which the applicability of Ritz and Galerkin methods was thoroughly studied by a number of authors [9, 10, 11, 12, 14, 22]. Let us point out that just as the concepts of K-symmetry and K-positivity generalize the concepts of symmetry and positivity of an operator so the concept of a non-*K-p.d.* operator is a generalization of the concept of "operators forming an acute angle." The exact relation between the latter two classes of operators will be indicated below. Let us add that these new generalized concepts enable us to apply the projection methods and other variational methods to the solution of non-selfadjoints ordinary differential equations of odd order (see, for example, [11, 12]) and to nonelliptic partial differential equations of odd and even order (see, for example, [11, 24, 26]).

Our investigation in this section consists in reducing a problem with unbounded operators to a problem with bounded operators to which the results of the preceding section apply. Let us add that we obtain here somewhat more general results than those contained in [11, 17, 21, 22, 24].

Operators forming an acute angle. Following Sobolevsky [24] we say that two densely defined operators A and D in H form an *acute angle* if $D(A) = D(D)$, A and D vanish only at zero element and

$$(13) \qquad |(Dx, Ax)| \geqq m\,||Ax||\,||Dx||, \qquad x \in D(A), \qquad 1 \geqq m > 0.$$

It will be shown here that the main theorem of Sobolevsky follows from our Corollary 1.

Lemma 1. *If A and D form an acute angle and are closed with $R(A) = R(D) = H$, then there exists constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$(14) \qquad ||Dx|| \leqq \gamma_1\,||Ax||, \qquad ||Ax|| \leqq \gamma_2\,||Dx||, \qquad x \in D(A).$$

Proof. Introduce in $D(D)$ a new inner product and norm by

$$(15) \qquad [x, y]_D = (Dx, Dy), \qquad |x|_D^2 = [x, x]_D\,, \qquad x, y \in D(D).$$

Since D^{-1} is closed and defined on all of H, it is bounded, *i.e.*, D is continuously invertible. Hence, $D(D)$ is a Hilbert space with respect to the metric (15) which we denote by H_D. Note now that A considered as a mapping from H_D to H is closed. Indeed, since A is defined on all of H_D it is sufficient to show that A admits in H_D a closed linear extension, *i.e.*, if $\{x_n\}$ is a sequence in H_D such that $|x_n - 0|_D \to 0$ and $||Ax_n - f|| \to 0$, as $n \to \infty$, then $f = 0$. Since D is continuously invertible, $|x|_D = ||Dx|| \geqq c\,||x||$ for all x in $D(D)$ and some $c > 0$. Hence $|x_n - 0|_D \to 0$ implies that $||x_n - 0|| \to 0$ and, since A is closed in H, $f = 0$. Thus A is closed in H_D and, being everywhere defined in H_D, it is bounded, *i.e.*, there exists a constant $\gamma_2 > 0$ such that $||Ax|| \leqq \gamma_2 |x|_D = \gamma_2 ||Dx||$, $x \in H_D$. Interchanging the role of A and D we obtain the first inequality in (14) and thus complete the proof of Lemma 1.

Lemma 1 shows that the mapping $G = D^{-1}A$ is bounded in H_D and that

$$(16) \qquad |[x, Gx]_D| = |(Dx, Ax)| \geqq m\gamma_1^{-1}\,|x|_D^2\,, \qquad x \in H_D\,.$$

Let $\{\varphi_i\} \subset D(A)$ be a complete orthonormal set in H. Then $\{H_n\} = \{\text{span of } \{\varphi_1, \cdots, \varphi_n\}\}$, satisfies (1) and (2) in H. Assume that each H_n is invariant under D. Then, since D is continuously invertible,

$$D_n = P_n D P_n = DP_n \quad \text{and,} \quad \text{for any} \quad x \in H_n\,,$$
$$||D_n x|| = ||DP_n x|| = ||Dx|| \geq c\,||x||.$$

Hence, D_n is a one-to-one mapping of H_n onto H_n. Furthermore, the set $\{\varphi_i\} \subset D(A)$ determined by $\varphi_i = D\psi_i$ is a complete orthonormal sequence in H_D such that for each n the set $\{\psi_i\}$, $1 \leqq i \leqq n$, belongs to H_n. Let $H_n^D (\subset H_D)$ denote the space H_n with the inner product and norm given by (15) and let π_n denote the orthogonal projection of H_D onto H_n^D.

Corollary 3. *Suppose that the closed operators A and D form an acute angle, and that $R(A) = R(D) = H$. Suppose also that the subspaces H_n are invariant under D. Then Eq.* (3) *is PS-solvable. Furthermore,* $||Ax_n - f|| \to 0$, as $n \to \infty$.

Corollary 3 is essentially Sobolevsky's Theorem 2, who stated it in [24] without proof under the additional assumption that D is self-adjoint.

Proof. It follows from our discussion that Eq. (3) in H is equivalent to

(17) $$Gx = g, \qquad g = D^{-1}f \in H_D ,$$

while Eq. (4) is equivalent to the approximate equation

(18) $$G_n x_n = \pi_n g, \qquad G_n = \pi_n G \pi_n , \qquad x_n \in H_n .$$

The first assertion is obvious while the second follows from the following argument. An element x_n in H_n is a solution of Eq. (4) if and only if x_n satisfies the system

(19) $$(Ax_n , \varphi_j) = (f, \varphi_j), \qquad 1 \leqq j \leqq n,$$

or, equivalently, the system

(18_0) $$[Gx_n , \psi_j]_D = [g, \psi_j]_D , \qquad 1 \leqq j \leqq n.$$

In virtue of the conditions satisfied by $\{\psi_i\}$, x_n satisfies the system (18_0) if and only if x_n is a solution of Eq. (18), *i.e.*, Eq. (4) is equivalent to Eq. (18). Since G is a bounded map of H_D into itself which is definite, Corollary 2 implies that, for each n, Eq. (18) has a unique solution $x_n \in H_n^D$, $x_n \to x$ in H_D and x is the unique solution of Eq. (17). Moreover, the equivalence relation between various equations and the continuous invertability of D show that Eq. (3) is PS-solvable. Finally, by (14), $||Ax_n - f|| = ||Ax_n - Ax|| \leqq \gamma_2 |x_n - x| \to 0$, as $n \to \infty$.

K-p.d. and non-K-p.d. operators. An operator B defined on a dense domain $D(B) \subset H$ is called *K-positive definite* (*K-p.d.*) if there exists a closeable operator K with $D(K) \supseteq D(B)$ such that $KD(B)$ is dense in H and

(20) $$(Bx, Kx) \geqq \alpha_3 ||x||^2$$
$$\text{and } ||Kx||^2 \leqq \alpha_4(Bx, Kx) \text{ for some } \alpha_3 > 0, \ \alpha_4 > 0 \text{ and all } x \in D(B);$$

B is *K-symmetric* if $(Bx, Ky) = (Kx, By)$ for all $x, y \in D(B)$. If B is K-symmetric and *K-p.d.* and H_0 denotes the completion of $D(B)$ in the metric

(21) $$[x, y] = (Bx, Ky), \qquad |x|^2 = [x, x],$$

then (for proofs of assertions made in this section see [17, 19]) H_0 can be regarded as a subset of H and B has a closed *K-p.d.* and K-symmetric extension B_0 which is continuously invertible.

Our problem is to investigate the operator equation

(22) $$Ax = Lx + Mx = f, \qquad f \in H,$$

where L is a complicated unbounded operator such that $D(L) = D(B)$ and

(23) $$|(Lx, Kx)| \geqq \eta_1(Bx, Kx) \quad \text{and} \quad |(Lx, Ky)|^2 \leqq \eta_2(Bx, Kx)(By, Ky)$$
$$\text{for } \eta_1 > 0, \ \eta_2 > 0 \ \text{ and all } \ x, y \in D(L),$$

and where $D(M) \supset D(L)$. Note that $W = B_0^{-1} L$, defined on $D(L)$, is bounded in H_0 and when $\bar{W}$ denotes its closure in H_0, then

$$(24) \qquad |[\bar{W}x, x]| \geqq \eta_1 |x|^2, \qquad x \in H_0 .$$

Hence, $\bar{W}$ is definite in H_0. Furthermore, L has a unique closed and continuously invertible extension $L_0 = T_0 W_0$, where W_0 is an operator in H_0 defined by relations $W \subset W_0 \subset \bar{W}$, $R(W_0) = D(B_0)$ and $D(L_0) = D(W_0)$. Assume that $S = B_0^{-1} M$, defined on $D(L)$, is bounded in H_0-norm and let $\bar{S}$ denote its extension to H_0. In what follows we regard a solution x of

$$(25) \qquad Wx + Sx = f_0 , \qquad f_0 = B_0^{-1} f \in H_0 ,$$

as a solution (possibly in generalized sense) of Eq. (22).

To construct a solution of Eq. (22) choose a linearly independent set $\{\varphi_i\} \subset D(L)$ which is complete in H_0 and construct $x_n = \sum_{i=1}^n a_i^n \varphi_i$ by solving

$$(26) \qquad (\bar{W}_n + \bar{S}_n)x_n = Q_n f_0 , \qquad \bar{W}_n = Q_n \bar{W} Q_n , \qquad \bar{S}_n = Q_n \bar{S} Q_n ,$$

where Q_n is the projection of H_0 onto $H_n = \text{span } \{\varphi_1 , \cdots , \varphi_n\}$. Eq. (26) is equivalent to the linear algebraic system

$$(26_0) \qquad \sum_{i=1}^{n} (L\varphi_i , K\varphi_j) + (M\varphi_i , K\varphi_j)a_i^n = (f, K\varphi_j), \qquad 1 \leqq j \leqq n,$$

which implies that we need not know B_0^{-1}, $\bar{W}$, and $\bar{S}$ to construct x_n.

The above discussion and Corollary 2 imply the validity of the following new result for the general Eq. (22).

Corollary 4. *If for each f in H, Eq. (22) has at most one solution and if $S = B_0^{-1}M$ can be extended to a completely continuous operator $\bar{S}$ in H_0, then Eq. (25) is PS-solvable, i.e., (26_0) has a unique solution $x_n \in H_n$ such that $x_n \to x$ in H_0 and x is the unique (possibly generalized) solution of Eq. (22).*

Remark 4. Let us remark that though Corollary 4 is new for the very general class of equations (22), the most familiar and extensive applications were already covered by a number of authors. Thus, if $K = I$ and $L = B$, then B_0 is a selfadjoint positive definite extension constructed by Friedrichs. In this case (26_0) reduces to the algebraic system determined by the method of Galerkin and Corollary 4 was first proved in this case by Mikhlin [14] (see also Polsky [22], Kravchuk [9], and Martyniuk [11]). If $L = B$, then L_0 is K-symmetric and K-positive definite and Corollary 4 was proved by the author [17] (see also Martyniuk [11, 12] and Lashko [10]). On the other hand, if $M = 0$ and K is closed with $D(K) = D(L)$ then, as was shown in [21], L forms an acute angle with K; under present conditions Corollary 4 was proved by the author [21] while under somewhat different conditions it was proved earlier by Sobolevsky [24].

(iii) Application to equations involving nonlinear operators. Let us first introduce some of the concepts to be used in the sequel. Let A be a nonlinear mapping of X into X. We say that A is *completely continuous* if A is continuous and maps bounded sets into precompact sets; A is *strongly continuous* if $x_n \rightharpoonup x$ implies $Ax_n \to Ax$; A is *weakly continuous* if $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$; A is *demicontinuous* if $x_n \to x$ implies $Ax_n \rightharpoonup Ax$; A is *bounded* if A maps bounded sets into bounded sets; A is *projectionally-compact* (*P-compact*) in a real space X if P_nA is continuous in X_n for all large n and if for any constant $p > 0$ and any bounded sequence $\{x_n\}$ in X with $x_n \in X_n$ the sequence $\{P_nAx_n - px_n\}$ is strongly convergent, then there exist a strongly convergent subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \to x$ and $P_{n_i}Ax_{n_i} \to Ax$; if $X = H$, a Hilbert space with the inner product $(\ ,\ )$ and norm $\|\ \|^2 = (\ ,\ )$, then we say that A is *strongly monotone* if for some constant $a > 0$

$$\text{Re}\,(Ax - Ay, x - y) \geqq a\,\|x - y\|^2, \qquad x, y \in H; \tag{27}$$

A is *complex monotone* if for some constant $b > 0$

$$|(Ax - Ay, x - y)| \geqq b\,\|x - y\|^2, \qquad x, y \in H. \tag{28}$$

In this section we consider certain classes of nonlinear operator equations in a Banach space X or a Hilbert space H for which all or some of the conditions of Theorem I are easily verifiable.

Corollary 5. *Let A be a continuous mapping of a real space X into itself such that condition* (Ib) *holds. Then Eq.* (3) *is* PS-*solvable if the operator $B = I\text{-}A$ is P-compact.*

Proof. First, since A is continuous, condition (Ia) is satisfied. Hence, by Theorem I, to prove the PS-solvability of Eq. (3) it suffices to show that the P-compactness of $B = I\text{-}A$ implies condition (Ic). Let $\{X_m\}$ be any sequence of finite dimensional subspaces in X satisfying (1) and (2) and let $\{x_m\}$ be a sequence in X with $x_m \in X_m$ such that $x_m \rightharpoonup x$ and $A_mx_m \to g$. Since $x_m \rightharpoonup x$ and $-A = B - I$, $\{x_m\}$ is bounded and $(P_mBx_m - x_m) \to -g$. Consequently, the P-compactness of B implies the existence of a subsequence $\{x_{m_i}\}$ which converges strongly (and necessarily) to x and $P_{m_i}Bx_{m_i} \to Bx$. This implies that $Ax = g$, *i.e.*, condition (Ic) holds. Hence, by Theorem I, Eq. (3) is PS-solvable.

As further consequences of our Theorem I we derive the important results concerning strongly monotone and complex monotone operators in H obtained recently in a nonconstructive way by Minty [15] and Zarantonello [27].

Corollary 6 (***Minty***). *If A is a continuous strongly monotone mapping in H, then Eq.* (3) *is PS-solvable.*

Proof. (Ia) follows from the continuity of A with $N = 1$, (Ib) follows from (27) with $\alpha(r) = ar$, while (Ic) follows from Browder's Lemma 3 [1]. Thus, Corollary 6 is deduced from Theorem I.

Corollary 7 (*Zarantonello*). *If A is continuous, bounded, and complex monotone in H, then Eq.* (3) *is PS-solvable.*

Proof. (Ia) and (Ib) follow from the continuity and complex monotonicity of A with $N = 1$ and $\alpha(r) = br$ while (Ic) follows from the following argument. Suppose that $\{H_m\}$ satisfies (1) and (2) and $\{x_m\} \subset H$ (with $x_m \in H_m$) is such that $x_m \rightharpoonup x$ and $A_m x_m = P_m A x_m \to g$. Then

$$b\,\|x_m - x\|^2 \leqq |(Ax_m - Ax, x_m - x)|$$
$$= |(P_m Ax_m, x_m) - (Ax, x_m) - (P_m Ax_m, x) - (Ax_m, (1 - P_m)x) + (Ax, x)|.$$

Since $x_m \rightharpoonup x$, $P_m Ax_m \to g$, and A is bounded the passage to the limit in the right-hand side of the above inequality implies that

$$|(f, x) - (Ax, x) - (f, x) - 0 + (Ax, x)| = 0,$$

i.e., $x_m \to x$. Since A is continuous, $P_m Ax_m \to Ax$. Thus, $Ax = g$ and (Ic) holds, *i.e.*, Corollary 7 follows from Theorem I.

Corollary 8. *Suppose that $A = T + S$, where T is a continuous mapping, which is either monotone or complex monotone in H, and where S is a strongly continuous mapping in H. Suppose further that $A_n = P_n A P_n$ satisfies* (Ib) *of Theorem I. Then the equation $Ax = Tx + Sx = f$ is PS-solvable.*

Proof. It is obvious that Corollary 8 will follow from Theorem I if we show that our conditions on T and S imply the validity of condition (Ic).

Suppose first that T is monotone and that $\{x_m\} \subset H$ (with $x_m \in H_m$) is such that $x_m \rightharpoonup x$ and $A_m x_m = P_m Ax_m \to g$. This and the strong continuity of S imply that

$$P_m T x_m = P_m A x_m - P_m S x_m \to g - Sx, \qquad (m \to \infty).$$

Hence, since T is monotone and continuous, Lemma 3 in [1] implies that $Tx = g - Sx$, *i.e.*, (Ic) holds.

Suppose now that T is complex monotone. Then (Ic) follows from the following useful lemma.

Lemma 2. *Let T be a demicontinuous complex monotone mapping of H into H and let $\{H_n\}$ satisfy* (1) *and* (2). *If $\{u_n\} \subset H$ is such that $u_n \in H_n$ for each n, u_n u_0 in H and $P_n\, Tu_n \to g_0$ in H, then $Tu_0 = g_0$.*

Proof. Let j be a fixed integer and let u be any element in H_j. Since $P_j u = u$, $P_n P_j = P_j$ for $n \geqq j$, $P_n u_n = u_n$ for u_n in X_n, $u_n \rightharpoonup u_0$ and $P_n T u_n \to g_0$ in H, we have for each $n \geqq j$ and each $u \in H_j$ the equality

$$(Tu_n - T(P_j u), u_n - P_j u) = (P_n T u_n, u_n - P_j u) - (T(P_j u), u_n - P_j u)$$

from which, on passage to the limit as $n \to \infty$, we obtain the relation

$$(Tu_n - T(P_ju), u_n - P_ju) \to (g_0, u_0 - P_ju) - (T(P_ju), u_0 - P_ju) = (g_0 - T(P_ju), u_0 - P_ju). \tag{29}$$

Since, by (28), for each n

$$|(Tu_n - T(P_ju), u_n - P_ju)| \geqq b\,||u_n - P_ju||^2 \tag{30}$$

and $\{u_n - P_ju\}$, which converges weakly to $(u_0 - P_ju)$, has the property that $||u_0 - P_ju|| \leqq \liminf_n ||u_n - P_ju||$, we derive from this, (29) and (30) the inequality

$$|(g_0 - Tu, u_0 - u)| \geqq b\,||u_0 - u||^2 \tag{31}$$

valid for all $u = P_ju$ in H_j. Since j is arbitrary, (31) is true for all u in a dense subset $\bigcup_j H_j \subset H$. This and the demicontinuity of T imply the validity of (31) for all u in H. Since, by Zarantonello-Browder theorem [1, 27], T is a one-to-one mapping of H onto H, there exists a unique v in H such that $g_0 = Tv$. Hence, it follows from (31) that

$$||u_0 - u|| \leqq ||Tv - Tu|| \quad \text{for all} \quad u \text{ in } H. \tag{32}$$

If we take $u = v$, (32) implies that $v = u_0$, *i.e.*, $Tu_0 = g_0$ and Lemma 2 is proved.

Now the validity of (Ic) for $A = T + S$, with T continuous and complex monotone and S strongly continuous, follows from the strong continuity of S and Lemma 2. Thus Corollary 8 follows from Theorem I.

Remark 5. A nonconstructive version of Corollary 8 for operator A satisfying more general conditions was given by Browder [4].

Added in proof. The following corollary, whose elementary proof employs neither Lemma 3 in [1] nor Lemma 2 above, provides a unification of Corollaries 6 and 7.

Corollary 7A. *If A is continuous and complex monotone, then Eq.* (3) *is PS-solvable.*

Proof. (Ia) and (Ib) follow from the continuity and complex monotonicity of A with $N = 1$ and $\alpha(r) = br$ while (Ic) follows from the following argument. To prove (Ic), suppose that $\{H_m\}$ satisfies (1) and (2) and $\{x_m \mid x_m \in H_m\}$ is such that $x_m \rightharpoonup x$ and $A_m x_m \to g$. Then, since $x_m \rightharpoonup x$, $P_m A x_m \to g$, $A_m x \to Ax$ and $P_m x_m = x_m$, (28) implies that as $m \to \infty$

$$\begin{aligned} b\,||x_m - P_m x||^2 &\leqq |(Ax_m - AP_m x, x_m - P_m x)| \\ &= |(P_m A x_m, x_m) - (P_m A P_m x, x_m) - (P_m A x_m, x) + (P_m A P_m x, x)| \\ &\to |(g, x) - (Ax, x) - (g, x) + (Ax, x)| = 0. \end{aligned}$$

Hence $x_m \to x$ and thus, $P_m A x_m \to Ax = g$ since A is continuous, *i.e.*, (Ic) holds. Consequently, Theorem I supplies a simple and constructive proof of Corollary 7A and, therefore, of Corollaries 6 and 7.

4. Equations which are projectionally and weakly solvable. In Section 3 we gave a constructional proof of the existence and uniqueness of solutions of Eq. (3) as strong limits of solutions of Eq. (4). In addition to a number of results for linear equations, we deduced also from Theorem I the important theorems for monotone and complex monotone continuous operators established previously by Minty [15], Browder [1], and Zarantonello [27]. However, if A is not assumed to be continuous but, say, demicontinuous or weakly continuous, then in view of conditions (Ia) and (Ib) we cannot in general expect Theorem I to be applicable to equations involving such operators.

The purpose of this section is to prove a second general theorem (*i.e.*, Theorem II) for a class of nonlinear equations in X which are projectionally and weakly solvable (PW-solvable). From our Theorem II we then, in particular, deduce in a constructive manner certain results of Browder [1, 3] and Browder and de Figueiredo [5] for monotone, complex monotone, and J-monotone demicontinuous and weakly continuous operators. At the same time we rederive some of our own results on complex monotone operators obtained in [19]. The significance of certain classes of operators introduced in this section will be indicated below.

Theorem II. *Let K be a (possibly nonlinear) mapping of X into X^* such that $Kx \neq 0$ if $x \neq 0$. Suppose that there exists an integer $N > 0$ and the function $\alpha(r)$ (as in Theorem I) such that the following conditions hold:*

(IIa) *A_n is continuous in X_n for each $n \geqq N$, and the sequences $\{A_n\}$ and $\{P_n^* KP_n\} = \{K_n\}$ are such that for each x and y we have $(K_nx, A_ny) \to (Kx, Ay)$ and $||K_{n_i} x|| \to \theta(x) \geqq ||Kx||$ for some subsequence $\{n_i\}$* of $\{n\}$ and *some real functional $\theta(x)$.*

(IIb) *For each $n \geqq N$ and all x and y in X_n*

$$|(K_n(x - y), A_nx - A_ny)| \geqq \alpha(||x - y||)\, ||K_n(x - y)||.$$

(IIc) *If $\{X_m\}$ satisfies* (1) *and* (2) *and $\{x_m\}$ is a sequence in X (with $x_m \in X_m$) such that $x_m \rightharpoonup x$ and $A_mx_m \to g$, then $Ax = g$.*

Then Eq. (3) *is PW-solvable.*

Corollary to Theorem II. *Condition* (IIa) *may be replaced by any one of the following slightly stronger but more practical conditions. A_n is continuous in X_n for $n \geqq N$ and*

(IIa$_1$) *$A_nx \rightharpoonup Ax$ in X and $K_nx \to Kx$ in X^* for each x in X.*

(IIa$_2$) *$A_nx \to Ax$ in X and $K_nx \rightharpoonup Kx$ in X^* for each x in X.*

(IIa$_3$) *$A_nx \to Ax$ in X and $K_nx \to Kx$ in X^* for each x in X.*

Proof of Theorem II. (*Uniqueness*). If $u \neq v$ and $Au = Av$, then by (IIb)

(IIb) $$|(K_n(u - v), A_nu - A_nu)| \geqq \alpha(||P_nu - P_nv||)\, ||K_n(u - v)||, \quad n \geqq N.$$

Hence (IIa) implies that the passage to the limit in the above inequality, with

n_i replacing n, yields the inequality

$$0 = |(K(u - v), Au - Av)| \geqq \alpha(||u - v||)\theta(u - v) \geqq \alpha(||u - v||)\ ||K(u - v)||$$

which contradicts the properties of $\alpha(r)$ and K. Thus A is one-to-one.

(*Existence*). First, $R(A_n)$ is a closed set in X_n. Indeed, if $\{y_m\} \subset R(A_n)$ and $y_m \to y$, then there exists $\{x_m\} \subset X_n$ such that $y_m = A_n x_m$ and, by II(b),

$$|(K_n(x_1 - x_k), A_n x_1 - A_n x_k)| \geqq \alpha(||x_1 - x_k||)\ ||K_n(x_1 - x_k)||$$

As before, this implies that $\{x_m\}$ is a Cauchy sequence in X_n. Thus there exists $x \varepsilon X_n$ such that $x_m \to x$ and $A_n x_m \to A_n x = y \varepsilon R(A_n)$, *i.e.*, $R(A_n)$ is closed in X_n for $n \geqq N$. Since IIb) implies that $||A_n u - A_n v|| \geqq \alpha(||u - v||)$ for all u and v in X_n and each $n \geqq N$, Brouwer's theorem on invariance of domain implies that $R(A_n)$ is also open. Hence $R(A_n) = X_n$ for $n \geqq N$. Thus for each f in X and $n \geqq N$ there exists a unique $x_n \varepsilon X_n$ such that $A_n x_n = P_n f$. For the sequence $\{x_n\}$ thus determined, (IIb) and (2) imply that $\{x_n\}$ is bounded. Since X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup x \quad \text{and} \quad A_{n_i} x_{n_i} = P_{n_i} f \to f \qquad (n \to \infty)$$

whence, in virtue of (IIc) we derive the equality $Ax = f$. Since, as was shown above, the solution of Eq. (3) (if it exists) is unique, it follows that the entire sequence $\{x_n\}$ converges weakly to x, *i.e.*, Eq. (3) is PW-solvable.

Remark 6. Theorem II and, in particular, the Corollary to Theorem II seem to shed more light on the place and the role played in the theory of nonlinear operators by the concept of *demicontinuity*. Just as the concept of continuity is associated with the norm of the space so the concept of demicontinuity seems to be very much associated with the inner product-like structure that can be introduced for the elements of the space. Furthermore, Theorem I shows that for continuous operators we can expect equations (3) to be PS-solvable while Theorem II shows that for demicontinuous operators we can, in general, expect the corresponding equations to be PW-solvable.

(j) Application to equations involving nonlinear monotone operators. An immediate consequence of Theorem II is the following result proved by the author in [19].

Corollary 9. *If A is a mapping of H into itself, then Eq. (3) is PW-solvable if A is demicontinuous and either strongly monotone or complex monotone and bounded.*

Proof. The continuity of A_n in H_n follows from the demicontinuity of A and the fact that in H_n the weak and strong convergence coincide. For $X = H$, X^* can be identified with H by the inner product and in view of our conditions on A the choice $K = I$ satisfies conditions (IIa) and (IIb) with $N = 1$ and $\alpha(r) = ar$ if A is strongly monotone or $\alpha(r) = br$ if A is complex monotone.

If A is strongly monotone, (IIc) follows from Lemma 3 in [1] while if A is complex monotone and bounded, (IIc) follows from the same simple argument as was used in the proof of Corollary 7. Thus, Corollary 9 follows from Theorem II.

Remark 7. It follows from the proof of Corollary 9 that Eq. (3) is in fact PS-solvable at least when A is bounded. Similar argument proves also the following corollary.

Corollary 10. *Eq.* (3) *is PS-solvable if A is a weakly continuous and either strongly monotone or complex monotone in H.*

Remark 8. For strongly monotone operators the nonconstructive version of Corollary 10 was given by Shinbrot [24]. For complex monotone operators the assertion of Corollary 10 seems to be new.

Remark 9. Let us observe that author's Theorem 4 in [19] which proves the applicability of the projection method to the solution of a certain class of equations in H with *densely* defined nonlinear operators can also be deduced as a special case of Theorem II. Since in applications we mostly deal with differential operators which are defined only on dense subsets of H it is often useful and advantageous to have the theory of the projection method which is applicable directly to such equations without the necessity of reducing it to integral or integrodifferential equations; furthermore, when we are able to construct solvable extensions (see [19]) of such operators then, at least in theory, we know something more about the solutions of our problem.

(jj) Applications to equations involving nonlinear J-monotone operators. In this section we deduce from our Theorems I and II some basic results for nonlinear equations in a real space involving the so-called J-monotone operators which were recently studied in [3, 5].

Let $\mu(r)$ be a continuous strictly increasing real-valued function on R^1 with $\mu(0) = 0$. A mapping J of X into X^* is called a *duality mapping* with gauge function μ if for each x in X

$$(Jx, x) = ||x||\, \mu(||x||), \qquad ||Jx|| = \mu(||x||). \tag{33}$$

It is known [3, 5] that if X^* is strictly convex, J is uniquely determined by its gauge function μ and exactly one duality mapping exists for each gauge function. Moreover, J is continuous from the strong topology of X to the weak topology of X^*. If X^* is uniformly convex, then J is also continuous. However, the uniform convexity does not imply the weak continuity of J for any duality mapping J of X into X^*. In fact, it was shown in [5] that for any space $L^p(0, 1)$ with $1 < p < \infty$ and $p \neq 2$ there are no weakly continuous duality mappings. But as was shown in [3], the class of spaces X having weakly continuous duality mappings includes the l^p spaces, $1 < p < \infty$.

A mapping A of X into X is called J-monotone if

$$(J(x - y), Ax - Ay) \geqq 0, \qquad x, y \in X. \tag{34}$$

The significance of J-monotone operators stems from the fact that certain results valid for Hilbert spaces can be extended to J-monotone operators acting in real Banach spaces having weakly continuous duality mappings, but, in general, not to other Banach spaces. For example, it was shown in [16], that if A is a nonexpansive mapping of a closed, bounded and convex set C ($\subset X$ with weakly continuous duality mappings) into itself and λ is any constant such that $0 < \lambda < 1$, then for any given x_0 in C the sequence $\{x_{n+1}\} = \{\lambda A x_n + (1 - \lambda)x_n\}$ converges weakly to a fixed point of A in C.

In our discussion of J-monotone operators we will need the following lemma.

Lemma 3. *Suppose that X has property $(\pi)_1$, X^* is strictly convex, and there exists a weakly continuous duality mapping J of X into X^*. Suppose also that A is J-monotone. If $\{x_n\}$ is a sequence in X with $x_n \in X_n$ such that $x_n \rightharpoonup x_0$ and $P_n A x_n \to g$, then $Ax_0 = g_0$ provided that either A is demicontinuous and J is continuous or A is continuous.*

Proof. Note that if P_n^* denotes the adjoint map of P_n, then P_n^* is an idempotent self-mapping of X^*; furthermore, as was shown in [5], if X has property $(\pi)_1$, X^* is strictly convex, and J is a duality mapping of X into X^*, then

$$P_n^* J x = Jx \quad \text{for all} \quad x \text{ in } H_n . \tag{35}$$

Let j be any fixed integer and let x be any element in X_j. Since $P_j x = x$, $P_n P_j x = P_j x$ for $n \geqq j$ and $P_n x_n = x_n$ for x_n in X_n, (35) implies that for each $n \geqq j$ and x in X_j

$$\begin{aligned} &(J(x_n - P_j x),\ Ax_n - A(P_j x)) \\ &\qquad = (P_n^* J(x_n - P_j x),\ Ax_n) - (J(x_n - P_j x),\ A(P_j x)) \\ &\qquad = (J(x_n - P_j x), P_n A x_n) - (J(x_n - P_j x),\ A(P_j x)). \end{aligned} \tag{36}$$

Since J is weakly continuous and

$$x_n - P_j x \rightharpoonup x_0 - P_j x,\ J(x_n - P_j x) \rightharpoonup J(x_0 - P_j x).$$

On the other hand, by hypothesis, $P_n A x_n \to g_0$; hence, in view of the J-monotonicity of V, the passage to the limit in (36) yields the relation

$$\begin{aligned} &(J(x_n - P_j x),\ Ax_n - A(P_j x)) \\ &\qquad \to (J(x_0 - P_j x),\ g_0) - (J(x_0 - P_j x),\ A(P_j x)) \geqq 0 \end{aligned}$$

valid for all $x = P_j x$ in X_j. Since j is arbitrary, the inequality

$$(J(x_0 - x),\ g_0 - Ax) \geqq 0 \tag{37}$$

is true for all x in a dense set $\bigcup_j X_j$. Furthermore, our conditions on A and J in both cases imply the validity of (37) for all x in X. Hence, by Lemma 4 in [3], $Ax_0 = g_0$.

Our object in this subsection is to deduce from our Theorems I and II certain interesting results for J-monotone operators. Our first assertion is the following apparently new result.

Corollary 11. *Suppose that X has property $(\pi)_1$ and X^* is strictly convex. Suppose further that there exists a weakly continuous duality mapping of X into X^*. If A is a continuous mapping of X into X such that*

$$(J(x - y), Ax - Ay) \geqq \alpha(\|x - y\|)\, \|J(x - y)\|, \qquad x, y \in X, \tag{38}$$

then Eq. (3) *is PS-solvable.*

Proof. In view of (35) and Lemma 3, the properties of A and J imply the validity of (Ia)-(Ic). Hence Theorem I applies.

Remark 10. Note that in contradistinction to Browder and de Figueiredo [5] we do not assume that J is continuous.

Corollary 12. *Suppose that X has property $(\pi)_1$ and X^* is strictly convex. Suppose also that there exists a duality map J of X into X^* which is both weakly continuous and continuous. If A is a demicontinuous map of X into X for which* (38) *holds, then Eq.* (3) *is PW-solvable.*

Proof. Corollary 12 follows from the Corollary to Theorem II if there we put $K = J$ and observe that $\|Kx\| = \|Jx\| = \mu\|x\|) \neq 0$ if $x \neq 0$ and, in view of (35), (IIa_1) and (IIb) follow from our conditions on J and A. Furthermore, (IIc) follows from Lemma 3.

Remark 13. A nonconstructive version of Corollary 12 under more general conditions on A was given in [5].

Finally let us remark that Theorem I and Lemma 3 imply that an assertion analogous to Corollary 8 is also valid for J-monotone operators.

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