Projection Methods
With Different Trial and Test Spaces*

By M. S. Mock

Abstract. We consider finite element projection methods for linear partial differential equations, in which the spaces of trial functions and test functions may be different. In addition to approximation and smoothness properties, conditions implying equality of dimensions and uniform coerciveness are required, the most important of which resembles a strong form of an inverse assumption.

Our results provide a mechanism for the difference in the rate of convergence of Galerkin procedures with cubic splines and Hermite cubics, applied to first order symmetric hyperbolic problems [13].

1. **Introduction.** In this paper we discuss the approximate solution of linear well-posed partial differential equations, using finite element projection methods in which the trial space, in which an approximate solution is obtained, may be different from the space of test functions. Perhaps the best known of such methods are the collocation schemes, which have been applied to two point boundary value problems in [4], [5], [23] and to parabolic problems in one space variable in [7]. Higher order methods utilizing different spaces have also been applied to one-dimensional problems in [4], [18], [22].

The requirement that the trial and test spaces have equal dimension is an important restriction on such methods. Equality of dimension is implied by the coerciveness conditions of [2]. In one dimension, the coerciveness conditions can be satisfied by taking one of the spaces (usually the test space, but the trial space in [22]) to be the image of the other under an ordinary differential operator of even order. This mapping is made elliptic by requiring the domain space to be conforming. This technique has recently been generalized to higher dimensions in [10], using an elliptic partial differential operator as the mapping between spaces.

In the following, we consider an abstract one-parameter family of elliptic pseudo-differential operators as the mapping between the two spaces. (A background reference for such operators is [29].) The coerciveness conditions are formulated as requirements on this family of operators; indeed, our basic results for elliptic and parabolic problems, Theorems 1 and 2 below, could be alternatively obtained by showing that for sufficiently small mesh size, our schemes are variational principles in the sense of [2]. Our results contain some examples of methods considered in [2], [10], [12], [18], [22], [27] as special cases.

Application to nonlinear elliptic problems was the motivation for this work.

Nonlinear problems can require more smoothness of the trial functions than linear problems of the same order, if for no other reason than for the discrete problem to be properly defined. Such problems are typically formulated, however, as that of finding a fixed point of a compact (nonlinear) operator [15], [16]. A nonlinear elliptic equation may assume the identity minus compact form on a relatively high Sobolev space, and this is the space from which an approximate solution is obtained. In [19], we show that for finite element methods, it suffices that the approximate solution satisfy the boundary conditions in the same sense as for linear problems; but we have not been able to relax the smoothness requirements.

In order to simplify the discrete equations requiring solution in such problems, it seems desirable for the test functions to have their support in as small a region as possible. Thus we are motivated to consider nonlocal operators between the trial and test space. Unfortunately, our analysis does not include collocation schemes, but does include schemes intermediate between collocation and Galerkin methods. The machinery we develop has also been applied in some subsequent work on hyperbolic problems [20].

The best known examples of spaces for which our methods are applicable are the smooth splines; other possibilities are discussed in Section 6. For smooth spline spaces, stronger results than ours are obtained for the heat equation in [26] and for periodic initial value problems in [27]. Also for one-dimensional problems, better results are obtained in [9] and references there cited.

2. Elliptic and Parabolic Problems. We first consider the problem

$$(2.1) \qquad\qquad Lu = f, \quad x \in R^n,$$

in which L is a uniformly and strongly elliptic positive definite operator of order $2m$ with smooth coefficients, f is a specified function in $H^{k-2m}(R^n) \equiv H^{k-2m}$, k a positive integer, and u is the desired solution in H^k. Our results can be readily extended to include the case of indefinite invertible L by the methods of [15], [24]. In the following, we use $(\ ,\ )$ for the real scalar product over R^n, C for a generic constant and $\|\cdot\|_j$ for the H^j norm,

$$(2.2) \qquad\qquad \|\phi\|_j^2 = \int_{R_\xi^n} |\hat\phi(\xi)|^2 (1 + |\xi^2|)^j \, d\xi,$$

where $\xi = (\xi_1, \dots, \xi_n)$ is the Fourier transform variable, and (2.2) applies for all real j.

In the following, $X_h^{(k)}$ and $Y_h^{(k-2s)}$ are one-parameter families of Hilbert spaces; $X_h^{(k)}$ denotes the trial space, assumed of order k, and $Y_h^{(k-2s)}$ denotes the test space, assumed of order $k - 2s$, where s is an integer or possibly half integer, and the parameter h is the mesh size. For all $j < k$ such that $X_h^{(k)} \subset H^j$, $g \in H^k$, we assume

$$\underset{g_x \in X_h^{(k)}}{\text{Inf}} \ \|g - g_x\|_j \leqslant Ch^{k-j} \|g\|_k;$$

and for all $j < k - 2s$ such that $Y_h^{(k-2s)} \subset H^j$, $g \in H^{k-2s}$,

$$\underset{g_y \in Y_h^{(k-2s)}}{\text{Inf}} \ \|g - g_y\|_j \leqslant Ch^{k-2s-j} \|g\|_{k-2s}.$$

We also assume that there exists a one-parameter (h) family of symmetric pseudo-differential operators, denoted by A_h, with symbol $a(h\xi)$, with the property that A_h^2 maps

$Y_h^{(k-2s)}$ into $X_h^{(k)}$. The Galerkin procedure is included as a special case, with $A_h \equiv I$, $s = 0$, $X_h^{(k)} = Y_h^{(k)}$.

Our approximation to the solution of (2.1) is denoted by $u_h \in X_h^{(k)}$ which satisfies

(2.3) $$(Lu - Lu_h, y) = 0 \quad \text{for all } y \in Y_h^{(k-2s)}.$$

Set $r = k - s - m$; $O(h^{2r})$ is the optimal rate of convergence that can be expected for the method (2.3) [2], [25]. We have the following result:

THEOREM 1. *Suppose*

(1) $X_h^{(k)}$ *and* $Y_h^{(k-2s)}$ *are subspaces of* H^m *(we will assume that this implies* $r > s$*);*

(2) A_h^2 *maps* $Y_h^{(k-2s)}$ *onto* $X_h^{(k)}$;

(3) $\|A_h\|_0 = \sup_{\phi \in H^0} \|A_h \phi\|_0 / \|\phi\|_0$ *is bounded independently of* h;

(4) *the symbol* $a(\cdot)$ *is Hölder continuous in* $h\xi$, *with exponent* $\gamma \in (0, 1]$;

(5) A_h *is inverse bounded on* $Y_h^{(k-2s)}$, *i. e. for all* $y \in Y_h^{(k-2s)}$, $\|y\|_m \leqslant C\|A_h y\|_m$, *with* C *independent of* h;

then for sufficiently small h, *there exists a unique solution to* (2.3), *and for all real* $j \leqslant m$,

(2.4) $$\|u - u_h\|_j \leqslant C(h^{k-j} + h^{2r})\|u\|_k.$$

Before proving the theorem, we comment on some implications of the hypotheses. We need test functions in H^m, so that collocation schemes are not covered. In fact, we could not expect (2.4) for collocation schemes, because $u \in H^k$ does not imply Lu continuous in sufficiently many dimensions. Hypotheses (2), (3), and (5) together imply the equality of dimension of the trial and test spaces. Hypothesis (5) may be viewed as a form of an inverse assumption on the space $X_h^{(k)}$, and for lack of a better name we shall refer to it as the "strong inverse assumption" below.

The proof of Theorem 1 requires a lemma, which is also in [29].

LEMMA 1. *Let* M *be a partial differential operator of order* p *with smooth coefficients, and let* $\phi \in H^j$ *and* $\psi \in H^{p-j}$. *Assume hypothesis* (4) *holds, then*

(2.5) $$|((MA_h - A_h M)\phi, \psi)| \leqslant Ch^\gamma \|\phi\|_j \|\psi\|_{p-j}, \quad 0 \leqslant j \leqslant p.$$

Proof. A typical term of M is of the form $b_\alpha(x) D^\alpha$, $|\alpha| \leqslant p$, with the usual multi-index notation. The contribution of such a term to the left side of (2.5) is given by

$$\left| \iint d\xi \, d\xi' \hat{\psi}(\xi) \hat{b}_\alpha(\xi' - \xi)(a(h\xi') - a(h\xi))(\xi')^\alpha \hat{\phi}(\xi') \right|$$

$$\leqslant Ch^\gamma \iint d\xi \, d\xi' |\hat{\psi}(\xi)| \, |\hat{b}_\alpha(\xi' - \xi)| \, |\xi' - \xi|^\gamma \, [1 + |\xi'|^j (|\xi|^{p-j} + |\xi' - \xi|^{p-j})] \, |\hat{\phi}(\xi')|$$

$$\leqslant Ch^\gamma \frac{\|\phi\|_j}{\|\psi\|_{p-j}} \iint d\xi \, d\xi' |\hat{\psi}(\xi)|^2 \, |b_\alpha(\xi' - \xi)| \, |\xi' - \xi|^\gamma$$

$$\cdot [1 + |\xi|^{2(p-j)} + |\xi' - \xi|^{2(p-j)}]$$

$$+ Ch^\gamma \frac{\|\psi\|_{p-j}}{\|\phi\|_j} \iint d\xi \, d\xi' |b_\alpha(\xi' - \xi)| \, |\xi' - \xi|^\gamma [1 + |\xi'|^{2j}] \, |\hat{\phi}(\xi')|^2$$

$$\leqslant Ch^{\gamma} \frac{\|\phi\|_j}{\|\psi\|_{p-j}} \iint d\xi \, d\xi' |\hat{\psi}(\xi)|^2 \, [1 + |\xi|^{2(p-j)}] \, |b_\alpha(\xi')| \, |\xi'|^{\gamma} \, [1 + |\xi'|]^{2(p-j)}$$

$$+ Ch^{\gamma} \frac{\|\psi\|_{p-j}}{\|\phi\|_j} \iint d\xi \, d\xi' |\hat{\phi}(\xi')|^2 \, [1 + |\xi'|^{2j}] \, |b_\alpha(\xi)| \, |\xi|^{\gamma}$$

$$\leqslant Ch^{\gamma} \|\psi\|_{p-j} \|\phi\|_j.$$

Proof of Theorem 1. Let u_x be some optimum approximation in $X_h^{(k)}$ to u, and choose y in (2.3) so that $A_h^2 y = u_x - u_h$, using hypothesis (2). Then (2.3) becomes

(2.6)
$$0 = (Lu - Lu_h, y) = (Lu - Lu_x, y) + (LA_h^2 y, y)$$
$$= (L(u - u_x), y) + ((LA_h - A_h L)A_h y, y) + (LA_h y, A_h y),$$

using the symmetric property of A_h. We use the positive definiteness of L and apply Lemma 1 with $M = L$, $p = 2m$, $j = m$, to obtain

(2.7)
$$\|A_h y\|_m^2 \leqslant Ch^{\gamma} \|y\|_m \|A_h y\|_m + C\|u - u_x\|_m \|y\|_m$$
$$\leqslant Ch^{\gamma} \|A_h y\|_m^2 + C\|u - u_x\|_m \|A_h y\|_m$$

using the strong inverse assumption, hypothesis (5). For sufficiently small h, the existence and uniqueness of y (and thus of u_h) is apparent from (2.7), by an application of the Lax-Milgram theorem, and we have

$$\|u - u_h\|_m \leqslant \|u - u_x\|_m + \|A_h^2 y\|_m \leqslant \|u - u_x\|_m + \|A_h y\|_m$$
$$\leqslant C\|u - u_x\|_m \leqslant Ch^{r+s}\|u\|_k$$

using the boundedness of A_h, hypothesis (3), and the approximation property of $X_h^{(k)}$.

A simplified form of the Nitsche trick [21], [25, p. 166] gives the result for lower norms,

$$\|u - u_h\|_{k-2r}^2 = (u - u_h, (1 - \Delta)^{k-2r}(u - u_h))$$
$$= (L(u - u_h), L^{-*}(1 - \Delta)^{k-2r}(u - u_h))$$
$$= (L(u - u_h), (I - P_0)L^{-*}(1 - \Delta)^{k-2r}(u - u_h))$$
$$\leqslant C\|u - u_h\|_m \|(I - P_0)L^{-*}(1 - \Delta)^{k-2r}(u - u_h)\|_m$$
$$\leqslant Ch^{r+s}\|u\|_k \, h^{k-2s-m} \|L^{-*}(1 - \Delta)^{k-2r}(u - u_h)\|_{k-2s}$$
$$\leqslant Ch^{2r}\|u\|_k \|u - u_h\|_{k-2r},$$

where L^{-*} is the formal adjoint of L^{-1} and P_0 is the projection into $Y_h^{(k-2s)}$ with respect to the H^m inner product.

COROLLARY. *Suppose $X_h^{(k)} \subset H^{m+2s}$ and A_h is elliptic of order $-s$, i. e.*

(2.8)
$$\|A_h \phi\|_{j+s} \leqslant Ch^{-s}\|\phi\|_j \quad \text{for all } \phi \in H^j.$$

Then in (2.4), j may be taken up to $m + 2s$.

Proof. Combining (2.7), (2.8) we obtain

$$\|u - u_h\|_{m+2s} \leqslant \|u - u_x\|_{m+2s} + \|A_h(A_h y)\|_{m+2s}$$
$$\leqslant Ch^{r-s}\|u\|_k + Ch^{-2s}\|y\|_m$$
$$\leqslant Ch^{r-s}\|u\|_k + Ch^{-2s}\|A_h y\|_m$$
$$\leqslant Ch^{r-s}\|u\|_k.$$

If u_h has any finite higher norms, estimates in these may be obtained by an inverse assumption in the usual way.

Estimates are also obtained for parabolic problems, using the above results and the general technique of Douglas and Dupont [6], [14]. We consider the problem

$$(2.9) \qquad \partial v/\partial t + Lv = f, \quad (x, t) \in R^n \times (0, T), \quad v(x, 0) = g(x),$$

where L is as above. For simplicity, we consider the continuous time approximation and assume that the coefficients in L do not depend on t. At each t, our approximation $v_h(t) \in X_h^{(k)}$ and satisfies

$$(2.10) \qquad (\partial v_h/\partial t + Lv_h - f, y) = 0 \quad \text{for all } y \in Y_h^{(k-2s)}.$$

We choose $v_h(0) = P_1 g$, where P_1 is a projection into $X_h^{(k)}$ induced by L, i.e.

$$(2.11) \qquad (L\phi, y) = (LP_1\phi, y) \quad \text{for all } y \in Y_h^{(k-2s)}.$$

THEOREM 2. *Suppose hypotheses* (1)–(5) *hold and h is sufficiently small; then for all* $T \geqslant 0$,

$$(2.12) \quad \|v(T) - v_h(T)\|_0 \leqslant C(h^k + h^{2r}) \left[\|v(T)\|_k + \left(\int_0^T \left\| \frac{\partial v(t)}{\partial t} \right\|_{k-m}^2 dt \right)^{1/2} \right].$$

Proof. In (2.10), choose y such that $A_h^2 y = v_h - P_1 v$. Subtracting (2.9) and using (2.11), we obtain

$$\left(A_h^2 \frac{\partial y}{\partial t} + LA_h^2 y, y \right) = \left(\frac{\partial}{\partial t} (v - P_1 v), y \right)$$

and since A_h commutes with time differentiation, we have

$$\left(A_h \frac{\partial y}{\partial t}, A_h y \right) = ((A_h L - LA_h) A_h y, y) - (LA_h y, A_h y) + \left((I - P_1) \frac{\partial v}{\partial t}, y \right)$$

$$\leqslant Ch^\gamma \|A_h y\|_m \|y\|_m - C\|A_h y\|_m^2 + \left\| (I - P_1) \frac{\partial v}{\partial t} \right\|_{-m} \|y\|_m$$

$$(2.13) \qquad \leqslant Ch^\gamma \|A_h y\|_m^2 - C\|A_h y\|_m^2 + C(h^k + h^{2r}) \left\| \frac{\partial v}{\partial t} \right\|_{k-m} \|A_h y\|_m$$

$$\leqslant C(h^{2k} + h^{4r}) \left\| \frac{\partial v}{\partial t} \right\|_{k-m}^2$$

using Lemma 1 for the commutator term, the regularity theory of [14] and Theorem 1 to estimate the last term. Thus

$$\frac{d}{dt} \|A_h y\|_0^2 \leqslant C(h^{2k} + h^{4r}) \left\| \frac{\partial v}{\partial t} \right\|_{k-m}^2,$$

and the initial condition was selected so that $\|A_h y(0)\|_0 = 0$. The proof is completed by a second application of Theorem 1, to obtain

$$\|v(t) - P_1 v(t)\|_0 \leqslant C(h^k + h^{2r}) \|v(t)\|_k$$

and using the boundedness of A_h, the triangle inequality, and the definition of y.

In Theorems 1, 2, the requirement of H^m test functions can be relaxed if a stronger assumption is made about the smoothing properties of A_h.

The following theorem is given for the case $m = 1$ for simplicity; a generalization to larger m is not difficult.

THEOREM 3. *Suppose $m = 1$, hypotheses (2), (3), (5) hold, and hypothesis (4) is replaced by the stronger requirement*

(4') *For each positive integer $j \leqslant n$, a Lipschitz condition of the form*

$$|\xi_j a(h\xi) - \xi_j' a(h\xi')| \leqslant C |\xi - \xi'|$$

holds for all $\xi, \xi' \in R^n$, with C independent of h.

Then the conclusions of Theorems 1, 2 remain valid if hypothesis (1) is replaced by the weaker requirement

(1') $$Y_h^{(k-2s)} \subset H^0, \quad A_h Y_h^{(k-2s)} \subset H^1.$$

Proof. It suffices to obtain a sharper estimate for the commutator term in (2.6), (2.13), specifically to show that

(2.14) $$|((LA_h - A_h L)A_h y, y)| \leqslant C \|A_h y\|_1 \|y\|_0$$

for all $y \in Y_h^{(k-2s)}$. A typical term in L is of the form $b(x)(\partial/\partial x_i)^2$, and the contribution of such a term to the left side of (2.14) is of the form

(2.15) $$\left| \iint d\xi \, d\xi' \hat{\overline{y}}(\xi') (a(h\xi) - a(h\xi')) \hat{b}(\xi' - \xi) \xi_i^2 a(h\xi) \hat{y}(\xi) \right|;$$

in (2.15), we use the estimate

(2.16) $$|\xi_i (a(h\xi) - a(h\xi'))| \leqslant |\xi_i a(h\xi) - \xi_i' a(h\xi')| + |\xi_i - \xi_i'| |a(h\xi')| \leqslant C |\xi - \xi'|$$

using hypotheses (3), (4'). Inserting (2.16) and estimating the expression (2.15) as in the proof of Lemma 1, (2.14) follows.

3. Tensor Products of Smooth Splines.

Let $S_h^{(j)}$ denote the space of smooth n-dimensional (tensor product) splines of order j, i.e. piecewise polynomials of degree $j - 1$, elements of C^{j-2}, on a uniform mesh of size h.

THEOREM 4. *Suppose $X_h^{(k)} \equiv S_h^{(k)}$, $Y_h^{(k-2s)} \equiv S_h^{(k-2s)}$, s is a nonnegative integer, and $k \geqslant m + 2s + 1$, then hypotheses (1)–(5), (4') are satisfied and Theorems 1–3 hold.*

Proof. The approximation properties of these spaces are well known [1], and will not be discussed. Hypothesis (1) is immediate from the conditions on s and k. We can choose A_h^2 to map B-splines in $S_h^{(k-2s)}$ into B-splines in $S_h^{(k)}$, in which case hypothesis (2) is apparent and the symbol $a(\cdot)$ is given by

(3.1) $$a(h\xi) = \prod_{i=1}^{n} \frac{\sin^s(h\xi_i / 2)}{(h\xi_i / 2)^s}.$$

Hypotheses (3), (4), (4'), and the ellipticity condition (2.8) are apparent from (3.1). Thus, it is only the strong inverse assumption which requires proof; and each dimension can be treated independently, as A_h is a product of one-dimensional mappings.

The theorem follows from the following lemma and the ordinary inverse assumption for splines.

LEMMA 2. *Suppose j, p are nonnegative integers with $0 \leqslant j \leqslant p - 2$, ϕ is a (one-dimensional) spline of order p, and A is a pseudo-differential operator with symbol $\sin(h\xi/2)/(h\xi/2)$. Then $A^{-1}\phi$ is defined and*

(3.2)
$$|A^{-1}\phi|_j \leqslant |\phi|_j + Ch|\phi|_{j+1},$$

where $|\cdot|_j$ denotes the H^j-seminorm.

   Proof. We set $z = h\xi$; noting that $\sin^2(z/2) = \frac{1}{2}(1 - \cos z)$, we have

(3.3)
$$|\phi|_j^2 = 4^p h^{-2j-1} \int_{-\infty}^{\infty} q(z) z^{-2(p-j)} \sin^2\left(\frac{z}{2}\right) dz,$$

(3.4)
$$|A^{-1}\phi|_j^2 = 4^{p-1} h^{-2j-1} \int_{-\infty}^{\infty} q(z) z^{-2(p-j-1)} dz,$$

in which $q(z)$ is a nonnegative definite trigonometric polynomial in z, with a sufficiently high order zero at $z = 0$ that the integrands are defined there. We may take $q(z)$ to be a finite polynomial, in which case both integrands are entire functions of z, and the integrals may be evaluated by the standard method of residues. Let $D \equiv d/dz$ and N be a linear operator on trigonometric polynomials such that

$$Ne^{i\alpha z} = \begin{cases} e^{i\alpha z}, & \alpha \geqslant 0, \\ 0, & \alpha < 0, \end{cases}$$

then we have from (3.3)

(3.5)
$$2\pi i |\phi|_j^2 = \frac{4^p h^{-2j-1}}{[2(p-j)-1]!} \left[D^{2(p-j)-1} N\left(q \sin^2\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0}.$$

   Because q is a trigonometric polynomial in z and $2(p-j)-1 \geqslant 3$, one factor of $\sin(z/2)$ can be placed to the left of N; then at least one of the derivatives must go on this factor, so that

$$2\pi i |\phi|_j^2 = \frac{4^p h^{-2j-1}}{[2(p-j)-1]!} \left[D^{2(p-j)-1} \sin\left(\frac{z}{2}\right) N\left(q \sin\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0}$$

$$= \frac{1}{2} \frac{4^p h^{-2j-1}}{[2(p-j)-2]!} \left[D^{2(p-j)-2} \cos\left(\frac{z}{2}\right) N\left(q \sin\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0}$$

$$+ h^{-2j-1} \sum_{\alpha=1,3,\ldots,2(p-j)-3} C_\alpha D^\alpha \left[\sin\left(\frac{z}{2}\right) N\left(q \sin\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0},$$

where the second term arises from the fact that any odd number of derivatives may be placed on the factor $\sin(z/2)$ outside N. Since $2(p-j)-2$ is still strictly positive, we can put the cosine factor back inside N and take the other sine factor out, obtaining

$$2\pi i |\phi|_j^2 = \frac{1}{2} \frac{4^p h^{-2j-1}}{[2(p-j)-2]!} \left[D^{2(p-j)-2} N\left(q \sin\left(\frac{z}{2}\right) \cos\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0}$$

$$+ h^{-2j-1} \sum_\alpha C_\alpha \left[D^\alpha N\left(q \sin^2\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0}$$

$$= \frac{1}{2} \frac{4^p h^{-2j-1}}{[2(p-j)-2]!} \left[D^{2(p-j)-2} \sin\left(\frac{z}{2}\right) N\left(q \cos\left(\frac{z}{2}\right) \right) \right] \bigg|_{z=0} + \sum_\alpha .$$

   As before, one or any odd number of the remaining derivatives has to go on the sine factor, so this becomes

$$2\pi i \, |\phi|_j^2 = \frac{4^{p-1}h^{-2j-1}}{[2(p-j)-3]!} \left[D^{2(p-j)-3} \cos\left(\frac{z}{2}\right) N\left(q \cos\left(\frac{z}{2}\right)\right) \right] \Big|_{z=0} + \sum_\alpha$$

$$+ \, h^{-2j-1} \sum_{\beta=1,3,\ldots,2(p-j)-5} C_\beta' \left[D^\beta \cos\left(\frac{z}{2}\right) N\left(q \cos\left(\frac{z}{2}\right)\right) \right] \Big|_{z=0},$$

where the Σ_β term is understood to be simply dropped if $p - j = 2$. Since $2(p - j) - 3 \geqslant 1$ and $\beta \geqslant 1$ if the last sum is present, we can put the cosine factors back inside N, obtaining finally

$$2\pi i \, |\phi|_j^2 = \frac{4^{p-1}h^{-2j-1}}{[2(p-j)-3]!} \left[D^{2(p-j)-3} N\left(q\left(1 - \sin^2\left(\frac{z}{2}\right)\right)\right) \right] \Big|_{z=0}$$

(3.6)
$$+ \, h^{-1-2j} \sum_{\alpha+1,3,\ldots,2(p-j)-3} C_\alpha \left[D^\alpha N\left(q \sin^2\left(\frac{z}{2}\right)\right) \right] \Big|_{z=0}$$

$$+ \, h^{-1-2j} \sum_{\beta=1,3,\ldots,2(p-j)-5} C_\beta' \left[D^\beta N\left(q\left(1 - \sin^2\left(\frac{z}{2}\right)\right)\right) \right] \Big|_{z=0}.$$

Using (3.3), (3.4), each term on the right side of (3.6) admits an interpretation as a seminorm of ϕ or $A^{-1}\phi$, in the same manner as (3.5) was obtained. We have

(3.7)
$$|\phi|_j^2 = |A^{-1}\phi|_j^2 - Ch^2 \, |\phi|_{j+1}^2 + \sum_{\alpha=1}^{p-j-1} C_\alpha h^{2\alpha} \, |\phi|_{j+\alpha}^2$$

$$+ \sum_{\beta=1}^{p-j-2} C_\beta' h^{2\beta} (|A^{-1}\phi|_{j+\beta}^2 - |\phi|_{j+\beta}^2),$$

in which α, β, C_α, C_β' have been redefined; but the last sum is still omitted for $p = j + 2$. Since $p - j \geqslant 2$, each of the terms in (3.7) is well defined. We set $j = p - 2$ initially, and let j descend to its desired value in steps of one. Applying the usual inverse assumption at each step, the result (3.2) follows.

4. A Superconvergence Mechanism. Suppose $Z_h^{(k-s)} = A_h Y_h^{(k-2s)}$ is also a space, of order $k - s$, as is the case for splines and some other spaces discussed in Section 6. We assume hereafter that s is positive, which is for our purposes the more interesting case. The results of Section 2 then suggest that for elliptic and parabolic problems, the accuracy of Galerkin's method in $Z_h^{(k-s)}$ can be improved by interpreting the approximate solution as an element of $X_h^{(k)} = A_h Z_h^{(k-s)}$ rather than as an element of $Z_h^{(k-s)}$. In this section we obtain conditions under which this is so. An alternative procedure for improving the order of accuracy by local averaging is described in [3].

We consider the elliptic case first. Let $u_z \in Z_h^{(k-s)}$ be the Galerkin approximation to the solution of (2.1),

(4.1) $\qquad (Lu - Lu_z, \psi) = 0 \quad \text{for all } \psi \in Z_h^{(k-s)},$

then it is well known [24] that

(4.2) $\qquad \|u - u_z\|_j \leqslant C(h^{k-s-j} + h^{2r}) \|u\|_{k-s}.$

THEOREM 5. *Let u_z be as above, suppose hypotheses (1)–(3) hold, hypothesis (4) holds with $\gamma = 1$ ($a(\cdot)$ Lipschitz), and hypothesis (5) holds in the stronger form*

(4.3) $\quad \|\phi\|_j \leqslant C\|A_h\phi\|_j, \qquad \phi \in Y_h^{(k-2s)} \text{ or } \phi \in Z_h^{(k-s)}, \quad j = 0, 1, \ldots, m.$

Suppose A_h is elliptic of order $-s$, i. e. (2.8) holds, and in addition

(4.4)

(6) $a(h\xi) \leqslant 1 + O((h|\xi|)^p)$, $p = maximum(2r, r + 2s - m - 2)$;

(7) $L = L_1 + L_2$, where L_1 has constant coefficients and L_2 is of order $2m + 1 - s$; then

(4.5) $\|u - A_h u_z\|_j \leqslant C(h^{k-j} + h^{2r})\|u\|_k$, $j \leqslant m + 2s$.

Several comments are made before proving this theorem.

The form (4.3) of the strong inverse assumption is implied by Lemma 2. Hypothesis (6) can presumably be achieved by a change of basis in $X_h^{(k)}$, for smooth splines this is done in [28]; such a change of basis must be stable, however, so that (4.3) is preserved. The hypothesis (6) is viewed simply as a practical requirement for the construction of a suitable A_h.

The hypothesis (7) is no restriction for $s = 1$; for larger s, it requires that some leading terms in L have constant coefficients. In this respect and in the sense that A_h defines a local mapping, our hypotheses resemble those of [8]. (Stronger results have been obtained for one-dimensional problems, however, in [9].) Our results are valid in an arbitrary number of dimensions, which is not the case for many superconvergence theorems; we note [11], however, as an interesting two-dimensional result, and [26], [27].

For $j > m + 2s$ in (4.5), the usual comments about inverse assumptions apply.

Proof of Theorem 5. As in Section 2, let $u_x \in X_h^{(k)}$ be an optimal approximation to u, let $y \in Y_h^{(k-2s)}$, and set $\psi = A_h y$ in (4.1); we obtain

(4.6) $(Lu - LA_h u_z, y) = (L(I - A_h)u, y) + ((LA_h - A_h L)(u - u_z), y)$.

This is almost the same discrete equation as satisfied by u_h, Eq. (2.3); we may identify the first right-hand term of (4.8) as due to the different treatment of the inhomogeneous term, and the second term as due to variable coefficients in L. In the special case where L has constant coefficients, this second term vanishes, and we observe the equivalence of the discrete equations.

Proceeding in the same way as to obtain (2.6), we choose y in (4.6) such that $A_h^2 y = u_x - A_h u_z$, then

(4.7)
$$(LA_h y, A_h y) = -(L(u - u_x), y) + (L(I - A_h)u, y)$$
$$+ ((LA_h - A_h L)(u - u_z), y) - ((LA_h - A_h L)A_h y, y)$$
$$= -(L(u - u_x), y) + (L(I - A_h)u, y) + ((LA_h - A_h L)(u - A_h^{-1}u_x), y);$$

in the last term we note that L_1 commutes with A_h, and we apply Lemma 1 with $M = L_2$, $p = 2m + 1 - s$, $j = p - m$, obtaining

(4.8)
$$\|A_h y\|_m^2 \leqslant C\|u - u_x\|_m \|y\|_m + C\|(I - A_h)u\|_m \|y\|_m + Ch\|u - A_h^{-1}u_x\|_{m+1-s}\|y\|_m$$
$$\leqslant C\|y\|_m [h^{k-m}\|u\|_k + h\|u - A_h^{-1}u_x\|_{m+1-s}]$$
$$\leqslant C\|A_h y\|_m [h^{k-m}\|u\|_k + h\|u - A_h^{-1}u_x\|_{m+1-s}].$$

Now let $\phi_z \in Z_h^{(k-s)}$ be some optimal approximation to u, and $y_1 \in Y_h^{(k-2s)}$ be such that

(4.9) $$A_h y_1 = \phi_z - A_h^{-1} u_x;$$

then for $j \leqslant m$,

(4.10)
$$
\begin{aligned}
\|u - A_h^{-1} u_x\|_j^2 &= ((1 - \Delta)^j (u - A_h^{-1} u_x), u - A_h^{-1} u_x) \\
&= ((1 - \Delta)^j (u - A_h^{-1} u_x), u - \phi_z + A_h y_1) \\
&\leqslant \|u - A_h^{-1} u_x\|_j \|u - \phi_z\|_j + ((1 - \Delta)^j (u - u_x), y_1) \\
&\quad - ((1 - \Delta)^j (I - A_h) u, y_1),
\end{aligned}
$$

since A_h commutes with $1 - \Delta$. First suppose $j \geqslant 0$, then we estimate (4.10) using (4.3),

(4.11) $$\|u - A_h^{-1} u_x\|_j^2 \leqslant Ch^{k-s-j} \|u\|_{k-s} [\|u - A_h^{-1} u_x\|_j + \|y_1\|_j];$$

from (4.3) and (4.9),

(4.12)
$$
\begin{aligned}
\|y_1\|_j &\leqslant C\|A_h y_1\|_j \leqslant C\|u - \phi_z\|_j + C\|u - A_h^{-1} u_x\|_j \\
&\leqslant Ch^{k-s-j} \|u\|_{k-s} + C\|u - A_h^{-1} u_x\|_j;
\end{aligned}
$$

combining (4.11) and (4.12) gives

(4.13) $$\|u - A_h^{-1} u_x\|_j \leqslant Ch^{k-s-j} \|u\|_{k-s}.$$

For negative j in (4.10), we proceed differently, obtaining

(4.14)
$$
\begin{aligned}
\|u - A_h^{-1} u_x\|_j^2 &\leqslant Ch^{k-s-j} \|u - A_h^{-1} u_x\|_j \|u\|_{k-s} \\
&\quad + C\|y_1\|_0 [\|u - u_x\|_{2j} + \|(I - A_h) u\|_{2j}];
\end{aligned}
$$

in (4.14) we use (4.12) and (4.13) with $j = 0$ to estimate $\|y_1\|_0$, and (4.4) to estimate the last term, obtaining

(4.15)
$$
\begin{aligned}
\|u - A_h^{-1} u_x\|_j^2 &\leqslant Ch^{k-s-j} \|u - A_h^{-1} u_x\|_j \|u\|_{k-s} \\
&\quad + C(h^{2(k-s-j)} + h^{2(k-m-1)}) \|u\|_{k-s}^2
\end{aligned}
$$

from which it follows that we may set $j = m + 1 - s$ in (4.13), regardless of the sign of j. Comparing this result with (4.8) then establishes (4.5) for $j = m$. As before,

$$
\begin{aligned}
\|u - A_h u_z\|_{m+2s} &\leqslant \|u - u_x\|_{m+2s} + \|u_x - A_h u_z\|_{m+2s} \\
&\leqslant Ch^{r-s}\|u\|_k + Ch^{-2s}\|A_h^{-2} u_x - A_h^{-1} u_z\|_m \leqslant Ch^{r-s}\|u\|_k + Ch^{-2s}\|u_x - A_h u_z\|_m \\
&\leqslant Ch^{r-s}\|u\|_k + Ch^{-2s}(\|u - u_x\|_m + \|u - A_h u_z\|_m) \leqslant Ch^{r-s}\|u\|_k.
\end{aligned}
$$

For lower norms, we again use the Nitsche trick; let $\chi = L^{-*}(1 - \Delta)^{k-2r}(u - A_h u_z)$, $\theta = L(I - A_h)u + (LA_h - A_h L)(u - u_z)$ and P_0 the projection operator into $Y_h^{(k-2s)}$ with respect to the H^m inner product as in Section 2. Then

(4.16)
$$
\begin{aligned}
\|u - A_h u_z\|_{k-2r}^2 &= (u - A_h u_z, (1 - \Delta)^{k-2r}(u - A_h u_z)) \\
&= (Lu - LA_h u_z, (I - P_0)\chi) + (Lu - LA_h u_z, P_0\chi) \\
&= (Lu - LA_h u_z, (I - P_0)\chi) + (\theta, P_0\chi),
\end{aligned}
$$

using (4.6) with $y = P_0\chi$. Since $(1 - \Delta)^m P_0$ is symmetric,

$$(\theta, P_0\chi) = ((1 - \Delta)^{-m}\theta, (1 - \Delta)^m P_0\chi) = ((1 - \Delta)^m P_0 (1 - \Delta)^{-m}\theta, \chi);$$

and (4.16) may be estimated

$$\|u - A_h u_z\|_{k-2r}^2 \leqslant C \|u - A_h u_z\|_m \|(I - P_0)\chi\|_m$$

$$(4.17) \qquad\qquad + \|(1 - \Delta)^m P_0 (1 - \Delta)^{-m}\theta\|_{2s-k} \|\chi\|_{k-2s}$$

$$\leqslant C h^{k-m} \|u\|_k \cdot h^{k-2s-m} \|\chi\|_{k-2s} + \|P_0 (1 - \Delta)^{-m}\theta\|_{k-2r} \|\chi\|_{k-2s}.$$

From the definition of χ, it follows that

$$(4.18) \qquad\qquad \|\chi\|_{k-2s} \leqslant C \|u - A_h u_z\|_{k-2r}.$$

Now we note that $k - 2r = k - 2s - 2(k - 2s - m)$, and so P_0 is stable in $\|\cdot\|_{k-2r}$; thus,

$$\|P_0 (1 - \Delta)^{-m}\theta\|_{k-2r} \leqslant \|(1 - \Delta)^{-m}\theta\|_{k-2r} + C h^{r-s} \|(1 - \Delta)^{-m}\theta\|_m$$

$$(4.19) \qquad \leqslant C\|\theta\|_{2s-k} + C h^{r-s} \|\theta\|_{-m} \leqslant C\|(I - A_h)u\|_{k-2r} + C h^{r-s} \|(I - A_h)u\|_m$$

$$+ C\|(LA_h - A_h L)(u - u_z)\|_{2s-k} + C h^{r-s} \|(LA_h - A_h L)(u - u_z)\|_{-m}$$

$$\leqslant C h^{2r} \|u\|_k + C h \|u - u_z\|_{m+1-r} + C h^{r+1-s} \|u - u_z\|_{m+1-s},$$

where we have used Lemma 1 twice in the last step, with $M = L_2$, $p = 2m + 1 - s$, $j = m + 1 - r$ and $m + 1 - s$, with $\psi = (1 - \Delta)^{2s-k}(L_2 A_h - A_h L_2)(u - u_z)$ and $(1 - \Delta)^{-m}(L_2 A_h - A_h L_2)(u - u_z)$. Finally, we can estimate the right side of (4.19) by (4.2); combining the result obtained with (4.17) and (4.18) establishes (4.5) for $j = k - 2r$, and so proves Theorem 5.

As expected, Theorem 5 leads easily to a similar theorem for parabolic problems. Let $v_z \in Z_h^{(k-s)}$ be the continuous time Galerkin approximation to the solution v of (2.9), i.e.

$$(4.20) \qquad (\partial v_z / \partial t + L v_z - f, \psi) = 0 \quad \text{for all } \psi \text{ in } Z_h^{(k-s)}.$$

For simplicity, we again assume that the coefficients of L do not depend on t, and we let P_2 denote the Galerkin projection operator into $Z_h^{(k-s)}$ induced by L,

$$(4.21) \qquad (L\phi - L P_2\phi, \psi) = 0 \quad \text{for all } \psi \in Z_h^{(k-s)},\ \phi \in H^m.$$

We choose as the initial condition

$$(4.22) \qquad\qquad v_z(0) = P_2 g,$$

in which case the following holds:

THEOREM 6. *Suppose the hypotheses of Theorem 5 hold, and v_z is determined as above. Then for all $T \geqslant 0$,*

$$(4.23) \quad \|v(T) - A_h v_z(T)\|_0 \leqslant C(h^k + h^{2r}) \left[\|v(T)\|_k + \int_0^T \left\| \frac{\partial v(t)}{\partial t} \right\|_{k-m} dt \right].$$

Proof.

$$(4.24) \qquad \|v(T) - A_h v_z(T)\|_0 \leqslant \|(I - A_h P_2)v(T)\|_0 + \|A_h(P_2 v(T) - v_z(T))\|_0$$

$$\leqslant C(h^k + h^{2r})\|v(T)\|_k + C\|P_2 v(T) - v_z(T)\|_0,$$

using Theorem 5 and the boundedness of A_h. From (2.9), (4.20), (4.21) it follows that for any $\psi \in Z_h^{(k-s)}$, $t \geqslant 0$,

$$\left(\frac{\partial}{\partial t}(v_z - P_2 v) + L(v_z - P_2 v), \psi \right) = \left((I - P_2)\frac{\partial v}{\partial t}, \psi \right);$$

choosing $\psi = v_z - P_2 v$, we obtain

$$\|v_z - P_2 v\|_0 \frac{d}{dt} \|v_z - P_2 v\|_0$$

$$\leqslant -C\|v_z - P_2 v\|_m^2 + \left(A_h(I - P_2) \frac{\partial v}{\partial t}, A_h^{-1}(v_z - P_2 v)\right)$$

$$\leqslant -C\|v_z - P_2 v\|_m^2 + \left(\left\|(I - A_h)\frac{\partial v}{\partial t}\right\|_{-m} + \left\|(I - A_h P_2)\frac{\partial v}{\partial t}\right\|_{-m}\right) \|A_h^{-1}(v_z - P_2 v)\|_m$$

$$\leqslant C(h^{2k} + h^{4r}) \left\|\frac{\partial v}{\partial t}\right\|_{k-m}^2$$

again using Theorem 5. Since $\|v_z(0) - P_2 v(0)\|_0 = 0$ by (4.22), (4.23) follows.

Pointwise superconvergence results are also obtained from Theorems 5 and 6. For the elliptic case, let $\Omega_h = \{x \in R^n \mid u_z(x) = A_h u_z(x)\}$. Ω_h may depend on h, but it is not empty, as $a(0) = 1$. We apply the elementary estimates

$$\|\cdot\|_{L_\infty(R^n)} \leqslant \begin{cases} C\|\cdot\|_0^{1/2} \|\cdot\|_1^{1/2}, & n = 1, \\ C\|\cdot\|_0^{1/4} \|\cdot\|_1^{1/2} \|\cdot\|_2^{1/4}, & n = 2, \\ \epsilon\|\cdot\|_2 + C\epsilon^{-1/2}\|\cdot\|_1, & n = 3, \end{cases}$$

to the function $u - A_h u_z$; in general,

$$\|u - u_z\|_{L_\infty(R^n)} \geqslant O(h^{k-s});$$

but

$$\text{Sup}_{x \in \Omega_h} |u(x) - u_z(x)| \leqslant C\|u\|_k \begin{cases} h^{k-1/2}, & n = 1, \\ h^{k-1}, & n = 2, \\ h^{k-4/3}, & n = 3, \end{cases}$$

so that for $s \geqslant 2$, pointwise superconvergence occurs in up to three dimensions. Similar results hold for the parabolic case.

5. First Order Hyperbolic Equations. It suffices for the present discussion to consider the scalar equation in one space dimension,

$$(5.1) \qquad \frac{\partial w}{\partial t} + b \frac{\partial w}{\partial x} = f, \quad (x, t) \in R \times (0, T), \quad w(\cdot, 0) = g;$$

the generalization of our results to symmetric hyperbolic systems in n space dimensions and including a linear undifferentiated term is immediate. In (5.1), b and f are smooth functions of (x, t) and g is a smooth function of x. We obtain continuous time approximations to w in $X_h^{(k)}$, a finite element space of order k, as above; the test space Y_h is not required to satisfy any approximation properties. We let A_h be a symmetric pseudo-differential operator as above, with symbol $a(h\xi)$. As above, we have in mind the case where $a(h\xi)$ is of the form (3.1) with positive integer s. The case $s = 0$, $A = I$ can be included by replacing the ellipticity hypothesis (iv) below by an inverse assumption on the space $X_h^{(k)}$.

THEOREM 7. *Let* $w_h \in X_h^{(k)} \times [0, T]$ *be determined from the relations*

(5.2) $\left(\dfrac{\partial w_h}{\partial t} + b\,\dfrac{\partial w_h}{\partial x} - f,\, y\right) = 0$ *for all* $y \in Y_h$, $t \in (0,\, T)$,

(5.3) $w_h(\,\cdot\,, 0) = P_3 g$,

where P_3 is the H^1 projection into $X_h^{(k)}$. *Suppose that*

  (i) $X_h^{(k)} \subset H^1$, $Y_h \subset H^0$;

  (ii) A_h^2 *maps* Y_h *onto* $X_h^{(k)}$;

  (iii) A_h *is bounded in* H^0;

  (iv) A_h *is elliptic of order* -1 ((2.8) *holds with* $s = 1$);

  (v) $a(\,\cdot\,)$ *is Lipschitz continuous in* $h\xi$;

  (vi) A_h *is inverse bounded on* Y_h ($\|\phi\|_0 \leqslant C\|A_h\phi\|_0$ *for all* $\phi \in Y_h$);

then for $t \geqslant 0$,

$$\|w(\,\cdot\,, t) - w_h(\,\cdot\,, t)\|_0$$

(5.4)
$$\leqslant C(t) h^{k-1} \left[\|w(\,\cdot\,, t)\|_{k-1} + \int_0^t [\|w(\,\cdot\,, t)\|_k + \|f(\,\cdot\,, t)\|_{k-1}]\, dt \right].$$

Proof. Subtracting (5.2) from (5.1) in the usual way and choosing y such that $A_h^2 y = P_3 w - w_h$, we obtain

$$\left(\dfrac{\partial A_h^2 y}{\partial t} + b\,\dfrac{\partial A_h^2 y}{\partial x} + \left(\dfrac{\partial}{\partial t} + b\,\dfrac{\partial}{\partial x}\right)(w - P_3 w),\, y\right) = 0,$$

$$\left(A_h y,\, \dfrac{\partial A_h y}{\partial t}\right) = -\left(A_h y,\, b\,\dfrac{\partial A_h y}{\partial x}\right) - \left(y,\, \left(b\,\dfrac{\partial}{\partial x} A_h - A_h b\,\dfrac{\partial}{\partial x}\right)A_h y\right)$$

$$\qquad - \left(y,\, \left(\dfrac{\partial}{\partial t} + b\,\dfrac{\partial}{\partial x}\right)(w - P_3 w)\right)$$

$$\leqslant \dfrac{1}{2}\left(A_h y,\, \left(\dfrac{\partial b}{\partial x}\right)A_h y\right) + Ch\,\|y\|_0\,\|A_h y\|_1$$

$$\qquad + C\|y\|_0 \left[\|(I - P_3)w\|_1 + \left\|(I - P_3)\dfrac{\partial w}{\partial t}\right\|_0\right]$$

using Lemma 1 with $\gamma = 1$, $M = b\partial/\partial x$, $p = 1$, $\phi = A_h y$, $\psi = y$. From the ellipticity and inverse bound on A_h, we have

$$\dfrac{d}{dt}\,\|A_h y\|_0^2 \leqslant C\|A_h y\|_0^2 + C\|A_h y\|_0 \left[\|(I - P_3)w\|_1 + \|(I - P_3)f\|_0\right];$$

since (5.3) implies $\|A_h y(\,\cdot\,, 0)\|_0 = 0$, this implies

$$\|P_3 w(\,\cdot\,, t) - w_h(\,\cdot\,, t)\|_0 = \|A_h^2 y\|_0 \leqslant C\|A_h y\|_0$$

$$\qquad\qquad \leqslant Ch^{k-1} \int_0^t [\|w(\,\cdot\,, t)\|_k + \|f(\,\cdot\,, t)\|_{k-1}]\, dt$$

from which (5.4) easily follows.

This is the same order of accuracy as obtained by Lesaint [17] for the Galerkin approximation. Theorem 6 shows that under suitable conditions, the same order of accuracy can be achieved with simple, e. g. piecewise constant test functions, and correspondingly simpler discrete equations.

The Galerkin approximation, however, is special. The following theorem contains Dupont's result for splines [13] as a special case (also implied by [27]).

THEOREM 8. *Let w_x be the Galerkin approximation in $X_h^{(k)} \times [0, T]$ to w, obtained from the relations*

(5.5) $\qquad \left(\dfrac{\partial w_x}{\partial t} + b \dfrac{\partial w_x}{\partial x} - f, \, \psi \right) = 0 \quad$ *for all $y \in Y_h$, $t \in (0, T)$,*

(5.6) $\qquad\qquad\qquad w_x(\,\cdot\,, 0) = A_h^{-1} P_4 g,$

where P_4 is the projection into $E_h^{(k+1)} = A_h X_h^{(k)}$ with respect to the H^1 inner product. Suppose that $E_h^{(k+1)}$ is a space of order up to $k + 1$, i. e., for $0 \leqslant j < m \leqslant k + 1$,

$$\mathop{\mathrm{Inf}}_{\chi \in E_h^{(k+1)}} \| \phi - \chi \|_j \leqslant C h^{m-j} \| \phi \|_m.$$

Let $F_h = A_h^{-1} X_h^{(k)}$; and suppose that

    (i) $F_h \subset H^0$, $X_h^{(k)} \subset H^1$;

    (ii) A_h^2 *maps F_h onto $E_h^{(k+1)}$;*

    (iii) A_h *is bounded in H^0;*

    (iv) $a(\,\cdot\,)$ *is Lipschitz continuous in $h\xi$;*

    (v) $|a(h\xi)| \leqslant 1 + O((h\xi)^k)$;

    (vi) A_h *is inverse bounded on F_h and on $X_h^{(k)}$, i. e. $\|\phi\|_0 \leqslant C \|A_h \phi\|_0$ for all $\phi \in F_h$ and $\|\psi\|_1 \leqslant C \|A_h \psi\|_1$ for all $\psi \in X_h^{(k)}$;*

then for all $t \geqslant 0$,

$$\| w(\,\cdot\,, t) - w_x(\,\cdot\,, t) \|_0$$

(5.7)
$$\leqslant C(t) h^k \left[\| w(\,\cdot\,, t) \|_k + \int_0^t \left[\| w(\,\cdot\,, t) \|_{k+1} + \| f(\,\cdot\,, t) \|_k \right] dt \right].$$

Remark. We have not assumed A_h to be inverse bounded on $F_h \cup X_h^{(k)}$, which is not true in general. For the case $a(h\xi)$ given by (3.1) multiplied by a trigonometric polynomial, hypotheses (ii)–(vi) are satisfied for s a nonnegative integer. The approximation property required of $E_h^{(k+1)}$ precludes the case $s = 0$. For $X_h^{(k)}$ the space of smooth splines of order k with $s = 1$, $E_h^{(k+1)}$ and F_h are the spaces of splines of order $k + 1$ and $k - 1$, respectively.

Proof. We subtract (5.1) from (5.5) and proceed essentially as before. The idea is that $A_h w_x$ satisfies similar discrete equations, with F_h thought of as the test space, as obtained in Theorem 6 for u_h. For any $y \in F_h$, we obtain

$$\left(\frac{\partial}{\partial t} (A_h w - A_h w_x) + b \frac{\partial}{\partial x} (A_h w - A_h w_x) + \left(A_h b \frac{\partial}{\partial x} - b \frac{\partial}{\partial x} A_h \right) (w - w_x), y \right) = 0;$$

choosing $A_h^2 y = P_4 w - A_h w_x$, this becomes

$$\left(\left(\frac{\partial}{\partial t} + b \frac{\partial}{\partial x} \right) A_h^2 y, \, y \right) + \left(\left(\frac{\partial}{\partial t} + b \frac{\partial}{\partial x} \right) (A_h w - P_4 w), \, y \right)$$

$$+ \left(\left(A_h b \frac{\partial}{\partial x} - b \frac{\partial}{\partial x} A_h \right) (w - w_x), \, y \right) = 0,$$

(5.8) $\quad \left(\dfrac{\partial}{\partial t} A_h y, \, A_h y \right) + \left(b \dfrac{\partial}{\partial x} A_h y, \, A_h y \right) + \left(\left(\dfrac{\partial}{\partial t} + b \dfrac{\partial}{\partial x} \right) (A_h w - P_4 w), \, y \right)$

$$+ \left(\left(A_h b \frac{\partial}{\partial x} - b \frac{\partial}{\partial x} A_h \right) (w - w_x), \, y \right)$$

$$- \left(\left(A_h b \frac{\partial}{\partial x} - b \frac{\partial}{\partial x} A_h \right) (A_h^{-1} P_4 w - w_x), \, y \right) = 0.$$

The two commutator terms can be combined; this is the special property of the Galerkin approximation. Integrating by parts and using Lemma 1 as above, we obtain from (5.8),

$$\left(\frac{\partial}{\partial t} A_h y, A_h y\right) = \frac{1}{2}\left(A_h y, \left(\frac{\partial b}{\partial x}\right) A_h y\right) - \left(\left(\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}\right)(A_h w - P_4 w), y\right)$$

$$+ \left(\left(A_h b \frac{\partial}{\partial x} - b\frac{\partial}{\partial x} A_h\right)(w - A_h^{-1} P_4 w), y\right),$$

$$(5.9) \quad \frac{d}{dt}\|A_h y\|_0^2 \leqslant C\|A_h y\|_0^2$$

$$+ C\|y\|_0\left[\left\|(I - A_h)\frac{\partial w}{\partial t}\right\|_0 + \left\|(I - P_4)\frac{\partial w}{\partial t}\right\|_0 + \|(I - A_h)w\|_1\right.$$

$$\left. + \|(I - P_4)w\|_1 + h\|(I - A_h^{-1}P_4)w\|_1\right]$$

$$\leqslant C\|A_h y\|_0^2 + C\|A_h y\|_0\left[h^k\|w\|_{k+1} + h^k\|f\|_k + h\|(I - A_h^{-1}P_4)w\|_1\right].$$

Let w_0 be an optimal approximation in $X_h^{(k)}$ to w, then

$$\|w - A_h^{-1}P_4 w\|_1 \leqslant \|w - w_0\|_1 + \|w_0 - A_h^{-1}P_4 w\|_1$$

$$(5.10) \qquad \leqslant Ch^{k-1}\|w\|_k + \|A_h w_0 - P_4 w\|_1$$

$$\leqslant Ch^{k-1}\|w\|_k + \|w - P_4 w\|_1 + \|w - A_h w_0\|_1$$

$$\leqslant Ch^{k-1}\|w\|_k + \|(I - A_h)w\|_1 + \|A_h(w - w_0)\|_1 \leqslant Ch^{k-1}\|w\|_k;$$

combining (5.9) and (5.10) and using (5.6) gives

$$(5.11) \qquad \|A_h y\|_0 \leqslant C(t)h^k \int_0^t \left[\|w(\cdot, t)\|_{k+1} + \|f(\cdot, t)\|_k\right] dt.$$

Finally,

$$\|w(\cdot, t) - w_h(\cdot, t)\|_0 \leqslant \|(I - A_h^{-1}P_4)w\|_0 + \|A_h^2 y\|_0;$$

an argument similar to that used to obtain (5.10) gives $\|(I - A_h^{-1}P_4)w\|_0 \leqslant Ch^k\|w\|_k$, and thus (5.7).

6. **Discussion.** There are spaces other than smooth splines to which our formalism applies. For example, we can take A_h^2 to be the inverse of a partial differential operator (cf. [12], [27], e.g. $a(h\xi) = (1 + h^2|\xi^2|)^{-1/2}$; with such a choice for A_h, our hypothesis (5) is simply an inverse assumption on $X_h^{(k)}$. Analogous to [12], [27] we would then choose $X_h^{(k)}$ a space of piecewise polynomials of order k, and $Y_h^{(k-2s)} = A_h^{-2} X_h^{(k)}$.

The results of [10], [22] correspond to A_h^2 a family of partial differential operators; many of our results can be directly extended to this case.

The choice of $a(\cdot)$ given by (3.1) is special if we wish to generate a hierarchy of spaces by successive application of A_h. This is the choice which minimizes the radius of support of the higher order basis elements in the following sense.

THEOREM 9. *Let $X_h^{(j)}$ be a space of order j, spanned by the uniform translates of tensor products of basis element(s) $\phi_j(x/h)$, the support of which is contained in*

$[- \rho_j, \rho_j]$ (ρ_j *is the smallest such number*). *Suppose that for all nonnegative integer* p, $X_h^{(j+p)} = A_h^p X_h^{(j)}$ *is a space of order* $j + p$. *Let* ρ_{j+p} *denote the radius of support of the basis elements of* $X_h^{(j+p)}$. *Then for sufficiently large* p, $\rho_{j+p+1} \geqslant \rho_{j+p} + \frac{1}{2}$, *and* $\rho_{j+p+1} = \rho_{j+p} + \frac{1}{2}$ *if and only if* $a(\,\cdot\,)$ *is given by* (3.1), *with* s *set equal to* 1.

Proof. In the x-variables, A_h is simply a convolution operator with kernel $\check{a}(x/h)$ (the inverse Fourier transform of $a(h\xi)$). In order that A_h successively generate spaces of all orders, $a(h\xi) = 0$ for all $h\xi = 2\pi j$, j any nonzero integer [25]. Thus, for any nonzero integer j,

$$(6.1) \qquad \int \check{a}\Big(\frac{x}{h}\Big) \exp\Big(2\pi i j\, \frac{x}{h}\Big)\, dx = 0.$$

If $\check{a}(x/h)$ has its support within $[-\frac{1}{2}, \frac{1}{2}]$, then by comparison of its Fourier series with (6.1), it follows that $\check{a}(x/h)$ is constant in $(-\frac{1}{2}, \frac{1}{2})$, which corresponds to $a(h\xi)$ given by (3.1) (with $s = 1$).

However, this choice of A_h maps piecewise polynomials into piecewise polynomials, and can be applied to other spaces. For example, if we start with the Hermite cubics, we obtain successively the C^2 piecewise quartics, the C^3 piecewise quintics, etc. Our results hold for such spaces (the modification of Lemma 2 in such cases is straightforward, but tedious).

Finally we consider the rates of convergence of Galerkin methods for first order symmetric hyperbolic systems. In [13], Dupont showed that in L^2, $O(h^3)$ was obtained for Hermite cubics and $O(h^4)$ for cubic splines. Our Theorem 8 holds for the cubic splines, but not for the Hermite cubics; for $X_h^{(k)}$ the space of Hermite cubics, $F_h = A_h^{-1} X_h^{(k)}$ is not a subspace of H^0. A different choice of A_h will not help, because the Fourier transforms of the two basis elements of the Hermite cubic space do not have a suitable common factor. Our results do imply the optimal order of accuracy for the higher spaces in the hierarchy beginning with the Hermite cubics, however, for example, $O(h^6)$ for the Galerkin procedure with the C^3 piecewise quintics.

Department of Mathematics
Rutgers University
New Brunswick, Jersey 08903

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