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PROJECTIONS AND APPROXIMATE IDENTITIES FOR IDEALS IN GROUP ALGEBRAS

BY

TENG-SUN LIU(1), ARNOUD VAN ROOIJ AND JU-KWEI WANG

ABSTRACT. For a locally compact group G with property (P_1), if there is a continuous projection of $L^1(G)$ onto a closed left ideal I, then there is a bounded right approximate identity in I. If I is further 2-sided, then I has a 2-sided approximate identity. The converse is proved for w^*-closed left ideals.

Let G be further abelian and let I be a closed ideal in $L^1(G)$. The condition that I has a bounded approximate identity is characterized in a number of ways which include (1) the factorability of I, (2) that the hull of I is in the discrete coset ring of the dual group, and (3) that I is the kernel of a closed element in the discrete coset ring of the dual group.

Introduction. Let G be a locally compact group, I a closed left ideal in $L^1(G)$ and P a continuous projection of $L^1(G)$ onto I. It is proved by W. Rudin [11, Theorem 1] that, if G is compact, there exists a continuous projection Q of $L^1(G)$ onto I such that

$$(*) \qquad f * Qg = Q(f * g) \qquad (f, g \in L^1(G)).$$

Further [11, Proof of Theorem 2], if in addition G is abelian, then there exists an idempotent measure μ on G such that

$$Qf = f * \mu \qquad (f \in L^1(G))$$

so that Q is actually an algebra homomorphism. It follows that I, considered as a Banach algebra, has a bounded approximate identity.

The purpose of Part I of this paper is to find out what happens if G is not compact or abelian. It turns out that if G has the property (P_1) (which it does if it is compact) then the projection P leads to a net of projections Q for which the formula (*) "almost" holds, and that I still has a bounded (right) approximate identity (Theorem 2).

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The converse problem (If I is a closed left ideal in $L^1(G)$ that has a bounded right approximate identity, does there necessarily exist a continuous projection of $L^1(G)$ onto I?) seems still to be open. Under an additional condition on I the question can be answered in the affirmative (Theorem 4).

Next we turn to the case where G is abelian. Let Γ be its dual group. Let X be a closed subset of Γ, and J a closed translation invariant linear subspace of $L^\infty(G)$ so that $X = \Gamma \cap J$. J. E. Gilbert [4], basing himself upon H. P. Rosenthal's paper [9], proves that there exists a continuous projection of $L^\infty(G)$ onto J if and only if X is an element of the coset ring of the discrete group Γ_d that has the same underlying group as Γ. Furthermore, he proves that then X must be what W. Rudin [12] calls a "C-set". Thus he brings about a connection between projections and approximate identities. (See also [11], [3], [10].) His most important tool is a theorem that describes all closed subsets of Γ that lie in the coset ring of Γ_d. (See also [13].)

In Part II we continue Gilbert's investigations, and characterize the closed ideals in $L^1(G)$ that have bounded approximate identities. At the same time we prove that these ideals are just the ideals that are factorable. (A commutative Banach algebra A is called *factorable* if there is a $c > 0$ such that for every $a \in A$ and $\epsilon > 0$ there exist $x, y \in A$ for which $a = xy$, $\|x\| < c$ and $\|a - y\| < \epsilon$.) This partly solves a problem raised by Hewitt and Ross [6, §39.40].

Preliminaries. For most notations we follow the conventions used by [6] and [12]. We use, however, the multiplicative notation even in the abelian case.

Let G be a locally compact group. By $C_0(G)$ we denote the Banach space of all continuous functions $G \to \mathbf{C}$ that vanish at infinity. The dual space, $C_0(G)^*$, of $C_0(G)$ is identified with the Banach space $M(G)$ of all bounded Radon measures on G. $M(G)$ is a Banach algebra under convolution. The w^*-topology on $M(G)$ is the weak topology determined by $C_0(G)$.

We select a left Haar measure on G. The Haar integral of a function f: $G \to \mathbf{C}$ is written $\int f(x)\,dx$. The Banach space $L^1(G)$ of equivalence classes of integrable functions is identified in the usual way with a subspace of $M(G)$. Thus, $L^1(G)$ is a two-sided ideal in $M(G)$.

Deviating from the notation of [6], for $f \in L^1(G)$ and $x \in G$ we define

$$f_x(y) = f(xy), \qquad f^x(y) = f(yx)\Delta(x) \qquad (y \in G),$$
$$f'(y) = f(y^{-1})\Delta(y^{-1}) \qquad (y \in G),$$

where Δ is the modular function of G. Then $f_x, f^x, f' \in L^1(G)$ and $\|f_x\| = \|f^x\| = \|f'\| = \|f\|$. One easily establishes the relations

$$(f * g)_x = f_x * g, \qquad (f * g)^x = f * f^x, \qquad f * g_x = f^x * g, \qquad (f * g)' = g' * f'.$$

Further,

$$(f * g, h) = (g, f' * h) \qquad (f, g \in L^1(G);\ h \in L^\infty(G)).$$

For every $f \in L^1(G)$ the formulas

$$x \mapsto f_x \quad \text{and} \quad x \mapsto f^x$$

define bounded continuous maps $G \to L^1(G)$. Using the terminology of vector-valued integration [1] one obtains

$$f * g = \int f(x)\, g_{x^{-1}}\, dx \qquad (f, g \in L^1(G)).$$

Similarly, for $j \in L^\infty(G)$ and $x \in G$ put $j_x(y) = j(xy)$ $(y \in G)$. For $j \in C_0(G)$, $x \mapsto j_x$ is a continuous map $G \to C_0(G)$, and

$$f * j = \int f(x)\, j_{x^{-1}}\, dx \qquad (f \in L^1(G);\ j \in C_0(G)).$$

G is said to have the property (P_1) if for all compact sets $C \subset G$ and $\epsilon > 0$ there exists an $h \in L^1(G)$, $h \geq 0$, $\|h\| = 1$, such that $\|h_x - h\| \leq \epsilon$ for all $x \in C$ (see [7, Chapter 8]). Equivalently, for all compact $C \subset G$ and $\epsilon > 0$ there exists an $h \in L^1(G)$, $h \geq 0$, $\|h\| = 1$, such that $\|h^x - h\| \leq \epsilon$ for all $x \in C$. If G has the property (P_1), then there exists a left invariant mean on $L^\infty(G)$ (i.e., an $M \in L^\infty(G)^*$ such that $\|M\| = 1$, $M1 = 1$ and $Mj_x = Mj$ for all $j \in L^\infty(G)$ and $x \in G$). For the proof of this statement, see [7, Chapter 8, §6] and [5], where it is also proved that the existence of a left invariant mean implies the property (P_1). All compact groups and all abelian locally compact groups have the property (P_1).

Part I. Projections onto ideals of group algebras. Our first lemma is a direct descendant of [11, Theorem 1]. By $\mathfrak{L}(L^1(G))$ we denote the space of all continuous linear maps $L^1(G) \to L^1(G)$.

Lemma 1. *Let G be a locally compact group. For every $T \in \mathfrak{L}(L^1(G))$ and $h \in L^1(G)$ the formula*

$$\text{(i)} \qquad T_h f = \int h(x)\,(T(f_x))_{x^{-1}}\, dx \qquad (f \in L^1(G))$$

defines a $T_h \in \mathfrak{L}(L^1(G))$ for which $\|T_h\| \leq \|h\|\,\|T\|$ and

$$\text{(ii)} \qquad \|f * T_h g - T_h(f * g)\| \leq \int |f(x)|\,\|h_x - h\|\, dx \cdot \|T\|\,\|g\| \qquad (f, g \in L^1(G)).$$

If G has the property (P_1), then for all $f_0, f_1, \cdots, f_n \in L^1(G)$ and $\epsilon > 0$ there exists an $h \in L^1(G)$, $h \geq 0$, $\|h\| = 1$, such that for each i,

$$\text{(iii)} \qquad \|f_i * T_h g - T_h(f_i * g)\| \leq \epsilon\,\|T\|\,\|g\| \qquad (g \in L^1(G);\ T \in \mathfrak{L}(L^1(G))).$$

Proof. *If $f \in L^1(G)$* then $x \mapsto (Tf_x)_{x^{-1}}$ is a continuous map $G \to L^1(G)$ [6, Theorem 20.4] and $\|(Tf_x)_{x^{-1}}\| \leq \|T\|\,\|f\|$ for all x. It follows [1, §1, Proposition 8] that the integral in (i) defines $T_h \in \mathfrak{L}(L^1(G))$ and that $\|T_h\| \leq \|h\|\,\|T\|$.

Take $g \in L^1(G)$. For every $a \in G$,

$$T_h(g_{a^{-1}}) = \int h(x)(Tg_{a^{-1}x})_{x^{-1}}\,dx = \int h(ax)(Tg_x)_{x^{-1}a^{-1}}\,dx$$
$$= \left[\int h_a(x)(Tg_x)_{x^{-1}}\,dx\right]_{a^{-1}} = (T_{h_a}g)_{a^{-1}}.$$

Then for all $f \in L^1(G)$,

$$f * T_h g - T_h(f * g) = \int f(x)(T_h g)_{x^{-1}}\,dx - T_h\left(\int f(x) g_{x^{-1}}\,dx\right)$$
$$= \int f(x)(T_h g)_{x^{-1}}\,dx - \int f(x)\,T_h(g_{x^{-1}})\,dx$$
$$= \int f(x)(T_h g - T_{h_x} g)_{x^{-1}}\,dx = \int f(x)(T_{h-h_x}(g))_{x^{-1}}\,dx.$$

(ii) follows, since $\|(T_{h-h_x}(g))_{x^{-1}}\| = \|T_{h-h_x}(g)\| \leq \|h - h_x\|\,\|T\|\,\|g\|$.

Now assume that G has the property (P_1). Let $f_0, \cdots, f_n \in L^1(G)$ and $\epsilon > 0$. We may assume $\|f_i\| \leq 1$ for each i. Take a compact set $C \subset G$ such that $\int_{G\setminus C}|f_i| \leq \frac{1}{4}\epsilon$ for each i, and an $h \in L^1(G)$, $h \geq 0$, $\|h\| = 1$ such that $\|h - h_x\| \leq \frac{1}{4}\epsilon$ for all $x \in C$. For every f_i,

$$\int_C |f_i(x)|\,\|h - h_x\|\,dx \leq \int_G |f_i(x)|\,dx \cdot \tfrac{1}{4}\epsilon \leq \tfrac{1}{4}\epsilon$$

and

$$\int_{G\setminus C} |f_i(x)|\,\|h - h_x\|\,dx \leq \int_{G\setminus C} |f_i(x)|\,dx \cdot 2\|h\| \leq \tfrac{1}{2}\epsilon;$$

(iii) follows.

Let A be a closed subalgebra of $L^1(G)$. A net (u_i) in A is a *left approximate identity* in A if $\lim u_i * f = f$ for all $f \in A$. It is called *bounded* if $\sup\|u_i\| < \infty$. A moment's reflection shows that A has a bounded left approximate identity if and only if there is a $c > 0$ such that for any $f_1, \cdots, f_n \in A$ and $\epsilon > 0$ there is a $u \in A$ such that $\|u\| \leq c$ and $\|u * f_i - f_i\| \leq \epsilon$ $(i = 1, 2, \cdots, n)$. Right and two-sided approximate identities are defined analogously. A linear map T: $L^1(G) \to A$ that leaves every element of A fixed is a *projection* of $L^1(G)$ onto A.

Theorem 2. *Let G be a locally compact group that has the property* (P_1). *Let I be a closed left* [*right*; *two-sided*] *ideal in $L^1(G)$, such that there exists a continuous projection P of $L^1(G)$ onto I. Then I has a right* [*left*; *two-sided*] *approximate identity of bound $\leq \|P\|$.*

Proof. Assume that I is a closed left ideal. For every $f \in L^1(G)$ we have $(Pf_x)_{x^{-1}} \in I$ $(x \in G)$; thus, every P_h maps $L^1(G)$ into I. Further, if $f \in I$, then for every x we have $f_x \in I$, so $(Pf_x)_{x^{-1}} = f_{xx^{-1}} = f$. Hence, if $\int h = 1$, then P_h is a projection of $L^1(G)$ onto I.

Now take $f_1, \cdots, f_n \in I$, $\epsilon > 0$. By Lemma 1(iii), and the above there exists a projection Q of $L^1(G)$ onto I such that $\|Q\| \leq \|P\|$ while for each i,

$$\|f_i * Qg - Q(f_i * g)\| \le \epsilon \, \|P\|^{-1}\|Q\| \, \|g\| \le \epsilon\|g\| \qquad (g \in L^1(G)).$$

Choose $g \in L^1(G)$ so that $\|g\| = 1$ while $\|f_i * g - f_i\| \le \epsilon \|Q\|^{-1}$ for each i. Then $Qg \in I$, $\|Qg\| < \|P\|$, and

$$\|f_i * Qg - f_i\| \le \|f_i * Qg - Q(f_i * g)\| + \|Q(f_i * g) - Qf_i\| \le 2\epsilon \quad (i = 1, \cdots, n).$$

In a similar way one can attack the right ideal. For two-sided ideals we prove, somewhat more generally

Theorem 3. *Let G be a locally compact group with the property* (P_1). *Let I be a closed two-sided ideal in $L^1(G)$ that has a left approximate identity of bound c. Then I has a two-sided approximate identity of bound c.*

Proof. Take $f_1, \cdots, f_n \in I$, $\epsilon > 0$. We construct a $w \in I$, $\|w\| \le c$, such that for each i, $\|f_i * w - f_i\| \le 3c\epsilon + 2\epsilon$ and $\|w * f_i - f_i\| \le 2c\epsilon + 2\epsilon$.

Take $f_0 \in L^1(G)$, $\|f_0\| \le 1$ so that $\|f_i * f_0 - f_i\| \le \epsilon$ and $\|f_0 * f_i - f_i\| \le \epsilon$ $(i = 1, \cdots, n)$. Let h be as in Lemma 1(iii). Take a compact $C \subset G$ such that $\int_{G \setminus C}|h| \le \epsilon$. There exist $x_1, \cdots, x_m \in C$ such that for each i and every $x \in C$ there is an x_j with $\|(f_i)_{x_j} - (f_i)_x\| \le \frac{1}{2}(1 + c)^{-1}\epsilon$. By our assumption on I we can find a $u \in I$, $\|u\| \le c$, such that

$$\|u * (f_i)_{x_j} - (f_i)_{x_j}\| \le \tfrac{1}{2}\epsilon \quad \text{for all } i, j.$$

Then

$$\|u * (f_i)_x - (f_i)_x\| \le \epsilon \qquad (x \in C; i = 1, \cdots, n).$$

Now put $Tf = u * f$ $(f \in L^1(G))$. From Lemma 1 we obtain

(i) $$\|f_i * T_h g - T_h(f_i * g)\| \le \epsilon c\|g\| \qquad (g \in L^1(G); i = 0, \cdots, n).$$

Further, for every $f \in L^1(G)$,

$$T_h f - f = \int h(x)[(u * f_x)_{x^{-1}} - f]\, dx = \int h(x)[u * f_x - f_x]_{x^{-1}}\, dx$$
$$= \int_C h(x)[u * f_x - f_x]_{x^{-1}}\, dx + \int_{G\setminus C} h(x)[u * f_x - f_x]_{x^{-1}}\, dx$$

so that

(ii) $$\|T_h f_i - f_i\| \le \epsilon(c + 2) \qquad (i = 0, \cdots, n).$$

Finally, observe that $x \mapsto u^x_{x^{-1}}$ is a bounded continuous map of G into I, so that $\int h(x)\, u^x_{x^{-1}} dx$ is an element of I. For all $f \in L^1(G)$,

$$T_h f = \int h(x)(u * f_x)_{x^{-1}}\, dx = \int h(x)(u_{x^{-1}} * f_x)\, dx$$
$$= \int h(x)(u^x_{x^{-1}} * f)\, dx = \left(\int h(x) u^x_{x^{-1}}\, dx\right) * f.$$

Consequently, T_h maps $L^1(G)$ into I, and

(iii) $$(T_h f) * g = T_h(f * g) \qquad (f, g \in L^1(G)).$$

Now put $w = T_b f_0$. Then $w \in I$, $\|w\| \le \|T\|\,\|b\|\,\|f_0\| \le \|u\| \le c$; and by applying (i), (ii) and (iii) above, we see that for $i = 1, \cdots, n$,

$$\|f_i * w - f_i\| \le \|f_i * T_b f_0 - T_b(f_i * f_0)\| + \|T_b(f_i * f_0 - f_i)\| + \|T_b f_i - f_i\|$$
$$\le \epsilon c\|f_0\| + \|T\|\epsilon + \epsilon(c+2) \le \epsilon(3c+2)$$

and

$$\|w * f_i - f_i\| = \|T_b(f_0 * f_i) - f_i\| \le \|T_b(f_0 * f_i - f_i)\| + \|T_b f_i - f_i\|$$
$$\le \|T\|\epsilon + \epsilon(c+2) \le \epsilon(2c+2).$$

Remarks. A closed left ideal I in $L^1(G)$ for which there exists a continuous projection of $L^1(G)$ onto I may fail to possess a left approximate identity—even an unbounded one. A sufficiently weird example seems to be the following. Let G be the group generated by two elements, a and b, with the relations $a^2 = b^3 = 1$, $a = bab$ (G is isomorphic to S_3). Consider the functions

$$f = \tfrac{1}{2}\chi_{\{1\}} + \tfrac{1}{2}\chi_{\{a\}} - 1/6,$$
$$g = \tfrac{1}{2}\chi_{\{b^2\}} + \tfrac{1}{2}\chi_{\{ab\}} - \tfrac{1}{2}\chi_{\{b\}} - \tfrac{1}{2}\chi_{\{ba\}},$$

where χ denotes characteristic function. The two-dimensional subspace I of $L^1(G)$ generated by f and g is a left ideal; obviously there is a continuous projection of $L^1(G)$ onto I. For f and g we have the relations

$$f * f = f,\ f * g = 0,\ g * f = f,\ g * g = 0.$$

It follow that f is a right unit, but I does not have a left approximate identity.

If G does not have the property (P_1), then $I^1(G) = \{f \in L^1(G)\colon \int f = 0\}$ is a two-sided ideal such that there is a continuous projection of $L^1(G)$ onto $I^1(G)$, while $I^1(G)$ does not have a (bounded or unbounded) right approximate identity. (See [8, Proof of Theorem II]. Note that Reiter's definition of a bounded right approximate identity is different from ours.)

Now we view $L^1(G)$ as a subspace of $M(G)$. The weak topology of $L^1(G)$, induced by $C_0(G)$, will be called the w^*-topology. It is now not hard to prove a converse to Theorem 2 for w^*-closed ideals.

Theorem 4. *Let G be a locally compact group with the property (P_1). Let I be a w^*-closed left ideal in $L^1(G)$ and $I^\perp = \{h \in L^\infty(G)\colon (f, h) = 0$ for all $f \in I\}$. The following conditions are equivalent:*

(α) *I has a bounded right approximate identity.*

(β) *There exists an idempotent $\mu \in M(G)$ such that $I = L^1(G) * \mu$.*

(γ) *There exists a continuous projection P of $L^1(G)$ onto I.*

(δ) *There exists a continuous projection Q of $L^\infty(G)$ onto $I^\perp$.*

Proof. We prove the implications $(\alpha) \Rightarrow (\delta) \Rightarrow (\beta) \Rightarrow (\gamma)$; the implication $(\gamma) \Rightarrow (\alpha)$ is contained in Theorem 2.

$(\alpha) \Rightarrow (\delta)$. Let Ω denote the natural map $L^1(G) \to L^\infty(G)^*$. Let (u_i) be a bounded right approximate identity in I. By the Alaoglu theorem the net (Ωu_i) has a w^*-convergent cofinal subnet; we may assume that it is itself w^*-convergent in $L^\infty(G)^*$. Then for every $h \in L^\infty(G)$, $\lim (u_i, h)$ exists. The formula

$$(f, Qh) = (f, h) - \lim (u_i, f' * h) \qquad (f \in L^1(G);\ h \in L^\infty(G))$$

defines a continuous linear map $Q\colon L^\infty(G) \to L^\infty(G)$. If $f \in I$, then for all h, $(f, Qh) = (f, h) - \lim (f * u_i, h) = 0$; so Q maps $L^\infty(G)$ into $I^\perp$. Conversely, for any $h \in I^\perp$ and $f \in L^1(G)$, $(f, Qh - h) = \lim (u_i, f' * h) = \lim (f * u_i, h) = 0$; so $Qh = h$. Hence Q is a projection of $L^\infty(G)$ onto $I^\perp$.

$(\delta) \Rightarrow (\beta)$. As G has the property (P_1) there exists a translation invariant mean M on the space of all bounded continuous functions on G. Instead of Mh we shall write $M_x h(x)$. Define a continuous linear $R\colon C_0(G) \to L^\infty(G)$ by

$$(f, Rj) = M_x(f_x, Q(j_x)) \qquad (f \in L^1(G);\ j \in C_0(G)).$$

For all $a \in G$, $(f, (Rj)_a) = (f_{a^{-1}}, Rj) = M_x(f_{a^{-1}x}, Qj_x) = M_x(f_x, Qj_{ax}) = (f, R(j_a))$. Hence, $(Rj)_a = R(j_a)$ for all $j \in C_0(G)$ and $a \in G$. It follows that, for all $g \in L^1(G)$ and $j \in C_0(G)$,

$$g * Rj = \int g(x)(Rj)_{x^{-1}}\,dx = \int g(x) R(j_{x^{-1}})\,dx = R\left(\int g(x) j_{x^{-1}}\,dx\right) = R(g * j).$$

Define $\mu \in M(G)$ by

$$(\mu, j) = j(1) - Rj(1) \qquad (j \in C_0(G)).$$

If $f \in L^1(G)$ and $j \in C_0(G)$, then $(f * \mu, j) = (\mu, f' * j) = (f' * j)(1) - (f' * Rj)(1) = (f, j - Rj)$. Therefore, $(f * \mu, j) = 0$ for all $f \in L^1(G)$ and $j \in C_0(G) \cap I^\perp$. By the w^*-closedness of I we may conclude that $f * \mu \in I$. On the other hand, if $g \in I$, then $(g_x, Qj_x) = 0$ for all $j \in C_0(G)$ and $x \in G$; so $(g, Rj) = 0$ and $(g * \mu, j) = (g, j)$ for all j. Therefore, $g = g * \mu$ for $g \in I$. Hence, $I = L^1(G) * \mu$. We also see that for all $f \in L^1(G)$ (take $g = f * \mu$), $f * \mu = f * \mu * \mu$. Consequently, μ is idempotent.

$(\beta) \Rightarrow (\gamma)$. Put $Pf = f * \mu$ $(f \in L^1(G))$.

Remarks. Observe that the w^*-closedness was not used in proving the implication $(\alpha) \Rightarrow (\delta)$. In fact, by [4] and Part II below, (α) and (δ) are equivalent for all closed ideals in $L^1(G)$, in case G is abelian.

If one applies the proof of $(\delta) \Rightarrow (\beta)$ to an I that is not w^*-closed, one finds an idempotent measure μ such that $L^1(G) * \mu$ lies in the w^*-closure of I, while I is contained in the w^*-closed set $\{g \in L^1(G)\colon g = g * \mu\}$. Thus, $L^1(G) * \mu = w^*\text{-}\mathrm{Cl}(I)$, the w^*-closure of I. Consequently, if I is any closed left ideal in $L^1(G)$ such that (δ) holds, then $w^*\text{-}\mathrm{Cl}(I)$ has a right approximate identity.

From the proof of the implication $(\beta) \Rightarrow (\gamma)$ it follows that actually P can be chosen so that

$$P(f * g) = f * Pg \qquad (f, g \in L^1(G)).$$

One can derive (δ) from (γ) by defining Q by

$$(f, Qh) = (f - Pf, h) \qquad (f \in L^1(G);\ h \in L^\infty(G)).$$

Thus, we can choose Q so that

$$Q(f * h) = f * Qh \qquad (f \in L^1(G);\ h \in L^\infty(G)).$$

Part II. Bounded approximate identities in ideals of commutative group algebras. In this part, G is a locally compact abelian group, Γ its dual group. Since G has the property (P_1) it follows from Theorem 2 that if I is a closed ideal in $L^1(G)$ and if there exists a continuous projection of $L^1(G)$ onto I, then I has a bounded approximate identity. It is as yet unknown whether the converse holds. The first few theorems of this part serve, partly as a preparation for our Main Theorem, partly as an illustration for the close parallel between continuous projections and bounded approximate identities.

Theorem 5. *Let $X, Y \subset \Gamma$ be closed.*

(i) *If kX and kY have bounded approximate identities, then so does $k(X \cup Y)$.*

(ii) *If there exist continuous projections P_X of $L^1(G)$ onto kX and P_Y of $L^1(G)$ onto kY, then there exists a continuous projection P of $L^1(G)$ onto $k(X \cup Y)$ provided that there exist continuous linear maps $S: kX \to k(X \cup Y)$ and $T: kY \to k(X \cup Y)$ such that $S + T = I$ on $k(X \cup Y)$. This condition is satisfied if there is a $\mu \in M(G)$ such that $\hat{\mu} = 1$ on $X \setminus Y$, $\hat{\mu} = 0$ on $Y \setminus X$.*

Proof. (i) Let (u_i), (v_j) be bounded approximate identities in kX and kY respectively. If $f_1, \cdots, f_n \in k(X \cup Y)$ and $\epsilon > 0$, there is a u_i such that $\|f_k * u_i - f_k\| \leq \frac{1}{2}\epsilon$ for each k, and there is a v_j such that $\|(f_k * u_i) * v_j - (f_k * u_i)\| \leq \frac{1}{2}\epsilon$ for each k. Then $u_i * v_j \in k(X \cup Y)$ while $\|f_k * (u_i * v_j) - f_k\| \leq \epsilon$ for each k.

(ii) Put $P = SP_X + TP_Y$; then $P: L^1(G) \to k(X \cup Y)$ and $P = I$ on $k(X \cup Y)$. If $\mu \in M(G)$, $\hat{\mu} = 1$ on $X \setminus Y$ and $\hat{\mu} = 0$ on $Y \setminus X$, then we can define S, T by

$$Sf = \mu * f \qquad (f \in kX),$$
$$Tf = f - \mu * f \qquad (f \in kY).$$

Corollary 6. *If there exists a continuous projection of $L^1(G)$ onto kX, and if $Y \subset \Gamma$ is finite, there exists a continuous projection of $L^1(G)$ onto $k(X \cup Y)$. (Trivially, there is a continuous projection of $L^1(G)$ onto kY, as kY has finite codimension.)*

Corollary 7. *Let X, Y be disjoint, closed subsets of Γ and assume that there exists a $\mu \in M(G)$ such that $\hat{\mu} = 1$ on X, $\hat{\mu} = 0$ on Y. (By [12, §2.6.2] this is true if either X or Y is compact.)*

(i) *There is a bounded approximate identity in $k(X \cup Y)$ if and only if there exist bounded approximate identities in kX and in kY.*

(ii) *There exists a continuous projection of* $L^1(G)$ *onto* $k(X \cup Y)$ *if and only if there exist continuous projections of* $L^1(G)$ *onto* kX *and* kY.

Proof. In both cases the sufficiency has been proved above.

(i) Let (u_i) be a bounded approximate identity in $k(X \cup Y)$ and (v_j) a bounded approximate identity in $L^1(G)$. Then $(u_i * \mu + v_j - \mu * v_j)$ and $(u_i - u_i * \mu + \mu * v_j)$ are bounded approximate identities in kX and kY, respectively. (Note that $\mu * f \in k(X \cup Y)$ if $f \in kX$, and $f - \mu * f \in k(X \cup Y)$ if $f \in kY$.)

(ii) Let P be a continuous projection onto $k(X \cup Y)$. Define $Q: L^1(G) \to L^1(G)$ by

$$Qf = P(\mu * f) + f - \mu * f \qquad (f \in L^1(G)).$$

Then $Q: L^1(G) \to kX$ and $Q = I$ on kX, so Q is a projection onto kX. Similarly, $f \mapsto P(f - \mu * f) + \mu * f$ is a projection onto kY.

At this stage the obvious question is whether Theorem 5 (ii) remains true if one drops the condition that the maps S and T exist. From Theorem 10 (i) it will become apparent that the existence of a bounded approximate identity in a closed ideal I of $L^1(G)$ would imply the existence of a continuous projection from $L^1(G)$ onto I if (and only if) the above question should be answered affirmatively.

The problem was raised first by H. P. Rosenthal who mentioned the following particular case. If $G = \mathbf{R}$, and α is an irrational real number, then there exist continuous projections of $L^1(G)$ onto $k\mathbf{Z}$ and onto $k(\alpha\mathbf{Z})$. (See [9] or Lemma 11 of this paper.) Does there exist a continuous projection onto $k(\mathbf{Z} \cup \alpha\mathbf{Z})$? The authors have not been able to answer this question.

Theorem 8. *Let* A *be a closed subalgebra of* $L^1(G)$. *If* $A \cap k(\{1\})$ *has a bounded approximate identity, so does* A.

Proof. We may assume $A \not\subset k(\{1\})$. Take $h \in A$; $\int h = 1$. If (u_i) is a bounded approximate identity for $A \cap k(\{1\})$, then $u_i - h * u_i + h$ is a bounded approximate identity for A (observe that $f - f * h \in A \cap k(\{1\})$ for all $f \in A$).

Theorem 9. *Let* A *be a closed subalgebra of* $L^1(G)$. *If there is a continuous projection of* $L^1(G)$ *onto* $A \cap k(\{1\})$, *then there is a continuous projection of* $L^1(G)$ *onto* A.

Proof. Let $P: L^1(G) \to A \cap k(\{1\})$ be a continuous projection. We may assume $A \not\subset k(\{1\})$. Choose $h \in A$ so that $Ph = 0$ and $\int h = 1$. Then $f \mapsto Pf + (\int f)h$ is a continuous projection from $L^1(G)$ onto A.

For the sake of easy citation, we recapitulate a number of results concerning the structure of closed sets in the coset ring of G_d, i.e., G with the discrete topology. From [6] we borrow the terms *Calderón set* and *spectral set* in preference to Rudin's *C-set* and *S-set* and to the terms *Wiener-Ditkin set* and *Ditkin set*, used by Reiter in [7]. Our Calderón sets are called *Ditkin sets* by Gilbert [4]. These results are essentially due to I. E. Gilbert (see [3] and also [13]).

Theorem 10. *Let G be an abelian topological group.*

(i) *If $a_1H_1, \cdots, a_nH_n$ are cosets of G, then there exists an open subgroup H of G such that* $\operatorname{int}\bigcup_i(a_iH_i)$ *is a union of finitely many cosets of H.*

(ii) *Let A be an element of the coset ring of G_d. Then $\bar{A}$ can be written as*

$$\bar{A} = \bigcup_{i=1}^{n} x_i\,(H_i\setminus F_iK_i),$$

where, for each i, x_i is an element of G, H_i is a closed subgroup of G, K_i is a relatively open subgroup of H_i and F_i is a finite subset of H_i. In particular, $\bar{A}$ lies in the coset ring of G_d.

(iii) *Let G, A, x_i, H_i, F_i and K_i be as above and let $I = \{i: H_i \text{ is open}\}$. Then*

$$\operatorname{int}\bar{A} = \bigcup_{i\in I} x_i(H_i\setminus F_iK_i).$$

In particular, $\operatorname{int} A$ *belongs to the coset ring of G.*

(iv) *If every infinite closed subgroup of G is open, then a subset X of G is a closed element of the coset ring of G_d if and only if it is the union of a finite set and an element of the coset ring of G.*

(v) *If G is locally compact with dual group Γ, then every closed element of the coset ring of Γ_d is a Calderón set, hence a spectral set.*

Now we turn to our main problem: describe the closed ideals in $L^1(G)$ that have bounded approximate identities. Our main tools are Theorem 2 and Theorem 10. The connection between them is the following lemma.

Lemma 11. *Let G be a locally compact abelian group, Γ its dual group. Let Λ be a closed subgroup of Γ and X an element of the coset ring of Λ. Then there exists a continuous projection of $L^1(G)$ onto kX.*

Proof. Let $\Lambda_\perp = \{x \in G: (x, y) = 1 \text{ for every } y \in \Lambda\}$. Let $G_1 = G/\Lambda_\perp$, let Γ_1 be the dual group of G_1, and π the natural map $G \to G_1$. π determines $\pi^*: \Gamma_1 \to \Gamma$ by the formula $\pi^*(\gamma_1) = \gamma_1 \circ \pi$ $(\gamma_1 \in \Gamma_1)$; this π^* is a topological isomorphism of Γ_1 onto Λ. By [7, §3.4.4] there is a natural homomorphism T of $L^1(G)$ onto $L^1(G_1)$ given by

$$Tf(\pi(x)) = \int f(xy)\,dm(y)$$

where m denotes a Haar measure on $\Lambda_\perp$.

T and π^* are related by

$$(\pi^*\gamma_1, f) = (\gamma_1, Tf) \qquad (\gamma_1 \in \Gamma_1;\ f \in L^1(G)).$$

There exists a linear isometry $S: L^1(G_1) \to L^1(G)$ such that TS is the identity map I_1 of $L^1(G_1)$. (See [7, Chapter 8, §2.7].)

The set $X_1 = \pi^{*-1}X$ is an element of the coset ring of Γ_1. By Cohen's Idempotent Measure Theorem [12, Chapter 3] there exists a $\mu_1 \in M(G_1)$ such that $\hat{\mu}_1 = \chi_{X_1}$. Then $P_1: f_1 \mapsto \mu_1 * f_1$ is a continuous projection of $L^1(G_1)$ onto kX_1. Now $S(I_1 - P_1)T$ is a continuous linear map of $L^1(G)$ into $L^1(G)$; it is idempotent because $TS = I_1$. Hence, $P = I - S(I_1 - P_1)T$ (where I is the identity map $L^1(G) \to L^1(G)$) is a continuous projection of $L^1(G)$ into $L^1(G)$; its range is $\operatorname{Ker} S(I_1 - P_1)T = \operatorname{Ker}(I_1 - P_1)T = T^{-1}(\operatorname{Im} P_1) = T^{-1}(kX_1) = kX$.

In H. Reiter's book [7], a commutative Banach algebra A is said to have a bounded approximate identity if there exists a number $c > 0$ such that for every $a \in A$ and $\epsilon > 0$ there is a $u \in A$, $\|u\| \le c$, for which $\|a - ua\| \le \epsilon$. We shall see that for closed ideals in $L^1(G)$ the presence of a bounded approximate identity in Reiter's sense is equivalent to that of a bounded approximate identity as we defined it before Theorem 2 of this paper.

Lemma 12. *Let A be a commutative Banach algebra. Assume that for every $a \in A$ there exists a bounded sequence (u_n) such that $\lim u_n a = a$. Then A has a bounded approximate identity in the sense of Reiter [7], i.e., there exists a $c > 0$ such that $a \in \operatorname{Cl}\{xa: \|x\| \le c\}$ for all $a \in A$. (The converse is trivial.)*

Proof. For $m \in \mathbf{N}$ let

$$A_m = \{a \in A: a \in \operatorname{Cl}\{xa : \|x\| \le m\}\}.$$

It is easy to see that A_m is a closed subset of A. By the given condition on A, $\bigcup A_m = A$. By the Baire Category Theorem one of the A_m, say A_{m_0}, contains a nonempty open ball B. Then $\mathbf{C}B - \mathbf{C}B$ is a linear subspace of A with nonempty interior, so $\mathbf{C}B - \mathbf{C}B = A$. Any $a \in A$ can be written as $a = a_1 - a_2$ where $a_1, a_2 \in \mathbf{C}B \subset A_{m_0}$. For any $\epsilon > 0$ there exist x_1, x_2 such that $\|x_i\| \le m_0$ and $\|a_i - x_i a_i\| \le \frac{1}{2}(1 + m_0)^{-1}\epsilon$ for $i = 1, 2$. Putting $x = x_1 + x_2 - x_1 x_2$ we obtain $\|x\| \le 2m_0 + m_0^2$ and $\|a - xa\| \le \epsilon$. Thus, we may take $c = 2m_0 + m_0^2$.

Now we turn to our main theorem.

Theorem 13. *Let G be a locally compact abelian group with dual group Γ, I a closed ideal in $L^1(G)$. The following conditions are equivalent.*

(α) *I has a bounded approximate identity.*

(β) *I is factorable, i.e., there exists a $c > 0$ such that for every $f \in I$ and $\epsilon > 0$ we can write $f = g_1 * g_2$ where $g_1, g_2 \in I$, $\|g_1\| \le c$, and $\|f - g_2\| \le \epsilon$.*

(γ) *For every $f \in I$ there is a bounded sequence (u_n) in I such that $\lim u_n * f = f$.*

(δ) *hI lies in the coset ring of Γ_d.*

(ϵ) *$I = kX$ for some Γ-closed element X of the coset ring of Γ_d.*

(ζ) *There exists a continuous projection of $L^\infty(G)$ onto $\{h \in L^\infty(G): (f, h) = 0$ for all $f \in I\}$.*

Proof. $(\alpha) \Rightarrow (\beta)$ is a special case of P. Cohen's Factorization Theorem [2].

$(\beta) \Rightarrow (\gamma)$. If c, f, ϵ, g_1, g_2 are as in (β), then $\|g_1 * f - f\| = \|g_1 * (f - g_2)\| \le c\epsilon$.

$(\gamma) \Rightarrow (\delta)$. By Lemma 12 there exists a $c > 0$ such that for every $\epsilon > 0$ and $f \in I$ we can find $u \in I$ for which $\|u\| \le c$ and $\|u * f - f\| \le \epsilon$.

Let $I_0 = \{f \in L^1(G)\colon \operatorname{Supp} \hat{f}$ is compact; $hI \cap \operatorname{Supp} \hat{f} = \varnothing\}$. By [12, §7.2.5] $I_0 \subset I$. Let $f_1, \cdots, f_n \in I_0$ and $\epsilon > 0$. There exists an $f \in I_0$ such that $\hat{f} = 1$ on $\operatorname{Supp} \hat{f}_i$ $(i = 1, \cdots, n)$ [12, §2.6.2]. Then for each i we have $\hat{f}\hat{f}_i = \hat{f}_i$, so $f * f_i = f_i$. As we have seen, there is a $u \in I$, $\|u\| \le c$, for which $\|u * f - f\| \le \epsilon(\max \|f_i\|)^{-1}$. Then for each i we have $\|u * f_i - f_i\| = \|(u * f - f) * f_i\| \le \epsilon$.

It follows from these considerations that I contains a bounded net (u_j) such that $\lim u_j * f = f$ for every $f \in I_0$. Let $\bar{G}$ be the Bohr compactification of G, and $\bar{\Gamma}$ its dual group. Every continuous almost periodic function f on G induces a continuous function $\bar{f}$ on $\bar{G}$. In particular, every $\gamma \in \Gamma$ induces $\bar{\gamma} \in \bar{\Gamma}$, and the map $\gamma \mapsto \bar{\gamma}$ is a surjective group isomorphism. The map $f \mapsto \bar{f}$ induces a continuous linear map $\Omega\colon L^1(G) \to M(\bar{G})$. For every $f \in L^1(G)$ and $\gamma \in \Gamma$, $(\gamma, f) = (\bar{\gamma}, \Omega f)$. The net (Ωu_j) is norm-bounded, hence has a w^*-limit point $\mu \in M(\bar{G})$. We may assume $w^*\text{-}\lim \Omega u_j = \mu$. In particular, for every $\gamma \in \Gamma$ we have $\lim(\gamma, u_j) = (\bar{\gamma}, \mu)$. As $u_j \in I$ it follows that $(\bar{\gamma}, \mu) = 0$ for $\gamma \in hI$. But if $\gamma \in \Gamma$ and $\gamma \notin hI$, we can choose $f \in I_0$, $(\gamma, f) = 1$; then $(\bar{\gamma}, \mu) = \lim(\gamma, u_j)(\gamma, f) = \lim(\gamma, u_j * f) = (\gamma, f) = 1$. Thus, $\hat{\mu}$ is the characteristic function of $\{\bar{\gamma}\colon \gamma \in \Gamma \backslash hI\}$. By Cohen's Theorem on Idempotent Measures we conclude that $\{\bar{\gamma}\colon \gamma \in hI\}$ lies in the coset ring of $\bar{\Gamma}$, so that hI lies in the coset ring of Γ_d.

$(\delta) \Rightarrow (\epsilon)$. By Theorem 10(v), hI is a spectral set, so $I = khI$.

$(\epsilon) \Rightarrow (\alpha)$. Applying Lemma 11 and Theorem 10 (ii), we see that X can be written as a finite union $X = \gamma_1 X_1 \cup \cdots \cup \gamma_n X_n$ where $\gamma_i \in \Gamma$ and X_i is such that there exists a continuous projection of $L^1(G)$ onto kX_i. The map $f \mapsto \gamma_i^{-1} f$ is a linear isometry of $L^1(G)$ onto $L^1(G)$ that maps kX_i onto $k(\gamma_i X_i)$; so there exist continuous projections of $L^1(G)$ onto the $k(\gamma_i X_i)$. Consequently, by Theorem 2 each $k(\gamma_i X_i)$ has a bounded approximate identity. Now use Theorem 5 (i).

$(\epsilon) \Leftrightarrow (\zeta)$. See [4].

Remark. The equivalence of (α) and (δ) was studied in [13, 2.8].

The following consequence of the above theorem is curious.

Corollary 14. *If Γ is a locally compact abelian group, if X is a closed element of the coset ring of Γ_d, and if Z is a compact, relatively open subset of X, then Z lies in the coset ring of Γ_d.*

One could apply the technique of the proof of the implication $(\gamma) \Rightarrow (\delta)$ in a slightly different way. View $L^1(G)$ as a subspace of $M(G)$. The net (u_i) has a w^*-limit point $\mu \in M(G)$. It is not difficult to prove that μ is an idempotent measure whose Fourier-Stieltjes transform is just the characteristic function of $\Gamma\backslash\text{int } hI$. (See Theorem 10(iii).)

The conditions (α)–(ζ) are not implied by the existence of a (possibly unbounded) approximate identity in I. As an example, let Γ be discrete. Then every $X \subset \Gamma$ is Calderón [6, 39.39 (b)], so every kX has an approximate identity.

Suppose that every infinite closed subgroup of Γ is open. Every closed element X of the coset ring of Γ_d then is the union of a finite set Φ and a set Y that can be written as

$$Y = \bigcup_{i=1}^{n} \gamma_i(\Lambda_i \backslash \Phi_i \Delta_i)$$

where $\gamma_i \in \Gamma$, Λ_i is an open subgroup of Γ, Δ_i is an open subgroup of Λ_i, and $\Phi_i \subset \Gamma$ is finite. Then Y lies in the coset ring of Γ, and by the Idempotent Measure Theorem there is an idempotent $\mu \in M(G)$ whose Fourier-Stieltjes transform is the characteristic function of $\Gamma\backslash Y$. Then $f \mapsto f * \mu$ is a continuous projection of $L^1(G)$ onto kY.

Applying Corollary 6, Theorem 10(iv) and Lemma 11 we obtain

Corollary 15. *Let G, Γ, I be as in Theorem 13. Assume that every infinite closed subgroup of Γ is open. Then the conditions (α)–(ζ) are equivalent to*

(η) *hI is the union of a finite set and an element of the coset ring of Γ.*

(ϑ) *There exists a continuous projection of $L^1(G)$ onto I.*

Added in proof. Most of Theorem 13 was proved independently by H. Reiter who published it as Theorem 2 in Chapter 17 of his book *L^1-algebras and Segal algebras*, Lecture Notes in Math., vol. 231, Springer-Verlag, Berlin and New York, 1971.

It was proved by M. Altman (*Contracteurs dans les algèbres de Banach*, C. R. Acad. Sci. Paris Sér. A 274 (1972), A399–A400) that for any Banach algebra the existence of a bounded left approximate identity in Reiter's sense is equivalent to the existence of a bounded left approximate identity as the term is used in this paper.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002 (Current address of Teng-sun Liu and Ju-kwei Wang)

DEPARTMENT OF MATHEMATICS, CATHOLIC UNIVERSITY, NIJMEGEN, THE NETHERLANDS (Current address of Arnoud van Rooij)