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Projective algebraic varieties whose universal covering spaces are biholomorphic to C^n

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Abstract. These varieties are conjectured to be abelian varieties up to finite étale coverings. This conjecture is derived from an affirmative answer to the abundance conjecture in minimal model theory. In particular, this is true for $n = 3$.

1. Introduction.

In [**I**], Iitaka posed the following:

CONJECTURE U_n. Let V be a nonsingular complex projective algebraic variety whose universal covering space is biholomorphic to an n-dimensional complex affine space $\boldsymbol{C}^n$. Then there exists an abelian variety which is a finite unramified covering manifold over V.

As he mentioned, U_1 is obvious and U_2 is solved by the classification theory of algebraic surfaces. The similar conjecture to U_n is considered for compact Kähler manifolds, which is in fact proved affirmatively in the case $n = 1, 2$. However there are many examples of non-Kähler compact complex manifolds whose universal covering spaces are biholomorphic to $\boldsymbol{C}^n$, if $n \geq 2$. For U_n, the Kodaira dimension $\kappa(V) < n$ is derived from the following fact concerning with hyperbolic geometry:

FACT 1.1 ([**KO1**], [**KO2,** Theorem 2]). Let X be an n-dimensional compact complex analytic manifold of general type, i.e., $\kappa(X) = n$. Let Z be a complex analytic manifold, B a proper closed analytic subset and let $h : Z \setminus B \to X$ be a generically smooth holomorphic mapping, i.e., the differential $dh : T_z(Z) \to T_{h(z)}(X)$ is surjective at a point $z \in Z \setminus B$. Then h extends to a meromorphic mapping from Z.

A nonsingular projective variety V with a finite unramified covering $A \to V$ from an abelian variety, is called a *para-abelian* variety. A hyperelliptic surface is a para-abelian variety, for example. For U_3, Iitaka proved in [**I**] that $\kappa(V) \neq 1$ and that the anti-Kodaira dimension $\kappa^{-1}(V) \neq 1$. By the theory of Kodaira dimension, he investigated the degenerations of para-abelian varieties and obtained a kind of canonical bundle formula to derive a contradiction to the both assumptions $\kappa(V) = 1$ and $\kappa^{-1}(V) = 1$. In that time, the classification theory of algebraic varieties had not enough contents to solve the conjecture. After more than twenty years, we have good information from minimal model theory of algebraic varieties and the structure theorem of compact Kähler manifolds with trivial first Chern classes. As is remarked in [**I**], V

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contains no rational curves. Thus by applying Mori's theory of extremal rays [**Mo**], we now see that the canonical divisor K_V is *nef*, i.e., $K_V \cdot \gamma \geq 0$ for any irreducible curve $\gamma \subset V$. In three dimensional case, we can say more by the abundance theorem (cf. [**Ka4**], [**Mi1**], [**Mi2**]): The canonical divisor K_V is *semi-ample*, i.e., some multiple of K_V is linearly equivalent to the pullback of a hyperplane of $\boldsymbol{P}^N$ for a morphism $V \to \boldsymbol{P}^N$. It is conjectured in any dimensions:

Abundance Conjecture. Let X be a *minimal* projective algebraic variety, i.e., X has only terminal singularities and K_X is nef. Then K_X is semi-ample.

Suppose that the canonical divisor K_V of a nonsingular projective variety V is numerically trivial. Then Bogomolov's decomposition theorem (cf. [**Bo**], [**Be**]) states that there is a finite unramified covering $A \times W \to V$, where A is an abelian variety, W is a simply connected nonsingular projective variety with trivial canonical bundle. Since the universal covering space of our V is $\boldsymbol{C}^n$, we have a finite unramified covering $A \to V$. Thus U_n is true for V with numerically trivial first Chern classes.

Therefore for the affirmative answer to U_3, we have only to eliminate the case $\kappa(V) = 2$. For this purpose, we need to know the structure of the elliptic fibration over a surface. The first version of this paper treats only the case and U_3 is solved affirmatively by applying the ∂-étale cohomology theory developed in [**N3**].

In this version, we shall generalize the argument of elliptic fibrations to the case of *para-abelian* fibrations, which are proper surjective morphisms whose smooth fibers are para-abelian varieties. We shall consider the following conditions (U1), (U2), (U3), (U4) for a complex analytic variety U:

Conditions. (U1) There exists a generically smooth holomorphic mapping $h : Z \to U$ from a Zariski-open subset Z of a compact complex analytic manifold.
(U2) Any prime divisor of U is not uniruled, i.e., it is not a union of rational curves.
(U3) U contains no positive dimensional nonsingular projective variety with numerically trivial first Chern classes.
(U4) U contains no positive dimensional compact complex analytic subsets.

The condition (U4) is stronger than (U2) and (U3). The affine space $\boldsymbol{C}^n$ satisfies the conditions (U1) and (U4). We shall prove:

Theorem 1.2. *Let $f : V \to S$ be a proper surjective morphism with connected fibers from a nonsingular projective variety V onto a normal projective variety S such that K_V is f-numerically trivial. Suppose that the universal covering space U of V satisfies the conditions* (U1), (U2), (U3). *Then there exist a para-abelian variety F and a generically finite surjective morphism $T \to S$ from a nonsingular projective variety such that*

(1) *the normalization of the main component of the fiber product $V \times_S T$ is isomorphic to $F \times T$ over T,*

(2) *the induced morphism $F \times T \to V$ is a finite unramified morphism.*

Especially U is isomorphic to $\boldsymbol{C}^d \times U_T$, where $d = \dim F = \dim V - \dim S$ and U_T is the universal covering space of T satisfying the conditions (U1), (U2), (U3).

Definition 1.3. A nonsingular projective variety V is called of type U, if its universal covering space satisfies the conditions (U1) and (U4).

As a corollary of Theorem 1.2, we have:

THEOREM 1.4. *Let V be a nonsingular projective algebraic variety of type U. Suppose that the canonical divisor* K_V *is semi-ample. Then V is a para-abelian variety.*

PROOF. By applying Theorem 1.2 to the Iitaka fibration $f : V \to S$, we have a nonsingular projective variety T of general type and of type U. Then by Fact 1.1, there is a dominant meromorphic mapping $Z \cdots \to T$ from a compact complex manifold which has a lift to U_T over a Zariski-open subset of Z. Thus there is also a lift $Z \cdots \to U_T$. Hence U_T is compact and is a point by the assumption. Thus V is a para-abelian variety. □

As a consequence, we see that Abundance Conjecture is stronger than Conjecture U_n. Further by a remark on Abundance Conjecture, we can prove: V is an n-dimensional nonsingular projective variety of type U for $n \geq 4$, then $\kappa(V) \leq n - 4$ (Theorem 6.4). Theorem 1.2 is proved in the following way: We first study an *abelian reduction* of a para-abelian variety, which is a finite unramified covering with minimal degree from an abelian variety. Further we similarly consider an abelian reduction for a para-abelian fibration with a section (cf. Section 2). By Bogomolov's decomposition theorem and by (U3), the fibration $f : V \to S$ is a para-abelian fibration. After a base change, we can take a section for the fibration. Then by the abelian reduction, we have an abelian fibration. The period mapping of the fibration is shown to be constant mainly from the condition (U1), in Theorem 5.1. Then we finish the proof by applying a key theorem: Theorem 4.2. There, to find a nice covering $T \to S$, we use an argument of [**V1**] instead of ∂-étale cohomology theory.

After this paper appeared in the preprint series of RIMS, the author was informed a result of Kollár [**Ko2, 6.3**] which is related to our problem. Combining the result with THEOREM 5.1, we have another proof of Theorem 1.4.

Next let us consider the Albanese mapping $\alpha : V \to \operatorname{Alb} V$ of a nonsingular projective variety V of type U. Then it is shown in Proposition 7.1 that α is a surjective morphism with connected fibers. Especially $q(V) \leq \dim V$. If $q(V) = \dim V$, then V is an abelian variety. Theorem 1.4 is generalized to the following:

THEOREM 1.5. *Let V be a nonsingular projective variety of type U. Suppose that the canonical divisor* K_F *of a general fiber F of the Albanese mapping* $\alpha : V \to \operatorname{Alb} V$ *is semi-ample. Then V is a para-abelian variety and* α *is an étale fiber bundle.*

2. Para-abelian variety.

DEFINITION 2.1 ([**I**]). A projective algebraic variety V is called a *para-abelian* variety, if there is a finite unramified covering $A \to V$ from an abelian variety A.

DEFINITION 2.2. An *abelian reduction* for a para-abelian variety V is a finite unramified covering $A \to V$ from an abelian variety A such that for any unramified covering $A' \to V$ from an abelian variety A', there is a morphism $A' \to A$ over V.

PROPOSITION 2.3. *For any para-abelian variety, an abelian reduction exists uniquely up to isomorphisms. The reduction is a Galois covering.*

PROOF. Let $C^n \simeq U \to V$ be the universal covering mapping of a para-abelian variety V and let us fix a point $v \in V$ and $u \in U$ over v. The fundamental group $\pi_1(V, v)$ is considered to act on U from left. Thus it also acts on the space of holomorphic global vector fields $\Gamma(U, \Theta_U)$ from left. Let $\pi_1^{\circ}(V, v)$ be the kernel of the homomorphism

$$\pi_1(V, v) \to \operatorname{Aut} \Gamma(U, \Theta_U).$$

Let $A \to V$ be a finite unramified covering from an abelian variety. We fix a point $a \in A$ over v. There is a morphism $U \to A$ over V which sends $u \mapsto a$. The $\pi_1(A, a)$ is a subgroup of $\pi_1(V, v)$ with finite index, which is contained in $\pi_1^{\circ}(V, v)$. Thus $\pi_1^{\circ}(V, v)$ is a normal subgroup of $\pi_1(V, v)$ with finite index. Let A_V be the quotient space of U by the action of $\pi_1^{\circ}(V, v)$. Then A_V is an abelian variety and there is a sequence of unramified coverings:

$$U \to A \to A_V \to V.$$

Thus $A_V \to V$ is the expected abelian reduction. □

REMARK. Let $A \to V$ and $A' \to V$ be two abelian reductions and let $a \in A$, $a' \in A'$ be points which are mapped to a same point in V. Then there is a unique isomorphism $A \to A'$ over V which sends $a \mapsto a'$.

DEFINITION 2.4. (1) A proper surjective morphism $f : X \to S$ of normal algebraic varieties is called a *para-abelian fibration*, if general fibers are para-abelian varieties.

(2) An abelian reduction for a para-abelian fibration $f : X \to S$ is a finite ramified covering $Y \to X$ such that the fiber $Y_s \to X_s$ over $s \in S$ is an abelian reduction, provided that X_s is smooth.

PROPOSITION 2.5. *Let $f : X \to S$ be a smooth para-abelian fibration admitting a section $\sigma : S \to X$. Then there exist an abelian reduction $Y \to X$ for f and a lift of the section σ to Y. If $Y' \to X$ is another abelian reduction with a lift of σ, then there is a unique isomorphism $Y \simeq Y'$ over X which preserves the lifts.*

PROOF. First we shall show the existence of the abelian reduction in local situation. Let us fix a point $s \in S$. Since f is a fiber bundle in C^{∞}-sense, there is a simply connected open neighborhood $\mathscr{U}$ such that $f^{-1}(\mathscr{U})$ is diffeomorphic to $\mathscr{U} \times X_s$. Then $\pi_1(X_s, \sigma(s)) \simeq \pi_1(f^{-1}(\mathscr{U}), \sigma(s))$. Let $f^{-1}(\mathscr{U})^{\sim} \to f^{-1}(\mathscr{U})$ be the universal covering mapping and let us take a lift $\tilde{\sigma} : \mathscr{U} \to f^{-1}(\mathscr{U})^{\sim}$ of the section σ over $\mathscr{U}$. Then for the universal covering space $X_s^{\sim}$ of X_s, there is a diffeomorphism $f^{-1}(\mathscr{U})^{\sim} \to \mathscr{U} \times X_s^{\sim}$, which sends $\tilde{\sigma}(s) \mapsto (s, \sigma(s))$. Thus $\pi_1(X_s, \sigma(s))$ acts on $f^{-1}(\mathscr{U})^{\sim}$ from left. Let $\mathscr{A}_{\mathscr{U}}$ be the quotient space of $f^{-1}(\mathscr{U})^{\sim}$ by the action of $\pi_1^{\circ}(X_s, \sigma(s))$. Then the fiber of $\mathscr{A}_{\mathscr{U}} \to f^{-1}(\mathscr{U})$ over the point s is an abelian reduction. Therefore other fibers are also abelian reductions. Next, we shall show the uniqueness. Let $Y_1 \to X$ and $Y_2 \to X$ be abelian reductions for f and let $\tilde{\sigma}_i : S \to Y_i$ be lifts of σ for $i = 1, 2$. Let us consider the fiber product $Y_1 \times_X Y_2$ and let Y_3 be the connected component containing the section $\tilde{\sigma}_1 \times_S \tilde{\sigma}_2$. Then Y_3 is isomorphic to Y_1 and Y_2, since it is so over every such open neighborhood as $\mathscr{U}$ above. Note that the isomorphism $Y_1 \to Y_2$ over X preserving the lifts of σ is uniquely determined. Therefore, finally, we can patch the local abelian reductions to have global one. □

Remark. Without the hypothesis of the existence of a section, it has no abelian reduction in general analytic situation.

Corollary 2.6. *A para-abelian fibration with a rational section has an abelian reduction.*

Proof. Let $f : X \to S$ be a para-abelian fibration with a rational section $\sigma : S \cdots \to X$. Let $S^* \subset S$ be the maximal Zariski open subset over which f is smooth and σ is a morphism. Then the restriction $f^*: X^* := f^{-1}(S^*) \to S^*$ is a smooth para-abelian fibration with a section $\sigma_{|S^*}$. Thus by Proposition 2.5, there is an abelian reduction $Y^* \to X^*$ for f^*. Since it is a finite morphism, it extends to a finite ramified covering $Y \to X$. □

Lemma 2.7. *Let $Y \to X$ be an abelian reduction for a para-abelian fibration $X \to S$. Suppose that $Y \to S$ is a smooth morphism having a group scheme structure, i.e., an abelian scheme. Then $X \to S$ is also a smooth morphism.*

Proof. We have only to show that the Galois group G of $Y \to X$ acts freely on Y. The action of $g \in G$ is written by:

$$Y \ni y \mapsto a_g(y + t_g) \in Y,$$

where a_g is an automorphism preserving the zero section and $t_g : S \to Y$ is a section. Then we have:

$$t_{gh} = t_h + a_h^{-1} t_g$$

for $g, h \in G$. Thus $G \ni g \mapsto a_g$ is a homomorphism to $\mathrm{GL}(d, \boldsymbol{C})$, where d is the dimension of the fiber. Let A be the group of sections $S \to Y$. Then A has a right G-module structure by $t \mapsto a_g^{-1} t$ for $t \in A$. We thus have a cohomology class in $H^1(G, A)$ by $\{t_g\}$. Since A is an abelian group, it is of finite order. Let m be the order and let $A_m \subset A$ be the set of sections of order m. The cohomology class is coming from $H^1(G, A_m)$. By changing the zero section, we may assume that $t_g \in A_m$ for any $g \in G$. For non-trivial $g \in G$, it has no fixed point over an open subset S^* *of* S. Thus the equation

$$y = a_g(y + t_g)$$

has no solution over S^*. Since the section t_g is now 'constant', it has no solution on every fibers. Thus G acts freely on Y. □

Corollary 2.8. *Let $Y \to X$ be an abelian reduction for a para-abelian fibration $X \to S$ such that Y is isomorphic to $A \times S$ over S for an abelian variety A. Then X is isomorphic to $F \times S$ for a para-abelian variety F.*

Proof. By the proof of Lemma 2.7, the Galois group of the abelian reduction acts on $Y \simeq A \times S$ by:

$$(y, s) \mapsto (a_g(y + t_g(s)), s)$$

and the morphism $t_g : S \to A$ is constant. Thus we are done. □

3. Smoothness of fibrations.

Lemma 3.1. *Let $\pi : Y \to C$ be a surjective morphism from a normal projective variety Y onto a smooth projective curve C, whose general fibers are abelian varieties. Suppose that there is a section $\sigma : C \to Y$ and that the every component in any fiber is not a ruled variety. Then π is an abelian scheme.*

Proof. We see that the Néron model $\pi' : Y' \to C$ of π is an abelian scheme. It is enough to show the birational mapping $Y \cdots \to Y'$ over C is actually an isomorphism. Let $M \to Y'$ be a succession of blowing-ups with nonsingular centers such that the induced $M \cdots \to Y$ is a morphism. Then every exceptional divisor for $M \to Y'$ is a ruled variety. Therefore it is also exceptional for $M \to Y$. Hence Y and Y' are isomorphic in codimension one. Let A be a π-ample Cartier divisor and let A' be its proper transform in Y'. Then A' is also π'-ample. Therefore $Y \simeq Y'$. □

Corollary 3.2. *Let $f : X \to C$ be a para-abelian fibration over a nonsingular projective curve C. Suppose that X is nonsingular and every components of any fiber are not uniruled varieties. Then any fibers are multiples of para-abelian varieties.*

Proof. There is a branched covering $\tau : C' \to C$ from a nonsingular projective curve C' such that $X \times_C C' \to C'$ has a section. We denote by X' the normalization of $X \times_C C'$. Let $\sigma : C' \to X'$ be the induced section and let $Y \to X'$ be an abelian reduction of $X' \to C'$ by σ. By Lemma 3.1, $Y \to C'$ is an abelian scheme. Hence $X' \to C'$ is also a smooth morphism by Lemma 2.7. Especially the support of every fiber of f is a para-abelian variety, since X is nonsingular. □

4. Ramification.

Let $f : X \to S$ be a surjective morphism from a nonsingular projective variety X onto a projective variety S. We shall consider the following three conditions for f:

(C1) Any uniruled prime divisor $\Gamma \subset X$ must dominate S.

(C2) Some multiple of the canonical divisor K_X is the pullback of a Cartier divisor of S.

(C3) There is a generically finite surjective morphism $T \to S$ such that the main component of $X \times_S T$ is birationally equivalent to $F \times T$ over T for a variety F.

Lemma 4.1. *Let $f : X \to S$ be a surjective morphism from a nonsingular projective variety satisfying the conditions* (C1) *and* (C2). *Then there is no prime divisor $B \subset X$ with* $\operatorname{codim} f(B) \geq 2$.

Proof. If such a prime divisor B exists, then there is a family of curves $\{C_\lambda\}$ in X such that $B \cdot C_\lambda < 0$, $f(C_\lambda)$ are points, and the closure of the union $\bigcup C_\lambda$ is B. Since $K_B \cdot C_\lambda = B \cdot C_\lambda < 0$, B is uniruled by [**MM**]. □

Theorem 4.2. *Let $f : X \to S$ be a para-abelian fibration from a nonsingular projective variety satisfying the three conditions* (C1), (C2) *and* (C3). *Then there is a generically finite surjective morphism $T' \to S$ from a nonsingular projective variety T' such that for the normalization $X_{T'}$ of the main component of $X \times_S T'$,*

(1) *$X_{T'}$ is isomorphic to $F \times T'$ for a para-abelian variety F, and*
(2) *$X_{T'} \to X$ is a finite unramified morphism.*

Proof. We may assume that S and T are normal and $\tau : T \to S$ is a finite Galois morphism with the Galois group G. Let X_T be the normalization of the main component of $X \times_S T$.

Claim 4.3. The rational mapping $F \times T \cdots \to X_T$ is an isomorphism over $T \backslash \operatorname{Sing} T$.

Proof. Suppose that T is nonsingular and let $T^* \subset T$ be the maximal Zariski open subset over which $f_T : X_T \to T$ is smooth. Then $F \times T$ is isomorphic to X_T over T^*, since any birational mapping to a para-abelian variety is a morphism. By Lemma 4.1, the rational mapping is an isomorphism in codimension one. Let H be an ample divisor of X_T and let H' be its proper transform in $F \times T$. Since the restriction of H' to $F \times \{t\}$ is ample for general point $t \in T$ and since F is a para-abelian variety, H' is relatively ample over T. Thus we have an isomorphism $F \times T \simeq X_T$ over T. □

Proof of Theorem 4.2 continued. Therefore by Lemma 4.1 and Claim 4.3, there is a Zariski open subset $S^\circ \subset S$ with the following properties:
(1) S° and $T^\circ := \tau^{-1}(S^\circ)$ are nonsingular.
(2) $\operatorname{codim}(S \backslash S^\circ) \geq 2$ and $\operatorname{codim} f^{-1}(S \backslash S^\circ) \geq 2$.
(3) $X_{T^\circ} \simeq F \times T^\circ$.

Let us denote the birational mapping by $\Psi : X_T \cdots \to F \times T$. The Galois group G acts also on X_T and its quotient morphism $\tau_X : X_T \to X$ is induced from $\tau : T \to S$. For a prime divisor $\Gamma \subset S$, let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_l\}$ be the set of all prime divisors of T contained in $\tau^{-1}(\Gamma)$. Then the Galois group G acts on the set transitively. Let $\Gamma_i' \subset X_T$ be the main component of $f_T^{-1}(\Gamma_i)$ which is the proper transform of $F \times \Gamma_i$ in X_T. For each $1 \leq i \leq l$, let G_i be the subgroup consisting of $g \in G$ such that $g(\Gamma_i) = \Gamma_i$. Then $\{G_i\}$ are conjugate to each others. For each i, G_i acts on Γ_i' and its quotient space is birationally equivalent to the main component of $f^{-1}\Gamma$. Let $R(\Gamma_i)$ be the subgroup of G_i consisting of $g \in G_i$ such that g acts trivially on Γ_i'. Then $\{R(\Gamma_i)\}$ are also conjugate to each others. Let $R(\Gamma)$ be the subgroup of G generated by all $R(\Gamma_i)$. Then it is a normal subgroup of G and let R be the subgroup generated by all the union of $R(\Gamma)$ for $\Gamma \subset S$. Note that the condition $R = \{1\}$ implies that $\tau_X : X_T \to X$ is a finite unramified covering.

Let $\operatorname{Aut}(F)$ be the group of holomorphic automorphisms of F and let $\operatorname{Aut}_0(F)$ be its identity connected component. Note that $\operatorname{Aut}_0(F)$ is an abelian variety. Let $A(T)$ be the set of all the rational mappings $T \cdots \to \operatorname{Aut}_0(F)$. We assume that G acts on T from left. For $g \in G$ and $\zeta : T \cdots \to \operatorname{Aut}_0(F)$, let us define

$$\zeta^g := \zeta \circ g : T \xrightarrow{g\times} T \cdots \xrightarrow{\zeta} \operatorname{Aut}_0(F).$$

Then $A(T)$ has a right G-module structure. Let $\sigma_g' : X_T \to X_T$ be the morphism induced from $(\mathrm{id}_X \times_S g) : X \times_S T \to X \times_S T$ and let $\sigma_g : F \times T \cdots \to F \times T$ be the rational mapping $\Psi \circ \sigma_g' \circ \Psi^{-1}$ for $g \in G$. Then

$$\phi_g := (\mathrm{id}_F \times g)^{-1} \circ \sigma_g : F \times T \cdots \to F \times T$$

is a birational automorphism over T. If $g \in R(\Gamma_i)$, then ϕ_g induces the identity on $F \times \Gamma_i$. Thus for $g \in R$, ϕ_g is given by

$$F \times T \ni (x, t) \mapsto (\zeta_g(t)(x), t),$$

for a rational mapping $\zeta_g \in A(T)$. We have the following cocycle condition for $g, h \in R$:

$$\zeta_{gh} = \zeta_h + \zeta_g^h.$$

Therefore we have a cohomology class in $H^1(R, A(T))$, which is of finite order. Let n be the order and let A_n be the kernel of the multiplication mapping by n:

$$\mathrm{Aut}_0(F) \xrightarrow{n\times} \mathrm{Aut}_0(F).$$

Then the group $A(T)_n$ of n-torsion points of $A(T)$ is just identified with $A_n \simeq (\boldsymbol{Z}/n\boldsymbol{Z})^{\oplus 2d}$, where $d = \dim \mathrm{Aut}_0(F)$. The cohomology class is coming from an element of $H^1(R, A_n)$. Thus by changing the birational mapping $\Psi : X_T \cdots \to F \times T$, we may assume that ζ_g is a constant mapping to A_n. Since $g \in R(\Gamma_i)$ fixes $F \times \Gamma_i$, the ζ_g must be zero. Hence if $g \in R$, then $(\mathrm{id}_F \times g) = \sigma_g$. Therefore the main component of $X \times_S (R\backslash T)$ is birationally equivalent to $F \times (R\backslash T)$ over $(R\backslash T)$. If we take $\tau : T \to S$ with the degree of τ minimal, then $R = \{1\}$. Hence we can take τ so that $X_T \to X$ is a finite unramified covering. The X_T is hence nonsingular. Especially the rational mapping $X_T \cdots \to F \times T$ is holomorphic, since F is a para-abelian variety.

Since some multiple of the canonical divisor K_{X_T} is the pullback of a divisor of T, and since $X_T \to F \times T$ is an isomorphism in codimension one, the $F \times T$ and hence T have only terminal singularities. Let $T' \subset X_T$ be the fiber of the composite $X_T \to F \times T \xrightarrow{p_1} F$ over a general point of F, where p_1 denotes the first projection. Then T' is nonsingular and the induced birational morphism $T' \to T$ is also crepant, i.e., $K_{T'}$ is the pullback of K_T. Therefore $T' \to T$ is an isomorphism in codimension one, i.e., there is no exceptional divisor, since T has only terminal singularities. Then there is a birational mapping $X_T \cdots \to F \times T'$ over $F \times T$ which is an isomorphism in codimension one between nonsingular varieties. We shall show it is actually an isomorphism.

Let A be an ample divisor on X_T, X_c the fiber of $X_T \to F \times T \to F$ over $c \in F$, A_c the restriction of A to X_c and let H be the proper transform of A in $F \times T'$. We denote the restriction of H to $\{c\} \times T'$ for $c \in F$ by H_c. For general $c \in F$, X_c is nonsingular and $X_c \to T$ is a birational morphism isomorphic in codimension one. The A_c is ample and H_c is its proper transform on T'. Note that all the H_c $(c \in F)$ are numerically equivalent to each other. Then all the nonsingular X_c are isomorphic to each other, since the proper transform of $H_{c'}$ in X_c also ample for any $c' \in F$. Similarly, the proper transform of $p_2^* H_c$ in X_T is relatively ample over $F \times T$, where p_2 is the second projection $F \times T' \to T'$. On the other hand, the proper transform in $F \times X_c$ is also relatively ample over $F \times T$. Therefore X_T is isomorphic to $F \times X_c$ over $F \times T$. Hence $X_T \simeq F \times T'$. Especially, the normalization $X_{T'}$ of the main component of $X \times_S T'$ is also isomorphic to $F \times T'$. □

5. Constant period mapping.

THEOREM 5.1. *Let $f : V \to S$ be a para-abelian fibration from a nonsingular projective variety satisfying the condition* (C1). *Suppose that the universal covering space U of V satisfies the condition* (U1). *Then every smooth fiber of f is isomorphic to a para-abelian variety F and there is a finite surjective morphism $T \to S$ such that the main component of $V \times_S T$ is birationally equivalent to $F \times T$ over T.*

PROOF. By Corollary 3.2, we can take a Zariski-open subset $S^\circ \subset S$ such that
(1) S° is nonsingular,
(2) $\operatorname{codim}(S \setminus S^\circ) \geq 2$,
(3) f is smooth over $S^\circ \setminus D^*$ for a smooth divisor $D^* \subset S^\circ$, and
(4) any fibers over D^* are multiples of para-abelian varieties.
By Lemma 4.1, we see also $\operatorname{codim} f^{-1}(S \setminus S^\circ) \geq 2$. Let $p_2 : V \times_S V \to V$ be the second projection and let W be the normalization of the main component of $V \times_S V$. Then the induced morphism $f_V : W \to V$ from p_2 is a para-abelian fibration with a rational section by considering the diagonal. Let $M \to W$ be the abelian reduction of f_V with respect to the rational section. By Corollary 3.2, we see that $M \to V$ and $W \to V$ are smooth over the open subset $V^\circ = f^{-1}S^\circ$.

Let us denote $U^\circ := U \times_V V^\circ$. The fiber product $M \times_V U \to U$ is a smooth abelian fibration over the simply connected manifold U°. Thus we have a period mapping of the abelian fibration from U° to a bounded domain. It extends to a holomorphic mapping from U and thus from Z. Since Z is a Zariski-open subset of a compact complex manifold, the holomorphic mapping must be constant. Therefore $M \times_V U^\circ$ is isomorphic to $A \times U^\circ$ over U° for an abelian variety A. Hence we have a finite unramified covering $V'^\circ \to V^\circ$ such that $M \times_V V'^\circ \simeq A \times V'^\circ$. The $W \times_V V'^\circ$ is also isomorphic to the product $F \times V'^\circ$ for a para-abelian variety F by Corollary 2.8. Especially, every smooth fiber of f is isomorphic to F. There is a finite covering $V' \to V$ whose restriction to V° is the previous unramified covering. Let $T' \subset V'$ be a general subvariety of the same dimension as S and let $T' \to T \to S$ be the Stein factorization. Then $T \to S$ is a finite surjective morphism such that $V \times_S T$ is birationally equivalent to $F \times T$. □

PROOF OF THEOREM 1.2. The general fiber F of $f : V \to S$ is a nonsingular projective variety with numerically trivial canonical divisor. Hence F is a para-abelian variety by Bogomolov's decomposition theorem (cf. [**Bo**], [**Be**]) and by the condition (U3). Hence f and V satisfy the condition of Theorem 5.1. Thus further by Theorem 4.2, we have a generically finite surjective morphism $T \to S$ and a finite unramified covering $F \times T \to V$. Let U_T be the universal covering space of T. Then $U \simeq \boldsymbol{C}^d \times U_T$, where $d = \dim F$. Let $Z \to U$ be the generically smooth holomorphic mapping from a Zariski-open subset of a compact manifold. Then the composite $Z \to U \to U_T$ is also generically smooth. Hence T also satisfies the conditions (U1), (U2), (U3). □

6. Abundance Conjecture.

For a nef Cartier divisor L on a projective variety X, we know the following invariant, which is called the numerical Iitaka dimension for L or the numerical L-

dimension:

$$\nu(L) = \nu(L, X) := \max\{0 \le k \le n \,|\, L^k \cdot H^{n-k} \neq 0\},$$

where $n = \dim X$ and H is an ample divisor of X. Kawamata [**Ka2**] has proved that if $\nu(K_X) = \kappa(X)$ for a minimal projective variety X, then K_X is semi-ample.

DEFINITION 6.1. A projective variety X is called a $\kappa 0$ variety, if for a nonsingular projective model M of X, the following two conditions are satisfied:
(1) $\kappa(M) = 0$,
(2) For any ample divisor A of M, $\dim H^0(M, mK_M + A)$ is bounded for $m > 0$.

REMARK. If a minimal model X' of X exists and Abundance Conjecture for X' is true, then $\kappa(X) = 0$ implies X is a $\kappa 0$ variety and some multiple of $K_{X'}$ is trivial.

LEMMA 6.2. *Let X be a minimal projective variety. Suppose that $\kappa(X) > 0$ and let $\Phi := \Phi_{|mK_X|} : X \cdots \to S$ be the Iitaka fibration. Suppose further that 'general' fibers of Φ are $\kappa 0$ varieties. Then K_X is semi-ample.*

PROOF. Let $\mu : X' \to X$ be a birational morphism from a nonsingular projective variety such that $h := \Phi \circ \mu : X' \to S$ is a morphism. Then $\mu^* K_{X'} \sim_{\boldsymbol{Q}} h^* A + E$ for some ample $\boldsymbol{Q}$-divisor A and effective $\boldsymbol{Q}$-divisor E. Since E is h-nef and its restriction to 'general' fibers are numerically trivial, it is the pullback of an effective $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor of S. Thus $\mu^* K_X \sim_{\boldsymbol{Q}} h^* L$ for some nef and big $\boldsymbol{Q}$-Cartier divisor L. Therefore $\nu(K_X) = \dim S = \kappa(X)$. By [**Ka2**], we are done. □

Since Abundance Conjecture is true for threefolds ([**Mi1**], [**Mi2**], [**Ka4**]), we have:

COROLLARY 6.3. *For a minimal projective variety X with $\kappa(X) \ge \dim X - 3$, K_X is semi-ample.*

Therefore we have:

THEOREM 6.4. *Let V be an n-dimensional projective variety of type U. If $n = 3$, then V is a para-abelian variety. If $n \ge 4$, then $\kappa(V) \le n - 4$.*

7. Albanese mapping.

Next, we shall consider the Albanese mapping $\alpha : V \to \mathrm{Alb}\, V$.

PROPOSITION 7.1. *Suppose that the universal covering space of a nonsingular projective variety V satisfies the condition* (U1), *then the Albanese mapping $\alpha : V \to \mathrm{Alb}\, V$ is a surjective morphism with connected fibers. Especially, the irregularity $q(V) \le \dim V$.*

PROOF. Let $V \to W \to A := \mathrm{Alb}\, V$ be the Stein factorization of α. Then by [**Ka1**, Theorem 13], there exist an abelian subvariety $B \subset A$ and a finite morphism $T \to A/B$ from a projective variety of general type such that W is isomorphic to a fiber bundle of B over T. There is a lift $U \to \boldsymbol{C}^q$ of the composite $V \to W \to T \to A/B$ from the universal covering space U of V to that of A/B, where $q = \dim A/B$. Let $Z \to U$ be a generically smooth morphism from a Zariski-open subset of a compact manifold $\bar{Z}$. By

Fact 1.1, $Z \to U \to \boldsymbol{C}^q \to A/B$ extends to a holomorphic mapping $\bar{Z} \to A/B$, which is constant. Therefore W is an abelian variety and $W \simeq \operatorname{Alb} V$. □

Now we shall prove Theorem 1.5.

Proof of Theorem 1.5. If $q(V) = 0$, then this is done by Theorem 1.4. Thus we may assume that $q(V) > 0$. The canonical divisor K_V is nef and $K_{V|F} \sim K_F$ is semi-ample. Thus the evaluation mapping:

$$\alpha^* \alpha_* \mathcal{O}(mK_V) \to \mathcal{O}(mK_V)$$

is surjective for some positive integer m by [**N1,** Theorem 5]. Thus we have a fiber space $\pi : V \to R$ over $\operatorname{Alb} V$ such that some multiple of K_V is the pullback of a Cartier divisor of R. Thus there exist a generically finite surjective morphism $V' \to R$ and a finite unramified covering $F' \times V' \to V$, where F' is a para-abelian variety and V' is a nonsingular projective variety of type U, by applying Theorem 1.2 to π. Hence the Albanese mapping for V' also satisfies the condition of Theorem 1.5. Here a general fiber of the Albanese mapping is of general type. However we can prove the F is a point provided that F is of general type as follows: Assume the contrary. Then by the addition theorem [**Ka3**], [**Ko1**], [**V2**], we have:

$$\kappa(V) \geq \kappa(F) = \dim F > 0.$$

Since $\kappa(V) > 0$, we have the Iitaka fibration:

$$f = \Phi_{|mK_V|} : V \cdots \to S,$$

where $\dim S = \kappa(V)$. Let us consider the rational mapping

$$V \cdots \xrightarrow{(f,\alpha)} S \times \operatorname{Alb} V$$

and let $V \cdots \to W \to S \times \operatorname{Alb} V$ be the Stein factorization. For a 'general' point $s \in S$, the fiber $W_s \subset W$ is a normal variety and has a finite morphism $W_s \to \operatorname{Alb} V$. Thus it has a finite unramified covering from a product of an abelian variety and a variety of general type by [**Ka3,** Theorem 13]. Let us consider the fiber $V_s \cdots \to W_s$ of the rational mapping $V \cdots \to W$ and let V_w be a 'general' fiber of $V \cdots \to W$. There is a factorization $V \cdots \to W \to \operatorname{Alb} V$, where K_V is relatively big over $\operatorname{Alb} V$. Thus by the same addition theorem, we see that

$$0 = \kappa(V_s) \geq \kappa(V_w) + \kappa(W_s) \geq 0.$$

Hence $\kappa(V_w) = 0$ and $V \cdots \to W$ is a birational mapping. Since $\kappa(W_s) = 0$, V_s is birationally equivalent to an abelian variety. Therefore by Lemma 6.2, K_V is semi-ample. Theorem 1.4 implies that $\kappa(V) = 0$. This is a contradiction.

Therefore $R \simeq \operatorname{Alb} V$ and $V' \to \operatorname{Alb} V$ is a generically finite surjective morphism. Hence $V' \to \operatorname{Alb} V$ is a finite unramified covering. Thus V is a para-abelian variety and the Albanese mapping is an étale fiber bundle. □

Corollary 7.2. *Let V be a nonsingular projective variety of type U. If $q(V) \geq \dim V - 3$, then V is a para-abelian variety.*

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