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PROJECTIVE FRAÏSSÉ LIMITS AND THE PSEUDO-ARC

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ABSTRACT. The aim of the present work is to develop a dualization of the Fraïssé limit construction from model theory and to indicate its surprising connections with the pseudo-arc. As corollaries of general results on the dual Fraïssé limits, we obtain Mioduszewski's theorem on surjective universality of the pseudo-arc among chainable continua and a theorem on projective homogeneity of the pseudo-arc (which generalizes a result of Lewis and Smith on density of homeomorphisms of the pseudo-arc among surjective continuous maps from the pseudo-arc to itself). We also get a new characterization of the pseudo-arc via the projective homogeneity property.

1. Introduction

In the first part of the paper we dualize the classical (injective) Fraïssé limit found in model theory [5]. The appropriate setting for it will be provided by topological L-structures, where L is a language of relational and functional symbols and where by a topological L-structure D we mean a compact, second countable, zero-dimensional space equipped with interpretations of relation symbols of L as closed subsets of D^k and of function symbols of L as continuous functions from D^k to D for various $k \in \mathbb{N}$. Morphisms between such structures, which will be defined in the next section, are always continuous.

We consider countable families of finite topological L-structures equipped with the discrete topology which satisfy certain "refinement" properties. These refinement properties will be stated precisely later. Whereas in the classical theory of Fraïssé limits one considers injective homomorphisms, here we take projective homomorphisms. We will make this concept precise with the definition of what we will call an epimorphism. In Theorem 2.4 we show that if Δ is a class satisfying these refinement properties, then there exists a topological L-structure $\mathbb{D}$, the projective Fraïssé limit of Δ, which is both projectively universal and projectively ultrahomogeneous with respect to Δ. This means that every member of Δ is an epimorphic image of $\mathbb{D}$, and given any epimorphisms ϕ_1, ϕ_2 from $\mathbb{D}$ to some $D \in \Delta$ there is an isomorphism ψ of $\mathbb{D}$ such that $\phi_2 = \phi_1 \circ \psi$. The topological L-structure $\mathbb{D}$ may be represented as an inverse limit of elements from Δ. We show that $\mathbb{D}$ is unique up to isomorphism. If Δ is infinite, then $\mathbb{D}$ is a non-discrete compact space. Contrast this with the classical Fraïssé construction where one obtains countable Fraïssé limits with no topology on them.

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In the second part, with an eye towards the application that follows in the final section of the paper, we introduce the family Δ_0 of finite linear (reflexive) graphs. We show in Theorem 3.1 that this class satisfies the refinement properties, and thus has a projective Fraïssé limit $\mathbb{P}$.

Finally in part three, we establish a connection between projective Fraïssé limits and the pseudo-arc. (For definitions related to continua and the pseudo-arc see the last paragraph of this section.) In Theorem 4.2 we show that by appropriately moding out the model theoretic content of $\mathbb{P}$, we obtain as a quotient space a hereditarily indecomposable chainable continuum, i.e., the pseudo-arc. By using the fact that the pseudo-arc is a quotient space of $\mathbb{P}$, we are able to transfer properties of $\mathbb{P}$ to the pseudo-arc. Thus with Theorem 4.4(i) we give a proof of Mioduszewski's universality theorem [9] that each chainable continuum is the continuous image of the pseudo-arc. In Theorem 4.4(ii) we establish a homogeneity result stating that for any two continuous surjections f_1, f_2 from the pseudo-arc onto the same chainable continuum there exists a homeomorphism h of the pseudo-arc with $f_2 \circ h$ as close to f_1 in the uniform topology as required. This extends a result of Lewis [7] and Smith [13]. Both these results are obtained as direct consequences of general properties of arbitrary projective Fraïssé limits. This indicates that the theorem of Lewis and Smith can be viewed as a homogeneity result for the pseudo-arc, and that the generalization of this result proved here and Mioduszewski's theorem can be seen as two phenomena (homogeneity and universality) linked at a deeper level. Informed by the analogy with projective Fraïssé limits, we show in Theorem 4.9 that the pseudo-arc is the unique chainable continuum fulfilling the conclusion of the homogeneity theorem. This gives a new characterization of the pseudo-arc.

Recall that a *continuum* is a compact connected metric space. For a compact metric space X, we say that an open cover $\mathcal{U}$ of X *refines* an open cover $\mathcal{V}$ if each element of $\mathcal{U}$ is contained in an element of $\mathcal{V}$. We call a continuum X *chainable* if each open cover of X is refined by an open cover $U_1, \ldots, U_n$ such that for $i, j \leq n$, $U_i \cap U_j \neq \varnothing$ if and only if $|i-j| \leq 1$. Such a cover of X is called a *chain*. A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable. The pseudo-arc is the unique hereditarily indecomposable chainable continuum. It is also the generic continuum: in the (compact) space of all subcontinua of $[0,1]^{\mathbb{N}}$ equipped with the Hausdorff metric, homeomorphic copies of the pseudo-arc form a dense G_δ set. Readers interested in learning more about the pseudo-arc should see [8].

2. The projective Fraïssé limit

2.1. **Definition and elementary lemmas.** Let L be a language consisting of relation symbols R_i, $i \in I$, with arity $m_i \in \mathbb{N}$, and function symbols f_j, $j \in J$, with arity $n_j \in \mathbb{N}$. By a *topological L-structure* we mean a zero-dimensional, compact, second countable space A together with closed sets $R_i^A \subseteq A^{m_i}$ and continuous functions $f_j^A : A^{n_j} \to A$ for all $i \in I$ and $j \in J$. Let A and B be two topological L-structures. By an *epimorphism* from A to B we mean a surjective continuous function $\phi : A \to B$ such that for any $j \in J$ and $x_1, \ldots, x_{n_j} \in A$ we have

$$f_j^B(\phi(x_1), \ldots, \phi(x_{n_j})) = \phi(f_j^A(x_1, \ldots, x_{n_j})) \tag{2.1}$$

and for any $i \in I$ and any $y_1, \ldots, y_{m_i} \in B$ we have

$$(2.2)\quad \begin{aligned} &(y_1, \ldots, y_{m_i}) \in R_i^B \\ &\Leftrightarrow \exists x_1, \ldots, x_{m_i} \in A\ (\phi(x_p) = y_p \text{ for all } p \leq m_i \text{ and } (x_1, \ldots, x_{m_i}) \in R_i^A). \end{aligned}$$

By an *isomorphism* we mean a bijective epimorphism. Since the topology on a topological L-structure is compact, each isomorphism is a homeomorphism. Note also that if $\phi : A \to B$ is an isomorphism, (2.2) is equivalent to

$$(\phi(x_1), \ldots, \phi(x_{m_i})) \in R_i^B \ \Leftrightarrow\ (x_1, \ldots, x_{m_i}) \in R_i^A.$$

We say that an epimorphism $\phi : A \to B$ between two topological L-structures A and B *refines* an open covering $\mathcal{U}$ of A if for each $y \in B$ there is a $U \in \mathcal{U}$ with $\phi^{-1}(y) \subseteq U$.

The following lemma encodes a crucial property of epimorphisms. Since its proof is simple diagram chasing, we leave it to the reader.

Lemma 2.1. *Let A, B, C be topological L-structures. Let $f : B \to A$, $g : C \to A$, $\phi : C \to B$ be functions such that $g = f \circ \phi$. Assume that ϕ is an epimorphism. Then f is an epimorphism iff g is an epimorphism.*

Let Δ be a family of topological L-structures. We say that Δ is a *projective Fraïssé family* if the following two conditions hold:

- (F1) for any $D, E \in \Delta$ there is an $F \in \Delta$ and epimorphisms from F onto D and onto E;
- (F2) for any $C, D, E \in \Delta$ and any epimorphisms $\phi_1 : D \to C$, $\phi_2 : E \to C$, there exists an $F \in \Delta$ with epimorphisms $\psi_1 : F \to D$ and $\psi_2 : F \to E$ such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$.

Let Δ be a family of topological L-structures. We say that a topological L-structure $\mathbb{D}$ is a *projective Fraïssé limit* of Δ if the following three conditions hold:

- (L1) (projective universality) for any $D \in \Delta$ there is an epimorphism from $\mathbb{D}$ to D;
- (L2) for any finite discrete topological space A and any continuous function $f : \mathbb{D} \to A$ there is a $D \in \Delta$, an epimorphism $\phi : \mathbb{D} \to D$, and a function $f' : D \to A$ such that $f = f' \circ \phi$;
- (L3) (projective ultrahomogeneity) for any $D \in \Delta$ and any epimorphisms $\phi_1 : \mathbb{D} \to D$ and $\phi_2 : \mathbb{D} \to D$ there exists an isomorphism $\psi : \mathbb{D} \to \mathbb{D}$ such that $\phi_2 = \phi_1 \circ \psi$.

The conclusion of the following lemma gives a convenient restatement of (L2).

Lemma 2.2. *Let Δ be a family of topological L-structures. Let $\mathbb{D}$ be a topological L-structure fulfilling* (L2)*. Then for each open covering of $\mathbb{D}$ there is an epimorphism from $\mathbb{D}$ to a structure in Δ refining the covering.*

Proof. Given an open cover of $\mathbb{D}$ find a finite cover A consisting of clopen sets and refining it. Then define $f : \mathbb{D} \to A$ by letting $f(x)$ be the element of A containing x. Now apply (L2). □

Lemma 2.3. *Let Δ be a projective Fraïssé family of finite topological L-structures and let $\mathbb{D}$ be a projective Fraïssé limit of Δ. Let $D, E \in \Delta$ and let $\phi : E \to D$ and $\psi : \mathbb{D} \to D$ be epimorphisms. Then there exists an epimorphism $\chi : \mathbb{D} \to E$ such that $\phi \circ \chi = \psi$.*

Proof. By (L1) there is an epimorphism $\alpha : \mathbb{D} \to E$. We now have two epimorphisms ψ and $\phi \circ \alpha$ from $\mathbb{D}$ to D. Thus, by (L3) there is an isomorphism $\beta : \mathbb{D} \to \mathbb{D}$ such that $(\phi \circ \alpha) \circ \beta = \psi$. Take χ to be $\alpha \circ \beta$. $\square$

2.2. **Existence and uniqueness.** In the following theorem we show that each countable projective Fraïssé family of finite topological L-structures has a projective Fraïssé limit which is unique.

Theorem 2.4. *Let Δ be a countable projective Fraïssé family of finite topological L-structures.*

(i) *There exists a topological L-structure which is a projective Fraïssé limit of Δ.*
(ii) *Any two topological L-structures which are projective Fraïssé limits of Δ are isomorphic.*

Proof. We prove (i) first. Inductively construct a sequence $(D_n)_{n\in\mathbb{N}}$ of structures in Δ and epimorphisms $\pi_n : D_{n+1} \to D_n$ so that the following conditions are satisfied where π_n^m, for $m > n$, stands for $\pi_n \circ \cdots \circ \pi_{m-1} : D_m \to D_n$:

(a) for any $D \in \Delta$ there is an n and an epimorphism from D_n to D;
(b) for any n, any pair $E, F \in \Delta$, and any epimorphisms $\phi_1 : F \to E$ and $\phi_2 : D_n \to E$ there exists $m > n$ and an epimorphism $\psi : D_m \to F$ such that
$$\phi_1 \circ \psi = \phi_2 \circ \pi_n^m.$$

This construction is easy to carry out by recursion. We use the countability of Δ and achieve point (a) by applying (F1) and point (b) by applying (F2).

Let $\mathbb{D} = \varprojlim(D_n, \pi_n)$. Let π_m^∞ be the natural projection from $\mathbb{D}$ to D_m. If R is a relation symbol in L of arity k and $x_1, \ldots, x_k \in \mathbb{D}$, we let $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$ precisely when for all $n \in \mathbb{N}$, $(\pi_n^\infty(x_1), \ldots, \pi_n^\infty(x_k)) \in R^{D_n}$. Similarly, if f is a function symbol of arity k and $x_1, \ldots, x_k \in \mathbb{D}$, we let $f^{\mathbb{D}}(x_1, \ldots, x_k) = y$ for the unique $y \in \mathbb{D}$ with $f^{D_n}(\pi_n^\infty(x_1), \ldots, \pi_n^\infty(x_k)) = \pi_n^\infty(y)$ for all n. Clearly $R^{\mathbb{D}}$ is closed and $f^{\mathbb{D}}$ is continuous.

We now check that $\mathbb{D}$ is a projective Fraïssé limit of Δ.

Claim 1. π_m^∞ is an epimorphism.

Proof of Claim 1. Let R be a k-ary relation symbol. If $x_1, \ldots, x_k \in \mathbb{D}$ and we have $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$, then by the definition of $R^{\mathbb{D}}$, $(\pi_m^\infty(x_1), \ldots, \pi_m^\infty(x_k)) \in R^{D_m}$. Now let $d_1, \ldots, d_k \in D_m$ be such that $(d_1, \ldots, d_k) \in R^{D_m}$. We need to find $x_1, \ldots, x_k \in \mathbb{D}$ such that $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$ and $d_i = \pi_m^\infty(x_i)$ for $i \leq k$. Since each π_n is an epimorphism, by recursion we choose $d_1^n, \ldots d_k^n \in D_n$ for $n \geq m$ so that for $i \leq k$
$$d_i^m = d_i,\ \pi_n(d_i^{n+1}) = d_i^n, \text{ and } (d_1^n, \ldots d_k^n) \in R^{D_n}.$$
It follows now from the definition of $\mathbb{D}$ that there are $x_1, \ldots, x_k \in \mathbb{D}$ with $\pi_n^\infty(x_i) = d_i^n$ for $n \geq m$ and $i \leq k$. In particular, we get $\pi_m^\infty(x_i) = d_i$ for $i \leq k$ and $(\pi_n^\infty(x_1), \ldots, \pi_n^\infty(x_k)) \in R^{D_n}$ for all $n \geq m$, and therefore for all n. The last formula implies that $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$.

Similarly we check the condition concerning function symbols and the claim is proved. $\square_{\text{claim}}$

Since π_m^∞ maps $\mathbb{D}$ onto D_m, we get (L1) by Claim 1 and by point (a) from the construction. To see (L2), note that if A is a finite space, then any continuous function from $\mathbb{D}$ to A factors through $\pi_n^\infty : \mathbb{D} \to D_n$ for some n; thus, (L2) follows from Claim 1 as well.

It remains to prove (L3). First we show the following claim.

Claim 2. Let $\chi : D_n \to D_k$, $n > k$, be an epimorphism. There exists an isomorphism ψ of $\mathbb{D}$ such that $\chi \circ \pi_n^\infty = \pi_k^\infty \circ \psi$.

Proof of Claim 2. We construct sequences (k_i) and (n_i) of natural numbers so that $k_0 = k$, $n_0 = n$, and $k_i < n_i \leq k_{i+1}$, for all i. We also construct epimorphisms $\chi_i : D_{n_i} \to D_{k_i}$ and $\alpha_i : D_{k_{i+1}} \to D_{n_i}$ so that $\chi_0 = \chi$ and

$$\chi_i \circ \alpha_i = \pi_{k_i}^{k_{i+1}} \text{ and } \alpha_i \circ \chi_{i+1} = \pi_{n_i}^{n_{i+1}}. \tag{2.3}$$

This is accomplished by induction. Set $\chi_0 = \chi$. Assume we have k_i, n_i, and χ_i. We will show how to get k_{i+1} and α_i. (One similarly produces n_{i+1} and χ_{i+1}.) We apply point (b) to the pair D_{k_i}, D_{n_i}, and the epimorphisms $\chi_i : D_{n_i} \to D_{k_i}$ and $\mathrm{id} : D_{k_i} \to D_{k_i}$ to get an $m > k_i$ and an epimorphism $\psi : D_m \to D_{n_i}$ such that $\chi_i \circ \psi = \pi_{k_i}^m$. Note that $m \geq n_i$. Let $k_{i+1} = m$ and $\alpha_i = \psi$. Now, it follows immediately from (2.3) and the fact that $\chi_0 = \chi$ that the sequence (χ_i) induces an isomorphism ψ on $\mathbb{D}$ as required by the claim. Indeed, equations (2.3) show that the sequence (α_i) induces a function from $\mathbb{D}$ to itself which is both the left and right inverse of ψ. Thus, ψ is a bijection. It is an epimorphism since each χ_i is an epimorphism. $\square_{\text{claim}}$

To get (L3), let ϕ_1 and ϕ_2 be epimorphisms from $\mathbb{D}$ to some $D \in \Delta$. There are $n_1, n_2 \in \mathbb{N}$ and functions $\phi_1' : D_{n_1} \to D$ and $\phi_2' : D_{n_2} \to D$ such that

$$\phi_1 = \phi_1' \circ \pi_{n_1}^\infty \text{ and } \phi_2 = \phi_2' \circ \pi_{n_2}^\infty. \tag{2.4}$$

By Claim 1 and Lemma 2.1, ϕ_1' and ϕ_2' are epimorphisms. Now apply point (b) to the pair D, D_{n_1}, and the epimorphisms $\phi_1' : D_{n_1} \to D$ and $\phi_2' : D_{n_2} \to D$ to get $m > n_2$ (we can assume $m > n_1$) and an epimorphism $\chi : D_m \to D_{n_1}$ such that

$$\phi_1' \circ \chi = \phi_2' \circ \pi_{n_2}^m. \tag{2.5}$$

Applying Claim 2 we obtain an isomorphism ψ of $\mathbb{D}$ with

$$\chi \circ \pi_m^\infty = \pi_{n_1}^\infty \circ \psi. \tag{2.6}$$

Now we have $\phi_1' \circ \chi \circ \pi_m^\infty = \phi_2' \circ \pi_{n_2}^\infty$ from (2.5) which yields $\phi_1' \circ \pi_{n_1}^\infty \circ \psi = \phi_2' \circ \pi_{n_2}^\infty$ by (2.6). From this it follows by (2.4) that ψ is as required by (L3).

Now we prove (ii). Let $\mathbb{D}_0$ and $\mathbb{D}_1$ be two projective Fraïssé limits of Δ. For $i = 0, 1$, fix a sequence of clopen sets $(U_n^i)_{n\in\mathbb{N}}$ which separates points in $\mathbb{D}_i$ with $U_0^0 = U_0^1 = \varnothing$. We construct a sequence $(D_n)_{n\in\mathbb{N}}$ of elements of Δ and epimorphisms $\phi_n^i : \mathbb{D}_i \to D_n$ and $\psi_n : D_{n+1} \to D_n$ such that

- (c) $\phi_n^i = \psi_n \circ \phi_{n+1}^i$, for $i = 0, 1$ and $n \in \mathbb{N}$;
- (d) ϕ_{2n}^0 refines the cover $\{U_n^0, \mathbb{D}_0 \setminus U_n^0\}$ for $n \in \mathbb{N}$;
- (e) ϕ_{2n+1}^1 refines the cover $\{U_n^1, \mathbb{D}_1 \setminus U_n^1\}$ for $n \in \mathbb{N}$.

Pick D_0 to be any element of Δ and, by (L1) for $\mathbb{D}_0$, let ϕ_0^0 be an epimorphism from $\mathbb{D}_0$ to D_0. Note that (d) holds since $U_0^0 = \varnothing$. We will now show how to proceed from $2n$ to $2n+1$. (The step from $2n+1$ to $2n+2$ is identical.) Assume we have D_j and ϕ_j^0 for $j \leq 2n$ and ψ_j and ϕ_j^1 for $j < 2n$. By Lemma 2.3 for $\mathbb{D}_1$ there is an epimorphism $\phi_{2n}^1 : \mathbb{D}_1 \to D_{2n}$ such that

$$\psi_{2n-1} \circ \phi_{2n}^1 = \phi_{2n-1}^1.$$

By Lemma 2.2, there is a $D_{2n+1} \in \Delta$ and an epimorphism $\phi^1_{2n+1} : \mathbb{D}_1 \to D_{2n+1}$ refining the open cover

$$\{(\phi^1_{2n})^{-1}(d) \cap U^1_n,\ (\phi^1_{2n})^{-1}(d) \cap (\mathbb{D}_1 \setminus U^1_n) : d \in D_{2n}\}.$$

The definition of the covering implies that there is a function $\psi_{2n} : D_{2n+1} \to D_{2n}$ such that $\psi_{2n} \circ \phi^1_{2n+1} = \phi^1_{2n}$. By Lemma 2.1 ψ_{2n} is an epimorphism. Thus, (c) holds for $2n$ and (e) for n. This finishes the inductive step.

We define a function $\phi : \mathbb{D}_0 \to \mathbb{D}_1$ as follows. Take an $x \in \mathbb{D}_0$. By (c) and the surjectivity of each ϕ^1_n, there is a $y \in \mathbb{D}_1$ such that for all n, $\phi^0_n(x) = \phi^1_n(y)$. By (e) this y is unique. Let $\phi(x) = y$. By (c) and the surjectivity of each ϕ^0_n, ϕ is onto. By (d), ϕ is an injection. The very definition of ϕ makes it continuous.

Now we need to check that for $x_1, \ldots, x_k, y$ from $\mathbb{D}_0$

$$\text{if } (x_1, \ldots, x_k) \in R^{\mathbb{D}_0},\ \text{then } (\phi(x_1), \ldots, \phi(x_k)) \in R^{\mathbb{D}_1}$$

and

$$\text{if } f^{\mathbb{D}_0}(x_1, \ldots, x_k) = y,\ \text{then } f^{\mathbb{D}_1}(\phi(x_1), \ldots, \phi(x_k)) = \phi(y)$$

for any k-ary relation symbol R and any k-ary function symbol f. This suffices to see that ϕ is an isomorphism since, by symmetry, these same two conditions are then fulfilled by ϕ^{-1}. Let us prove only the first condition for ϕ. Assume towards contradiction that for some $x_1, \ldots, x_k \in \mathbb{D}_0$

$$(x_1, \ldots, x_k) \in R^{\mathbb{D}_0} \ \text{ and } \ (\phi(x_1), \ldots, \phi(x_k)) \notin R^{\mathbb{D}_1}.$$

Since $R^{\mathbb{D}_1}$ is closed, by conditions (c) and (e) we can find n and $d_1, \ldots, d_k \in D_n$ such that

$$(\phi(x_1), \ldots, \phi(x_k)) \in (\phi^1_n)^{-1}(d_1) \times \cdots \times (\phi^1_n)^{-1}(d_k) \subseteq (\mathbb{D}_1)^k \setminus R^{\mathbb{D}_1}. \tag{2.7}$$

Then by the definition of epimorphism, $\big(\phi^1_n(\phi(x_1)), \ldots, \phi^1_n(\phi(x_k))\big) \notin R^{D_n}$. On the other hand, from (2.7) and the definition of epimorphism, $\big(\phi^0_n(x_1), \ldots, \phi^0_n(x_k)\big) \in R^{D_n}$ which immediately leads to a contradiction since $\phi^1_n(\phi(x_j)) = \phi^0_n(x_j)$ for all $j \le k$. □

2.3. **Additional properties.** The content of the following lemma is that if D is a topological L-structure each of whose open covers is refined by an epimorphism onto a structure from a family of finite topological L-structures Δ, then D can be regarded as an inverse limit of structures from Δ.

Lemma 2.5. *Let Δ be a family of finite topological L-structures. Let D be a topological L-structure such that each open cover of D is refined by an epimorphism from D onto a structure in Δ. Then there is a sequence $(D_n) \subseteq \Delta$ and epimorphisms $\psi_n : D_{n+1} \to D_n$ and $\phi_n : D \to D_n$ such that*

(i) *$\psi_n \circ \phi_{n+1} = \phi_n$;*
(ii) *each open cover of D is refined by ϕ_n for some $n \in \mathbb{N}$.*

Proof. Let $\mathcal{U}_n$, $n \in \mathbb{N}$, enumerate all clopen covers of D. Let $D_0 \in \Delta$ and let $\phi_0 : D \to D_0$ be an epimorphism refining $\mathcal{U}_0$. If $D_n \in \Delta$ and $\phi_n : D \to D_n$ has been defined, consider the cover

$$\{U \cap \phi_n^{-1}(d) : d \in D_n \text{ and } U \in \mathcal{U}_n\}.$$

Let $D_{n+1} \in \Delta$ and let $\phi_{n+1} : D \to D_{n+1}$ be an epimorphism refining the above cover. By the definition of the cover, there exists a function $\psi_n : D_{n+1} \to D_n$ such that $\psi_n \circ \phi_{n+1} = \phi_n$. By Lemma 2.1, ψ_n is an epimorphism. □

Proposition 2.6. *Let Δ be a projective Fraïssé family of finite topological L-structures and let $\mathbb{D}$ be the projective Fraïssé limit of Δ. Let D be a topological L-structure such that any open covering of D is refined by an epimorphism onto a structure in Δ. Then there is an epimorphism from $\mathbb{D}$ onto D.*

Proof. For D fix D_n, ϕ_n, and ψ_n as in Lemma 2.5. By (L1) we can find an epimorphism $\phi_0' : \mathbb{D} \to D_0$, and using Lemma 2.3 we can inductively find $\phi_{n+1}' : \mathbb{D} \to D_{n+1}$ such that

$$\psi_n \circ \phi_{n+1}' = \phi_n'. \tag{2.8}$$

We define a function $\phi : \mathbb{D} \to D$ as follows. Take an $x \in \mathbb{D}$. By (2.8) and since each ϕ_n' is onto, there is a $y \in D$ such that for all n, $\phi_n'(x) = \phi_n(y)$. By condition (ii) of Lemma 2.5 for the sequence (ϕ_n), this y is unique. Let $\phi(x) = y$. By (2.8) and since each ϕ_n' is onto, ϕ is onto. The definition of ϕ makes it continuous.

An argument as in the proof of Theorem 2.4(ii) justifies that for $x_1, \ldots, x_k, y \in \mathbb{D}$

$$\text{if } (x_1, \ldots, x_k) \in R^{\mathbb{D}}, \text{ then } (\phi(x_1), \ldots, \phi(x_k)) \in R^D$$

and

$$\text{if } f^{\mathbb{D}}(x_1, \ldots, x_k) = y, \text{ then } f^D(\phi(x_1), \ldots, \phi(x_k)) = \phi(y)$$

for any k-ary relation symbol R and any k-ary function symbol f. We only need to verify that if $(y_1, \ldots, y_k) \in R^D$, then for some $x_1, \ldots, x_k \in \mathbb{D}$ with $\phi(x_i) = y_i$, for $i \le k$, we have $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$. Note that for each n there are $x_1^n, \ldots, x_k^n \in \mathbb{D}$ such that $\phi_n'(x_i^n) = \phi_n(y_i)$, for $i \le k$, and $(x_1^n, \ldots, x_k^n) \in R^{\mathbb{D}}$. This is because ϕ_n and ϕ_n' are epimorphisms. By compactness of $\mathbb{D}$, we can assume that the sequence $(x_1^n, \ldots, x_k^n)_n$ converges to some $(x_1, \ldots, x_k) \in \mathbb{D}^k$. By definition of ϕ, we have $\phi(x_i) = y_i$, for $i \le k$. Since $R^{\mathbb{D}}$ is closed, we get $(x_1, \ldots, x_k) \in R^{\mathbb{D}}$, as well. □

2.4. **Remarks.** The classical Fraïssé construction is a method of taking a direct limit of a family of finite models of a language, provided the family fulfills certain conditions. The limit is a (countable) model of the same language which can be characterized by its (injective) homogeneity and universality with respect to the initial family of models. The standard example here is the family of finite linear orders. In this case the Fraïssé limit is the set of all rational numbers with the usual ordering. See [5] for a treatment.

The results in this section can be viewed as dual to this classical theory. Two points about the dualization need to be emphasized. First, our use of topology in the projective Fraïssé limit has no parallel in the traditional injective Fraïssé theory. Of course, the structures in the injective Fraïssé theory could just as well be equipped with the discrete topology, however, the topology would play no evident role. The situation is different when considering projective limits. If the projective limit is infinite, then it is non-discrete compact, and the topology plays an important role in the uniqueness of the projective Fraïssé limit and is crucial in applications. Second, readers familiar with the classical Fraïssé limit will notice a marked similarity between two of the conditions from that theory and the conditions in our definition of projective Fraïssé family. There is, however, in the classical theory a third condition which is absent from our definition, namely the Hereditary Property. In our case it is essential that we omit (the projective version of) this property since the families in the applications we have in mind do not fulfill it.

3. The family of finite linear graphs

Let L_0 be the language consisting of one binary relation R. Given a topological L_0-structure A, we will henceforth write $R^A(a,b)$ to mean $(a,b) \in R^A$. Let Δ_0 be the class of all finite (reflexive) linear graphs, i.e., the class of all finite sets A with at least two elements so that R^A has the following properties:

(1) R^A is reflexive;
(2) R^A is symmetric;
(3) every element of A has at most three (including itself) R^A-neighbors;
(4) there are exactly two elements of A with less than three R^A-neighbors;
(5) R^A is connected, i.e., for every $a, b \in A$ there exists $a_0, \ldots, a_n \in A$ such that $a = a_0$, $b = a_n$, and $R^A(a_i, a_{i+1})$ for $0 \leq i < n$.

Given $A \in \Delta_0$, a *labeling* of A is a one-to-one function $l : A \to \mathbb{N}$ such that for each $a, b \in A$ with $a \neq b$ we have $R^A(a,b) \Leftrightarrow |l(a) - l(b)| = 1$ and one endpoint, i.e., an element satisfying (4), of A gets mapped by l to 0. Note that there are precisely two labelings on A. Once a labeling of A has been given, for convenience we will regularly identify the points of A with their labels. If $I \subseteq A$, then $\max(I)$ and $\min(I)$ will respectively mean the element of I (or its label) with the maximum and minimum label. Note that labelings are for convenience only; in particular, a labeling of A is not a part of the structure.

If $R^A(a,b)$ we will say that a and b are R-neighbors. We call $I \subseteq A$ an *interval* if for any $a, b \in I$, there are $a_0, \ldots, a_n \in I$ such that $a_0 = a$, $a_n = b$, and $R^A(a_i, a_{i+1})$ for all $i < n$. An element a of I is called an *endpoint of* I if there are at most two elements $b \in I$ with $R^A(a,b)$ (one of which is, of course, a itself). We say that two intervals I and J are *adjacent* if they are disjoint, and there is an endpoint a of I and an endpoint b of J such that $R^A(a,b)$.

Theorem 3.1. *Δ_0 is a projective Fraïssé class.*

Proof. We need to show that Δ_0 satisfies (F1) and (F2). Property (F1) is quite simple, and we leave it to the reader to check. We will prove (F2).

We start with describing the construction of an unfolding. Let A be in Δ_0 and let I be a proper subinterval of A. The pair $(\tilde{A}, f_A)$, where $\tilde{A} \in \Delta_0$ and $f_A : \tilde{A} \to A$ is an epimorphism, will be called the *unfolding of A by I* if $|\tilde{A}| = |A| + 2(|I| - 1)$, and for some labelings on A and $\tilde{A}$ we have

$$f_A(i) = \begin{cases} i, & i \leq \max(I), \\ 2\max(I) - i, & \max(I) < i < 2\max(I) - \min(I), \\ i - (2|I| - 2), & 2\max(I) - \min(I) \leq i. \end{cases} \tag{3.1}$$

Note that given a labeling of A if $(\tilde{A}, f_A)$ is an unfolding of A, then there is a labeling on $\tilde{A}$ such that (3.1) holds. Note also that if $J \subseteq A$ is an interval disjoint from I, then $f_A^{-1}(J)$ is an interval. If $I_1, \ldots, I_n$ are pairwise disjoint proper subintervals of A, then $(\tilde{A}, f_A)$ is the *unfolding of A by $I_1, \ldots, I_n$* if there is a sequence A_i, $0 \leq i \leq n$, of structures from Δ_0 and epimorphisms $f_i : A_i \to A_{i-1}$ such that for each $i \in \{1, \ldots, n\}$, (A_i, f_i) is an unfolding of A_{i-1} by the interval $f_{i-1}^{-1} \circ \cdots \circ f_1^{-1}(I_i)$, $A_0 = A$, $A_n = \tilde{A}$, and $f_A = f_1 \circ \cdots \circ f_n$. Note that the unfolding $(\tilde{A}, f_A)$ does not depend on the order of the intervals $I_1, \ldots, I_n$. Also note that if a labeling is given on A, then there is a labeling on $\tilde{A}$ such that $f_A(0) = 0$. We will call such labelings *compatible*.

Claim 1. Let A and B be in Δ_0. Let $\phi : B \to A$ be an epimorphism. Let $J \subseteq A$ be a proper subinterval and let $(\tilde{A}, f_A)$ be the unfolding of A by J. Then there exists a $\tilde{B} \in \Delta_0$ with epimorphisms $\pi : \tilde{B} \to B$ and $\tilde{\phi} : \tilde{B} \to \tilde{A}$ such that

$$f_A \circ \tilde{\phi} = \phi \circ \pi. \tag{3.2}$$

Proof of Claim 1. If $|J| = 1$, note that $f_A : \tilde{A} \to A$ is an isomorphism, so identify $\tilde{A}$ with A and let $\tilde{B} = B$, let π be the identity on B, and let $\tilde{\phi} = \phi$.

Now suppose $|J| \geq 2$. Let $I_1, \ldots, I_n$ be the maximal subintervals of $\phi^{-1}(J)$, so they are pairwise disjoint, not adjacent, and their union is $\phi^{-1}(J)$. Let $(\tilde{B}, f_B)$ be the unfolding of B by $I_1, \ldots, I_n$. Let $\pi = f_B$.

Label A and $\tilde{A}$, and B and $\tilde{B}$ with compatible labelings. There are three maximal subintervals of $\tilde{A}$ contained in $f_A^{-1}(J)$ on which f_A is one-to-one. Let $\tilde{J}_{\rm bot}$ be the one that contains $\min(f_A^{-1}(J))$, let $\tilde{J}_{\rm top}$ be the one that contains $\max(f_A^{-1}(J))$, and let $\tilde{J}_{\rm mid}$ be the one that contains neither $\min(f_A^{-1}(J))$ nor $\max(f_A^{-1}(J))$. For each I_i with $|I_i| \geq 2$, let $\tilde{I}_{i\,{\rm bot}}$, $\tilde{I}_{i\,{\rm mid}}$, and $\tilde{I}_{i\,{\rm top}}$ be subintervals of $\tilde{B}$ defined in an analogous fashion using f_B.

We now define $\tilde{\phi}$. If $b \in \tilde{B} \setminus f_B^{-1}(\phi^{-1}(J))$, then let $\tilde{\phi}(b) = a$, where $a \in \tilde{A}$ is unique such that $\phi \circ f_B(b) = f_A(a)$. That is, on the non-folded parts of B, $\tilde{\phi}$ and ϕ agree. For the folded parts there are three cases.

Case 1. No endpoint of I_i is mapped by ϕ to $\max(J)$. If $b \in f_B^{-1}(I_i)$, then let

$$\tilde{\phi}(b) = (f_A \upharpoonright \tilde{J}_{\rm bot})^{-1} \circ \phi \circ f_B(b).$$

Case 2. No endpoint of I_i is mapped by ϕ to $\min(J)$. If $b \in f_B^{-1}(I_i)$, then let

$$\tilde{\phi}(b) = (f_A \upharpoonright \tilde{J}_{\rm top})^{-1} \circ \phi \circ f_B(b).$$

Case 3. The endpoints of I_i are mapped by ϕ onto the endpoints of J. Note that in this case $|I_i| \geq 2$. If $\phi(\min(I_i)) = \min(J)$, then let

$$\tilde{\phi}(b) = \begin{cases} (f_A \upharpoonright \tilde{J}_{\rm bot})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm bot}}, \\ (f_A \upharpoonright \tilde{J}_{\rm mid})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm mid}}, \\ (f_A \upharpoonright \tilde{J}_{\rm top})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm top}}. \end{cases}$$

If $\phi(\min(I_i)) = \max(J)$, then let

$$\tilde{\phi}(b) = \begin{cases} (f_A \upharpoonright \tilde{J}_{\rm bot})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm top}}, \\ (f_A \upharpoonright \tilde{J}_{\rm mid})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm mid}}, \\ (f_A \upharpoonright \tilde{J}_{\rm top})^{-1} \circ \phi \circ f_B(b), & b \in \tilde{I}_{i\,{\rm bot}}. \end{cases}$$

We leave it to the reader to check that $\tilde{\phi} : \tilde{B} \to \tilde{A}$ is an epimorphism. Formula (3.2) is then obvious from the definition of $\tilde{\phi}$. $\square_{\rm claim}$

We will now prove (F2). We proceed by induction on the size of D. If $|D| = |C|$, then ϕ_1 is an isomorphism: let $F = E$, let ψ_2 be the identity, and let $\psi_1 = \phi_1^{-1} \circ \phi_2$. Assume now that $|D| > |C|$. It will suffice to show the following claim.

Claim 2. Let $D, C \in \Delta_0$ be such that $|D| > |C|$. Let an epimorphism $\phi : D \to C$ be given. There exists a $D' \in \Delta_0$ with $|D'| < |D|$ and an epimorphism $\pi' : D \to D'$ such that

(a) $\phi' \circ \pi' = \phi$ for some epimorphism $\phi' : D' \to C$;

(b) for any $F' \in \Delta_0$ and any epimorphism $\psi' : F' \to D'$ there is an $F \in \Delta_0$ with epimorphisms $\psi : F \to D$ and $\pi : F \to F'$ such that

$$\psi' \circ \pi = \pi' \circ \psi.$$

To see that this is enough, let $\phi_1 : D \to C$ and $\phi_2 : E \to C$ be epimorphisms. Applying Claim 2 to ϕ_1, we obtain D', π' and also an epimorphism $\phi_1' : D' \to C$ as in (a), that is, we have $\phi_1' \circ \pi' = \phi_1$. Then by our inductive assumption, there is an $F' \in \Delta_0$ with epimorphisms $\psi_1' : F' \to D'$ and $\psi_2' : F' \to E$ so that

$$\phi_1' \circ \psi_1' = \phi_2 \circ \psi_2'. \tag{3.3}$$

Now, condition (b) allows us to find $F \in \Delta_0$, and epimorphisms $\psi_1 : F \to D$ and $\pi : F \to F'$ such that

$$\psi_1' \circ \pi = \pi' \circ \psi_1. \tag{3.4}$$

Now from (a), (3.4), and (3.3) we get

$$\phi_1 \circ \psi_1 = \phi_1' \circ \pi' \circ \psi_1 = \phi_1' \circ \psi_1' \circ \pi = \phi_2 \circ (\psi_2' \circ \pi).$$

If we let $\psi_2 = \psi_2' \circ \pi$, then $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$ as required.

Proof of Claim 2. For $A, B \in \Delta_0$, let us call an epimorphism $\alpha : B \to A$ *simple* if for each $x, y \in B$ we have that if $R^B(x, y)$ and $\alpha(x) = \alpha(y)$, then $x = y$.

We first consider the case where ϕ is simple.

We claim that there is a $D' \in \Delta_0$ and a proper subinterval $J \subseteq D'$ such that

$$|D'| < |D|,$$

and if $(\tilde{D}', f_{D'})$ is the unfolding of D' by J, then there are epimorphisms $\alpha_1 : \tilde{D}' \to D$ and $\alpha_2 : D \to D'$ such that

$$f_{D'} = \alpha_2 \circ \alpha_1, \tag{3.5}$$

and there is a $\phi' : D' \to C$ with

$$\phi' \circ \alpha_2 = \phi. \tag{3.6}$$

This will suffice to prove the claim for when ϕ is simple. To see it, let $\pi' = \alpha_2$. Given $F' \in \Delta_0$ and an epimorphism $\psi' : F' \to D'$, by Claim 1, there is an $F \in \Delta_0$ with epimorphisms $\pi : F \to F'$ and $\tilde{\psi} : F \to \tilde{D}'$ such that $\psi' \circ \pi = f_{D'} \circ \tilde{\psi}$, that is, by (3.5), $\psi' \circ \pi = \alpha_2 \circ (\alpha_1 \circ \tilde{\psi})$. Thus, we can let $\psi = \alpha_1 \circ \tilde{\psi}$.

Therefore, it remains to prove the existence of D' and J as above. For the remainder of the proof fix some labelings on D and C. Consider all maximal subintervals $J \subseteq D$ such that $\phi \restriction J$ is one-to-one. Note that, since ϕ is simple, each such subinterval has at least two elements. Let J_1 be (one of) the shortest among such subintervals. Since $|D| > |C|$, $J_1 \neq D$, and therefore we have one of the following cases.

Case 1. Neither endpoint of J_1 is an endpoint of D.

By the assumptions on J_1 and ϕ, there are two disjoint subintervals J_0 and J_2 which are adjacent to J_1 and such that $|J_0| = |J_2| = |J_1| - 1$, and $\phi \restriction J_0$ and $\phi \restriction J_2$ are one-to-one. Let $D' \in \Delta_0$ be such that $|D'| = |D| - 2(|J_1| - 1)$ and, for some labeling on D', let $f_{D'} : D \to D'$ be given by

$$f_{D'}(i) = \begin{cases} i, & 0 \le i \le \min(J_1), \\ 2\min(J_1) - i, & \min(J_1) < i < \max(J_1), \\ i - 2(|J_1| - 1), & \max(J_1) < i \le \max(D). \end{cases}$$

Put $J = f_{D'}(J_1)$. Thus, we specified D' and J. Since $|J_1| \geq 2$, $|D'| < |D|$. Note also that J is a proper subinterval of D' and that $(D, f_{D'})$ is the unfolding of D' by J. Therefore, we can let α_1 be the identity. It is easy to see that $\phi' : D' \to C$ given by the formula

$$\phi'(x) = \phi(y) \text{ for any } y \in D \text{ with } x = f_{D'}(y)$$

is well defined and is an epimorphism. Once this is established, it is clear that $\phi = \phi' \circ f_{D'}$. Thus, by letting $\alpha_2 = f_{D'}$, we get (3.5) and (3.6).

Case 2. Precisely one endpoint of J_1 is an endpoint of D.

Suppose without loss of generality that $\min(J_1)$ is an endpoint of D. By the case assumption there exists a subinterval J_2 of D which is adjacent to J_1 such that $|J_2| = |J_1| - 1$ and $\phi \restriction J_2$ is one-to-one.

Let $D' \in \Delta_0$ be such that $|D'| = |D| - |J_1| + 1$. Note that since $|J_1| \geq 2$, $|D'| < |D|$. Define an epimorphism $\alpha_2 : D \to D'$ so that for some fixed labeling on D'

$$\alpha_2(i) = \begin{cases} \max(J_1) - i, & 0 \leq i \leq \max(J_1), \\ i - \max(J_1), & \max(J_1) < i \leq \max(D), \end{cases}$$

Put $J = \alpha_2(J_1)$. Note that J is a proper subinterval of D. Now let $\tilde{D}' \in \Delta_0$ be such that $|\tilde{D}'| = |D| + |J_1| - 1$. Define an epimorphism $\alpha_1 : \tilde{D}' \to D$ so that for some fixed labeling on $\tilde{D}'$

$$\alpha_1(i) = \begin{cases} \max(J_1) - i, & 0 \leq i \leq \max(J_1), \\ i - \max(J_1), & \max(J_1) < i \leq \max(\tilde{D}'). \end{cases}$$

Let $f_{D'} = \alpha_2 \circ \alpha_1$. Note that with this definition $(\tilde{D}', f_{D'})$ is the unfolding of D' by J. Thus, (3.5) holds. Now define $\phi' : D' \to C$ via

$$\phi'(x) = \phi(y) \text{ for any } y \in D \text{ with } x = \alpha_2(y).$$

We leave it to the reader to check the simple fact that ϕ' is a well-defined epimorphism. Clearly (3.6) is fulfilled.

We proved the claim for simple $\phi : D \to C$. Assume now that ϕ is not simple and fix $d_1, d_2 \in D$ such that $R^D(d_1, d_2)$, $d_1 \neq d_2$ and $\phi(d_1) = \phi(d_2)$. Let $D' \in \Delta_0$ be such that $|D'| = |D| - 1$ and let $\pi : D \to D'$ be an epimorphism with $\pi(d_1) = \pi(d_2)$. So $\pi \restriction (D \setminus \{d_i\})$, $i = 1, 2$, is injective. Define $\phi'(x) = \phi(y)$ for any $y \in D$ with $x = \pi(y)$. Then $\phi' : D' \to C$ is a well-defined epimorphism. Clearly (a) holds. Checking (b) is straightforward, and we leave it to the reader.

This proves the claim and, therefore, also the theorem. □

4. The pseudo-arc

4.1. **The projective Fraïssé limit of Δ_0 and the pseudo-arc.** Let L_0 be the language consisting of one binary relation R. Let Δ_0 be the class of finite L_0-structures which are linear graphs as defined in the previous section.

Lemma 4.1. *Let $\mathbb{P}$ be the projective Fraïssé limit of Δ_0. Then $R^{\mathbb{P}}$ is an equivalence relation each of whose equivalence classes has at most two elements.*

Proof. It suffices to show that $R^{\mathbb{P}}$ is reflexive, symmetric, and that for any $x \in \mathbb{P}$ there is at most one $y \in \mathbb{P}$ distinct from x such that $R^{\mathbb{P}}(x, y)$.

Let $x \in \mathbb{P}$. Since $R^{\mathbb{P}}$ is closed, to show that $R^{\mathbb{P}}(x,x)$, it will suffice to find in any clopen neighborhood U of x points y_1 and y_2 with $R^{\mathbb{P}}(y_1,y_2)$. Fix a clopen set U containing x. Use Lemma 2.2 to get $D \in \Delta_0$ and an epimorphism $\phi : \mathbb{P} \to D$ refining $\{U, \mathbb{P} \setminus U\}$. Since $R^D(\phi(x),\phi(x))$, there are y_1, y_2 such that $\phi(y_1) = \phi(x) = \phi(y_2)$ and $R^{\mathbb{P}}(y_1,y_2)$. Clearly $y_1, y_2 \in U$.

Let $x, y \in \mathbb{P}$ be such that $R^{\mathbb{P}}(x,y)$. Again to see that $R^{\mathbb{P}}(y,x)$, it suffices to prove the following. For any clopen sets V, U containing y and x, respectively, there is a $y_1 \in V$ and an $x_1 \in U$ so that $R^{\mathbb{P}}(y_1,x_1)$. Given such V and U, let $D \in \Delta_0$ and $\phi : \mathbb{P} \to D$ be such that ϕ is an epimorphism refining $\{U, V, \mathbb{P} \setminus (U \cup V)\}$. Then $R^D(\phi(x),\phi(y))$, hence $R^D(\phi(y),\phi(x))$. It follows that there is a y_1 and an x_1 such that $R^{\mathbb{P}}(y_1,x_1)$, $\phi(y_1) = \phi(y)$, and $\phi(x_1) = \phi(x)$. These x_1 and y_1 are as required.

Assume now towards contradiction that there are distinct $x, y, z \in \mathbb{P}$ such that $R^{\mathbb{P}}(x,y)$ and $R^{\mathbb{P}}(y,z)$. Let U, V, W be clopen disjoint sets containing x, y, z, respectively. Let $D \in \Delta_0$ and $\phi : \mathbb{P} \to D$ be such that ϕ is an epimorphism refining $\{U, V, W, \mathbb{P} \setminus (U \cup V \cup W)\}$. Let $a, b, c \in D$ be the images via ϕ of x, y, z, respectively. Let $D' \in \Delta_0$ and $\psi : D' \to D$ be an epimorphism such that $\psi^{-1}(\{a,b,c\}) = \{a', b', b'', c'\}$ with a', b', b'', c' distinct and such that $\psi(a') = a$, $\psi(b') = \psi(b'') = b$, and $\psi(c') = c$ and $R^{D'}(a',b')$, $R^{D'}(b',b'')$, and $R^{D'}(b'',c')$. Note that then $\neg R^{D'}(b',c')$ and $\neg R^{D'}(a',b'')$. Now, by Lemma 2.3, there exists an epimorphism $\chi : \mathbb{P} \to D'$ such that $\psi \circ \chi = \phi$. Note that we have $\chi(x) = a'$, $\chi(z) = c'$, and $\chi(y)$ is equal either to b' or to b''. But in the first case $\neg R^{D'}(\chi(y),\chi(z))$ and in the second case $\neg R^{D'}(\chi(x),\chi(y))$, leading to a contradiction. □

The above lemma allows us to consider $\mathbb{P}/R^{\mathbb{P}}$ as a topological space with the quotient topology.

Theorem 4.2. *Let $\mathbb{P}$ be the projective Fraïssé limit of Δ_0. Then $\mathbb{P}/R^{\mathbb{P}}$ is a chainable hereditarily indecomposable continuum.*

Thus by Bing's characterization of the pseudo-arc [2], $\mathbb{P}/R^{\mathbb{P}}$ is the pseudo-arc.

We start with a lemma.

Lemma 4.3. *Let D be a topological L_0-structure such that R^D is an equivalence relation. Assume that each open cover of D is refined by an epimorphism from D to an element of Δ_0. Then D/R^D is a chainable continuum.*

Proof. Let $\rho : D \to D/R^D$ denote the quotient map. We first show that D/R^D is compact, second countable, and connected, that is, that D/R^D is a continuum. Compactness of R^D easily implies that D/R^D is Hausdorff. Then, continuity of ρ gives that D/R^D is compact with a countable basis since D is such; see [4, Corollary 3.3.7]. That D/R^D is connected will follow if we only show that given a non-empty clopen set $U \subsetneq D$ there is an $x \in U$ and a $y \in D \setminus U$ such that $R^D(x,y)$. There is a $D_0 \in \Delta_0$ and an epimorphism $\phi : D \to D_0$ refining the cover $\{U, D \setminus U\}$. Since $\phi(U) \cup \phi(D \setminus U) = D_0$, there is a $d_1 \in \phi(U)$ and a $d_2 \in \phi(D \setminus U)$ such that $R^{D_0}(d_1,d_2)$. This last condition ensures that there are $x, y \in D$ with $\phi(x) = d_1$, $\phi(y) = d_2$ and $R^D(x,y)$ since ϕ is an epimorphism. Since ϕ refines $\{U, D \setminus U\}$, we also have $x \in U$ and $y \in D \setminus U$.

Now we check chainability of D/R^D. Let d be a compatible metric on D/R^D. Fix $\varepsilon > 0$ and cover D/R^D with a finite number of ε-balls $\{B_0, \ldots, B_n\}$. Now $\left\{\rho^{-1}(B_0), \ldots, \rho^{-1}(B_n)\right\}$ is an open cover of D. Let $\phi : D \to D_0$, for some $D_0 \in \Delta_0$, be an epimorphism refining this cover. Let $U_d = \phi^{-1}(d)$ for $d \in D_0$. Note that each U_d is clopen and that $\rho(U_{d_1}) \cap \rho(U_{d_2}) \neq \varnothing$ if and only if $R^{D_0}(d_1, d_2)$. Thus, by slightly enlarging each element of $\{\rho(U_d) : d \in D_0\}$, we get a chain on D/R^D refining $\{B_0, \ldots, B_n\}$, so the d-diameter of each of its elements is less than ε. Hence D/R^D is chainable. □

Proof of Theorem 4.2. The following notions will turn out to be useful in the proof. Let D be a topological L_0-structure. A set $A \subseteq D$ is called *R-invariant* if for any $x \in A$ and $y \in D$, if $R^D(x, y)$ or $R^D(y, x)$, then $y \in A$. It is called *R-connected* if it is not a disjoint union of two non-empty, relatively closed sets A_1 and A_2 such that for all $x \in A_1$ and $y \in A_2$ neither $R^D(x, y)$ nor $R^D(y, x)$.

Note that by Lemma 2.2 each open covering of $\mathbb{P}$ is refined by an epimorphism onto an element of Δ_0. Thus, by Lemma 4.3, $\mathbb{P}/R^{\mathbb{P}}$ is a chainable continuum. It remains to check that it is hereditarily indecomposable.

Let $\rho : \mathbb{P} \to \mathbb{P}/R^{\mathbb{P}}$ denote the quotient map. Let $X \subseteq \mathbb{P}/R^{\mathbb{P}}$ be a subcontinuum and suppose $X = X_1 \cup X_2$ is the union of two proper subcontinua. Let $F = \rho^{-1}(X)$, $F_1 = \rho^{-1}(X_1)$, and $F_2 = \rho^{-1}(X_2)$. Note that F_1 and F_2 are R-connected and R-invariant. Furthermore, since $X_1 \cap X_2 \neq \varnothing$, F_1 and F_2 are not disjoint. It will suffice to show that either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$, since this will imply that either $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$. Since X, X_1, X_2 are arbitrary, it will follow that $\mathbb{P}/R^{\mathbb{P}}$ is hereditarily indecomposable.

Suppose that $F_1 \not\subseteq F_2$ and $F_2 \not\subseteq F_1$; then there is an $x_1 \in F_1 \setminus F_2$ and an $x_2 \in F_2 \setminus F_1$. Since F_1 and F_2 are R-invariant, $\neg R^{\mathbb{P}}(x_1, y_2)$ for any $y_2 \in F_2$ and $\neg R^{\mathbb{P}}(x_2, y_1)$ for any $y_1 \in F_1$. Since $R^{\mathbb{P}}$ is closed and F_2 is compact, we can find clopen sets U_1 and U_2 such that $x_1 \in U_1$, $F_2 \subseteq U_2$ and $\neg R^{\mathbb{P}}(y_1, y_2)$ for any $y_1 \in U_1$ and $y_2 \in U_2$. Similarly we can find clopen sets V_1 and V_2 such that $F_1 \subseteq V_1$, $x_2 \in V_2$ and $\neg R^{\mathbb{P}}(y_1, y_2)$ for any $y_1 \in V_1$ and $y_2 \in V_2$. Let $\chi : \mathbb{P} \to D$, for some $D \in \Delta_0$, be an epimorphism refining the partition by the atoms of the algebra of sets generated by U_1, U_2, V_1, V_2.

Note now that since F is R-connected, its image $\chi(F)$ is R-connected in D and similarly for F_1 and F_2. Also note that since F_1 and F_2 are not disjoint, we have $\chi(F_1) \cap \chi(F_2) \neq \varnothing$. Obviously, we also have $\chi(F) = \chi(F_1) \cup \chi(F_2)$. Since $\chi^{-1}(\chi(x_1)) \subseteq U_1$ and $\chi^{-1}(\chi(F_2)) \subseteq U_2$ and since χ is an epimorphism, we see that $\chi(x_1) \notin \chi(F_2)$. (In fact, $\chi(x_1)$ is not even a neighbor of an element of $\chi(F_2)$.) Thus, $\chi(F_1) \setminus \chi(F_2) \neq \varnothing$. Similarly we obtain $\chi(F_2) \setminus \chi(F_1) \neq \varnothing$. By composing χ with an epimorphism from D to an element of Δ_0, we can assume that $\chi(F) = \{a_0, a_1, a_2\}$, where $R^D(a_i, a_j) \Leftrightarrow |i - j| \leq 1$, and that $\chi(F_1) = \{a_0, a_1\}$ and $\chi(F_2) = \{a_1, a_2\}$.

Let $B \in \Delta_0$ and $\phi : B \to D$ be chosen so that ϕ is an epimorphism, $\phi^{-1}(\chi(F)) = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6\}$, where $R^B(b_i, b_j) \Leftrightarrow |i - j| \leq 1$, and

$$\begin{aligned} \phi(b_0) = \phi(b_4) &= a_0, \\ \phi(b_1) = \phi(b_3) = \phi(b_5) &= a_1, \\ \phi(b_2) = \phi(b_6) &= a_2. \end{aligned}$$

By Lemma 2.3, there is an epimorphism $\psi : \mathbb{P} \to B$ such that $\phi \circ \psi = \chi$. We now have that $\psi(F_1) \supseteq \{b_0, b_4\}$, but this contradicts the fact that $\psi(F_1)$ is R-connected and that $a_2 \notin \chi(F_1)$. □

4.2. **Applications.** In this subsection P denotes the pseudo-arc and $\mathbb{P}$ denotes the projective Fraïssé limit of the family Δ_0.

This subsection contains applications of our results to the theory of the pseudo-arc. First, in Theorem 4.4(i), we obtain from Theorem 4.2 and Proposition 2.6 a result of Mioduszewski [9] that each chainable continuum is a continuous image of the pseudo-arc. (Mioduszewski notes in [9] that his result seems to be derivable from an earlier theorem of Bing [1].) We then, in Theorem 4.4(ii), give a generalization of a result of Lewis [7] and Smith [13] who proved that homeomorphisms of the pseudo-arc are dense in the space of all continuous surjections from the pseudo-arc to itself with the uniform convergence topology. (To see that this is a special case of (ii), note that given a continuous surjection $f : P \to P$, we can apply Theorem 4.4(ii) to $X = P$, $f_1 = f$, $f_2 =$ identity to get a homeomorphism ε close to f.) These results are obtained as direct consequences of the general properties of projective Fraïssé limits. Moreover, following this line of thought it becomes clear that Theorems 4.2 and 2.4 suggest a new characterization of the pseudo-arc. We indeed establish such a characterization in Theorem 4.9.

Theorem 4.4. (i) (Mioduszewski) *Each chainable continuum is a continuous image of the pseudo-arc.*
(ii) *Let X be a chainable continuum with a metric d on it. If f_1, f_2 are continuous surjections from the pseudo-arc onto X, then for any $\varepsilon > 0$ there exists a homeomorphism h of the pseudo-arc such that $d(f_1(x), f_2 \circ h(x)) < \varepsilon$ for all x.*

Note that (L1) corresponds to (i) in the theorem above, (L2) corresponds to chainability of the pseudo-arc (see Lemma 4.3), and (L3) corresponds to (ii).

Lemma 4.3 suggests the following definition. A topological L_0-structure D is called *special* if it fulfills the following conditions:

(α) each open cover of D is refined by an epimorphism onto an element of Δ_0;
(β) R^D is an equivalence relation with each equivalence class having not more than two elements.

The following definition and lemma will allow us to transfer results about $\mathbb{P}$ to results about P. Let D_1 and D_2 be special topological L_0-structures. Let $\rho_i : D_i \to D_i/R^{D_i}$, $i = 1, 2$, denote the quotient functions. An epimorphism $\phi : D_1 \to D_2$ induces a function $\phi^* : D_1/R^{D_1} \to D_2/R^{D_2}$ by $\phi^*(\rho_1(x)) = \rho_2(\phi(x))$. The fact that ϕ is an epimorphism implies that ϕ^* is well defined. We now have a lemma whose proof requires only checking definitions, so we leave the proof to the reader.

Lemma 4.5. *Let D_1 and D_2 be special. Let $\phi : D_1 \to D_2$ be an epimorphism. Let $\rho_i : D_i \to D_i/R^{D_i}$, $i = 1, 2$, denote the quotient functions. Then the following hold:*

(i) *ϕ^* is a continuous surjection from D_1/R^{D_1} to D_2/R^{D_2} and $\rho_2 \circ \phi = \phi^* \circ \rho_1$;*
(ii) *if ϕ is an isomorphism, then ϕ^* is a homeomorphism.*

We will now review some elementary facts about chainable compact metric spaces. For a metric space (X, d) and non-empty sets $A, B \subseteq X$ we write

$$\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$$

and

$$\operatorname{diam}(A) = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}.$$

Let (X, d) be a chainable compact metric space. Let $\delta > 0$. A chain $U_1, \dots, U_N$ on X is called a *δ-fine chain* if

(C1) $\operatorname{dist}(U_i, U_j) > \delta$ if $|i - j| > 1$;
(C2) $\operatorname{dist}(\{x\}, \bigcup_{j \neq i} U_j) > \delta$ for some $x \in U_i$;
(C3) for each $A \subseteq X$ with $\operatorname{diam}(A) < \delta$ there exists $i \leq N$ with $A \subseteq U_i$.

A chain is *fine* if it is δ-fine for some $\delta > 0$. A chain $U_1, \dots, U_N$ *closure refines* a covering $\mathcal{V}$ if the closure of each U_i is included in some element of $\mathcal{V}$. The following fact is easy to check.

Lemma 4.6. *If (X, d) is a chainable continuum, then each open cover of X is closure refined by a fine chain.*

The following lemma gives a converse to Lemma 4.3.

Lemma 4.7. *If X is a chainable continuum, then there is a special topological L_0-structure C such that X is homeomorphic to C/R^C.*

Proof. If $\mathcal{C}$ is a chain on X, we will denote the k-th element of $\mathcal{C}$, which we call its k-th link, by $\mathcal{C}(k)$. In particular, $\mathcal{C}(i) \cap \mathcal{C}(j) \neq \varnothing$ holds if and only if $|i - j| \leq 1$. Whenever we write $\mathcal{C}(k)$ we assume that $\mathcal{C}$ has at least k links. If $\mathcal{C}$ is a chain, then the *mesh* of $\mathcal{C}$ is $\operatorname{mesh}(\mathcal{C}) = \max\{\operatorname{diam}(\mathcal{C}(i)) : 1 \leq i \leq N\}$, where N is the number of links in $\mathcal{C}$.

We will construct a sequence of chains $(\mathcal{C}_n)_{n=0}^{\infty}$ on X such that for each $n \in \mathbb{N}$

(1) $\mathcal{C}_{n+1}$ closure refines $\mathcal{C}_n$;
(2) $\operatorname{mesh}(\mathcal{C}_n) < \frac{1}{n}$;
(3) $\mathcal{C}_n$ is fine;
(4) if $\mathcal{C}_{n+1}(i) \subseteq \mathcal{C}_n(k)$, $\mathcal{C}_{n+1}(j) \subseteq \mathcal{C}_n(l)$, and $|k - l| > 1$, then $|i - j| > 2$;
(5) for each k there exists an i such that $\mathcal{C}_{n+1}(i) \subseteq \mathcal{C}_n(k) \setminus \bigcup_{l \neq k} \mathcal{C}_n(l)$.

Let $\mathcal{C}_0$ be a fine chain on X. Now suppose we have $\mathcal{C}_n$ for $n \geq 0$; we will construct $\mathcal{C}_{n+1}$. Let $\delta > 0$ be such that $\mathcal{C}_n$ is δ-fine. Cover X by balls of radius less than $\min(1/2(n+1), \delta/6)$. Let $\mathcal{C}_{n+1}$ be a fine chain closure refining this cover. Observe that for each link $\mathcal{C}_{n+1}(i)$ of $\mathcal{C}_{n+1}$

$$\operatorname{diam}(\overline{\mathcal{C}_{n+1}(i)}) < \delta/3 \tag{4.1}$$

so, by (C3), $\mathcal{C}_{n+1}$ closure refines $\mathcal{C}_n$. Condition (2) is clear. We also observe that by (4.1) and (C1) we have (4). Condition (5) follows from (C2) and (4.1).

For each $n \in \mathbb{N}$ let $C_n = \{1, \dots, N\}$, where $\mathcal{C}_n$ consists of N links, and we take $R^{C_n}(i, j)$ precisely when $|i - j| \leq 1$, so in particular $R^{C_n}(i, j)$ iff $\mathcal{C}_n(i) \cap \mathcal{C}_n(j) \neq \varnothing$. For each $n \in \mathbb{N}$ define $\phi_n : C_{n+1} \to C_n$ via

$$\phi_n(i) = \min\left\{k : \overline{\mathcal{C}_{n+1}(i)} \subseteq \mathcal{C}_n(k)\right\}.$$

The function ϕ_n is well defined by (1). By (5), ϕ_n is onto, and it is an epimorphism by (4).

Let $C = \varprojlim(C_n, \phi_n)$. We take each C_n with the discrete topology and C with the inverse limit topology. If $x \in C$, we will denote the natural projection of x onto C_n by $x(n)$, so in particular one can write $x = (x(0), x(1), x(2), \dots)$. Define R^C by letting $R^C(x, y)$ if and only if $R^{C_n}(x(n), y(n))$ for all n.

Claim 1. C is a special topological L_0-structure.

Proof of Claim 1. Clearly C is a topological L_0-structure. We check that R^C is an equivalence relation with each equivalence class having not more than two elements. It is clear that R^C is reflexive and symmetric. Suppose $x, y, z \in C$ are such that $R^C(x, y)$ and $R^C(y, z)$ with $x \neq z$. First note that by (4), for every $n \in \mathbb{N}$, $\phi_n^{-1}(y(n))$ contains $y(n+1)$ along with an R-neighbor of this element distinct from it. Since $R^{C_{n+1}}(x(n+1), y(n+1))$ and $R^{C_{n+1}}(y(n+1), z(n+1))$, unless $x(n+1) = z(n+1)$ we have $z(n+1) \in \phi_n^{-1}(y(n))$ or $x(n+1) \in \phi_n^{-1}(y(n))$. Therefore, we have $x(n) = y(n)$ or $z(n) = y(n)$ or $x(n) = z(n)$ for each n.

Now since $x \neq z$, for all but finitely many n, $x(n) \neq z(n)$. It must then be that for all but finitely many n $x(n) = y(n)$ or $z(n) = y(n)$. Thus, one of these possibilities is realized for all n, whence either $x = y$ or $z = y$. This proves the claim. $\square_{\text{claim}}$

Claim 2. X is homeomorphic to C/R^C.

Proof of Claim 2. Define $f : C \to X$ by letting $f(x)$ be the unique, by (1) and (2), element in $\bigcap_n \overline{\mathcal{C}_n(x(n))}$. It is a routine check that f is continuous. To see that it is surjective fix $y \in X$. Consider the set T_y of all sequences $(m_0, \dots, m_n) \in \prod_{j \le n} C_j$ such that $y \in \mathcal{C}_n(m_n)$ and $\phi_j(m_{j+1}) = m_j$ for $j < n$, where n ranges over $\mathbb{N}$. Equip T_y with the partial order of extension. Then T_y is a tree. It is obviously finitely branching. As is easily seen, for each n there exists an element of T_y of length n; thus, T_y is infinite. It follows now from König's lemma that there exists $x = (m_0, m_1, \dots)$ such that for each n we have $(m_0, \dots, m_n) \in T_y$. Clearly then $x \in C$ and $f(x) = y$. Thus, f is surjective.

We now check that

$$f(x_1) = f(x_2) \Leftrightarrow R^C(x_1, x_2). \tag{4.2}$$

The implication $\Leftarrow$ follows immediately from (2). To see $\Rightarrow$, note that $f(x_1) = f(x_2)$ is in $\bigcap_n \overline{\mathcal{C}_n(x_i(n))} = \bigcap_n \mathcal{C}_n(x_i(n))$ for $i = 1, 2$. Thus, for each n, $\mathcal{C}_n(x_1(n))$ and $\mathcal{C}_n(x_2(n))$ have a point in common. It follows that for each n, $R^{C_n}(x_1(n), x_2(n))$, and we are done.

Now (4.2) allows us to define $\overline{f} : C/R^C \to X$ by letting $\overline{f}(\rho(x)) = f(x)$ for $x \in C$. By [4, Proposition 2.4.2], $\overline{f}$ is a homeomorphism.

This completes the proof of the claim and hence the lemma. $\square$

Lemma 4.8. *Identify P with $\mathbb{P}/R^{\mathbb{P}}$. Let X be a chainable continuum with a metric d on it. Let f_1 and f_2 be continuous surjections from P to X. Then, for any $\varepsilon > 0$, there is an isomorphism $\phi : \mathbb{P} \to \mathbb{P}$ such that $d(f_1(x), f_2(\phi^*(x))) < \varepsilon$ for any $x \in P$.*

Proof. Let $\rho : \mathbb{P} \to P$ be the quotient map.

By Lemma 4.6, we can find a δ-fine chain $U_1, \dots, U_N$ for some $\delta > 0$ which refines the covering of X by balls of radius less than $\varepsilon/2$. Thus, we have

$$\operatorname{diam}(U_i) < \varepsilon \quad \text{for } i \le N. \tag{4.3}$$

By Lemma 2.2 and the fact that $f_1 \circ \rho$ and $f_2 \circ \rho$ are uniformly continuous on $\mathbb{P}$, there are $E_i \in \Delta_0$, $i = 1, 2$, and epimorphisms $\phi_i : \mathbb{P} \to E_i$ such that

$$\operatorname{diam}(f_i \circ \rho(\phi_i^{-1}(e))) < \delta \quad \text{for } e \in E_i. \tag{4.4}$$

Let $D = \{1, \ldots, N\}$ with $R^D(i,j)$ precisely when $|i-j| \leq 1$. Then D with R^D is an element of Δ_0. Now define $\psi_i : E_i \to D$, $i = 1, 2$, by letting

$$\psi_i(e) = \min\{k \in \{1, \ldots, N\} : f_i \circ \rho(\phi_i^{-1}(e)) \subseteq U_k\}.$$

By (4.4) and (C3), ψ_i is well-defined. By (4.4), the fact that f_i is a surjection, and (C2), ψ_i is a surjection. To see that ψ_i is an epimorphism, it suffices to show that if $e_1, e_2 \in E_i$ and $R^{E_i}(e_1, e_2)$, then $R^D(\psi_i(e_1), \psi_i(e_2))$, that is, $|\psi_i(e_1) - \psi_i(e_2)| \leq 1$. If $R^{E_i}(e_1, e_2)$, then, since ϕ_i is an epimorphism, there are $x_1, x_2 \in \mathbb{P}$ with $\phi_i(x_1) = e_1$, $\phi_i(x_2) = e_2$, and $R^{\mathbb{P}}(x_1, x_2)$. This last condition gives $\rho(x_1) = \rho(x_2)$ from which it follows that

$$f_i \circ \rho(\phi_i^{-1}(e_1)) \cap f_i \circ \rho(\phi_i^{-1}(e_2)) \neq \varnothing.$$

Thus, by (4.4) and (C1), $|\psi_i(e_1) - \psi_i(e_2)| \leq 1$.

Now by (L3), there exists an isomorphism $\phi : \mathbb{P} \to \mathbb{P}$ such that for all $x \in \mathbb{P}$ we have

$$\psi_1 \circ \phi_1(x) = \psi_2 \circ \phi_2 \circ \phi(x).$$

This means that for each $x \in \mathbb{P}$ there exists $i \leq N$ such that $f_1 \circ \rho(x), f_2 \circ \rho(\phi(x)) \in U_i$ which combined with (4.3) and Lemma 4.5(i) gives the conclusion of the lemma. □

Proof of Theorem 4.4. (i) Let X be a chainable continuum and, by Lemma 4.7, let C be a special topological L_0-structure such that C/R^C is homeomorphic to X. Then by Proposition 2.6 there is an epimorphism $\phi : \mathbb{P} \to C$. Now, by Theorem 4.2 and Lemma 4.5(i), ϕ^* is a continuous surjection from the pseudo-arc onto X.

(ii) is an immediate consequence of Lemma 4.8 and Lemma 4.5(ii). □

As an analogue of Theorem 2.4 we have the following result. It gives an apparently new characterization of the pseudo-arc.

Theorem 4.9. *The pseudo-arc is the unique non-degenerate chainable continuum X such that for any chainable continuum Y with a metric d, any continuous surjections f_1, f_2 from X onto Y, and any $\varepsilon > 0$ there exists a homeomorphism $h : X \to X$ such that $d(f_1(x), f_2 \circ h(x)) < \varepsilon$ for all $x \in X$.*

If X and Y are compact metric spaces with metrics d_X and d_Y respectively, then a continuous map $f : X \to Y$ is called a δ-*map* if $\operatorname{diam}(f^{-1}(f(x))) < \delta$ for each $x \in X$. A basic fact about δ-maps is the following: if $f : X \to Y$ is a δ-map, then there is a $\zeta > 0$ such that $\operatorname{diam}(f^{-1}(A)) < \delta$ whenever $\operatorname{diam}(A) < \zeta$ for any $A \subseteq Y$, so in particular if $d_X(x_0, x_1) \geq \delta$, then $d_Y(f(x_0), f(x_1)) \geq \zeta$. It is well known that a non-degenerate continuum (X, d_X) is chainable if and only if for every $\delta > 0$ there is a δ-map from X onto the closed unit interval; see [11, Theorem 12.11].

Below $|\cdot|$ stands for the absolute value on the reals.

Lemma 4.10 ([11, Lemma 12.17])**.** *If X is a compact metric space, $f : X \to [0,1]$ is a continuous surjection, and $\varepsilon > 0$, then there exists a $\delta = \delta(f, \varepsilon)$ so that if $g : X \to [0,1]$ is any onto δ-map, then there is a continuous surjection*[1] $\phi : [0,1] \to [0,1]$ *so that $|\phi \circ g(x) - f(x)| < \varepsilon$ for every $x \in X$.*

[1]This lemma in [11] explicitly states that ϕ need not be onto, but an analysis of the proof shows that if f is onto, then ϕ is onto.

Proof of Theorem 4.9. The pseudo-arc fulfills the conclusion of the theorem by Theorem 4.4(ii).

Let (X, d_X) and (Y, d_Y) be chainable continua satisfying the condition from the statement of Theorem 4.9. For two functions f and g defined on the same set Z with values in $[0,1]$, we write

$$\|f - g\| = \sup_{z\in Z} |f(z) - g(z)|.$$

Below I_n is the unit interval for $n \in \mathbb{N}$. We will construct continuous surjections $\phi_n : I_{n+1} \to I_n$, continuous surjections $f_n : X \to I_n$ and $g_n : Y \to I_n$, and $\varepsilon_n > 0$ such that for $n \geq 0$

$(a)_n$ $\varepsilon_n < \frac{1}{n+1}$;
$(b)_n$ $\|\phi_{k,n-1} \circ f_n - \phi_{k,m-1} \circ f_m\| < \varepsilon_m$ for all $k \leq m \leq n$;
$(c)_n$ $\|\phi_{k,n-1} \circ g_n - \phi_{k,m-1} \circ g_m\| < \varepsilon_m$ for all $k \leq m \leq n$;
$(d)_n$ if n is even and if $d_X(x, y) \geq \frac{1}{n+1}$, then $|f_n(x) - f_n(y)| > 2\varepsilon_n$, for $x, y \in X$;
$(e)_n$ if n is odd and if $d_Y(x, y) \geq \frac{1}{n+1}$, then $|g_n(x) - g_n(y)| > 2\varepsilon_n$, for $x, y \in Y$,

where we adopt the following notation: $\phi_{i,j} = \phi_i \circ \cdots \circ \phi_j : I_{j+1} \to I_i$, for $i \leq j$, with the convention that $\phi_{i,i-1}$ is the identity map on I_i.

Let $f_0 : X \to I_0$ be a continuous surjection and let ε_0 be any positive real less than 1 such that for any $x, y \in X$ with $d_X(x, y) \geq 1$, $|f_0(x) - f_0(y)| > 2\varepsilon_0$. Clearly such an ε_0 can be found and, obviously, $(a)_0$–$(e)_0$ hold ($(c)_0$ and $(e)_0$ hold vacuously no matter how g_0 is chosen).

Assume now that we have chosen ε_i and f_i for $i \leq 2n$ and ϕ_i and g_i for $i < 2n$ so that they fulfill $(a)_i$, $(b)_i$ and $(d)_i$ for $i \leq 2n$ and $(c)_i$ and $(e)_i$ for $i < 2n$. We show now how to find g_{2n}, ϕ_{2n}, g_{2n+1}, and ε_{2n+1}. (This is a half of the inductive step. The other half producing f_{2n+1}, ϕ_{2n+1}, f_{2n+2} and ε_{2n+2} is essentially identical.) Since Y satisfies the condition from the statement of Theorem 4.9, we have the following claim which is seen to hold by considering the surjections $f_1 = g : Y \to [0,1]$ and $f_2 = \phi \circ g : Y \to [0,1]$.

Claim. For any $\varepsilon > 0$, given any continuous surjections $\phi : [0,1] \to [0,1]$ and $g : Y \to [0,1]$, there is a continuous surjection $g' : Y \to [0,1]$ such that $\|\phi \circ g' - g\| < \varepsilon$.

Now note that by our inductive assumption $(c)_{2n-1}$, we can find $\varepsilon > 0$ such that for all $k \leq m \leq 2n - 1$ we have

$$\|\phi_{k,2n-2} \circ g_{2n-1} - \phi_{k,m-1} \circ g_m\| + \varepsilon < \varepsilon_m. \tag{4.5}$$

By the claim, we can choose $g_{2n} : Y \to I_{2n}$ so that $\|\phi_{2n-1} \circ g_{2n} - g_{2n-1}\|$ is as small as we wish, and we wish it to be small enough to fulfill

$$\begin{aligned}\|\phi_{k,2n-1} \circ g_{2n} &- \phi_{k,2n-2} \circ g_{2n-1}\| \\ &= \|\phi_{k,2n-2} \circ (\phi_{2n-1} \circ g_{2n}) - \phi_{k,2n-2} \circ g_{2n-1}\| < \varepsilon\end{aligned} \tag{4.6}$$

for all $k \leq 2n - 1$. This is possible by uniform continuity of $\phi_{k,2n-2}$. But then (4.5), (4.6), and the triangle inequality give

$$\|\phi_{k,2n-1} \circ g_{2n} - \phi_{k,m-1} \circ g_m\| < \varepsilon_m$$

for all $k \leq m \leq 2n-1$. Since this inequality obviously also holds for $m = 2n$, we see that g_{2n} fulfills $(c)_{2n}$. We now define g_{2n+1}, ϕ_{2n}, and ε_{2n+1}. Using the inductive assumption $(c)_{2n}$, we can find $\varepsilon > 0$ such that for all $k \leq m \leq 2n$,

$$\|\phi_{k,2n-1} \circ g_{2n} - \phi_{k,m-1} \circ g_m\| + \varepsilon < \varepsilon_m. \tag{4.7}$$

We will find g_{2n+1} and ϕ_{2n} so that for all $k \leq 2n$

$$\begin{aligned}(4.8) \qquad &\|\phi_{k,2n} \circ g_{2n+1} - \phi_{k,2n-1} \circ g_{2n}\| \\ &\qquad = \|\phi_{k,2n-1} \circ (\phi_{2n} \circ g_{2n+1}) - \phi_{k,2n-1} \circ g_{2n}\| < \varepsilon.\end{aligned}$$

Since Y is chainable, we can pick $g_{2n+1} : Y \to I_{2n+1}$ to be a δ-map with $\delta < 1/(2n+2)$ and small enough so that, by Lemma 4.10, we can choose a continuous surjection $\phi_{2n} : I_{2n+1} \to I_{2n}$ making the quantity $\|\phi_{2n} \circ g_{2n+1} - g_{2n}\|$ as small as we wish; in particular, small enough for (4.8) to hold. Now by the triangle inequality, using (4.7) and (4.8), we get that for all $k \leq m \leq 2n$,

$$\|\phi_{k,2n} \circ g_{2n+1} - \phi_{k,m-1} \circ g_m\| < \varepsilon_m.$$

Since as long as we choose ε_{2n+1} positive this inequality also holds for $m = 2n+1$, we see that g_{2n+1} and ϕ_{2n} fulfill $(c)_{2n+1}$. Since g_{2n+1} is a δ-map for some $\delta < 1/(2n+2)$, there exists $\varepsilon_{2n+1} > 0$ for which $(e)_{2n+1}$ and $(a)_{2n+1}$ hold. Thus, the construction is completed.

We will now show that X and Y are homeomorphic by showing that they are both homeomorphic to $Z = \varprojlim(I_n, \phi_n)$. We prove this for X. Note that by $(a)_n$ and $(b)_n$ the sequence of functions $(\phi_{k,n-1} \circ f_n)_{n \geq k}$ converges uniformly to a continuous function $\bar{f}_k : X \to I_k$. Moreover, by $(b)_n$, for all k and all $m \geq k$,

$$(4.9) \qquad \|\bar{f}_k - \phi_{k,m-1} \circ f_m\| \leq \varepsilon_m.$$

From the identity $\phi_k \circ \phi_{k+1,n-1} = \phi_{k,n-1}$ for $n \geq k+1$ and from continuity of the functions ϕ_k, we get that the image of the function $F : X \to \prod_n I_n$ given by $F(x) = (\bar{f}_0(x), \bar{f}_1(x), \dots)$ is included in Z. Since each $\bar{f}_k$ is continuous, F is continuous as well. To check that F is onto it suffices to see that the range of F is dense in Z, which will follow if we show that for each k the range of $\bar{f}_k$ is dense in I_k. But this is immediate from surjectivity of $\phi_{k,m-1} f_m$, (4.9), and $(a)_m$. Thus, since X is compact, it remains to show that F is injective. Fix $x, y \in X$. Find k_0 with $d_X(x,y) \geq 1/(2k_0+1)$. From $(d)_{2k_0}$ we have that $|f_{2k_0}(x) - f_{2k_0}(y)| > 2\varepsilon_{2k_0}$. It follows now from (4.9) for $k = m = 2k_0$ that $\bar{f}_{2k_0}(x) \neq \bar{f}_{2k_0}(y)$, whence $F(x) \neq F(y)$. □

4.3. **Remarks. 1.** We would like to point out certain similarities between the theorems proved in this section and the theory developed by Bing and Moise. One can view this latter theory as follows. A chainable, hereditarily indecomposable continuum P_1 is constructed. Two results are proved about it: it is homogeneous [1], [10], i.e., for any two points $x, y \in P_1$ there is a homeomorphism of P_1 mapping x to y (and even more homogeneous by [6]), and it is approximately universal among chainable continua [2], i.e., each chainable continuum can be approximated by P_1 in the Hausdorff metric (which, when combined with the obvious observation that any continuum can be approximated by a chainable one, gives that each continuum can be approximated by P_1). Furthermore, the homogeneity property characterizes P_1 among chainable continua [3].

The results in this section can be seen as dual to the above theory, much as the projective Fraïssé limit is dual to the Fraïssé limit. A chainable, hereditarily indecomposable continuum P_2 is constructed (Theorem 4.2). Two results are proved. First, P_2 is approximately projectively homogeneous (Theorem 4.4(ii)) and, second, it is projectively universal among chainable continua (Theorem 4.4(i)). Moreover, the approximate projective homogeneity of P_2 characterizes it among chainable continua (Theorem 4.9).

The link between the Bing–Moise theory and its dual is provided by the old theorem of Bing [2] saying that up to a homeomorphism there is at most one chainable, hereditarily indecomposable continuum. Thus, P_1 and P_2 are homeomorphic.

2. It seems very likely that, using methods similar to the ones developed in the present paper, one will be able to prove existence, surjective universality, surjective homogeneity, and uniqueness of a universal pseudo-solenoid. This would extend the work of Rogers [12]. In this context, it will be important to modify the notion of epimorphism and appropriately change the definitions of surjective universality and surjective homogeneity.

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