Tôhoku Math. J.
49 (1997), 299–321

PROJECTIVE MANIFOLDS SWEPT OUT BY LARGE DIMENSIONAL LINEAR SPACES

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(Received August 28, 1995, revised February 10, 1997)

Abstract. The author studies the deformation of a flag variety and investigates the structure of a smooth closed subvariety in a projective space which is swept out by large dimensional linear spaces. Then a sufficient condition is given for the subvariety to be isomorphic to one of a projective bundle, a hyperquadric and a Grassman variety.

In this paper we investigate some deformations of $\boldsymbol{P}(\Omega_{\boldsymbol{P}^m}(2))$ and determine varieties swept out by large dimensional linear spaces.

We get the following:

MAIN THEOREM. *Let X be an $n(\geq 2)$-dimensional smooth projective variety in a projective space $\boldsymbol{P}^N$. Assume that for each point x in X, there exists an m-dimensional linear subspace P_x in X containing the point x with $2m \geq n$. (From now on an m-dimensional linear subspace is abbreviated as an m-plane.) Moreover, suppose that the characteristic of the base field is zero. Then for a general point x the normal bundle $N_{P_x/X}$ of P_x in X is isomorphic to one of the following:*

(1) $\mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$ *(a and b are non-negative integers)*

(2) $\Omega_{\boldsymbol{P}^m}(2)$ $(m \geq 2)$,

(3) $T_{\boldsymbol{P}^m}(-1)$ $(m \geq 2)$.

Moreover, corresponding to the above cases, X is respectively as follows:

(1) *a $\boldsymbol{P}^{n-a}$-bundle over an a-dimensional smooth projective variety S where a general m-plane P_x is in the fiber of the canonical projection $(n \geq a \geq n/2)$.*

(2) *An even-dimensional smooth hyperquadric.*

(3) *The Grassmann variety* Gr$(m+1, 1)$ *parametrized by lines in $\boldsymbol{P}^{m+1}$ with $n = 2m$, if m is even.*

The main theorem is shown in §6. Combined with Theorem 3.1, it yields:

COROLLARY. *Let X be an $n(\geq 2)$-dimensional smooth projective variety in a projective space $\boldsymbol{P}^N$. Assume that there exists an m-dimensional linear space P in X such that the normal bundle $N_{P/X}$ of P in X is isomorphic to one of the following:*

(1) $\mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$ *(a and b are non-negative integers)*

1991 *Mathematics Subject Classification*. Primary 14J40.

Partly supported by the Grants-in-Aids for Scientific as well as Co-operative Research, The Ministry of Education, Science and Culture, Japan.

(2) $\Omega_{\boldsymbol{P}^m}(2)$ $(m \geq 2)$,

(3) $T_{\boldsymbol{P}^m}(-1)$ $(m \geq 2)$.

Moreover assume $n \leq 2m$. *Then in characteristic zero,* X *in the above cases corresponds, respectively, to the one in the latter conclusion of the main theorem, where in the case* (3) *the assumption that* m *is even is added.*

What is most important and difficult in treating the main theorem is to determine the structure of the normal bundle $N_{P_x/X}$ and to show that in one case X in question is a Grassmann variety. For the former we have to classify a vector bundle on $\boldsymbol{P}^n$ which is 'uniform at a point' in §1. For the latter, which is most central in this paper we have to show in §4 and §5 that the structure of $\boldsymbol{P}(\Omega_{\boldsymbol{P}^m})$ is deformation-invariant under suitable conditions as follows:

THEOREM 1. *Let* E *be a vector bundle of rank* m *on* $\boldsymbol{P}^n$ *with* $n \geq 2$. *Assume that there is a point* A *on* $\boldsymbol{P}^n$ *so that for all the lines* l *through the point* A, $E_{|l}$ *is* $\mathcal{O}^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b}$ $(a+b=m)$. *Moreover assume that when* $m=n$, *the* n*-th Chern class* $c_n(E)=0$ *or* 1 *modulo rational equivalence.*

Then E *is* $\mathcal{O}_{\boldsymbol{P}^n}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus b}$ $(a+b=m)$ *if* $m>n$, *while if* $m=n$, *then* E *is isomorphic to one of* $\mathcal{O}_{\boldsymbol{P}^n}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus b}$ $(a+b=m)$, $T_{\boldsymbol{P}^n}(-1)$, *and* $\Omega_{\boldsymbol{P}^n}(2)$.

This theorem is used not only to classify the normal bundle in the main theorem but also to show that some deformation of $\boldsymbol{P}(\Omega_{\boldsymbol{P}^m})$ has the structure of $\boldsymbol{P}(\Omega_{\boldsymbol{P}^m})$ as follows:

THEOREM 4.2. *Let* V *be a vector bundle of rank* $m+1$ *on a scheme* T *and* f: $\boldsymbol{P}(V) \to T$ *a* $\boldsymbol{P}^m$*-bundle on* T. *Set* $W := \boldsymbol{P}(V)$. *Let* $\mathscr{E}$ *be a vector bundle of rank* m *on* W *and* $g : \boldsymbol{P}(\mathscr{E}) \to W$ *a* $\boldsymbol{P}^m$*-bundle on* W. *Set* $Q := \boldsymbol{P}(\mathscr{E})$ *and* $E_t := \mathscr{E}_{|f^{-1}(t)}$. *Assume that* E_t *is generated by global sections for each point* t *in* T *and* $E_t \simeq \Omega_{\boldsymbol{P}^m}(2)$ *for a generic point* t *in* T. *If* m *is even, then* $E_t \simeq \Omega_{\boldsymbol{P}^m}(2)$ *for each point* t *in* T.

Thus Theorem 4.2 enables us to prove in §6 that if an $2m$-dimensional smooth projective variety X in $\boldsymbol{P}^N$ contains an m-plane P with $N_{P/X} \simeq T_{\boldsymbol{P}^m}(-1)$, then X is isomorphic to $\mathrm{Gr}(m+1, 1)$.

Thanks to the developement of the contraction theory and the adjunction mapping theory, the structures of large dimensional Fano varieties continue to be studied actively under some conditions described in terms of coindex, length and so on. There seem to exist few results about large dimensional Fano manifolds with the Picard group $\boldsymbol{Z}$, even about Grassmann varieties, except projective spaces, hyperquadrics, Del Pezzo manifolds and Mukai manifolds, whose coindex are very small (cf. [CS], [Mo], [Mu]). Moreover the condition like the semi-ampleness of the tangent bundle does not give enough imformation to determine the structure so far (though understood well in dimensions 3 and 4. cf. [CP1], [CP2]). This is the reason why the author treats the varieties as in the main theorem. More precisely, we consider an n-dimensional projective variety X in $\boldsymbol{P}^N$ satisfying the following condition:

(♯) X is swept out by m-planes P with $2m \geq n$.
As for such a variety Ein [E, Theorem 1.7] and Wiśniewski [W, Theorem 1.4] already showed that an n-dimensional smooth projective variety X containing an m-plane with a trivial normal bundle has a projective bundle structure. (See also Remark 3.4.) We remark that the condition "$N_{P/X}$ is a trivial vector bundle" is a sufficient condition for (♯).

This paper is organized as follows:

§1. Uniform vector bundles on $\boldsymbol{P}^n$ at a point.

§2. The normal bundle $N_{P_y/X}$.

§3. Smooth projective varieties with $N_{P/X} \simeq \mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$, $\Omega_{\boldsymbol{P}^m}(2)$.

§§4 and 5. The deformation of $\boldsymbol{P}(\Omega_{\boldsymbol{P}^m}(2))$ under some conditions, (I) (II).

§6. Smooth projective varieties with $N_{P/X} \simeq T_{\boldsymbol{P}^m}(-1)$.

Conventions and Notation. We work over an algebraically closed field k of any characteristic in general. In §§4, 5 and 6 it is supposed that the characteristic of the base field is zero. We freely use the customary terminology in algebraic geometry. For simplicity $\mathcal{O}(a)$ means the line bundle $\mathcal{O}_{\boldsymbol{P}^1}(1)^{\otimes a}$ on $\boldsymbol{P}^1$. For a vector bundle E on a scheme S, $\check{E}$ denotes the vector bundle dual to E. In the N-dimensional projective space $\boldsymbol{P}^N$ an m-plane means an m-dimensional linear subspace in $\boldsymbol{P}^N$. $\mathrm{Gr}(N, m)$ denotes the Grassmann variety parameterizing m-planes in $\boldsymbol{P}^N$. $E(N, m)$ or $U(N, m)$ are the rank $(m+1)$ universal subbundle of $\mathcal{O}_{\mathrm{Gr}(N,m)}^{\oplus(N+1)}$ or the rank $(N-m)$ universal quotient bundle on $\mathrm{Gr}(N, m)$, respectively. $F(N, m, 0)$ means the incidence subvariety $\{(x, y) \in \boldsymbol{P}^N \times \mathrm{Gr}(N, m) \mid x \in L_y\}$, where L_y is an m-plane corresponding to the point y.

The author would like to express his gratitude to the referee for valuable comments and advice.

1. Uniform vector bundles on $\boldsymbol{P}^n$ at a point. In this section we work over an algebraically closed field k of any characteristic. Let E be a rank m vector bundle on $\boldsymbol{P}^n$ with $m \leq n$. We consider a vector bundle satisfying the following conditions:
(1.0) There is a point A on $\boldsymbol{P}^n$ so that for all the lines l on $\boldsymbol{P}^n$ through the point A, $E_{|l}$ is $\mathcal{O}^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b}$ $(a+b=m)$. Moreover when $m=n$, the n-th Chern class $c_n(E)=0$ or 1 modulo rational equivalence.

Then our aim in this section is to study the structure of such a vector bundle. As a consequence we have the following:

Theorem 1. *Let a vector bundle E be as in* 1.0. *Assume $n \geq 2$. Then E is isomorphic to $\mathcal{O}_{\boldsymbol{P}^n}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus b}$ $(a+b=m)$ if $m<n$. If $m=n$, then E is isomorphic to one of $\mathcal{O}_{\boldsymbol{P}^n}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^n}(1)^{\oplus b}$ $(a+b=m)$, $T_{\boldsymbol{P}^n}(-1)$ and $\Omega_{\boldsymbol{P}^n}(2)$.*

The following result is already known:

Theorem 1.1 (cf. Main Theorem, Remark 1.1 and Remark 2.1 in [Sa]). *Let E be a rank r vector bundle on $\boldsymbol{P}^n$. Assume that there is a point A on $\boldsymbol{P}^n$ so that for all* *the lines l through the point A, $E_{|l}$ is independent of the choice of the lines l and is isomorphic to $\bigoplus_{i=1}^{s} \mathcal{O}(a_i)^{\oplus r_i}$ with $n \geq 2$, $r \geq 2$ and $a_1 > a_2 > \cdots > a_s$. Then E is isomorphic to $\bigoplus_{i=1}^{s} \mathcal{O}_{P^n}(a_i)^{\oplus r_i}$ if either of the following two conditions is satisfied*:

(1) $n > r$.

(2) $n = r$, $r_i \geq 2$ *for* $i = 1, s$ *and* n *is* 2 *or odd*.

REMARK 1.1.1. If $s = 1$, the above conclusion holds without the restriction on n and r as stated in the first part of the main theorem in [Sa].

Now let us consider the condition (1.0).

Let $\varphi : \bar{P} \to P^n$ be the blow-up of P^n at the point A. Then $\bar{P}$ is isomorphic to $\mathrm{Proj}(\mathcal{O}_{P^{n-1}} \oplus \mathcal{O}_{P^{n-1}}(1))$ where $\pi : \bar{P} \to P^{n-1}$ is a P^1-bundle. Let D be the exceptional locus of φ which is isomorphic to P^{n-1}. Then D is a section of π.

By virtue of the base change theorem the condition 1.0 yields a canonical homomorphism $\pi^*\pi_*\varphi^*\check{E} \to \varphi^*\check{E}$ which is injective as vector bundles where $\pi_*\varphi^*\check{E}$ is a rank a vector bundle on P^{n-1}. Let F_1 be the vector bundle dual to $\pi_*\varphi^*\check{E}$. Taking the same procedure for the quotient vector bundle, we get the following exact sequence on $\bar{P}$:

$$0 \longrightarrow \pi^*F_2 \otimes \varphi^*\mathcal{O}_{P^n}(1) \longrightarrow \varphi^*E \longrightarrow \pi^*F_1 \longrightarrow 0\,. \tag{1.2}$$

Here F_2 is a rank b vector bundle on P^{n-1}. Moreover restricting the above sequence to the exceptional locus D, we have an exact sequence on D:

$$0 \longrightarrow F_2 \longrightarrow \mathcal{O}_{P^{n-1}}^{\oplus m} \longrightarrow F_1 \longrightarrow 0\,. \tag{1.3}$$

This yields a morphism $f \colon P^{n-1} \to \mathrm{Gr}(m-1, a-1)$ so that $f^*E(m-1, a-1) \simeq F_2$ and $f^*U(m-1, a-1) \simeq F_1$ where $E(m-1, a-1)$ is the rank b universal subbundle of $\mathcal{O}_{\mathrm{Gr}(m-1,a-1)}^{\oplus m}$ on the Grassmann variety $\mathrm{Gr}(m-1, a-1)$, and $U(m-1, a-1)$ is the universal quotient vector bundle of rank a.

Now we investigate the Chern class of the vector bundle E. First let h be the class of hyperplane of P^{n-1} in the first Chow group of P^{n-1}. Then the i-th Chow group $\mathrm{CH}^i(P^{n-1})$ of P^{n-1} is equal to $\boldsymbol{Z}h^i$ where $h^{n-1} = 1$ and $h^p = 0$ for $p > n-1$. Thus the Chern polynomial $c_1(F_1)$ of F_1 is equal to $c_0 + c_1ht + \cdots c_a h^a t^a$ and $c(F_2) = d_0 + d_1 ht + \cdots d_b h^b t^b$ where $c_0 = d_0 = 1$. Since F_1 and $\check{F}_2$ are generated by global sections from 1.3 we have:

(1.4) each Chern class of F_1 and $\check{F}_2$ is numerically positive, namely, $c_i \geq 0$ and $(-1)^j d_j \geq 0$. Moreover $f^*\mathcal{O}_{\mathrm{Gr}(m-1,a-1)}(1) = c_1(F_1) = c_1(\check{F}_2)$. (See, for example, Proposition 2.1 in [Ta].)

Let us begin the proof of Theorem 1. When $n > m$, there is nothing to show by Theorem 1.1.

Next we consider the case $n = m$. We may assume $ab \neq 0$ by Remark 1.1.1. Let ξ be the tautological line bundle of the vector bundle $\mathcal{O}_{P^{n-1}} \oplus \mathcal{O}_{P^{n-1}}(1)$ $(= G)$. Letting $\bar{h} = \pi^*h$, we have the equality $\xi^2 = \xi\bar{h}$ and therefore the equality $\xi^i\bar{h}^{b-i} = \xi\bar{h}^{b-1}$ with $1 \leq i \leq b$. Thus remarking that $\varphi^*\mathcal{O}_{P^n}(1) = \xi$, we see that

$$c_b(\pi^*F_2 \otimes \varphi^*\mathcal{O}_{P^n}(1)) = \sum_{i=0}^{b} d_{b-i}\bar{h}^{b-i}\xi^i = d_b\bar{h}^b + \left(\sum_{i=1}^{b} d_{b-i}\right)\xi\bar{h}^{b-1}.$$

Therefore since $\xi\bar{h}^{n-1}=1$ and $\bar{h}^n=0$, we have

$$c_n(E)=c_a\left(\sum_{i=1}^{b} d_{b-i}\right)=c_a\left(\sum_{i=0}^{b-1} d_i\right).$$

Hence the latter condition of (1.0) and (1.4) provide us with two cases:

(1.5) (I) $c_a=0$ or $\sum_{i=0}^{b-1} d_i=0$.

(II) $c_a=\sum_{i=0}^{b-1} d_i=1$.

Since $c(\mathcal{O}_{P^{n-1}}^{\oplus n})=c(F_1)c(F_2)$ from 1.3, we get the followings:

$$1=(1+c_1ht+\cdots c_ah^at^a)(1+d_1ht+\cdots d_bh^bt^b).$$

Thus to evaluate integers c_i, d_j, we can consider the following equality of polynomials of one variable x:

(1.6) $$1+\alpha x^n=(1+c_1x+\cdots c_ax^a)(1+d_1x+\cdots d_bx^b)$$

with $a+b=n$ and

(1.6.1) $$\alpha=c_ad_b.$$

First we treat the case (I).

Assume $c_a=0$. Since $\alpha=0$, we have $c_1=d_1=0$ from 1.6. Therefore we see that the morphism f: $P^{n-1}\to \mathrm{Gr}(n-1, a-1)$ in 1.3 is constant by 1.4 and therefore that F_1 and F_2 are trivial vector bundles. Hence E is isomorphic to $\mathcal{O}_{P^n}^{\oplus a}\oplus\mathcal{O}_{P^n}(1)^{\oplus b}$.

Next we treat the case $\sum_{i=0}^{b-1} d_i=0$.

We substitute 1 for x of 1.6 to obtain

(1.7) $$1+\alpha=(1+c_1+\cdots+c_a)(1+d_1+\cdots+d_b).$$

Thus by 1.6.1 and our assumption we have $1+c_ad_b=(1+c_1+\cdots+c_a)d_b$, which yields two equalities $1+c_1+\cdots+c_{a-1}=1$ and $d_b=1$ by 1.4. If $a\geq 2$, we have $c_1=\cdots=c_{a-1}=0$ by 1.4 and therefore the morphism f is constant. Thus F_2 is a trivial vector bundle and therefore $d_b=0$, which yields a contradiction to $d_b=1$. Hence $a=1$. Moreover from 1.4, $(-1)^bd_b=(-1)^b$ is non-negative. Thus we get

(1.8) If $\sum_{i=0}^{b-1} d_i=0$, then F_1 is a line bundle. Moreover n is odd and $b=n-1$.

Again from 1.6, 1.6.1 and $d_{n-1}=1$ we have

(1.9) $$1+c_1t^n=(1+c_1t)(1+d_1t+\cdots+t^{n-1}).$$

Thus substituting -1 for t and noting from 1.4 that $1+d_1(-1)+\cdots+1\geq 2$, we easily get $c_1=1$ for odd n, which implies that $\mathcal{O}_{P^{n-1}}(1)=c_1(F_1)=f^*\mathcal{O}_{P^{n-1}}(1)$ by 1.3 and therefore the morphism f: $P^{n-1}\to P^{n-1}$ is an isomorphism. Hence we see that $F_1\simeq\mathcal{O}_{P^{n-1}}(1)$ and $F_2\simeq\Omega_{P^{n-1}}(1)$. Moreover from 1.2 we obtain the following exact sequence on $\bar{P}$:

$$(1.10)\qquad 0\longrightarrow\pi^*\Omega_{P^{n-1}}(1)\otimes\varphi^*\mathcal{O}_{P^n}(1)\longrightarrow\varphi^*E\longrightarrow\pi^*\mathcal{O}_{P^{n-1}}(1)\longrightarrow 0\,.$$

It is easy to see that the vector bundle φ^*E corresponds to a non-zero element in $H^1(\bar{P}, \mathrm{Hom}(\pi^*\mathcal{O}_{P^{n-1}}(1), \pi^*\Omega_{P^{n-1}}(1)\otimes\varphi^*\mathcal{O}_{P^n}(1))$ which is isomorphic to $H^1(P^{n-1}, \Omega_{P^{n-1}}\otimes\pi_*\varphi^*\mathcal{O}_{P^{n-1}}(1))$. Hence it is isomorphic to k by Leray's spectral sequence and by the fact that $\pi_*\varphi^*\mathcal{O}_{P^n}(1)\simeq\mathcal{O}_{P^{n-1}}(1)\oplus\mathcal{O}_{P^{n-1}}$. On the other hand for $E=\Omega_{P^n}(2)$, the exact sequence (1.10) holds good. Thus we infer that E with $\sum_{i=0}^{b-1}d_i=0$ is $\Omega_{P^n}(2)$ (n is odd).

Thus we finish the case 1.5.I.

Secondly let us consider the case 1.5.II. First since $c_a=1$, the morphism f: $P^{n-1}\to\mathrm{Gr}(n-1, a-1)$ of (1.3) is not constant. From (1.7) and $\alpha=d_b$ we have $1+d_b=(1+c_1+\cdots c_a)(1+d_b)$. By $c_i\geq 0$, we have $d_b=-1$ and $\alpha=-1$. Thus b is odd. Now noting that $\mathrm{Gr}(n-1, a-1)\simeq\mathrm{Gr}(n-1, b-1)$ with $a+b=n$, we can apply the following:

THEOREM 1.11 (Tango [Ta2]). *There exists no non-constant morphism from P^n to* $\mathrm{Gr}(n, d)$ $(n\geq 3)$ *if one of the following conditions holds*:
(1) *n is even and $n-1>d>0$.* (2) *d is even and $n-1>d>0$ and $(n, d)\neq(5, 2)$.*

The morphism f is not constant by $c_a=1$. Hence recalling that b is odd, we have the following possibilities for n and b:
(I) $b=1$ or $n-1$ (n is even). (II) $n=2$. (III) $(n, b)=(6, 3)$.

As for the case (I) let us consider the morphism f: $P^{n-1}\to P^{n-1}$. When $b=1$, we have $d_1=-1$. Thus f is an isomorphism by (1.4). Thus we infer that

$$(F_1, F_2)\ \text{is}\ (T_{P^{n-1}}(-1), \mathcal{O}_{P^{n-1}}(-1))\,.$$

Next when b is $n-1$, we have $a=1$. By $c_1=1$ and (1.4), we get, in the same way as in the case of $b=1$,

$$(F_1, F_2)\ \text{is}\ (\mathcal{O}_{P^{n-1}}(1), \Omega_{P^{n-1}}(1))\quad (n\ \text{is even})\,.$$

Moreover, in the same way as in the latter part of 1.5 (I), we immediately see that

$$E\ \text{is isomorphic to}\ T_{P^n}(-1)\ (n\ \text{is arbitrary})\quad\text{or}\quad \Omega_{P^n}(2)\ (n\ \text{is even})\,.$$

As for (II) what we have to study is the case $a=b=1$ by Theorem 1.1. Then by the assumption $c_1=1$, $d_1=-1$ we have the exact sequence

$$0\longrightarrow\pi^*\mathcal{O}_{P^1}(-1)\otimes\varphi^*\mathcal{O}_{P^1}(1)\longrightarrow\varphi^*E\longrightarrow\pi^*\mathcal{O}_{P^1}(1)\longrightarrow 0\,.$$

Thus in the same manner as in 1.5.II we have $E\simeq T_{P^2}(-1)$.

Finally we show that the case $(n, b)=(6, 3)$ does not occur. In view of $\alpha=-1$ it is shown in [Ta1, Lemma 3.1] that when f is a non-constant morphism, both c_i and

$(-1)^j d_j$ are positive. Thus by the equality

$$1-t^6=(1-t)(1+t)(1-t+t^2)(1+t+t^2)$$

we infer that

$$c_1=2\,,\quad c_2=2\,,\quad c_3=1\quad\text{and}\quad d_1=-2\,,\quad d_2=2\,,\quad d_3=-1\,.$$

Thus since the vector bundle F_1 is generated by global sections and $c_3=1$, the zero locus of a general section s of F_1 is isomorphic to a 2-plane P_2. Taking the dual of the homomorphism $\mathcal{O}\to F_1$ induced by the sections s, we get a surjective homomorphism $\check{F}_1\to I\to 0$ where I is the ideal sheaf of the 2-plane, which induces a surjective homomorphism $\check{F}_1\to I/I^2\to 0$. Thus since I/I^2 is isomorphic to $\mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 3}$, it follows that $\check{F}_1\simeq\mathcal{O}_{\boldsymbol{P}^2}(-1)^{\oplus 3}$. Comparing the Chern classes of two vector bundles, we get a contradiction.

Thus we complete the proof of Theorem 1.

2. The normal bundle $N_{P_y/X}$. Our aim in this section is to show the following which is proved in (2.10):

THEOREM 2.0. *Let X be an n-dimensional smooth projective variety defined over an algebraically closed field of characteristic zero. Assume that for a general point x in X ($\subset\boldsymbol{P}^N$) there exists an m-plane P in the variety X containing x. Then there is an irreducible component Y of the Hilbert scheme of m-planes in X so that the normal bundle $N_{P_y/X}$ of an m-plane P_y in X is generically generated by global sections for a general point y in Y where P_y is an m-plane corresponding to the point y. Moreover assume $2m\geq n$ and $n\geq 2$. Then for a sufficiently general point y in Y, the normal bundle $N_{P_y/X}$ is isomorphic to one of the following*:

$$\mathcal{O}_{\boldsymbol{P}^m}^{\oplus a}\oplus\mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}\,,\quad T_{\boldsymbol{P}^m}(-1)\quad\text{and}\quad\Omega_{\boldsymbol{P}^m}(2)\,.$$

For this purpose let us consider an n-dimensional smooth projective variety X in an N-dimensional projective space $\boldsymbol{P}^N$ with the following condition:

$$\text{(2.1)}\qquad X \text{ contains an } m\text{-plane } P\,.$$

Then we have an exact sequence

$$\text{(2.2)}\qquad 0\longrightarrow N_{P/X}\longrightarrow N_{P/\boldsymbol{P}^N}\longrightarrow N_{X/\boldsymbol{P}^N|P}\longrightarrow 0\,.$$

Here we note that $N_{P/\boldsymbol{P}^N}\simeq\mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus(N-m)}$.

Moreover taking the dual of (2.2) and tensoring $\mathcal{O}_{\boldsymbol{P}^m}(1)$, we have

$$\text{(2.3)}\qquad 0\longrightarrow\check{N}_{X/\boldsymbol{P}^N|P}(1)\longrightarrow\check{N}_{P/\boldsymbol{P}^N}(1)\longrightarrow\check{N}_{P/X}(1)\longrightarrow 0\,.$$

Since $\check{N}_{P/\boldsymbol{P}^N}(1)$ is $\mathcal{O}_{\boldsymbol{P}^m}^{\oplus(N-m)}$, $\check{N}_{P/X}(1)$ is generated by global sections. Here for a vector bundle E on $\boldsymbol{P}^m$, we denote $E(a)=E\otimes\mathcal{O}_{\boldsymbol{P}^m}(a)$ from now on.

Now we study the following easy lemma:

LEMMA 2.4. *Let E be a vector bundle on P^n. Assume that E is generically generated by global sections. Then there is an open set U in P^n such that $E_{|l}$ is generated by global sections for each line l with $l \cap U \neq \emptyset$.*

PROOF. When $n=1$, the lemma is trivial since E is $\oplus \mathcal{O}(a_i)$. Assume $n>1$. The assumption yields a canonical homomorphism $\varphi : \mathcal{O}^{\oplus r} \to E$ on P^n with $r = \dim H^0(P^n, E)$. Let U be the set $\{x \in P^n \mid \varphi \otimes k(x) \text{ is surjective}\}$, which is a non-empty open set in P^n. Taking a line l with $l \cap U \neq \emptyset$, we have a generically surjective homomorphism $\varphi_{|l}$: $\mathcal{O}^{\oplus r} \to E_{|l}$ induced by φ. In view of the result for $n=1$, we are done. q.e.d.

PROPOSITION 2.5. *Let P be the m-plane in* (2.1). *Assume that $N := N_{P/X}$ is generically generated by global sections. Then there exists an open set U in P^n so that $N_{|l}$ is of type $\mathcal{O}^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b}$ on any line l intersecting U. Moreover the pair (a, b) is independent of the choice of an l.*

PROOF. Take a line l intersecting the open set U in Lemma 2.4. Then we get $a_i \geq 0$ where $N_{|l}$ is isomorphic to $\oplus \mathcal{O}(a_i)$ by Lemma 2.4. On the other hand since $\check{N}(1)$ is generated by global sections by (2.3), we get $1 - a_i \geq 0$, as desired. Finally let $c_1(N) = cH$ in the Chow ring where H is the class of hyperplanes. Then $b = c$. q.e.d.

The following is an immediate consequence of the above and Theorem 1.1.

COROLLARY 2.6. *Let the condition and assumption be as in* (2.5). *Moreover assume that $n \geq 2$, $2m \geq n$ and that the $(n-m)$-th Chern class of $N := N_{P/X}$ is zero or H^{n-m} in the $(n-m)$-th Chow group of P^m where H is the class of hyperplanes of P^m in the Chow group. Then N is isomorphic to one of $\mathcal{O}_{P^m}^{\oplus a} \oplus \mathcal{O}_{P^m}(1)^{\oplus b}$, $T_{P^m}(-1)$ and $\Omega_{P^m}(2)$ where the last two cases occur only when $n = 2m$.*

From now on we study the condition for the normal bundle $N_{P/X}$ to satisfy the assumptions of Proposition 2.5 and Corollary 2.6.

We impose the following condition which will be assumed till the end of this section.
(2.7) For a general point x in X ($\subset P^N$) there exists an m-plane P in the variety X containing x.

Then we consider the Hilbert scheme Y of m-planes in X. Let Z be the universal scheme of Y and $p : Z \to X$, $q : Z \to Y$ the canonical projections. Moreover in view of the projectivity of Y it follows from the condition (2.7) that
(2.7.1) there is an irreducible component Y_0 of Y so that the canonical projection $p : q^{-1}(Y_0) \to X$ is surjective.

Now we take the reduced part $(Y_0)_{red}$ of Y_0 if Y_0 is non-reduced and use Z and Y in place of $q^{-1}(Y_0)$ and Y_0, respectively. Thus we have the following diagram:

(2.8)
$$\begin{array}{ccccc} & & F(N,m,0) & & \\ & & \cup & & \\ & & Z & & \\ & \swarrow^{q} & & \searrow^{p} & \\ G(N,m)\supset Y & & & & X\subset \boldsymbol{P}^N . \end{array}$$

Note that $pq^{-1}(y)$ is an m-plane for each point y in Y. From now on set $P_y = pq^{-1}(y)$. Moreover when $\bar{Y}$ is the smooth part of Y, we set $\bar{Z} := q^{-1}(\bar{Y})$. The morphism $\bar{p} := p_{|Z}$ induces a homomorphism on $\bar{Z}$:

(2.9)
$$\bar{p}^* : T_{\bar{Z}} \longrightarrow p^* T_X .$$

On the other hand when $T_{Z/Y}$ denotes the relative tangent bundle of the $\boldsymbol{P}^m$-bundle q, there is a canonical homomorphism on $\bar{Z}$:

(2.9.1)
$$0 \longrightarrow T_{\bar{Z}/\bar{Y}} \longrightarrow T_{\bar{Z}} \longrightarrow \bar{q}^* T_{\bar{Y}} \longrightarrow 0 .$$

Hence we have a canonical homomorphism

$$\bar{p} : T_{Z/Y} \longrightarrow p^* T_X$$

on $\bar{Z}$, which is injective as a homomorphism of vector bundles.

We remark the following:

(2.9.2) The cokernel $\mathcal{N}$ of $\bar{p}$ is a rank $n-m$ vector bundle on $\bar{Z}$ and $\mathcal{N}_{|q^{-1}(y)} \simeq N_{P_y/X}$ for each y in Y.

For each point y in $\bar{Y}$ taking the restriction of the exact sequence (2.9.1) to $q^{-1}(y)$, we get

$$0 \longrightarrow T_{P_y} \longrightarrow T_{\bar{Z}|P_y} \longrightarrow \mathcal{O}^{\oplus m} \longrightarrow 0 .$$

Then we infer that $T_{\bar{Z}|P_y}$ is isomorphic to $T_{P_y} \oplus \mathcal{O}^{\oplus m}$, which is generated by global sections. Thus if the characteristic of the base field is zero, for a general point y in $\bar{Y}$ the restriction of the homomorphism p^* to $q^{-1}(y)$ is generically surjective by the surjectivity of p and Sard's Theorem.

Hence we have:

PROPOSITION 2.9.3. *Let X be a smooth projective variety satisfying the condition* (2.7) *and let the notation be as in* (2.8). *Assume that the characteristic of the base field is zero. Then for a general point y in Y, the normal bundle $N_{P_y/X}$ is generically generated by global sections.*

We begin the proof of Theorem 2.0.

(2.10) Let us assume that $n \leq 2m$ and that $N_{P_y/X}$ is generically generated by global sections. Then we have:

Claim: $c_m(N_{P_y/X}) = cH^{m-n}$ in the sense of (2.6) where c is 0 or 1.

Indeed, consider the intersection of P_y and $P_{y'}$ for two points y, y' in Y. If $\dim Y = d$, then $\dim Z = m+d$. Thus for a general point y in Y we have $\dim p^{-1}(P_y) = 2m+d-n$. In particular if $n=2m$, then there is a point y in Y' such that either $P_y \cap P_{y'}$ is empty or P_y properly intersects $P_{y'}$. Consequently since $c_m(N_{P_y/X}) = P_y \cdot P_{y'}$ it is 0 or 1. The case $n \leq 2m-1$ is reduced to the one $n=2m$ by successive general hyperplane sections of X in $\boldsymbol{P}^N$. Thus we get the claim.

Therefore we get Theorem 2.0 from Corollary 2.6.

REMARK 2.11. Let us consider the following condition: there exists an m-plane P in X ($\subset \boldsymbol{P}^N$) so that the normal bundle $N_{P/X}$ is generically generated by global sections and $H^1(P, N_{P/X}) = 0$. Then in any characteristic, Y in (2.7) is smooth at the point $[P]$ by virtue of the deformation theory of Grothendieck. Thus there exists a unique component Y_o containing the point $[P]$ and so let us put $Y = Y_o$. Moreover since the morphism p is surjective and separable by assumption, the condition (2.7) holds.

Therefore we get:

PROPOSITION 2.12. *Let P be an m-plane in an n-dimensional smooth projective variety X contained in $\boldsymbol{P}^N$. Assume that $2m \geq n$ and $n \geq 2$ and that the normal bundle $N_{P/X}$ is isomorphic to one of $\mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$, $T_{\boldsymbol{P}^m}(-1)$ and $\Omega_{\boldsymbol{P}^m}(2)$. Then X has the diagram as in* 2.8 *and for a general point y in Y the normal bundle $N_{P_y/X}$ is isomorphic to $N_{P/X}$.*

PROOF. Each vector bundle as above is generated by global sections and the first cohomology group vanishes. Hence by Remark 2.11 the normal bundle $N_{P_y/X}$ is generically generated by global sections for a general y in Y. Thus $N_{P_y/X}$ is one of the vector bundles stated above. Moreover in view of the Chern polynomial, we get the desired result. q.e.d.

3. Smooth projective variety with $N_{P/X} \simeq \mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$, $\Omega_{\boldsymbol{P}^m}(2)$. In this section we state the following:

THEOREM 3.1. *Let X be an n-dimensional smooth projective variety in $\boldsymbol{P}^N$ and P an m-plane in X. Assume that $2m \geq n$ and $n \geq 3$ and that the characteristic of the base field is zero. Then we have*:

(1) *if the normal bundle $N_{P/X}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$, then X is a $\boldsymbol{P}^{n-a}$-bundle over an a-dimensional smooth projective variety where the m-plane P is contained in some fiber of the projection.*

(2) *If $N_{P/X}$ is isomorphic to $\Omega_{\boldsymbol{P}^m}(2)$, then X is a hyperquadric and $n=2m$.*

First we treat the case (1) where $N_{P/X} \simeq \mathcal{O}_{\boldsymbol{P}^m}^{\oplus a} \oplus \mathcal{O}_{\boldsymbol{P}^m}(1)^{\oplus b}$. Let us recall the following fact.

REMARK 3.2 ([E, Theorem 1.7] and [W2, Theorem 2.4]). Let the condition be as in Theorem 3.1. Assume that the normal bundle $N_{P/X}$ is a trivial vector bundle ($b=0$). Ein showed (1) in the case $2m > n$ (in 'any' characteristic) and Wiśniewski showed it in the case $2m \geq n$ when the characteristic of the base field is zero.

Moreover the argument in [E, §4] shows:

REMARK 3.3. Let the condition be as in Theorem 3.1. Assume that the normal bundle $N_{P/X}$ is $\mathcal{O}_{P^m}^{\oplus a} \oplus \mathcal{O}_{P^m}(1)^{\oplus b}$ with $b>0$. Moreover assume that $2m \geq n$ and $m \geq 3$. Then in any characteristic there exists an $(m+b)$-plane $\bar{P}$ in X containing P such that the normal bundle $N_{\bar{P}/X}$ is $\mathcal{O}_{P^{m+b}}^{\oplus a}$.

Now we consider the proof of Theorem 3.1, (1). By Remarks 3.2 and 3.3 we have only to treat the cases $(m, n)=(2, 4)$, $(2, 3)$, $(1, 2)$. First consider the case $(m, n)=(2, 4)$ such that $N_{P/X} \simeq \mathcal{O}^{\oplus a} \oplus \mathcal{O}(1)^{\oplus b}$ with $(a, b)=(1, 1)$, $(0, 2)$. For a line l in a 2-plane P, we have $-K_X \cdot l = 4$ or 5. Thus X is a smooth hyperquadric 4-fold Q, a P^3-bundle over a curve or P^4 by virtue of the adjunction mapping theory. (See, for example, [Fu].) The other cases are easy to check.

Finally consider the case (2) with $N_{P/X} \simeq \Omega_{P^m}(2)$. Taking a line l in P, we see that $N_{l/X} \simeq \mathcal{O} \oplus \mathcal{O}(1)^{\oplus n-2}$ and therefore $K_{X|l} \simeq \mathcal{O}(-n)$. Thus X is a smooth hyperquadric or a scroll over a smooth projective curve. In the latter case the m-plane is contained in a fiber, which implies that the normal bundle $N_{P/X}$ is $\mathcal{O}_{P^m} \oplus \mathcal{O}_{P^m}(1)^{\oplus m-2}$. Thus the latter case is ruled out.

Thus we get Theorem 3.1.

In the remainder of this section, we give a proposition which holds in any characteristic.

(3.4) Using the fact that for a line l on $p(P_y)$ and a point A on the line l, the normal bundle $N_{l/X}(-A)$ is $\mathcal{O}(-1)^{\oplus a} \oplus \mathcal{O}^{\oplus n-a-1}$, Ein [E] showed in the proof of Theorem 1.7 that the union of lines through the point A coincides with the plane $p(P_y)$ together with the following:

FACT 3.5. In the diagram (2.8) assume that $N_{P_y/X}$ is trivial for each point y in Y. Then for any pair y, y' of points in Y we have $\dim(p(P_y) \cap p(P_{y'})) \leq 0$ in any characteristic.

Consequently we have:

PROPOSITION 3.6. *Let the condition be as in Theorem* 3.1. *Assume that* $n \geq 3$, $2m \geq n$ *and that the normal bundle* $N_{P/X}$ *is* $\mathcal{O}_{P^m}^{\oplus a} \oplus \mathcal{O}_{P^m}(1)^{\oplus b}$ *with* $b>0$. *Moreover if* $n=2m$ *and* $b=0$, *assume in addition that in the diagram* (2.8) $N_{P_y/X}$ *is trivial for each point* y *in* Y. *Then, in any characteristic,* X *is a* P^{n-a}*-bundle over an a-dimensional smooth projective variety and* P *is contained in a fiber.*

PROOF. We have only to show the case $n=2m$ and $b=0$ by [E, Theorem 1.7] and Proposition 3.4, In view of Fac 3.5 suppose that there are two points y, y' in Y satisfying $\dim(p(P_y) \cap p(P_{y'}))=0$. This implies that the intersection number of $p(P_y)$ and $p(P_{y'})$ is equal to one, which means $c_m(N_{P_y/X})=1$. On the other hand by assumption $c_m(N)$ is 0, a contradiction. Thus we infer that for any pair y, y' of points in Y the intersection $p(P_y) \cap p(P_{y'})$ is empty. The rest of the proof is found in [E, Theorem 1.7]. q.e.d.

4. The deformation of $P(\Omega_{P^m}(2))$ under some conditions, (I). We prove Theorem 4.2 in this section and Section 5.

We consider the following condition:

(4.1) Let V be a vector bundle of rank $m+1$ on a scheme T and $f: P(V)\to T$ a P^m-bundle on T. Set $W:=P(V)$. Let $\mathscr{E}$ be a vector bundle of rank m on W and $g: P(\mathscr{E})\to W$ a P^{m-1}-bundle on W. Set $Q:=P(\mathscr{E})$ and $E_t:=\mathscr{E}_{|f^{-1}(t)}$.

Then the main result in this section is the following:

THEOREM 4.2. *Let us maintain the notation* (4.1). *Assume that E_t is generated by global sections for each point t in T and $E_t\simeq\Omega_{P^m}(2)$ for a general point t in T. If m is even, then $E_t\simeq\Omega_{P^m}(2)$ for each point t in T.*

The above theorem immediately yields the following:

COROLLARY 4.3. *Let $\mathscr{N}$ be as in* (2.9.2). *Assume that m is even. If $\mathscr{N}_{|q^{-1}(y)}$ is isomorphic to $T_{P^m}(-1)$ for a general point y in Y, then so is $\mathscr{N}_{|q^{-1}(y)}$ for every point y in Y.*

Theorem 4.2 and Corollary 4.3 are discussed in Section 5. We start with the following condition.

(4.4) T is a smooth curve and o is a closed point in T. Moreover for every point t in $T-\{o\}$, E_t is isomorphic to $\Omega_{P^m}(2)$ and E_o is generated by global sections.

(4.4.1) Letting $Q_t:=(fg)^{-1}(t)$ we see that Q is a family $\{Q_t\}_{t\in T}$ where $g_t: Q_t\to f^{-1}(t)$ is a P^{m-1}-bundle on P^m. Let $\mathcal{O}_Q(1)$ be the tautological line bundle of $\mathscr{E}$. Then $\mathcal{O}_Q(1)_{|Q_t}=\mathcal{O}_{P(E_t)}(1):=\mathcal{O}_{Q_t}(1)$. $\mathcal{O}_{Q_t}(1)$ is base point free.

PROPOSITION 4.5. *Let $\mathscr{L}$ be the dual line bundle $K_{Q/T}^{-1}$ of the relative canonical line bundle of the morphism fg. Then the line bundle $\mathscr{L}$ on Q is fg-ample. Moreover $h^0(Q_t, \mathcal{O}_{Q_t}(1))$ is independent of the choice of t.*

PROOF. Since $K_{Q_t}^{-1}$ is isomorphic to $\mathcal{O}_{Q_t}(1)^{\otimes m}\otimes g_t^*\mathcal{O}_{P^m}(2)$ and since $\mathcal{O}_{Q_t}(1)$ is generated by global sections, we see that $K_{Q_t}^{-1}$ is ample and therefore $\mathscr{L}$ is fg-ample. As for the latter part, by virtue of the Kodaira vanishing theorem we infer that $H^i(Q_t, \mathcal{O}_{Q_t}(1))=H^{2m-1-i}(Q_t, \mathcal{O}_{Q_t}(1)^{-1}\otimes K_{Q_t})=0$ for $1\le i\le 2m-1$ because $\mathcal{O}_{Q_t}(1)^{-1}\otimes K_{Q_t}$ is a negative line bundle. Thus we get an equality $\chi(Q_t, \mathcal{O}_{Q_t}(1))=h^0(Q_t, \mathcal{O}_{Q_t}(1))$. Moreover since $\chi(Q_t, \mathcal{O}_{Q_t}(1))$ is independent of the choice of t, we get the desired result. q.e.d.

Since $\mathcal{O}_{Q_t}(1)$ is base point free, Proposition 4.5 yields a surjective homomorphism $(fg)^*(fg)_*\mathcal{O}_Q(1)\to\mathcal{O}_Q(1)\to 0$, which induces a T-morphism $\bar{h}: Q\to P((fg)_*\mathcal{O}_Q(1))$. Let $\bar{R}=\bar{h}(Q)$ and $\tilde{R}\to\bar{R}$ the normalization of $\bar{R}$. Moreover we take the Stein factorization $Q\xrightarrow{h}R\xrightarrow{j}\tilde{R}$ of the induced morphism $\tilde{h}: Q\to\tilde{R}$ where every fiber of h is connected and j is a finite morphism.

(4.6) Let $a: F(m,1,0)\to P^m$ and $b: F(m,1,0)\to \mathrm{Gr}(m,1)$ be the canonical projections as in the introduction.

REMARK 4.7. (1) Let $h_t : Q_t \to P^{N-1}$ be a morphism induced by the line bundle $\mathcal{O}_{Q_t}(1)$ with $N = h^0(Q_t, \mathcal{O}_{Q_t}(1))$. Then $h_{|Q_t}$ is equal to h_t for each $t \neq o$ and it coincides with the morphism b in (4.6). Thus since j is a birational morphism, j is an isomorphism and in particular the identity map.

(2) For each point $t\ (\neq o)$ in T, every fiber G_t of h_t is P^1. Moreover the intersection number $(G_t \cdot g_t^* \mathcal{O}_{P^m}(1)) = 1$ in Q_t.

From now on we fix the notation $h : Q \to R$ and $Q_o := (fg)^{-1}(o)$. Let h_o be the restriction of $h : Q \to R$ to Q_o. We would like to study the property of $h_o : Q_o \to h_o(Q_o)$, in particular, the behavior of the fiber of h_o, to show that Q_o is canonically isomorphic to Q_t $(t \neq o)$ and h_o to h_t. For the purpose we need the following tool:

(4.8) Let M be a vector bundle of rank $r+1$ which is generated by global sections on P^m. Let $L = \mathcal{O}_{P(M)}(1)^{\otimes 2} \otimes e^* \mathcal{O}_{P^m}(1)$ where $e : P(M) \to P^m$ is the canonical projection. Then L is an ample line bundle on $P(M)$. Moreover let $\bar{H} = \mathrm{Hilb}^{p(\bar{t})}_{P(M)}$ with respect to L where $p(\bar{t}) = \chi(\mathcal{O}_{P^1}(\bar{t}))$ and $\mathcal{H}$ the universal space of $\bar{H}$. Let $\alpha : \mathcal{H} \to P(M)$ and $\beta : \mathcal{H} \to \bar{H}$ be the canonical projections.

LEMMA 4.9. *Assume that $\bar{H}$ is not empty. Then we have:*

(1) *$\beta : \mathcal{H} \to \bar{H}$ is a P^1-bundle.*

(2) *For a closed point y in $\bar{H}$, $\alpha\beta^{-1}(y) := L_y$ is a smooth curve in $P(M)$ so that $e : L_y \to e(L_y)$ is an isomorphism and $e(L_y)$ is a line l_y in P^m.*

(3) *L_y is a section corresponding to the trivial quotient line bundle of $M_{|l_y}$ where $M_{|l_y} \simeq \bigoplus \mathcal{O}(a_i)$ with $a_1 = 0 \leq a_2 \leq \cdots \leq a_{r+1}$.*

Consequently there is a one-to-one correspondence between the set $\bar{H}(k)$ of k-rational points and the set of trivial quotient line bundles of $M_{|l}$ where l is a line on P^m.

PROOF. By assumption we get an equality $1 = (L_y \cdot L) = (L_y \cdot \mathcal{P}_{P(M)}(1)^{\otimes 2} \otimes e^* \mathcal{O}_{P^m}(1))$. Since $\mathcal{O}_{P(M)}(1)$ is base point free, $(L_y \cdot \mathcal{O}_{P(M)}(1)) = 0$ and $(L_y \cdot e^* \mathcal{O}_{P^m}(1)) = 1$. From the latter we get deg $e(L_y) = 1$ and deg $e_{|L_y} = 1$. Hence l_y is a line and therefore L_y is smooth. The reminder is easily checked. q.e.d.

The above lemma immediately yields the following:

COROLLARY 4.9.1. *The normal bundle $N_{L_y/P(M)}$ appears in the exact sequence:*

$$0 \longrightarrow N_{L_y/e^{-1}(l_y)} \longrightarrow N_{L_y/P(M)} \longrightarrow N_{e^{-1}(l_y)/P(M)|L_y} \longrightarrow 0\,,$$

where $N_{L_y/e^{-1}(l_y)} \simeq \bigoplus_{i \geq 2} \mathcal{O}(-a_i)$ and $N_{e^{-1}(l_y)/P(M)|L_y} \simeq \mathcal{O}(1)^{\oplus(m-1)}$.

We apply Lemma 4.9 and Corollary 4.9.1 to the T-morphism $h : Q \to R$.

(4.10) Let $\bar{\mathcal{L}} = \mathcal{O}_Q(2) \otimes g^* \mathcal{O}_{P(V)}(1)$ in (4.1). Note that $\bar{\mathcal{L}}$ is fg-ample. We consider the relative Hilbert T-scheme $H^{P(\bar{t})}_{Q/T}$ of Q over T with respect to $\bar{\mathcal{L}}$ where $P(\bar{t})$ is the Hilbert polynomial $\chi(P^1, \mathcal{O}_{P^1}(t))$ of the fiber G_t in (2) of Remark 4.7. $H^{P(\bar{t})}_{Q/T}$ is abbreviated as H. Let $H = \bigcup_{i=1}^s H^i$ be the decomposition of H into the irreducible components.

REMARK 4.10.1. (1) We have a canonical isomorphism by virtue of the universality of the Hilbert scheme:

$$H \times_T \operatorname{Spec} k(t) \simeq H^{P(t)}_{Q \times_T \operatorname{Spec} k(t)} \,(=: H_t)\,.$$

(2) For $t \neq o$, H_t is equal to $\mathrm{Gr}(m, t)$ by Remark 4.7.1.

To investigate the deformation of $G_t =: l$ of a fiber of h_t $(t \neq o)$ in Q in detail, we use the exact sequence

$$(4.10.2) \qquad 0 \longrightarrow N_{l/Q_t} \longrightarrow N_{l/Q} \longrightarrow N_{Q_t/Q|l} \longrightarrow 0\,.$$

Noting that $N_{l/Q_t} \simeq \mathcal{O}^{\oplus(2m-2)}$ by Remark 4.7.1, we see that $N_{l/Q} \simeq \mathcal{O}^{\oplus 2m-1}$. Since $H^1(l, N_{l/Q})$ vanishes, H is smooth at the point l and $\dim_l H = 2m-1$.

Summing up the above, we get:

PROPOSITION 4.11. *Let H be as in* (4.10) *and let us denotes* $T^{o} := T - \{o\}$. *Then we have*:

(1) *There is a unique irreducible component H' of H containing H_t for all $t \in T^{o}$.*

(2) *$H \times_T T^{o}$ is equal to $H^1 \times_T T^{o}$ which is isomorphic to a* $\mathrm{Gr}(m, 1)$*-bundle over T^{o}.*

(3) *For each $i (\geq 2)$ (if it exists), $H^i \cap H_t$ is empty for each t in T^{o}.*

(4.11.1) Let $\mathscr{H}$ be the universal scheme of H, and $\pi : \mathscr{H} \to Q$ and $\rho : \mathscr{H} \to H$ the canonical projections. Let $\mathscr{H}^1 := \rho^{-1}(H^1)$ and $\pi^1 := \pi_{|\mathscr{H}^1}$. Then π and ρ are T-morphisms. Thus for t in T let us set $H_t =: H \times_T \operatorname{Spec} k(t)$, $H^1_t =: H^1 \times_T \operatorname{Spec} k(t)$, $\mathscr{H}_t = \rho^{-1}(H_t)$, $\mathscr{H}^1_t = \rho^{-1}(H^1_t)$ and $\pi^1_t = \pi_{|\mathscr{H}^1_t}$.

Then we have the following diagram:

$$\begin{array}{ccccc}
\mathscr{H} & \xrightarrow{\ \pi\ } & Q & \xrightarrow{\ \ g\ \ } & P(V) \\
{\scriptstyle\rho}\downarrow & & \downarrow & \overset{f}{\swarrow} & \\
H & \xrightarrow{\ \theta\ } & T & &
\end{array}$$

PROPOSITION 4.12. (1) *$\pi^1 : \mathscr{H}^1 \to Q$ is surjective. Moreover for each t in T, $\pi^1_t : \mathscr{H}^1_t \to Q_t$ is birational. In particular for $t \neq o$ π^1_t is an isomorphism. Moreover* $\dim H^1 = 2m-1$.

(2) *Every fiber of ρ is a $\boldsymbol{P}^1$-bundle. For every u in H, $\pi\rho^{-1}(u)$ is $\boldsymbol{P}^1$, $g : \pi\rho^{-1}(u) \to g(\pi\rho^{-1}(u))$ is an isomorphism with a line as its image in $\boldsymbol{P}^m$.*

(3) $\dim H^1_t = 2m-2$ *for each t in T.*

(4) *For $t \neq o$, H_t is equal to H^1_t and is isomorphic to* $\mathrm{Gr}(m, 1)$, *and* $\mathscr{H}_t \simeq F(m, 1, 0)$. *Moreover ρ_t is a $\boldsymbol{P}^1$-bundle and is equal to b.*

PROOF. Since $N_{l/Q}$ is generated by global sections, we infer that $\pi^1_t(\mathscr{H}^1_t) = Q_t$ for $t \neq o$ and $\dim H = \dim H^0(l, N_{l/Q}) = 2m-1$. Next we show $\pi^1_o(\mathscr{H}^1_o) = Q_o$. For a point z in Q_o take a local section C (in Q) of $fg : Q \to T$ passing through the point z. Then for each point c in $C - \{z\}$, there exists a line l_c in $Q_{fg(c)}$ where l_c corresponds to an element

$\bar{c}$ in $H_{fg(c)}$ and where $\pi(\rho^{-1}(\bar{c}))=l_c$. Then we have only to take the limit of $\{l_c\}$. Thus from Remark 4.7 (1) we get (1) except the assertion that π_o is birational. Hence we have only to show:

SUBLEMMA 4.13. *Let h_o, Q_o and G_t (Remark 4.7 (2)) be as above. Then there is an open set U in $h_o(Q_o)$ so that for each point u in U a fiber G_u of the morphism h_o is $\boldsymbol{P}^1$ and $g_o(G_u)$ is a line in $\boldsymbol{P}^m$.*

PROOF. Indeed, since $fg: Q\to T$ is a smooth morphism, the $(2m-1)$-ple self-intersection number $(\mathcal{O}_{Q_t}(1)^{2m-1})$ of $\mathcal{O}_{Q_t}(1)$ is equal to 0 and similarly $(\mathcal{O}_{Q_t}(1)^{2m-2})\neq 0$ for each point t in T modulo rational equivalence. Thus the morphism $h_o: Q_o\to h_o(Q_o)$ has a general fiber which is of dimension one. As was shown, the deformation $\mathscr{H}^1=\{C_s\}_{s\in H^1}$ of $l:=G_t$ spans the total space Q with the parameter space H^1. Let $A=\{s\in H^1 \mid C_s\subset Q_o\}$. Since $fg(C_s)$ is a point for a general point s in H^1, A is equal to $H^1\times_T \operatorname{Spec} k(\mathrm{o})$ and Q_o is covered by the family of closed subschemes parameterized by A. Noting that for any s, C_s is a smooth rational curve and that $(\mathcal{O}_Q(1), l)=0$, we see that $(\mathcal{O}_Q(1), C_s)=0$ in Q_o. Recalling that $h_o(C_s)$ is a point and that every fiber of h_o is connected by the observation after Proposition 4.5, we get the desired result by Lemma 4.9 (2).

Moreover (2) follows from Lemma 4.9, while (4) follows from Proposition 4.11. The rest is trivial. q.e.d.

REMARK 4.12.1. From Proposition 4.12 (2) there is a morphism $\sigma: H_o\to \mathrm{Gr}(m, 1)$ by virtue of the universality of deformation theory.

The sublemma 4.13 yields:

COROLLARY 4.13.1. *Let $\rho_o^1: \mathscr{H}_o^1\to H_o^1$ be the canonical morphism induced by $\rho: \mathscr{H}\to H$. Then there exist two open sets U in $h(Q_o)$ and $\bar{U}$ in H_o^1 and two isomorphisms $i: U\to \bar{U}$, $j: h_o^{-1}(U)\to \rho^{-1}(\bar{U})$ such that $h_{o|h_o^{-1}(U)}i=j\rho^1_{o|(\rho_o^1)^{-1}(\bar{U})}$.*

Now let us recall the notation:

$$a: F(m, 1, 0)\longrightarrow \boldsymbol{P}^m \quad \text{and} \quad b: F(m, 1, 0)\longrightarrow \mathrm{Gr}(m, 1) \quad \text{in} \quad (4.6)\,.$$

The following is well-known:

FACT 4.13.2. For each point x in $\boldsymbol{P}^m$, we have:

(1) $ba^{-1}(x)$ is $\boldsymbol{P}^{m-1}$ in $\mathrm{Gr}(m, 1)$.

(2) $b^{-1}ba^{-1}(x)$ $(=\bar{P})$ is isomorphic to $P(\mathcal{O}_{\boldsymbol{P}^{m-1}}\oplus\mathcal{O}_{\boldsymbol{P}^{m-1}}(1))$ which is a $\boldsymbol{P}^1$-bundle on $\boldsymbol{P}^{m-1}$. Moreover $\bar{P}$ contains $a^{-1}(x)$.

(3) $\bar{P}\to \boldsymbol{P}^m$ is the blow-up of $\boldsymbol{P}^m$ at the point x.

(4) (4.α) $(b^*\mathcal{O}_{\mathrm{Gr}(m,1)}(1)\cdots b^*\mathcal{O}_{\mathrm{Gr}(m,1)}(1)\cdot\bar{P})=0$ (m-times).

(4.β) $(a^*\mathcal{O}_P(1)\cdots a^*\mathcal{O}_P(1)\cdot\bar{P})=1$ (m-times).

Now for a point x in $\boldsymbol{P}_t^m$, P_x denotes the closed subscheme $\pi^1(\rho^1)^{-1}(\rho^1)(\pi^1)^{-1}g^{-1}(x)$

in Q_t where $\rho^1=\rho_{|\mathscr{H}^1}:\mathscr{H}^1\to H^1$ and $\pi^1=\pi_{|\mathscr{H}^1}:\mathscr{H}^1\to Q$. When x runs through $\boldsymbol{P}(V)$, we get an algebraic family $\mathscr{P}=\{P_x\}_{x\in\boldsymbol{P}(V)}$ $(\subset\boldsymbol{P}(V)\times_T Q)$.

PROPOSITION 4.14. (1) *For x in $P(V_{|T-\{o\}})$, P_x is canonically isomorphic to $\bar{P}$ in* (2) *of Fact* 4.13.2.

(2) *There is an open set S in $\boldsymbol{P}^m_o$ such that for each point x in $\boldsymbol{P}^m_o$, we have the following*:

(α) $\dim P_x=m$.

(β) $(P_x\cdot g_o^*\mathcal{O}_P(1)\cdots g_o^*\mathcal{O}_P(1))=1$ *(m-times), namely, $g_{o|P_x}:P_x\to\boldsymbol{P}^m_o$ is surjective.*

(γ) *P_x contains $g^{-1}(x)$.*

(3) *The restricted map $\sigma_{|H^1_o}:H^1_o\to\mathrm{Gr}(m,1)$ (cf. Remark* 4.12.1) *is surjective.*

PROOF. (1) is trivial. As for (2), $\dim H_o=2m-2$ and $\mathscr{H}_o\to Q_o$ is surjective and generically finite, as required. (3) is obvious by (2). q.e.d.

REMARK 4.14.1. If H^1_o is irreducible, so is P_x for a general point x in $\boldsymbol{P}^m_o$.

Further detailed observation continues in the next section.

5. The deformation of $P(\Omega_{\boldsymbol{P}^m}(2))$ under some conditions, (II). This section is a continuation of the previous section. The main aim is to show (I), (II), (III) stated below, which yields Theorem 4.2 and Corollary 4.3. The condition (4.4) is maintained.

We would like to prove:

(I) Let $i:U\to\bar{U}$ be an isomorphism as in Corollary 4.13.1 and $\sigma:H^1_o\to\mathrm{Gr}(m,1)$ a surjective morphism as in Proposition 4.14.3. Then $\sigma(i(U))$ is a dense open set in $\mathrm{Gr}(m,1)$ or $r=1$. (It is treated in (5.1) and (5.2).)

(II) If the conclusion of (I) holds, then for a general line l passing through each point x in the open set S in $\boldsymbol{P}^m_o$ ($\simeq\boldsymbol{P}^m$) (see Proposition 4.14 (2)), $E_{o|l}$ is isomorphic to $\mathcal{O}\oplus\mathcal{O}(1)^{\oplus m-1}$. (It is treated in (5.3).)

(III) If the conclusion of (II) holds and m is even, then $E_o\simeq\Omega_{\boldsymbol{P}^m}(2)$. (It is treated in (5.4) and (5.5).)

Among the three assersions above, (I) is the central part for the proof of Theorem 4.2. We now make preparation for the argument of (I). We keep the notation $a:F(m,1,0)\to\boldsymbol{P}^m$ and $b:F(m,1,0)\to\mathrm{Gr}(m,1)$. Moreover l_y denotes the line in $\boldsymbol{P}^m$ corresponding to the point y in $\mathrm{Gr}(m,1)$.

(5.1) From now on until the end of this section we denote $E:=E_o$. Recalling that E is generated by global sectios, we see that for a line l on $\boldsymbol{P}^m$ $E_{|l}$ is $\oplus\,\mathcal{O}(a^l_i)$ with $a^l_i\geq 0$. Set $r(l):=\#\{i\,|\,a^l_i=0\}$. Let $r:=r(E)=\min\{r(l)\,|\,l$ is a line in $\boldsymbol{P}^m\}$. Moreover set $D:=\{y\in\mathrm{Gr}(m,1)\,|\,r(l_y)=r\}$. Then D is an open set in $\mathrm{Gr}(m,1)$. Since $\det E=\mathcal{O}_{\boldsymbol{P}^m}(m-1)$,

(5.1.1) $r=r(E_o)$ is a positive integer.

In the notation (4.6), we have a canonical homomorphism $\varphi:b^*b_*a^*\check{E}\to a^*\check{E}$. Thus we see by virtue of the base change theorem that $b_*a^*\check{E}$ is a torsion-free sheaf of rank r and φ is an injection on $b^{-1}(D)$ as a subbundle. Let $\bar{E}$ be the dual vector bundle of $b_*(a^*\check{E}_{|D})$ and let $\phi : \boldsymbol{P}(b^*\bar{E}) \to \boldsymbol{P}(E)$ be the composite morphism of the canonical immersion: $\boldsymbol{P}(b^*\bar{E}) \to \boldsymbol{P}(a^*E)$ and the canonical projection $\boldsymbol{P}(a^*E) \to \boldsymbol{P}(E)$. Letting $\bar{b}$: $\boldsymbol{P}(b^*\bar{E}) \to \boldsymbol{P}(\bar{E})$ to be the canonical projection. We infer that for a point e in $P(\bar{E})$, we get $(\mathscr{L} \cdot \phi\bar{b}^{-1}(e)) = 1$ with $\mathscr{L}$ in (4.10), which implies that $\boldsymbol{P}(\bar{E})$ is the parameter space of lines on $Q_o := \boldsymbol{P}(E)$ with respect to $\mathscr{L}$. Now recall that H is the Hilbert scheme $H^{p(t)}_{Q/T}$ and $H_o = H \times_T \operatorname{Spec} k(o)$. Thus by virtue of the universality of the Hilbert scheme H_o we have:

(5.1.2) a morphism $\theta : \boldsymbol{P}(\bar{E}) \to H_o$ which is injective by construction. Moreover $\dim \boldsymbol{P}(\bar{E}) = \dim \theta(\boldsymbol{P}(\bar{E})) = 2m - 3 + r$.

Now we study the behavior of the above line on $\boldsymbol{P}(E)$ in terms of the normal bundle.

For a point y in D, $\boldsymbol{P}(E_{|l_y})$ contains $\boldsymbol{P}(\mathcal{O}_{l_y}^{\oplus r})$ which is $l_y \times \boldsymbol{P}^{r-1}$. Let $L = l_y \times \{A\}$ in $l_y \times \boldsymbol{P}^{r-1}$ ($\subset \boldsymbol{P}(E)$) where A is a point in $\boldsymbol{P}^{r-1}$. Then the normal bundle N_{L/Q_o} appears in the exact sequence

$$0 \longrightarrow N_{L/\Pi} \longrightarrow N_{L/Q_o} \longrightarrow N_{\Pi/Q_o|L} \longrightarrow 0$$

where $\Pi = g_o^{-1}(g_o L)$. Then $N_{L/\Pi} \simeq \bigoplus_{i \geq 2} \mathcal{O}(-a_i)$ and $N_{\Pi/Q_o|L} \simeq \mathcal{O}(1)^{\oplus(m-1)}$ which yields that $h^0(N_{L/\Pi}) = r - 1$ and $h^0(N_{\Pi/Q_o|L}) = 2m - 2$. Hence $h^0(N_{L/Q_o}) \leq 2m - 3 + r$, which implies that the dimension ($= \dim_c H_o$) of H_o at a point c in $\theta(P(\bar{E}))$ is not greater than $2m - 3 + r$. On the other hand by (5.1.2) we get:

PROPOSITION 5.1.3. (1) *The Hilbert scheme H_o of lines in Q_o is smooth at each point c of $\theta(\boldsymbol{P}(\bar{E}))$ and therefore $\theta : \boldsymbol{P}(\bar{E}) \to H_o$ is an open immersion.*

(2) $\sigma(\theta(\boldsymbol{P}(\bar{E}))) = D$.

Let J be the closure of $P(\bar{E})$ *in* H_o. Then J is an irreducible component of H_o which contains $\boldsymbol{P}(\bar{E})$ as an open set and $\dim J = 2m - 3 + r$.

REMARK 5.1.4. If $r \geq 2$, then $\dim J = \dim \boldsymbol{P}(\bar{E}) \geq 2m - 1$. On the other hand $\dim H_o^1 = 2m - 2$. Hence by the construction of $\boldsymbol{P}(\bar{E})$ there exist an irrducible component H^j ($j \geq 2$) of H so that $\operatorname{supp} H^j = \operatorname{supp} J$ (Remark 4.10.1, (2)).

(5.2) Proof of (I).

(5.2.1) Assume that H_o^1 is irreducible.

In our notation, $\mathscr{H}_o^1$ is a variety and $\pi_o : \mathscr{H}_o^1 \to Q_o$ is birational by (4.12.1). Thus we get the desired result by the commutativity stated in Corollary 4.13.1 and by the surjectivity $\sigma(H_o^1) = \mathrm{Gr}(m, 1)$ in Proposition 4.14.3.

Next we assume:

(5.2.2) H_o^1 is not irreducible.

Let I be the closure of $\bar{U}$ (of Corollary 4.13.1) in H_o^1. Since $\dim H_o^1 = \dim \bar{U} = 2m - 2$, I is an irreducible component of H_o^1. Let $H_o^1 = I \cup (\bigcup_i I_i)$ be the decomposition of H_o^1 into irreducible components. Since the morphism $\sigma : H_o \to \mathrm{Gr}(m, 1)$ is surjective, we have two cases.

(α) $\sigma(I)=\mathrm{Gr}(m, 1)$.

Then we get the assertion I in the same way as in the case of (5.2.1). (However, this situation turns out not to occur.)

(β) $\sigma(I)$ is a proper closed set in $\mathrm{Gr}(m, 1)$. (We will deduce a contradiction.)

Hence as for the component I_i of H_o^1 we have:

Claim 5.2.3. $\theta(\boldsymbol{P}(\bar{E})) \cap I_i$ is not empty.

Indeed, take a point c in $(\mathrm{Gr}(m, 1)-\sigma(I)) \cap D$ (see (5.1) for D). By the property of D, $E_{|l_c}$ is $\mathcal{O}^{\oplus r} \bigoplus_{i \geq r+1} \mathcal{O}(a_i^l)$ with $a_i^l > 0$. ($r := r(E)$ as in (5.1)). On the other hand from $\sigma(H_o^1)=\mathrm{Gr}(m, 1)$ and the characterization of I_i there exist a point $\bar{c}$ in I_i and a point c in $\mathrm{Gr}(m, 1)$ so that $g(\pi_o(\rho^{-1}(\bar{c})))$ is a line l_c on $\boldsymbol{P}^m$ and that the line $\pi_o(\rho^{-1}(\bar{c}))$ on Q_o corresponds to a trivial quotient line bundle $\mathcal{O}_{l_c}$ of $E_{|l_c}$ by Lemma 4.9 (3). Moreover a section $\boldsymbol{P}(\mathcal{O}_{l_c})$ corresponds to an element in $\boldsymbol{P}(\bar{E}_{|b^{-1}(c)})$. Thus we get the claim.

We have come to the final stage of the proof of (I). Assume $r \geq 2$. Then we see by virtue of the universality of the Hilbert scheme (Remark 4.10.1, (1)) and by (5.1.4) that H_o has, as a component, the closure J of $\theta(\boldsymbol{P}(\bar{E}))$ with the embedded closed subscheme $H_o \cap H^1 \cap J\,(\neq \emptyset)$ scheme-theoretically. Thus we infer from (5.2.3) that H_o is not smooth at each point c in $\theta(\boldsymbol{P}(\bar{E})) \cap I_i$, a contradiction to (5.1.3).

Thus we complete the proof of (I).

(5.3) Proof of (II). If $r=1$ in (5.1.1), there is nothing to show. Thus assuming $r \geq 2$, we get a contradiction. We maintain the notation (5.1).

The assumption of (II) yields:

(5.3.1) the morphism $\phi : \boldsymbol{P}(b^*\bar{E}) \to \boldsymbol{P}(E)$ is dominant.

We study the property of a general fiber of ϕ.

In the notation (5.1.2) we fix a general point y in D and choose a curve L as in (5.1.2). Since $\dim \boldsymbol{P}(b^*\bar{E})=2m+r-2$ and $\dim \boldsymbol{P}(E)=2m-1$, we see that:

(5.3.2) for each point c in L, we have $\dim \phi^{-1}(c) \geq r-1$. Namely, for such a general curve L as above, there exists a closed subscheme Z in J and a family of smooth rational curves $\{L_z\}_{z \in Z}$ in $\boldsymbol{P}(E)$, where for a general point z in Z we denote $L_z := l_{\bar{y}} \times \{A'\}$ with $\bar{y} \in D$, $A' \in \boldsymbol{P}^{r-1}$. Moreover $\dim Z=r-1$ and each curve L_z passes through the point c. Therefore $\dim \bigcup_{z \in Z} L_z \geq r$. Recalling that L and each L_z go to a point via the morphism h_o we get:

(5.3.3) The subscheme $\bigcup_{z \in Z} L_z$ in $\boldsymbol{P}(E)$ collapses to a point via h_o. On the other hand since $\dim P(E)=\dim h_o(\boldsymbol{P}(E))+1$, we get $r=1$.

Hence we have shown (II).

(5.4) Proof of (III). We first show:

PROPOSITION 5.4.1. *Let F be a rank m vector bundle on $\boldsymbol{P}^m$ satisfying the following*:

(1) $c_m(F(-1)) \neq 0$.

(2) $\check{F}(1)$ *is generated by global sections.*

(3) $F_{|l} \simeq \mathcal{O}^{\oplus(m-1)} \oplus \mathcal{O}(1)$ *for a general line l in $\boldsymbol{P}^m$.*

Then $\dim H^0(\boldsymbol{P}^m, F) \leq m+1$. *Moreover if* $\dim H^0(\boldsymbol{P}^m, F)=m+1$, *then F is generically* *generated by global sections.*

PROOF. If $h^0(F)=0$, there is nothing to prove. Next let s be a non-zero section of F, $\check{s}: \check{F}\to \mathcal{O}$ the homomorphism induced by the non-zero section s of F and I the ideal sheaf defining of $(s)_o$. Then the homomorphism $\check{s}$ induces a surjective homomorphism $\check{F}\to I\to 0$, and therefore $\check{F}\to I/I^2\to 0$. By (2), we see that $I\otimes\mathcal{O}(1)$ is generated by global sections and there exists an injective homomorphism $0\to I\otimes\mathcal{O}(1)\to\mathcal{O}_P(1)$ induced by the canonical injective homomorphism: $0\to I\to\mathcal{O}_P$. Hence the following is shown easily:

FACT 5.4.1.1. Assume that the zero locus $(s)_o$ of a non-zero section s is of dimension $t\geq 1$. Then $(s)_o$ is a linear space in $\boldsymbol{P}^m$ and $I/I^2\simeq\mathcal{O}(-1)^{\oplus(m-t)}$.

Now assume that $\dim H^0(\boldsymbol{P}^m, F)\geq m+2$. Then letting J to be the ideal sheaf of l in (3), we have an exact sequence:

(5.4.1.1) $$0\longrightarrow J\longrightarrow\mathcal{O}_{\boldsymbol{P}^m}\longrightarrow\mathcal{O}_l\longrightarrow 0\,.$$

Tensoring F, we get

$$0\longrightarrow J\otimes F\longrightarrow F\longrightarrow F_{|l}\longrightarrow 0\,.$$

Then since $\dim H^0(P^1, \mathcal{O}^{\oplus(m-1)}\oplus\mathcal{O}(1))=m+1$, there exists a non-zero section s in F so that $s_{|l}=0$. Moreover $(s)_o$ is a t-dimensional linear subspace in $\boldsymbol{P}^m$ containing l by the above fact. If $\operatorname{codim}(s)_o=m-t\geq 2$, then by Fact 5.4.1.1 $\check{F}_{|l}$ has a quotient vector bundle $\mathcal{O}(-1)^{\oplus(m-t)}$, a contradiction to (3). Thus we infer that $(s)_o$ is a hyperplane in $\boldsymbol{P}^m$. Hence we get a nowhere-vanishing section $\bar{s}$ in $F(-1)$, which implies that $c_m(F(-1))=0$, a contradiction. Hence we have shown the former part. Next we consider the latter part. When F is not generically generated by global sections, neither is $F_{|l}$ for each line l. Since $h^0(l, \mathcal{O}^{\oplus(m-1)}\oplus\mathcal{O}(1))=m+1$, by the exact sequence (5.4.1.2) we have a non-zero section s in F such that $s_{|l}=0$ for a generic line on $\boldsymbol{P}^m$. In the same way as in the former part, we get a contradiction. q.e.d.

We note that

$$c_m(T_{\boldsymbol{P}^m}(-2))=\begin{cases} 0 & \text{if } m \text{ is odd} \\ -1 & \text{if } m \text{ is even}\,. \end{cases}$$

Thus the above proposition immediately yields:

COROLLARY 5.4.2. *Let F be a rank m vector bundle on $\boldsymbol{P}^m$ satisfying the following*:

(1′) *The Chern polynomial of F is equal to that of $T_{\boldsymbol{P}^m}(-1)$.*

(2) *$\check{F}(1)$ is generated by global sections.*

(3) *$F_{|l}\simeq\mathcal{O}^{\oplus(m-1)}\oplus\mathcal{O}(1)$ for a general line l in $\boldsymbol{P}^m$.*

(4′) $\dim H^0(\boldsymbol{P}^m, F)\geq m+1$.

If m is even, then $F\simeq T_{\boldsymbol{P}^m}(-1)$.

PROOF. (1′) and the above remark yield (1) in Proposition 4.13. At the same time by (4′) we see that F is generically generated by global sections. Moreover (1′) yields $c_m(F)=1$. Therefore combining (3), Lemma 2.4 (1.0) and Theorem 1, we infer that F is isomorphic to either $T_{\mathbf{P}^m}(-1)$ or $\mathcal{O}^{\oplus(m-1)}\oplus\mathcal{O}(1)$. Moreover in view of the assumption (1′) we get the desired result. q.e.d.

Thus we have proved (III).

As a consequence we can prove Theorem 4.2.

(5.5) Finally we show Corollary 4.3. In 4.1, take $T:=Y$, $W:=Z$ and $V:=\check{\mathcal{N}}\otimes p^*(\mathcal{P}_{\mathbf{P}^N}(1)_{|X})$ on Z. Then for each point y in Y, $Y_y:=V_{|q^{-1}(y)}$ is generated by global sections by (2.3). Moreover by assumption, V_y for a general point y in Y is isomorphic to $\Omega_{\mathbf{P}^m}(2)$. Thus Theorem 4.2 immediately yields Corollary 4.3. q.e.d.

6. Smooth variety with $N_{P/X}\simeq T_{\mathbf{P}^m}(-1)$. In this section we work over the field of characteristic zero.

(6.0) Let X be a $2m$-dimensional smooth projective variety containing an m-plane P with the normal bundle $N_{P/X}\simeq T_{\mathbf{P}^m}(-1)$.

Then we have the diagram (2.8) from Remark 2.11. Moreover by Proposition 2.12 and Corollary 4.3 if m is even, then we have $N_{P_y/X}\simeq T_{\mathbf{P}^m}(-1)$ for each point y in Y. Thus our aim in this section is to show the following:

THEOREM 6. *Let X be as in* (6.0). *If m is odd, we suppose that $N_{P_y/X}\simeq T_{\mathbf{P}^m}(-1)$ for each point y in Y. Then in characteristic zero X is isomorphic to* $\mathrm{Gr}(m+1, 1)$.

We begin with the following:

PROPOSITION 6.1. *Let the notation be as in* (2.8). *Then Y (resp. Z) is a smooth projective variety of dimension $m+1$ (resp. $2m+1$). Moreover $p: Z\to X$ is a smooth morphism with one-dimensional fiber.*

PROOF. Recall the observation in (2.9.1) and (2.9.2). Then since $H^1(\mathbf{P}^m, T_{\mathbf{P}^m}(-1))=0$ and $\dim H^0(\mathbf{P}^m, T_{\mathbf{P}^m}(-1))=m+1$, Y is an $(m+1)$-dimensional smooth projective variety. Moreover since $T_{\mathbf{P}^m}(-1)$ is generated by global sections, the homomorphism $p_*: T_Z\to p^*T_X$ induced by p is surjective. Thus we get the latter part. q.e.d.

For a point y in Y, P_y denotes the m-plane $pq^{-1}(y)$ in X.

REMARK 6.2. Since $N_{P/X}=T_{\mathbf{P}^m}(-1)$, we have $c_m(N)=1$. Thus for any y, y' in Y the intersection number of P_y and $P_{y'}$ is 1 and $P_y\cap P_{y'}$ is not empty.

PROPOSITION 6.3. *$p: Z\to X$ is a $\mathbf{P}^1$-bundle.*

PROOF. Assuming that the fiber of p is of genus ≥ 1, we get a contradiction as we now show.

First the following facts are well-known (see [BPV, §§3 and 5]):

(I) Let S be a compact complex surface, B a projective curve and $f: S\to B$ a surjective holomorphic map without singular fibers. Assume that B is rational or elliptic.

Then f is locally trivial.

(II) Any elliptic fiber bundle over $\boldsymbol{P}^1$ is a product or a Hopf surface.

Let us take a line l on X and study the smooth morphism $p_l : p^{-1}(l) \to l$. Letting C to be a fiber of p_l, we have:

Claim 1. $p^{-1}(l)$ is a product of l and C and p_l is the first projection.

Indeed, the above (I) implies that p_l is locally trivial. First assume that C is an elliptic curve. Since Z is projective, so is $p^{-1}(l)$. Noting that a Hopf surface is not Kähler, we see that $p^{-1}(l) \simeq l \times C$ by (II). Next assume that C is of genus ≥ 2. Then the period map $f: l \to D$ to the classifying space D is constant, hence by Torelli's Theorem f is a fiber bundle. Moreover since the automorphism of C is a finite group, $p^{-1}(l)$ is isomorphic to $l \times C$.

On the other hand Claim 1 yields:

Claim 2. $qp^{-1}(l) = qp^{-1}(A)$ for each point A in l and it is a point or a curve.

Indeed, let y be a point in Y, l a line in an m-plane P_y and $C_1 = p^{-1}(l) \cap q^{-1}(y)$. Then since $p^{-1}(l) \simeq C \times l$ by Claim 1, C_1 is a fiber of $p^{-1}(l) \to l$ or an ample divisor in $p^{-1}(l)$.

Now take another line $\bar{l}$ in P_y such that $l \cap \bar{l}$ is a point A. Then Claim 2 implies that $qp^{-1}(\bar{l}) = qp^{-1}(A)$ and therefore is equal to $qp^{-1}(P_y)$. On the other hand for any two points y, y' in Y, P_y intersects P_y by Remark 6.2. Consequently, we infer that $q(Z)$ is a point or a curve, a contradiction. Thus we get $C = \boldsymbol{P}^1$. q.e.d.

Next we show:

(6.4) $$Y \simeq \boldsymbol{P}^{m+1}.$$

Since p is a smooth morphism with the fiber $\boldsymbol{P}^1$, $p^{-1}(P_y)$ is of dimension $m+1$. For a general point y in Y let $P = P_y$ and let us consider the morphism $\bar{q} : p^{-1}(P) \to Y$ induced by the morphism q. Then we see that $\bar{q}$ is surjective. Indeed, if the image of $p^{-1}(P)$ via the morphism q is a proper set in Y, P does not intersect $P_{y'}$ for a general point y' in Y. This contradicts Remark 6.2. At the same time $\bar{q}$ is birational by (6.2). Since $p^{-1}(P)$ is a $\boldsymbol{P}^1$-bundle over P containing a section $q^{-1}(y)$, it can be written as $\boldsymbol{P}(E)$ where E is a rank 2 bundle on P. Then by $m \geq 2$, we infer that $E \simeq \mathcal{O}_P \oplus \mathcal{O}_P(a)$ and that $a \neq 0$ by the surjectivity of the morphism $\bar{q}$. Moreover by taking $\boldsymbol{P}(\mathcal{O}_P)$ as the section of $\boldsymbol{P}(E)$ which collapses to the point y via q, we get $a > 0$. Letting $\bar{Y}$ to be the normal cone obtained by the blowing-down of $p^{-1}(P)$ along the section $P(\mathcal{O})$. Then $\operatorname{Pic} p^{-1}(P) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$ and therefore $\operatorname{Pic} \bar{Y} \simeq Z$. Thus we get a canonical morphism $h : \bar{Y} \to Y$ which is finite and birational. By Zariski's main theorem h is an isomorphism. Let v be the vertex of the cone Y. Then we get

(6.4.1) $$a = 1.$$

Indeed, let $(R, \mathfrak{m})$ be the local ring of Y at the point v with $R = \mathcal{O}_{Y,y}$ and $\mathfrak{m}$ the maximal ideal of R. Then the Zariski tangent space of Y at v is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$. On the other hand we infer that $\dim \mathfrak{m}/\mathfrak{m}^2 \geq h^0(P, \mathcal{O}_P(a))$ and therefore $a = 1$. Hence $Y \simeq \boldsymbol{P}^{m+1}$.

Summarizing the above argument, we get:

PROPOSITION 6.5. *We have $Y=\boldsymbol{P}^{m+1}$. For a general point y in Y, we have $p^{-1}(P_y)\simeq \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^m}\oplus\mathcal{O}_{\boldsymbol{P}^m}(1))$, and $\bar{q}: p^{-1}(P_y)\to Y(\simeq \boldsymbol{P}^{m+1})$ is the blowing-up of Y at the point y. Hence for each point x in X, we have $qp^{-1}(x)=\boldsymbol{P}^1$, hence in particular, a line in Y.*

We come to the final stage of the proof of Theorem 6.0.

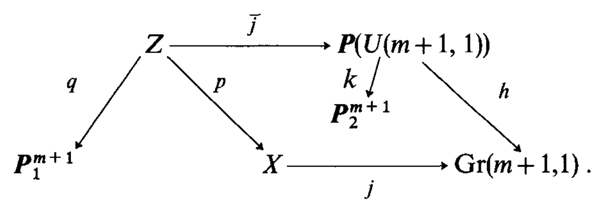
(6.6) Recalling that Z is a closed variety of $X\times \boldsymbol{P}^{m+1}$ by the construction (6.5) we see that X is the parameter space of lines in $\boldsymbol{P}^{m+1}$ and Z is the universal space of X. By the universality of Hilbert scheme we have a morphism $j: X\to \mathrm{Gr}(m+1, 1)$ which induces an X-morphism $\bar{j}^*: Z\to F(m+1, 1, 0)$.

(6.7) Claim. $\mathrm{Pic}\, Z\simeq \boldsymbol{Z}\oplus\boldsymbol{Z}$ and $\mathrm{Pic}\, X\simeq \boldsymbol{Z}\mathcal{O}_X(1)$ where $\mathcal{O}_X(1)$ is the ample generator. Indeed, $Z=\boldsymbol{P}(U(N, m)_{|\boldsymbol{P}^{m+1}})$ where $U(N, m)$ is the rank $m+1$ universal quotient bundle on $G(N, m)$, which yields the former. On the other hand $p: Z\to X$ is a $\boldsymbol{P}^1$-bundle, which yields the latter part.

Thus j is finite surjective.

We have the following diagram:

(6.8)



Let $M:=\bar{j}^*k^*\mathcal{O}_{\boldsymbol{P}_2^{m+1}}(1)$. Then we note that

(#) M is not ample but spanned.

(6.9) Claim. $M_{|q^{-1}(y)}$ is trivial for each point y in $\boldsymbol{P}_1^{m+1}$. Thus $M=q^*\mathcal{O}_{\boldsymbol{P}^{m+1}}(1)$. Consequently the morphism $\bar{j}k$ induced by M collapses each fiber of q.

Indeed, we see that the restriction of $k^*\mathcal{O}_{\boldsymbol{P}_2^{m+1}}(1)$ to the fiber of k is $\mathcal{O}(1)$ and therefore $M_{|p^{-1}(x)}\simeq\mathcal{O}_{\boldsymbol{P}^1}(1)$. Moreover $q^*\mathcal{O}_{\boldsymbol{P}^{m+1}}(1)_{|p^{-1}(x)}\simeq\mathcal{O}_{\boldsymbol{P}^1}(1)$ by Proposition 6.5. Thus we have an equality:

(##) $M:=q^*\mathcal{O}_{\boldsymbol{P}^{m+1}}(1)\otimes p^*\mathcal{O}_X(b)$

with an integer b by virtue of the base change theorem. Restricting (##) to the fiber $q^{-1}(y)$, we infer that $b\geq 0$ since $M_{|q^{-1}(y)}$ is generated by global sections. Assume that b is positive. Since Z is a subvariety of $\boldsymbol{P}^{m+1}\times X$, M is an ample line bundle, a contradiction to (#). Therefore we get Claim 6.9.

By Claim 6.9 the morphism $\bar{j}$ induces a morphism $j': \boldsymbol{P}_1^{m+1}\to\boldsymbol{P}_2^{m+1}$ with $qj'=\bar{j}k$. Thus we have $j'^*\mathcal{O}_{\boldsymbol{P}_2^{m+1}}(1)\simeq\mathcal{O}_{\boldsymbol{P}_1^{m+1}}(1)$, which means that j' is an isomorphism and $\bar{j}$ is a $\boldsymbol{P}^m$-bundle map with respect to the two projections q and k. On the other hand noting that there exist canonical isomorphism: $F(m+1, 1, 0)\simeq\boldsymbol{P}(U(m+1, 1))\simeq\boldsymbol{P}(\Omega_{\boldsymbol{P}^{m+1}}(2))$, we infer that $Z\simeq\boldsymbol{P}(j'^*\Omega_{\boldsymbol{P}^{m+1}}(2))$. Moreover since $\Omega_{\boldsymbol{P}^{m+1}}(2)$ is a homogeneous vector bundle on $\boldsymbol{P}^{m+1}$, we infer that $j'^*\Omega_{\boldsymbol{P}^{m+1}}(2)\simeq\Omega_{\boldsymbol{P}^{m+1}}(2)$ and therefore $\bar{j}$ is an isomorphism. On the other hand since $\bar{j}h=pj$, j is an isomorphism and X is $\mathrm{Gr}(m+1, 1)$ as desired.

Thus we complete the proof of Theorem 6.

Combining Theorem 2.10, Theorem 3.1 and Theorem 6.0, we get the main theorem.

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